N=4 Meets N=8 at

2 Loop
5 Leg

Lance Dixon (SLAC)
S. Abreu, LD, E. Herrmann, B. Page and M. Zeng, 1812.08941, 1901.08563
LD, E. Herrmann, K. Yan, H.-X. Zhu, 1912.nnnnn

“QCD Meets Gravity”
UCLA, December 13, 2019
But: $\hbar \neq 0$, $m = 0$, $N_{\text{susy}} = 8$, and symbol level

Yet $N = 8$, $m = 0$ still useful for comparison to ACV!
Talk by Julio Parra-Martinez

L. Dixon  N=4 meets N=8 @ 2loop 5leg

UCLA - 2019.12.13
QCD Loop Amplitude Bottleneck

- **NLO**: Efficient, “prescriptive” unitarity-based methods for computing one-loop amplitudes at high multiplicity, e.g. the 8-point process $pp \rightarrow W + 5 \text{ jets}$

- **NNLO**: Two-loop QCD amplitudes unknown beyond 2 → 2 processes, except for recent all massless 2 → 3 cases:
  
  $gg \rightarrow ggg, \ qg \rightarrow qgg, \ \bar{q}q \rightarrow \gamma \gamma \gamma$ in large $N_c$ (planar) limit

Gehrmann, Henn, Lo Presti, 1511.05409;
Badger, Brønnum-Hansen, Hartanto, Peraro, 1712.02229, 1811.11699;
Abreu, Dormans, Febres Cordero, Ita, Page, Zeng, Sotnikov, 1712.03946, 1812.04586, 1904.00945
Chawdhry, Czakon, Mitov, Poncelet, 1911.00479

Bern, LD, et al., 1304.1253, BlackHat 1.0
+ 256,264 more diagrams

L. Dixon    N=4 meets N=8 @ 2loop 5leg    UCLA - 2019.12.13
Why is two loops so hard?

- Primarily because two-loop integrals are intricate, transcendental, multi-variate functions.
- In contrast, at one loop all integrals are reducible to scalar box integrals + simpler combinations of dilogarithms.

\[ \text{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1 - t) \]

+ logarithms and rational terms

`t Hooft, Veltman (1974)
Our favorite toy model(s)

- Explore **nonplanar** multi-loop, multi-leg amplitudes in **N=4 super-Yang-Mills theory (SYM)**. Gauge group **SU(Nc)**, **NOT** in the large **Nc** (planar) limit
- First two-loop 2 → 3 amplitude with **full color** dependence – albeit still at level of **symbol**
- **Spinoff**: same amplitude in **N=8 supergravity**
- Space of functions encountered here also enters two-loop 5-point amplitudes in **full-color QCD**.
- Soft limit understood at **function level**; complicated **tripole emission** is same in **QCD** as in **N=4 SYM**
Two-loop color decomposition

Bern, Rozowsky, Yan, hep-ph/9702424

\[ A_5^{(2)} = \left( \frac{N_c g^2 e^{-\gamma_E}}{(4\pi)^2} \right)^2 \left\{ \sum_{S_5/D_5} (\text{Tr}[12345] - \text{Tr}[54321]) (A^{\text{ST}}[12345] + \frac{A^{\text{SLST}}[12345]}{N_c^2}) + \sum_{S_5/(S_3 \times Z_2)} \frac{\text{Tr}[15](\text{Tr}[234] - \text{Tr}[432])}{N_c} A^{\text{DT}}[15|234] \right\} \]

\[ \text{Tr}[12345] \equiv \text{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4} T^{a_5}] \]

- **Leading color coefficient** \( A^{\text{ST}} \) obeys ABDK/BDS ansatz, Anastasiou, Bern, LD, Kosower, hep-th/0309040, Bern, LD, Smirnov, hep-th/0505205,
- **Verified numerically long ago**
  Cachazo, Spradlin, Volovich, hep-th/0602228; Bern, Czakon, Kosower, Roiban, Smirnov, hep-th/0604074
- **Given by exponential of one-loop amplitude (need \( O(\epsilon^2) \) terms)**
  Bern, LD, Dunbar, Kosower, hep-th/9611127
Color trace relations

Kleiss, Kuijf (1989); Bern, Kosower, (1991); Del Duca, LD, Maltoni, hep-ph/9910563; Edison, Naculich, 1111.3821; talk by Fei Teng

• Tree-level: $A_n[1,\ldots,n,\ldots]$ given in terms of permutations of $(n-2)!$ independent $A_n[1,\ldots,n]$ [Kleiss-Kuijf relations]
• One loop: subleading-color $A^{DT}$ completely determined by permutations of $A^{ST}$
• Both follow from applying Jacobi relations for structure constants $f^{abc}$ to all-adjoint color structures.
• Two loops: Same method $\rightarrow$ Edison-Naculich relations, which we solve as:

$$A^{SLST}[12345] = 5A^{ST}[13254]$$

$$+ \sum_{\text{cyclic}} \left[ A^{ST}[12435] - 2A^{ST}[12453] + \frac{1}{2} (A^{DT}[12|345] - A^{DT}[13|245]) \right]$$

L. Dixon      N=4 meets N=8 @ 2loop 5leg
Integrands

- First obtained Carrasco, Johansson, 1106.4711 in a “BCJ” form Bern, Carrasco, Johansson, 1004.0476 which simultaneously gives the integrand for N=8 supergravity as a “square” of the N=4 SYM integrand. This integrand is manifestly $D$-dimensional.
- Integrand also given in a four-dimensional form Bern, Herrmann, Litsey, Stankowicz, Trnka, 1512.08591 which exposes the expected rational prefactors for pure transcendental functions $g^{DT}$ as 6 “KK” independent Parke-Taylor factors,

$$A^{DT}[15|234] = \sum_{\sigma(234) \in S_3} \frac{\delta^8(Q)}{\langle \sigma_1 \sigma_2 \rangle \langle \sigma_2 \sigma_3 \rangle \langle \sigma_3 \sigma_4 \rangle \langle \sigma_4 \sigma_5 \rangle \langle \sigma_5 \sigma_1 \rangle} \cdot \text{pure}$$
• Linear in loop momentum for N=4 SYM: multiply $N(x)$ by $f^{abc}$ structures

• Quadratic for N=8 SUGRA: square the $N(x)$

\[
N^{(a,b)} = \frac{1}{4} \left( \gamma_{12} (2s_{45} - s_{12} + \tau_{2\ell_1} - \tau_{1\ell_1}) + \gamma_{23} (s_{45} + 2s_{12} - \tau_{2\ell_1} + \tau_{3\ell_1})
+ 2\gamma_{45} (\tau_{5\ell_1} - \tau_{4\ell_1}) + \gamma_{13} (s_{12} + s_{45} - \tau_{1\ell_1} + \tau_{3\ell_1}) \right),
\]

\[
N^{(c)} = \frac{1}{4} \left( \gamma_{15} (\tau_{5\ell_1} - \tau_{1\ell_1}) + \gamma_{25} (s_{12} - \tau_{2\ell_1} + \tau_{5\ell_1}) + \gamma_{12} (s_{34} + \tau_{2\ell_1} - \tau_{1\ell_1} + 2[s_{15} + \tau_{1\ell_2} - \tau_{2\ell_2}])
+ \gamma_{45} (\tau_{4\ell_2} - \tau_{5\ell_2}) - \gamma_{35} (s_{34} - \tau_{3\ell_2} + \tau_{5\ell_2}) + \gamma_{34} (s_{12} + \tau_{3\ell_2} - \tau_{4\ell_2} + 2[s_{45} + \tau_{4\ell_1} - \tau_{3\ell_1}]) \right),
\]

\[
N^{(d-f)} = \gamma_{12}s_{45} - \frac{1}{4} \left( 2\gamma_{12} + \gamma_{13} - \gamma_{23} \right) s_{12},
\]

\[
s_{ij} = (k_i + k_j)^2 = 2k_i \cdot k_j, \quad \tau_{i\ell_j} = 2k_i \cdot \ell_j
\]
Most topologies were known previously, e.g.

- **planar (a)** Papadopoulous, Tommasini, Wever, 1511.09404; Gehrmann, Henn, Lo Presti, 1511.05409, 1807.09812;
- **hexabox (b)** Chicherin, Henn, Mitev, 1712.09610
- **planar (d)** Gehrmann, Remiddi, hep-ph/000827
- **nonplanar (e,f)** Gehrmann, Remiddi, hep-ph/0101124
Integrals (cont.)

• Use IBP reduction method of Abreu, Page, Zeng, 1807.11522 building off earlier work based on generalized unitarity and computational algebraic geometry Gluza, Kajda, Kosower, 1009.0472; Ita, 1510.05626; Larsen, Zhang, 1511.01071; Abreu, Febres Cordero, Ita, Page, Zeng, 1712.03946

• Reduction performed numerically, at numerous rational phase space points, over a prime field to avoid enormous intermediate expressions

• Quite sufficient for full analytic reconstruction when structure of the rational function prefactors is so heavily constrained, as in N=4 SYM

• Even works for planar QCD Abreu, Dormans, Febres Cordero, Ita, Page, 1812.04586,…

• Our results for the integrals and the amplitude confirmed by Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, 1812.11057, 1812.11160
Iterated integrals

Chen; Goncharov; Brown

• Generalized polylogarithms, or $n$-fold iterated integrals, or weight $n$ pure transcendental functions $f$.

• Define by derivatives:

$$ df = \sum_{s_k \in S} f^{s_k} d \ln s_k $$

where $S$ = finite set of rational expressions, “symbol letters”, and $f^{s_k}$ are also pure functions, weight $n-1$

• Iterate: $d f^{s_k} \Rightarrow f^{s_j}, s_k \equiv \{n - 2, 1, 1\}$ component

• Symbol = $\{1,1,\ldots,1\}$ component (maximally iterated)

Goncharov, Spradlin, Vergu, Volovich, 1006.5703
Example: Harmonic Polylogarithms of one variable (HPLs \{0,1\})

Remiddi, Vermaseren, hep-ph/9905237

- Generalization of classical polylogs:
  \[ \text{Li}_n(u) = \int_0^u \frac{dt}{t} \text{Li}_{n-1}(t), \quad \text{Li}_1(t) = -\ln(1-t) \]

- Define HPLs by iterated integration:
  \[ H_{0,\bar{w}}(u) = \int_0^u \frac{dt}{t} H_{\bar{w}}(t), \quad H_{1,\bar{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\bar{w}}(t) \]

- Or by derivatives
  \[ dH_{0,\bar{w}}(u) = H_{\bar{w}}(u) d\ln u \quad dH_{1,\bar{w}}(u) = -H_{\bar{w}}(u)d\ln(1-u) \]

- Just two symbol letters: \( S = \{u, 1-u\} \)

- Weight \( n = \) length of binary string \( \bar{w} \)

\[
S[\text{Li}_n(u)] = - (1-u) \otimes u \otimes u \otimes \cdots \otimes u \]

\[ n - 1 \]
Symbol alphabet for **planar 5-point**

\[ 5 \times 5 + 1 = 26 \text{ letters} \]

\[ \mathcal{S} = \{s_{i,i+1}, s_{i-1,i} + s_{i,i+1}, s_{i,i+1} - s_{i+2,i+3}, s_{i+3,i+4} - s_{i,i+1} - s_{i+1,i+2}, o_i, \Delta\} \]

\[ s_{i,i+1} \equiv (k_i + k_{i+1})^2 \]

\[ o_1 = \frac{[12] \langle 23 \rangle [34] \langle 41 \rangle}{\langle 12 \rangle [23] [34] [41]} \]

\[ \Delta = \text{tr}[\gamma_51234] = [12] \langle 23 \rangle [34] \langle 41 \rangle - \langle 12 \rangle [23] [34] [41] \]

Closed under dihedral permutations, \( D_5 \), subset of \( S_5 \)

\( o_i \) are odd under parity, \( \langle ab \rangle \Leftrightarrow [ab] \)

- Most letters seen already in one-mass four-point integrals
- But not \( o_i \) or \( \Delta \)
Symbol alphabet for nonplanar 5-point

\[ 10 + 15 + 5 + 1 = 31 \text{ letters} \]

\[ S = \{ s_{i,j}, \ s_{i,j} - s_{k,l}, \ o_i, \ \Delta \} \]

\[ o_1 = \frac{[12\langle 23|34\rangle41]}{\langle 12\rangle23\langle 34\rangle41} \]

\[ \Delta = \text{tr}[\gamma_51234] = [12\langle 23|34\rangle41] - \langle 12\rangle23\langle 34\rangle41 \]

- Obtained by applying full \( S_5 \) to planar alphabet; only generates 5 new letters
- However, function space is much bigger because branch-cut condition now allows 10 first entries,
  \[ s_{i,j} \equiv (k_i + k_j)^2 \]
- In planar case there were only 5,
  \[ s_{i,i+1} \equiv (k_i + k_{i+1})^2 \]
Numerical reduction and assembly

- Given decomposition into 6 PT factors, suffices to perform reduction to master integrals at 6 numerical kinematic points
- Use mod \( p \) arithmetic with \( p \) a 10-digit prime; reconstruct rational numbers using Wang’s algorithm [Wang (1981); von Manteuffel, Schabinger, 1406.4513; Peraro, 1608.01902]
- Inserting symbols of all master integrals, we obtain symbols of all the pure functions
- Basic result is for \( g_{234}^{DT} \), but also recover \( M^{BDS} \), where \( A^{ST}[12345] = PT[12345] M^{BDS} \)
- Also computed \( A^{SLST}[12345] \), so color algebra could be checked via Edison-Naculich relations
Validation

- Five-point gauge theory amplitudes have stringent set of limiting behaviors as one gluon becomes soft or two partons become collinear.
- E.g. as legs 2 and 3 become collinear:

\[ A^{(2)}_5 \rightarrow \begin{array}{c}
\text{splitting amplitudes} \\
\end{array} \]

- We checked the collinear limit, as well as the soft limit, and the IR poles in \( \varepsilon \) which are predicted by

Soft gluon emission

• Compute from Wilson lines → only depends on rescaling invariant combinations of velocities \( \beta_{m}^{\mu} = p_{m}^{\mu}/p_{m}^{0} \)

\[
S^{\pm}(\{\beta_{m}\}, q) = \sum_{L=0}^{\infty} g_{s} \alpha_{L}^{L} S^{\pm,(L)}(\{\beta_{m}\}, q)
\]

• (+) gluon emission at tree level:

\[
S^{+,0}_{a}(\{\beta_{m}\}) = + \frac{1}{2n} \sum_{i \neq j} (T_{i}^{a} - T_{j}^{a}) \frac{\langle i,j \rangle}{\langle iq \rangle \langle qj \rangle}
\]

• At 1 loop, still only dipole emission:

\[
S^{+,1}_{a} = \frac{1}{2} C_{1}(\epsilon) \sum_{i \neq j} f_{aa_{i}a_{j}} T_{i}^{a_{i}} T_{j}^{a_{j}} V_{ij}^{q} \frac{\langle i,j \rangle}{\langle iq \rangle \langle qj \rangle}
\]

\( V_{ij}^{q} = \left[ \frac{\mu^{2}(-s_{ij})}{(-s_{iq})(-s_{qj})} \right]^{\epsilon} \)
Soft emission at two loops

LD, E. Herrmann, K. Yan, H.-X. Zhu, 1912.nnnnn

- Have to distinguish dipole terms

\[
S_{a,ij}^{+(2)} = C_2(\epsilon) f_{aa_ia_j} T_i^{a_i} T_j^{a_j} \left( V_{ij}^q \right)^2 \frac{\langle ij \rangle}{\langle iq \rangle \langle qj \rangle}
\]

(matter dependent, simple kinematic dependence, but not uniform transcendental)

from tripole terms

\[
S_{a_{ijk}}^{+(2)} = 2 \left( S_{a_ikj}^{+(2)} + S_{a_jki}^{+(2)} + S_{a_ijk}^{+(2)} \right)
= 2 T_i^{a_i} T_j^{a_j} T_k^{a_k} \left\{ \frac{\langle ik \rangle}{\langle iq \rangle \langle qk \rangle} \left( V_{ik}^q \right)^2 \left[ f_{aa_ib} f_{ba_ik} D_1(z, \bar{z}) + f_{aa_i} f_{ba_ik} D_2(z, \bar{z}) \right] + \{i \leftrightarrow j\} \right\},
\]
Tripole emission

- Subleading color, same in any gauge theory, including QCD, and N=4 SYM
- Hence expect weight 4 transcendentality
- Nontrivial dependence on rescaling invariant ratio,
  \[ z = z_k^{ij} = \frac{\langle kj \rangle \langle ij \rangle}{\langle ij \rangle \langle kq \rangle} \quad \overline{z} = \overline{z}_k^{ij} = \frac{[kj][iq]}{[ij][kq]} \]

- \( D_1, D_2 \) are weight 4 SVHPLs
  \[ D_1(z) = -\frac{1}{e^2}(\mathcal{L}_1)^2 - \frac{1}{e}(\mathcal{L}_1)^3 - \frac{7}{12}(\mathcal{L}_1)^4 + 4\mathcal{L}_{1,0,1,0} + 2\mathcal{L}_{1,0,1,1} + 2\mathcal{L}_{1,1,1,0} \]
  \[ D_2(z) = \frac{1}{e^2}\mathcal{L}_0\mathcal{L}_1 + \frac{1}{e}\mathcal{L}_0(\mathcal{L}_1)^2 + \frac{2}{3}\mathcal{L}_0(\mathcal{L}_1)^3 + 6\zeta_2(\mathcal{L}_{0,1} - \mathcal{L}_{1,0}) + 2(\mathcal{L}_{0,0,0,1} - \mathcal{L}_{0,0,1,0} + \mathcal{L}_{0,1,0,0} + \mathcal{L}_{0,1,0,1} - \mathcal{L}_{1,0,0,0}) \]

- Checks symbol terms, and constrains beyond-symbol terms, in full 2 loop 5 point amplitude
Structure of SYM result for full kinematics

- Symbols are large: $M^{BDS}$ (planar) has “only” 2,365 terms, while $g^{DT}_{234}$ (nonplanar) has 24,653 terms.
- How many functions are there in the full amplitude?
- Take linear span of all 120 permutations of $g^{DT}_{234}$ and $M^{BDS}$
- At order $\mathcal{E}^0$, there are 52 weight 4 functions. Naively there should be $72 = 12$ (planar) + $6 \cdot 10$ (nonplanar)
- So there are 20 relations among the permutations, e.g.
  
  \[
  g[12345] + g[12453] + g[12534] + g[21345] + g[21453] + g[21534] \\
  -g[12435] - g[12543] - g[12354] - g[21435] - g[21543] - g[21354] = 0
  \]
- The 20 relations are also obeyed by the lower-weight $1/\mathcal{E}$ pole terms.
- What do they mean? Do they reflect a nonplanar version of dual conformal invariance or integrated BCJ relations?
Glimpse of 24,653 term symbol

- Simplicity of weight 3 odd space lets us present the odd part of the derivative of the odd part of the basic double trace function:

\[
\left. \frac{\partial}{\partial x_i} \left[ g_{234}^{\text{DT,odd}} \right] \right|_{\text{odd}} = \sum_{j=1}^{12} \mathcal{I}_5^{d=6} (\Sigma_j) \sum_\gamma m_{j\gamma} \frac{\partial \log W_\gamma}{\partial x_i}
\]

- \( \gamma \in \{1, ..., 5,16, ..., 20,31\} = \{s_{ij}, \Delta\} \) only, \( \Sigma_j \in S_5/D_5 \)

- \{3,1\} coproduct matrix \( m_{j\gamma} \):

\[
\begin{pmatrix}
-\frac{17}{4} & -\frac{5}{4} & -6 & -\frac{17}{4} & -\frac{7}{2} & -\frac{17}{4} & -\frac{7}{4} & \frac{1}{2} & -1 & -\frac{17}{4} & 10 \\
\frac{17}{4} & \frac{5}{4} & \frac{5}{4} & \frac{17}{4} & \frac{4}{4} & \frac{17}{4} & \frac{11}{4} & \frac{17}{4} & \frac{1}{2} & \frac{1}{2} & -10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\
\frac{17}{4} & -\frac{5}{4} & -\frac{17}{4} & -\frac{7}{2} & \frac{1}{2} & -\frac{7}{4} & -\frac{17}{4} & -\frac{17}{4} & -\frac{17}{4} & -1 & 10 \\
\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\
0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\
\frac{17}{4} & 6 & 6 & \frac{17}{4} & 9 & -\frac{1}{2} & 4 & -\frac{1}{2} & -\frac{5}{4} & -\frac{5}{4} & -10
\end{pmatrix}
\]

rank 8 \(\rightarrow\) only 8 independent linear combinations of final entries appear. Why?
N=8 supergravity

Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, 1901.05932; Abreu, LD, Herrmann, Page, Zeng, 1901.08563

- Same integration methods can be applied to the double-copy N=8 supergravity integrand
  Carrasco, Johansson, 1106.4711
- Loop-momentum numerator is quadratic instead of linear in the loop momentum (QCD would be ~ ninth order)
- Richer set of rational function prefactors
- 40 prefactors can be inferred from four-dimensional leading singularities computed from on-shell diagrams
- 5 more require \( d \)-dimensional leading singularities
After reductions for > 45 phase space points, discover 5 additional rational structures ($d$-dim’l leading sing’s)

Result has uniform transcendentality

Because there is no color, there are exactly 45 pure function components to the amplitude

5 of the 45 are removed by a natural IR subtraction.

Compare the 45 functions to the 52 for N=4 SYM: They overlap a lot; their span has dimension 62
N=8 Validation

- Five-point gravity amplitudes have stringent set of limiting behavior as a graviton becomes soft
  Weinberg (1965); Berends, Giele, Kuijf (1988); Bern, LD, Perelstein, Rozowsky, hep-th/9811140
  or two gravitons become collinear
  Bern, LD, Perelstein, Rozowsky, hep-th/9811140

- E.g. as legs 2 and 3 become collinear:
  four-point two-loop amplitude only
  tree splitting amplitude only

- Checked collinear limit as well as soft limit, and IR poles in $\epsilon$ which are correctly predicted by
  Weinberg (1965); Naculich, Nastase, Schnitzer, 0805.2347
Conclusions

- Two-loop five-point nonplanar amplitudes now available at symbol level in maximally supersymmetric theories.
- Soft limit known at function level, including intricate tripole terms.
- Need to promote symbols $\rightarrow$ functions, beyond soft limit.
- All required master integrals needed for QCD now known at symbol level.
- Opens door to full-color $2 \rightarrow 3$ massless QCD amplitudes for e.g. NNLO 3 jet production at hadron colliders.
- Can these results give insight into classical gravity too?
Extra Slides
Nonplanar 5-point function space

Chicherin, Henn, Mitev, 1712.09610

- Also empirical constraint on first 2 entries of the symbol.
- Imposing this condition and integrability, dimension of even | odd part of function space is:

<table>
<thead>
<tr>
<th>Weight</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td># of integrable symbols for $A_P$ after 2nd entry condition</td>
<td>5</td>
<td>0</td>
<td>25</td>
<td>0</td>
</tr>
<tr>
<td># of integrable symbols for $A_{NP}$ after 2nd entry condition</td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>9</td>
</tr>
</tbody>
</table>

- SYM and SUGRA amplitudes both lie in this space
Structure (cont.)

- Take first derivatives, i.e. \{3,1\} coproducts.
- How many functions are there?
- Weight 3 even: 362 (out of a possible 505).
- But only 40 of them have (two) odd letters. Rest simple.
- Weight 3 odd is even more restricted: only 12 (out of a possible 111)
- They are just the \(12 \frac{S_5}{D_5}\) permutations of the \(D=6\) one-loop pentagon integral!!
- Weight 2 is not restricted at all; the \{2,1,1\} coproducts include all 70 even and 9 odd functions obeying the second entry condition.
Comparing function spaces

<table>
<thead>
<tr>
<th>functions</th>
<th>{1, 1, 1, 1}</th>
<th>{2, 1, 1}</th>
<th>{3, 1}</th>
<th>weight 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>P odd space</td>
<td>0</td>
<td>9</td>
<td>111</td>
<td>1191</td>
</tr>
<tr>
<td>no. from (N = 8)</td>
<td>0</td>
<td>9</td>
<td>11</td>
<td>45</td>
</tr>
<tr>
<td>no. from (N = 4)</td>
<td>0</td>
<td>9</td>
<td>12</td>
<td>52</td>
</tr>
<tr>
<td>no. from both</td>
<td>0</td>
<td>9</td>
<td>12</td>
<td>62</td>
</tr>
<tr>
<td>P even space</td>
<td>10</td>
<td>70</td>
<td>505</td>
<td>3736</td>
</tr>
<tr>
<td>no. from (N = 8)</td>
<td>10</td>
<td>70</td>
<td>285</td>
<td>40</td>
</tr>
<tr>
<td>no. from (N = 4)</td>
<td>10</td>
<td>70</td>
<td>362</td>
<td>52</td>
</tr>
<tr>
<td>no. from both</td>
<td>10</td>
<td>70</td>
<td>367</td>
<td>56</td>
</tr>
<tr>
<td>P even with odd letters</td>
<td>0</td>
<td>0</td>
<td>45</td>
<td>711</td>
</tr>
<tr>
<td>no. from (N = 8)</td>
<td>0</td>
<td>0</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>no. from (N = 4)</td>
<td>0</td>
<td>0</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>no. from both</td>
<td>0</td>
<td>0</td>
<td>40</td>
<td>44</td>
</tr>
</tbody>
</table>
Structure

• Same odd, odd \{3,1\} coproduct matrix as in N=4 SYM, but now for a component of the N=8 finite remainder:

\[
m_{j\alpha_1} = \frac{1}{12} \\
\begin{pmatrix}
-3 & -2 & 2 & 2 & -2 & 1 & 0 & 1 & 0 & 1 & 0 \\
3 & 1 & -1 & -3 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
-3 & -2 & 0 & 0 & -2 & 1 & 0 & 5 & 0 & 1 & 0 \\
3 & 0 & -3 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\
3 & 0 & 0 & -2 & 4 & -3 & 0 & -3 & 0 & 1 & 0 \\
-3 & -1 & 1 & 3 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
-3 & 2 & -1 & 3 & -3 & 2 & 0 & 3 & 0 & 3 & 0 \\
3 & 4 & -2 & 0 & 0 & 1 & 0 & -3 & 0 & -3 & 0 \\
3 & -1 & 0 & 0 & -1 & 2 & 0 & -5 & 0 & 2 & 0 \\
-3 & 0 & 3 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\
-3 & -3 & 3 & -1 & 2 & -3 & 0 & 3 & 0 & 2 & 0 \\
3 & 2 & -2 & -2 & 2 & -1 & 0 & -1 & 0 & -1 & 0
\end{pmatrix}
\]

rank 5 \rightarrow only 5 independent linear combinations of final entries!
N=8 SUGRA 4-dim’l leading singularities

\[ \frac{[12][23][45]^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle \langle 13 \rangle} \rightarrow \]

\[ \frac{s_{12}[12][23][34][45][51]}{\text{tr}_5 \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \]

5 more

linear span has dimension 40

L. Dixon  N=4 meets N=8 @ 2loop 5leg
Shift from “Euclidean” to Minkowski region

\[
\text{disc}_{A_1} D_1(z) = 2\pi \left\{ 8 \left[ \text{Li}_3(z) + \text{Li}_3 \left( \frac{-z}{1-z} \right) \right] \\
- \log(1-z) \left[ 4 \left( \text{Li}_2(z) - \text{Li}_2(\bar{z}) \right) + \log^2(1-z) - \log^2(1-\bar{z}) \right] \right\}
\]

\[
\text{disc}_{A_1} D_2(z) = 2\pi \left\{ \frac{\log |1-z|^2}{\epsilon^2} - \frac{\log^2 |1-z|^2}{\epsilon} + 8\text{Li}_3(z) - 4\text{Li}_3(\bar{z}) + 4\text{Li}_3 \left( \frac{-\bar{z}}{1-\bar{z}} \right) \\
- 2 \left[ \text{Li}_2(z) - \text{Li}_2(\bar{z}) \right] \left[ 2 \log |z|^2 - \log(1-\bar{z}) \right] + 2\zeta_2 \log \left( \frac{1-z}{1-\bar{z}} \right) \\
- \log \left( \frac{1-z}{1-\bar{z}} \right) \log^2 |z|^2 + 2 \log(1-z) \log(1-\bar{z}) \log |1-z|^2 \\
+ \frac{2}{3} \log^3(1-z) \right\} \\
- 4\pi^2 \left\{ 2 \left[ \text{Li}_2(z) - \text{Li}_2(\bar{z}) \right] + \log \left( \frac{1-z}{1-\bar{z}} \right) \log |z|^2 \right\} .
\]

\ldots