Solitons and Magnetic Monopoles in Unified Gauge Theories

So far in this course we have discussed only the properties of spontaneously broken gauge theories at the level of perturbation theory. However, these theories often contain additional structure that is essential non-perturbatively. In this lecture I will introduce this structure with some examples in 1 and 3 dimensions. [A beautiful reference: E.B. Bogomolny, Sov.J.Nucl.Phys. 24, 449 (1976).]

Begin with the simple model of $\phi^4$ theory with a real scalar field in $1+1$ dimensions

$$L = \frac{1}{2}(\partial \phi)^2 - \frac{1}{2}m^2 \phi^2 + \frac{\lambda}{4!} \phi^4$$

The potential of this theory has the form:

The vacuum states are

$$\langle \phi \rangle = \pm \nu = \pm \sqrt{\frac{m^2}{\lambda}}$$

There are two degenerate vacuum states. Note that $\nu \sim \frac{1}{\sqrt{\lambda}}$, so $\nu \gg 1$ at the vacua are well separated for small $\lambda$. 
Now consider a classical field configuration of the form:

\[ \phi \]

\[ \begin{array}{c}
\text{This is a domain wall: For } x < x_0, \phi \text{ is in the vacuum } \\
\langle \phi \rangle = -\nu. \text{ For } x > x_0, \phi \text{ is in the vacuum } \langle \phi \rangle = +\nu.
\end{array} \]

It is not difficult to find the classical field configuration explicitly. The equation of motion for \( \phi \) is

\[ -\partial^2 \phi + \mu^2 \phi - 2 \phi^3 = 0 \]

Let \( \phi = \nu f(z) \) where \( z = m \phi(x-x_0) = \sqrt{2} \mu (x-x_0) \) (time-independent). Then

\[ 2 \frac{d^2 f}{dz^2} f + f - f^3 = 0 \]

The solution

\[ f(z) = \tanh \frac{z}{\sqrt{2}} \]

satisfies this equation with the correct boundary conditions

\[ \phi(x) \to -\nu \text{ as } x \to -\infty \text{ and } \phi(x) \to +\nu \text{ as } x \to +\infty. \]

Explicitly,

\[ \phi = \nu \tanh \left[ \frac{m \phi(x-x_0)}{2} \right] \]
there is an arbitrary center $x_0$. Any term $x_0$, the deviation from the vacuum falls off expontially

$$x > x_0 \quad \phi \sim \nu = \Theta(e^{-m\phi(x-x_0)})$$

like a QFT correlation function. The energy of the confinment is

$$E > \int dx \left[ \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} \mu^2 \phi^2 + \frac{3}{4} \phi^4 \right]$$

$$= \int dx \left( \frac{2\nu^2}{4} \right) \left( \frac{2}{\cosh^2 m\phi x/2} - 1 \right)$$

we say of the broken symmetry vacuum

$$= \frac{2\nu^2}{3} \mu \nu^2 - E_0$$

the domain wall has localized energy $\sim \frac{1}{\sqrt{a}}$. This is very large as $a \rightarrow 0$. Such a massive, localized field configuration is a nonlinear field theory called a soliton.

This state should be interpreted as a localized massless particle with mass given by the energy above. The perturbation corrections to the energy are small, shown by one power of $\nu$. So the result is unambiguous.

There are states in the field theory that contain many of these particles:
The massive particles carry a $\mathbb{Z}_2$ charge number; a pair of these particles can annihilate to purely perturbative excitations. An isolated particle, however, is absolutely stable. We say that the classical field configuration is "topologically stable": It cannot decay to small fluctuations about a perturbative vacuum consistent with the topology of its boundary conditions.

I will now show that similar topologically stable massive particles can be found in spontaneously broken gauge theories. The construction is due to 't Hooft and Polyakov.

Consider the Georgi-Glashow model: an $SO(3)$ gauge theory spontaneously broken by a Higgs field in the vector representation of $SO(3)$.

$$\langle \phi^a \rangle = \nu (\hat{n})^a \quad a = 1, 2, 3$$

Two of the three vector bosons of $SO(3)$ receive mass

$$m_w = g\nu$$

The third vector boson $A^a_\mu \hat{n}^a$ remains massless; this
is the photon of a unified theory of weak and electromagnetic interactions.

Now consider an alternative Higgs field configuration of non-trivial topology:

\[ \phi^a(x) = \alpha f(r) \hat{r}^a \]

This is a non-singular configuration if \( f(0) \rightarrow 0 \)
\[ \alpha \rightarrow 0. \]

The boundary condition for the gap field at large distances can be found by analyzing the Higgs kinetic energy term: \( \frac{1}{2} (D_i \phi)^2 \).

\[ D_i \phi^a = \partial_i \phi^a + g \epsilon^{abc} A_i^b \phi^c \]

\[ \partial_i \hat{r}^a = \partial_i \frac{r^a}{(r^2)^{1/2}} = \frac{\partial_i \hat{r}^a}{r} = \epsilon^{abc} \epsilon_{ibk} \hat{r}^k \hat{r}^c \]

If \( D_i \phi^a \sim \frac{1}{r} \), we obtain \( \int d^3x (D_i \phi)^2 \sim \int d^3x \frac{1}{r^2} = \infty \).

So the \( D_i \phi^a \) term must cancel this leading \( \frac{1}{r} \) term. We accomplish this if

\[ A_i^b = - \frac{\epsilon_{ibk} \hat{r}^k}{gr} \chi(r), \quad \chi(r) \rightarrow 1 \text{ as } r \rightarrow \infty. \]
For a non-singular solution $h(r) \sim r^2$ as $r \to 0$

This ansatz

$$\phi^q(t) = v f(r) \mathcal{A}^q \quad \mathcal{A}^q = \frac{e^{ibk}}{gr} \hat{r}^k h(r)$$

with $f(t) \to 1$, $h(r) \to 1$ as $r \to \infty$, leads to a topologically stable solution.

It is interesting to compute the Yang-Mills field tensor for this configuration:

$$F_{0i}^q = 0 \quad \text{since} \quad \frac{d}{dt} = 0 \quad A_0 = 0$$

but $F_{ij}^q$ is nonzero:

$$F_{ij}^q = \partial_i A^q_j - \partial_j A^q_i + g \varepsilon^{abc} A^b_i A^c_j$$

$$= \partial_i \left[ -\frac{e^{jal}}{gr} \hat{r}^l h \right] - (iy) + \frac{e^{abc}}{gr^2} e^{ibl} \hat{r}^l e^{jcm} \hat{r}^m h^2$$

$$= \left( -\frac{e^{jal}}{g} \frac{g^{il} \hat{r}^i \hat{r}^l}{r^2} h - \frac{\hat{r}^i e^{jal} \hat{r}^l}{gr} h' \right) - (i \leftrightarrow j)$$

$$+ \frac{1}{gr^2} (\delta^{ia} \delta^{bj} - \delta^{ib} \delta^{aj}) e^{jcm} \hat{r}^l \hat{r}^m h^2$$

$$= \left( -\frac{(e^{j} - 2 \hat{r}^i e^{jal} \hat{r}^l)}{gr^2} h - \frac{\hat{r}^i e^{j} e^{jal} \hat{r}^l}{gr} h' \right) - (i \leftrightarrow j)$$

$$+ \frac{1}{gr^2} \varepsilon^{ilm} \hat{r}^a \hat{r}^m h^2$$
This is a little hard to parse. It will be easier if we construct

\[ B^k_a = -\frac{i}{e} \epsilon^{ijk} F_{ij}^a \]

the YM magnetic-field

\[ B^k_a = \left[ \frac{\delta^k_a}{gr^2} h + (\delta^k_a \delta^i_j - \delta^a_i \delta^k_j) \frac{h^i \hat{r}^k}{gr^2} + \frac{1}{2} (\delta^k_a \delta^i_j - \delta^a_i \delta^k_j) \frac{\hat{r}^i \hat{r}^j}{gr^2} \right] \]

\[ \times 2 \]

\[ - \frac{1}{gr^2} \delta^k_m \hat{r}^a \hat{f}^m h^2 \]

\[ = \frac{2}{gr^2} \delta^k_a h - \frac{2}{gr^2} \delta^k_a h + \frac{2 \hat{r}^a \hat{r}^k}{gr^2} h \]

\[ + \frac{\delta^k_a - \hat{r}^a \hat{r}^k}{gr^2} h' - \frac{\hat{r}^a \hat{r}^k}{gr^2} h^2 \]

so

\[ B^k_a = \frac{\hat{r}^a \hat{r}^k}{gr^2} (2h - h^2) + \left( \frac{\delta^k_a - \hat{r}^a \hat{r}^k}{gr^2} \right) h' \]

Assuming that \( h \to 1 \) very rapidly and \( h' \to 0 \) as \( r \to \infty \)

\[ B^k_a \to \frac{\hat{r}^a \hat{r}^k}{gr^2} \]

The physical electromagnetic \( \tilde{B} \) field is the field along the direction \( \hat{r}^a \hat{r}^k \):

\[ B^k = \frac{\phi^a}{r} B^k_a \to \frac{\hat{r}^k}{gr^2} \]
This is a radical \( A \) field: a magnetic monopole!

An electric monopole is then also one

\[
E^k = \frac{e}{4\pi r^2} \delta^k_0 \quad \text{where} \quad \int_{\text{sph}} \delta^k_0 \cdot E = e \delta^k_0
\]

For a unit charge \( e = g \). The soliton solution here

\[
\int_{\text{sph}} B = g M \quad \text{where} \quad g M = \frac{4\pi}{3} \frac{e^2}{g}
\]

Note that the Y-M-Maxwell equations are satisfied everywhere; in particular

\[
\nabla \cdot B = 0
\]

But because \( \phi^n \) changes direction on the sphere at \( \infty \),

\[
\nabla \cdot B = \frac{\phi^n}{\nabla \phi^n}
\]

can have a nonzero integral.

It is possible to solve explicitly for \( \phi(x) \) and \( h(x) \) in a certain limit. The energy of the field configuration is

\[
E = \int_{\text{sph}} \left[ \frac{1}{2} (B^k)^2 + \frac{1}{2} (\nabla \phi^n)^2 + V(\phi^n) \right]
\]

assuming that the configuration is time-independent: \( \dot{\phi} = 0 \), \( \dot{\phi}^n = 0 \).

We can write \( V \) as:

\[
V(\phi^n) = \frac{\lambda}{4} \left[ (\phi^n)^2 - \nu^2 \right]^2 + \text{const.}
\]
Now I would like to consider the case $a \rightarrow 0$ (the
"Bogomol'nyi-Prasad-Sommerfield (BPS) limit").
In this case, we can rewrite $E$ as

$$E = \int d^3x \ \frac{1}{2} (D_i \phi^a - B_i^a)^2 + (D_i \phi^a) B_i^a$$

Since $\partial_i (\phi^a B_i^a) = (D_i \phi^a) B_i^a + \phi^a (D_i B_i^a)$

$$= 0 \text{ by } e^{\mu \lambda \sigma} F_{\mu \lambda} = 0$$

the second term is a total divergence.

So if

$$D_i \phi^a = B_i^a \quad (*)$$

then

$$E = \int d^3x \ \partial_i (\phi^a B_i^a) = \int d^3x \ \phi^a B_i^a$$

$$= \nu \cdot \phi_m$$

$(*)$ is a set of 1st-order differential equations for $\phi$ and $h$!

$$D_i \phi^a = \partial_i (\nu^a f(r)) + g e^{abc} (-\frac{e^{ibk}}{\partial r} \hat{r}^k \cdot h)(\nu^c f)$$

$$= \nu \hat{r} \cdot \frac{\delta^a \hat{r}^i \phi^a}{r} + \nu \hat{r}^a \hat{r}^i f'$$

$$- \left(\frac{\delta^a \hat{r}^i \phi^a}{r}\right) \nu \hat{r} \cdot f$$

$$= \hat{r} \cdot f \nu f' + \frac{\delta^a \hat{r}^i \phi^a}{r} \nu (1-h) f(r)$$
Equal this to $\alpha \theta \tau$ we find

$$g' = \frac{2h - h^2}{gr^2} \quad h' = \frac{v(1-h)g}{r}$$

We can scale out $go$ by with $z = gr = mw r$

$$\frac{d}{dz}g = \frac{h(2-h)}{z^2} \quad \frac{d}{dz}h = (1-h)g$$

The solution to these equations is

$$g(z) = \frac{\cosh z}{\sinh^2 z} - \frac{1}{z} \quad h(z) = 1 - \frac{z}{\sinh z}$$

Notice that $g'(z) \sim 2ze^{-z} \to 0$ as $z \to \infty$ as promised. In fact, all of the vanishing parts of the solution die of exponentially outside of the core of the monopole. The decay is $e^{-mwz}$

similar to that for the domain wall. Outside the core, we have a pure magnetic monopole.

The mass of the monopole is

$$E = v g m = \frac{4\pi s}{g} = \frac{4\pi}{g^2} mw$$

This mass is $\frac{1}{2} \times$ mass of small os oscillating again, as we found for the domain wall.
It is worth discussing a bit the quantization of the magnetic charge. Namely, it seems that, in a pure Abelian geometry with $\overrightarrow{B} = \nabla \times \overrightarrow{A}$, $\nabla \cdot \overrightarrow{B} = 0$ strictly. However, Dirac pointed out that it is possible to have properly quantized magnetic charges. The trick is to treat space the way you treat a curved manifold: Cover it with coordinate patches in which $\overrightarrow{A}$ is nonsingular; relate $\overrightarrow{A}$ in these patches by say transformations in the overlap region.

In particular, for a magnetic monopole of strength $gM$:

$$\overrightarrow{B} = \frac{\vec{r}}{4\pi r^2} \cdot gM$$

The space field

$$\overrightarrow{A}_+ = \frac{gM}{4\pi} \frac{1 - \cos \theta}{r \sin \theta} \hat{\phi}$$

yields $\nabla \times \overrightarrow{A}_+ = \overrightarrow{B}$ and is nonsingular for $\cos \theta > -\alpha > -1$, and

$$\overrightarrow{A}_- = -\frac{gM}{4\pi} \frac{1 + \cos \theta}{r \sin \theta} \hat{\phi}$$

yields $\nabla \times \overrightarrow{A}_- = \overrightarrow{B}$ and is nonsingular for $1 > \alpha > \cos \theta$.

In the overlap region

$$-\alpha < \cos \theta < \alpha$$

both $\overrightarrow{A}_+$ and $\overrightarrow{A}_-$ are well-defined and

$$\overrightarrow{A}_+ - \overrightarrow{A}_- = \frac{gM}{4\pi} \frac{2}{r \sin \theta} \hat{\phi} = \nabla \left( \frac{gM}{2\pi} \hat{\phi} \right)$$
Locally, this is a gauge transformation. However, globally, in the overlap region, this gauge transformation might not be well defined. Let $\Psi(x)$ be a field changed under the Abelian gauge by the electric charge $qE$. Then a gauge transformation transforms

$$\Psi(x) \rightarrow e^{i q E \cdot \beta(x)} \Psi(x)$$
$$A \rightarrow A + \nabla \beta(x)$$

so we can transform away the difference between $A_+$ and $A_-$.

So, for every such $\Psi(x)$, the transformation

$$\Psi(x) \rightarrow e^{i \frac{qE_\text{mag}}{2\pi} \cdot \Phi} \Psi(x)$$

is well-defined. This requires

$$\frac{qE_\text{mag}}{2\pi} = \pi n$$

for every possible value of $qE$ in the theory. Dirac said, if there is a magnetic monopole, electric charge is quantized; but we could also start with a theory with quantized electric charge — such as the Georgi-Glashow model — and derive the quantization of magnetic charge. We find in this case

$$q_M = 2\pi n \frac{1}{q_{E0}}$$

where $q_{E0}$ is the minimum electric charge in the theory.
For the Georgi-Glashow model, this is found by copiously field in the spinor (2) representation of $SU(2)$ to the vector fields; that gives

$$q_F = \frac{g}{2}, \quad q_M = 2\pi n \cdot \frac{2}{g}$$

Indeed, the 4'khoft-Polyakov monopole has $q_M = \frac{4\pi}{g}$.

The 4'khoft-Polyakov solution can be generalised to a soliton with both magnetic and electric charge (the "Julia-Zee dyon"). To do this, consider a field-dependent field configuration with

$$\Phi(x), \quad A_i(x), \quad A_0(x) \neq 0.$$ 

Despite the fact that we have no time dependence,

$$D_0 \Phi = \partial_0 \Phi + g \epsilon^{abc} A_0 \Phi^c = g \epsilon^{abc} A_0 \Phi^c$$

$$E_i^a = -\partial_i A_0^a - \epsilon_i A_i^a - g \epsilon^{abc} A_i A_0^c$$

$$= -\nabla_i A_0^a$$

may be nonzero. The energy of the configuration is

$$E = \int d^3x \left[ \frac{1}{2} (E_i^a)^2 + \frac{1}{2} (B_i^a)^2 + \frac{1}{4} (D_0 \Phi)^2 + \frac{1}{4} (\nabla \Phi)^2 \right]$$
Let's go to the BPS limit $\beta \to 0 \quad v(\phi) \to 0$.

Then we can rewrite $E$ as

$$E = \int d^2 \phi \sum \left[ \frac{1}{2} (D_0 \Phi_a)^2 + \frac{1}{2} (E^{ia} - C D_0 \Phi_a)^2 + \frac{1}{2} (B^{ia} - (1-C)^2 D_0 \Phi_a)^2 \right]$$

$$+ C E^{ia} D_0 \Phi_a + (1-C)^2 B^{ia} D_0 \Phi_a$$

for an as-yet-arbitrary constant $C$. The last two terms are fine surface terms:

$$\int d^2 \phi \sum (1-C)^2 B^{ia} D_0 \Phi_a = \int d^2 \phi \sum \bar{E}^{ia} \Phi_a (1-C)^2 - \int d^2 \phi \sum (1-C)^2 (\bar{D}_0 \bar{\Phi}_a)^2$$

$$= (1-C)^2 \text{unqm}$$

$$\int d^2 \phi \sum C E^{ia} D_0 \Phi_a = \int d^2 \phi \sum \bar{E}^{ia} \Phi_a C - \int d^2 \phi \sum C (D_0 \Phi_a) \Phi_a$$

$$= C \text{unqm}$$

since $\Phi_a D_0 E^{ia} = \Phi_a^2 \sim \Phi_a \epsilon^{abc} \phi^b D_0 \phi^c = 0$. Then we can minimize $E$ subject to the boundary condition of fixed electric and magnetic charges by solving

$$D_0 \Phi_a = 0$$

$$E^{ia} = C D_0 \Phi_a$$

$$B^{ia} = (1-C)^2 D_0 \Phi_a$$

Now $D_0 \Phi_a = \epsilon^{abc} A_0^b \Phi_c$ so if $A_0^b = a^b$

$$E^{ia} = -D_i A_0^a = -a D_i \Phi_a$$

so the second equation is solved if $a = -C$.
Finally, the third equation can be solved from the field contribution of a pure monopole:

\[ \phi^a_r(x) = A^a_{\alpha_0}(1-C^2)^{1/4}x \]

by unity

\[ \phi^a_r(x) = \phi^a_r \left((1-C^2)^{1/4}x \right) \]

\[ A^a_{\alpha_0}(1-C^2)^{1/4} \]

Then

\[ \text{D}_r \phi^a = (1-C^2)^{1/4} \left( \text{D}_r \phi \right)_r \left((1-C^2)^{1/4}x \right) \]

\[ B^{ia} = [(1-C^2)^{1/4}]^2 \left( B^{ia} \right)_r \left((1-C^2)^{1/4}x \right) \]

and we may use \( \left( \text{D}_r \phi^a \right)_r = \left( B^{ia} \right)_r \) !

The B field tends as \( r \to \infty \) to

\[ B^{ia} \sim (1-C^2) \frac{\hat{r}^{i} \hat{r}^{a}}{gr^2 (1-C^2)} \sim \frac{\hat{r}^{i} \hat{r}^{a}}{gr^2} = \frac{IM}{4\pi r^2} \hat{r}^{i} \hat{r}^{a} \]

as before, and

\[ E^{ia} \sim C \frac{\hat{r}^{i} \hat{r}^{a}}{(1-C^2)^{1/4} gr^2} \sim \frac{qe}{4\pi r^2} \hat{r}^{i} \hat{r}^{a} \]

so we must identify

\[ C = \frac{qe}{(1-C^2)^{1/4} \sqrt{q^2 + qm^2}} \]

for consistency. From this we can determine:
\[ E = \nu \cdot (C q e + (1-c^2)^{1/2} q m) \]

\[ E = \nu \cdot [q e^2 + q m^2]^{1/2} \]

It can be shown (this is a somewhat advanced analysis) that \( q e \) is quoted in units of \( q \) when the soliton is considered in its youth field theory, then the dyon of charge \( m \) has mass

\[ M = \nu \cdot \left[ (m q)^2 + \left( \frac{4\pi e}{q^2} \right)^2 \right]^{1/2} \]

There is a more surprising further along this road. It turns out that the BPS limit of this model can be embedded into an N=2 superconformal theory. In that theory, the formula

\[ E = \nu \cdot [q e^2 + q m^2]^{1/2} = \nu \cdot \left[ (m q)^2 + \left( \frac{4\pi e}{q} \right)^2 \right]^{1/2} \]

is an exact consequence of superconformal symmetry. If true to all orders of perturbation theory beyond. Other solutions in superconformal theories also have this "BPS" property. For more details, see J. Harvey, hep-th/9603086 (1996).