Grand Unification

Are you disappointed that the Yang-Mills gauge group of the Standard Model is $SU(3) \times SU(2) \times U(1)$? It seems that there are too many components, both in the gauge field and in the fermion matter representation. Can we make it simpler?

Here is a proposal: The group $SU(3) \times SU(2) \times U(1)$ is, in quite natural way, a subgroup of $SU(5)$. Maybe we can write a theory of spontaneously broken $SU(5)$ Yang-Mills theory that leads to the Standard Model. In 1973, Georgi and Glashow showed how to do this, and Pati and Salam constructed a very similar theory based on the group $SO(10)$ that naturally has $SU(5)$ as a subgroup.

To begin, assume that we have an $SU(5)$ Yang-Mills theory with a Higgs field in the adjoint representation of $SU(5)$. This Higgs field $\Phi$ is a traceless Hermitian matrix. A possible potential for $\Phi$ is

$$V(\Phi) = -\mu^2 \Phi^T \Phi + \frac{1}{4} (\Phi^T \Phi)^2 + \frac{1}{2} \kappa \Phi^T \Phi$$

The symmetric state $\langle \Phi \rangle = 0$ is unstable. $\Phi$ can always be diagonalized:

$$\langle \Phi \rangle = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$
with \( \sum_i a_i = 0 \). Add the constraint as a Lagrange multiplier, the extreme \( \Phi \) and \( \Psi \) are given by

\[
\frac{\delta}{\delta \Phi} \left( -\mu^2 + \Phi^2 + 2 \lambda_1 (\Phi \Phi^2) + 2 \lambda_2 \Phi^4 + \eta \Phi^5 \right) = 0
\]

\[
\eta_2 + [-2\lambda_1 + 4\lambda_2 \Phi \Phi^2] a_i + 4\lambda_2 a_i^3 = 0
\]

So the \( a_i \) take at most three different values. Actually, the minimum of \( \Phi \) can be constructed in which the \( a_i \) take two values:

\[
<\Phi> = A \left( \begin{array}{c} 2 \lambda_2 \\ -2 \lambda_2 \\ \eta \end{array} \right) \quad \text{or} \quad <\Phi> = B \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ \eta \end{array} \right)
\]

depend on the value of \( \lambda_1 \) and \( \lambda_2 \). The first choice gives a \( <\Phi> \) that commutes with the matrices

\[
\left( \begin{array}{c} t^a \\ \end{array} \right) \quad t^a \quad 3 \times 3 \quad \text{Hermitian}
\]

\[
\left( \begin{array}{c} c^a \\ \end{array} \right) \quad c^a \quad 2 \times 2 \quad \text{Hermitian}
\]

\[
\left( \begin{array}{c} \sigma^a \\ \end{array} \right) \quad c^a \quad \text{Pauli matrices}
\]

The covariant derivative on \( \Phi \) is:

\[
D\Phi = \partial \Phi - i g_5 A^A \left[ t^A, \Phi \right]
\]

so the generators of \( SU(5) \) correspond to these matrices are left unbroken and the associated \( A^A \) are left massless.
The remaing generators of SU(5) \( \left( \frac{X}{5} \right) \) are spontaneously broken, ad the associated gauge boson recieve mass \( m_S = 5g_S A \). We'll come back to these bosons later.

The unbroken SU(3) \times SU(2) \times U(1) group. To define \( g_S \) consistently, all generators should be normalised to a common convention

\[
4 \left[ t^A, t^B \right] = \frac{1}{2} \delta^{AB}
\]

Then \( t^a, c^a \) will be the standard SU(3) and SU(2) matrices, and the last matrix will be

\[
\sqrt{\frac{3}{5}} \begin{pmatrix} -y_3 & y_2 & -y_1 \\ -y_2 & y_1 & -y_3 \\ y_1 & -y_3 & y_2 \end{pmatrix} = \sqrt{\frac{3}{5}} Y
\]

In a moment, I will identify \( Y \) with the hypercharge of the GSW model structure thing. So, the part of the SU(5) covariant derivative will involve gauge fields is

\[
D_r = \partial_r - ig_S c^a A^a_r - ig_S A^a_r c^a - i \sqrt{\frac{3}{5}} g_S B_r Y
\]

so we will predict an SU(3) \times SU(2) \times U(1) gauge thing with

\[
g_S = g \quad g' = \sqrt{\frac{3}{5}} g
\]

\[
\tan \theta_W = \frac{g'}{g} = \sqrt{\frac{3}{5}} \quad \sin^2 \theta_W = \frac{3}{8}
\]

Both predictions
Now couple fermions to this model. The simplest choice is a multiplet of left-handed fermions in the $\overline{5}$ of $SU(5)$. This is acted on by the lepton generators as

\[
\begin{bmatrix}
\frac{1}{3} \\
-1
\end{bmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{bmatrix}
\frac{1}{3} \\
1
\end{bmatrix}
\]

These do not correspond to any Standard Model fermions. However, the $\overline{5}$ of $SU(5)$ gives:

\[
\begin{bmatrix}
\frac{1}{3} \\
-1
\end{bmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{bmatrix}
\frac{1}{3} \\
1
\end{bmatrix}
\]

These are the $(\overline{1})_L = \text{antiparticles of } \nu$.

\[\nu = (\gamma_2)\]

The other fermions of the Standard Model can be found in the
5 x 5 antisymmetric tensor representation of SU(5)

\((5 \times 4 = 10 - \text{dimensional})\)

\(1 \oplus SU(2) \quad (3 \times 3)_{\text{antisym}} = 3 \oplus SU(3) \quad Y = -\frac{1}{3} + \frac{1}{3} = 0\)

\(3 \oplus SU(3), 2 \oplus SU(2) \quad Y = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}\)

\(1 \oplus SU(3) \quad (2 \times 2)_{\text{antisym}} = 1 \oplus SU(2) \quad Y = \frac{1}{3} + \frac{1}{3} = 1\)

There are the \((\bar{u})_L, \quad Q = (\bar{d})_L, \quad (e^+)_L\). So, left-handed fermions in the \(5 + 10\) representation of SU(5) account for 1 full generation of fermion in the Standard Model. Since \(5 + 10\) is not a real representation of SU(5), the model includes parity violation — but (by arguments in Pekeris-Schrieffer, section 20.3) only in the weak interaction.

Notice that, because SU(5) is a simple group, the U(1) that results from spontaneous symmetry breaking is quantized. After SU(2) x U(1) breaking, they will give

\[Q(\text{proton}) = -Q(e)\quad \text{exactly}!\]

The SU(5) structure also gives all of the bizarre Y assignments for the various quarks and leptons.
Return now to the question of the coupling constants. What can we do about this? If the grand unified group is broken at a very high mass scale, there might be substantial re-arrangements of coupling constants between these and accessible particle physics scales. The effect goes in the right direction: the SU(3) couplings should get stronger, the U(1) couplings should get weaker. To be more quantitative, write

\[ a_i = \frac{g_i^2}{4\pi} \quad g_i = \frac{\sqrt{3}}{3} g' \]
\[ g_2 = g \]
\[ g_3 = g_s \]

\[ \frac{d}{d\ln Q} \left( \frac{a_i}{a_1} \right) = \frac{1}{d\ln M} + \frac{b_i}{2\pi} \ln \left( \frac{Q}{M} \right) \]

we computed

\[ b = \frac{11}{3} C_2(g) = \sum \frac{3}{2} c(g_f) - \sum \frac{3}{2} c(g_f) \]

Let \( m \) be the scale of grand unified ("GUT") symmetry breaking. For \( Q < m \), only the particles that do not obtain masses \( < m \) are relevant. A first hypothesis would be that these are only the particles of the Standard model. Then we can evaluate the \( b_i \). In the following,
\[ n_g \text{ is the number of fermion generations:} \]

\[ \text{SU}(3): \quad b_3 = \frac{11}{3} \cdot \frac{3}{2} \cdot \frac{1}{3} = \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot n_g \]

\[ \text{SU}(2): \quad b_2 = \frac{11}{3} \cdot \frac{3}{2} \cdot \frac{1}{3} = \frac{3}{2} \cdot \frac{1}{3} \cdot (\frac{3}{2} + \frac{3}{2}) \cdot n_g \]

\[ \text{U}(1): \quad b_1 = 0 - \frac{3}{2} \cdot \frac{3}{5} \left( 2 \cdot \frac{1}{2} \cdot \left( \frac{3}{2} \right)^2 + 3 \cdot \frac{1}{3} \cdot \left( \frac{3}{2} \right)^2 + 3 \cdot \left( -\frac{3}{2} \right)^2 \right) + 2 \cdot \left( \frac{1}{2} \right)^2 + 1^2 \right) \cdot n_g \]

\[ \sum \frac{3}{5} \gamma^2 \]

\[ a = \frac{3}{5} \left( \frac{1}{2} + \frac{1}{3} + \frac{4}{3} + \frac{1}{2} + 1 \right) \]

\[ = \frac{3}{5} \cdot \frac{10}{3} = 2 \]

\[ b_3 = 11 - \frac{4}{3} \cdot n_g \]

\[ b_2 = \frac{22}{3} - 4 \cdot i \cdot n_g \]

\[ b_1 = -\frac{4}{3} \cdot n_g \]

So \( a_3 \) gets stronger, \( a_1 \) gets weaker for \( a \ll m_u \).

The Higgs doublet makes a small additional contribution:

\[ b_3: \quad 0 \]

\[ b_2: \quad -\frac{1}{3} \cdot \frac{1}{2} = -\frac{1}{6} \]

\[ b_1: \quad -\frac{1}{3} \cdot \frac{3}{5} \cdot 2 \cdot \left( \frac{1}{2} \right)^2 = -\frac{1}{10} \]

Can this be made quantitative? The values \( a_1, a_2, a_3 \) are
well measured at $Q = m_Z$:

\[
\alpha_3(m_Z) = \frac{1}{8.3} \quad \alpha_2 = \frac{1}{29.6} \quad \alpha_1 = \frac{5}{3} \alpha' = \frac{1}{59.1} \quad (\alpha' = \frac{1}{q_{8.5}})
\]

The equation

\[
\alpha_i^{-1}(m_Z) = \alpha_i^{-1}(m_U) - \frac{b_i}{2\pi} \log \frac{m_U}{m_Z} \quad \text{with } \alpha_i^{-1}(m_U) \quad \text{indp. of } i
\]

are 3 equations for 2 unknown $\alpha(m_U), m_U$.

So we get 1 prediction, plus the values of $\alpha(m_U), m_U$.

Take $\alpha_1, \alpha_2$ as the reference values

\[
\log \frac{m_U}{m_Z} = 2\pi \left( \frac{\alpha_1^{-1} - \alpha_2^{-1}}{b_2 - b_1} \right) \quad \alpha_1^{-1} = \frac{b_2 \alpha_1^{-1} - b_1 \alpha_2^{-1}}{b_2 - b_1}
\]

and

\[
\alpha_3^{-1}(m_Z) = \left( \frac{b_2 - b_3}{b_2 - b_1} \right) \alpha_1^{-1} - \left( \frac{b_1 - b_3}{b_2 - b_1} \right) \alpha_2^{-1}
\]

Note that the formulae for $\alpha_3$ and $m_U$ depend only on differences of the $b_i$ -- $m_Z$ cancels out. Evaluating for the Standard Model below $m_U$:

\[
\alpha_3^{-1} = 14.2 \quad \text{a} \quad \alpha_3(m_Z) = 0.07
\]

\[
\log \frac{m_U}{m_Z} = 25.5 \quad \text{a} \quad m_U = 1 \times 10^{13} \text{ GeV}
\]

The output value of $Q, m_U$ is very large, but, as we will see, not nearly large enough. The output for $\alpha_3$ is quite incorrect, so we have the quadratic physics, but not the right
It turns out that we do much better by assuming that the constant of the model for $m_2 < \alpha < m_0$ is the supersymmetric generalization of the Standard Model. Supersymmetry adds:

- one left-handed Majorana fermion in the adjoint rep. for each $g_2$ boson
- one spin-zero complex field for each left-handed fermion
- two Higgs doublets, each associated fermion.

\[ b = \frac{1}{3} C(A) \cdot \frac{2}{3} C(A) - \sum_{\text{with reps.}} \left( \frac{2}{3} C(V) + \frac{1}{3} C(R) \right) \]
\[ = 3 C(A) - \sum_{\text{ferm.}} C(R) \]

Then:

\[ b_3 = 3 \cdot 3 - 2 \cdot \frac{5}{2} \cdot 2 \cdot n_g \quad \text{Higgs sector} \]
\[ b_2 = 3 \cdot 2 - \frac{1}{2} (3+1) \cdot n_g - \frac{1}{2} \cdot 2 \text{ ferm.} \]
\[ b_1 = \sum_{\text{ferm.}} \cdot n_g - \frac{3}{5} \cdot 2 \cdot 2 \left( \frac{1}{2} \right)^2 \]

So:

\[ b_3 = 9 - 2n_g \]
\[ b_2 = 6 - 2n_g - 1 \]
\[ b_1 = -2n_g - \frac{3}{5} \]
Then \[ a_3^{-1} = 8.5 \quad \text{and} \quad a_3(m_Z) = 0.12 \]

\[ \log \frac{m_0}{m_2} = 33. \quad \text{at} \quad m_0 = 2 \times 10^{16} \text{ GeV} \]

In general, we get the measured value for \( a_3(m_Z) \) if

\[ \frac{b_3 - b_2}{b_2 - b_1} \approx \frac{5}{7} \]

and this is the value in the supersymmetric SM!

Grand unification has several important implications, of which I will discuss two. First, since grand unification puts quarks and leptons into the same multiplets, it predicts new gauge interactions that violate lepton and quark number. The massive \( W \) boson (p.3) mediates the transition

\[
\begin{pmatrix}
\bar{d} \\
\bar{L}
\end{pmatrix} \rightarrow \begin{pmatrix}
\bar{d} \\
\bar{L}
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
\bar{d} \\
\bar{L}
\end{pmatrix} \rightarrow \begin{pmatrix}
\bar{d} \\
\bar{L}
\end{pmatrix}
\]

lead to

\[
\mathcal{L}_{\text{eff}} = \frac{g_5^2}{2m_W^2} (\bar{d}^+ \gamma^\mu L) (\bar{\ell}^+ \gamma^\mu \nu L)
\]

with all indices explicit, the operator is

\[
\epsilon_{ijk} \epsilon^{abc} (d_R)^i (U_R)^a (d_L)^b (U_L)_c
\]

with \( U \) color SU(2)

\[
\gamma = -\frac{1}{2} + \frac{1}{3} + \frac{1}{6} - \frac{1}{2} = 0 \quad \text{\checkmark}
\]
This operator mediates processes:

\[ d u u \rightarrow e^+ n \quad u d \rightarrow \bar{u} e^+ \]

which leads to proton decay processes such as:

\[ p \rightarrow e^+ \pi^0 \]

The rate of this process is:

\[ \Gamma \sim \alpha_s \frac{1}{M^4} \frac{1}{m_p^5} \sim \frac{1}{24} \left(1 \text{ GeV}\right) \cdot \left(\frac{m_p}{M}\right)^4 \]

\[ \tau \sim \frac{1.6 \times 10^{-23} \text{ sec} \cdot \left(\frac{M}{m_p}\right)^4}{5 \times 10^{-31} \left(\frac{M}{m_p}\right)^4} \text{ yr.} \]

Please note that \( M \sim 10^{13} \text{ GeV} \) gives \( \tau \sim 10^{22} \text{ yr.} \)

That sounds like a long time, but it is not nearly long enough. It is easy to do the following experiment: Take a small tank of water, put it in a very well shielded place, wash it very carefully for a year; this would give:

\[ \tau (p \rightarrow e^+ \pi^0) > 10^{24} \text{ yr.} \]

In fact, the current limit is:

\[ \tau (p \rightarrow e^+ \pi^0) > 1.6 \times 10^{33} \text{ yr} \]

From the Super-Kamiokande experiment (100 kT of water in an underground cavern). This requires \( M \gtrsim 10^6 \text{ GeV} \).

In supersymmetric grand unified theories, the most
important proton decay modes are actually mediated by heavy particles — the Higgs boson multiplets. The dominant decay is the odd process

$$p \rightarrow \nu K^+$$

However, $$\tau(p \rightarrow \nu K^+) > 6.7 \times 10^{32} \text{ yr.}$$ from super-K. This number is roughly the same size as the prediction. It is interesting to push the search further.

Grand unification theories can also lead to relations among the Higgs singlets that lead to the grand ed (lepton masses). In SU(5) grand unification, the Higgs field comes from a 5 of SU(5)

$$\Phi = \left( \frac{X}{\phi} \right) \frac{3}{5} \text{ Higgs doublet}$$

The $$d, u, d, L$$ come from a $$\frac{10}{3}$$, while the $$Q$$ and $$e_R$$ come from a $$\frac{10}{3}$$. The SU(5) - invariant coupling

$$\mathcal{L} = - \frac{1}{4} \Phi^a \Phi_b \Phi^a \Phi_b + h_c$$

contains

$$- \frac{1}{2} \Phi^a \Phi^a + \Phi^a \Phi^a$$

which predicts $$A_d = A_L$$ or $$m_d = m_L \sim$$ each generation. For the $$e$$ and $$b$$, we have

$$m_b \approx 4.3 \text{ GeV} \quad m_t = 177 \text{ GeV}$$
but quark masses receive a QCD correction of about a factor 3 from introducing the effect

\[ \frac{m_s}{m_d} \sim 15 \quad \frac{m_s}{m_e} \ll 200 \]

are clearly inconsistent with the prediction. It is possible to build models in which small corrections to the
Higgs coupling from higher-derivative operators spoil the prediction for \( s/\mu \), the which keeps the prediction for \( b/\tau \).

To go further only within the study of proton decay or that of quark and lepton mass relations, we would have to get
much more technical and, in particular, we would have to
include the technical properties of supersymmetric GUT's.

In addition to the ambiguously phenomenological status of grand unification, grand unification has an important purely theoretical difficulty. Let’s go back to the Higgs boson GUT multiplet:

\[
\Phi = \left( \begin{array}{c}
\phi \\
\bar{\phi}
\end{array} \right) \quad \text{3 of such,}
\]

A mediator began under-weak processes, so it must have a mass $\lesssim 10^{15}$ GeV. By various strategies, one can arrange that $\phi$ has zero mass. However, we do not want $m^2_{\phi} = 0$, we want $m^2_{\phi} = -\mu^2$ where $\mu \approx 100$ GeV. How could this huge ratio of scales come into the story?

There is no good answer. In addition, the Yukawa field they help connect to $m^2_{\mu}$ are of order (the gauge hierarchy problem)

\[
\frac{g^2}{4\pi} m^2_{\mu} \sim (100 \text{ GeV})^2 \times (10^{27} !)
\]

so we need to understand why these corrections are absent. There are several answers to this quite hard and have been proposed in the literature; in particular, supersymmetry gives a mechanism. All of these mechanisms require new physics at energies of a few hundred GeV to actually generate $m^2_{\phi} = -\mu^2$ at that scale.
Hopefully, the experiments that will be done soon at the
CERN LHC will give evidence for this new physics.
Those experiments might indicate the presence of supersymmetry
or other new ingredients that have been
referred to in this lecture.