Hexagon Scattering Amplitude at the Origin

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UC Davis, January 16, 2020
Why is multi-loops so hard?

- Primarily because multi-loop integrals are intricate, transcendental, multi-variate functions.
- In contrast, at one loop all integrals are reducible to scalar box integrals + simpler combinations of dilogarithms.

\[
\text{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1 - t)
\]

+ logarithms and rational terms.

‘t Hooft, Veltman (1974)
Planar N=4 SYM, toy model for QCD amplitudes

- QCD’s maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group $SU(N_c)$, in the large $N_c$ (planar) limit
- Structure very rigid:
  $$\text{Amplitudes} = \sum_i \text{rational}_i \times \text{transcendental}_i$$
- For planar N=4 SYM, we understand rational structure quite well, focus on the transcendental functions.
- Space of functions is so restrictive, and physical constraints are so powerful, one can write $L$ loop answer as linear combination of known weight $2L$ polylogarithms.
- Unknown coefficients found by solving (a large number of) linear constraints
• Use analytical properties of perturbative (six) point amplitudes in planar N=4 SYM to determine them directly, without ever peeking inside the loops
• Step toward doing this nonperturbatively (no loops to peek inside) for general kinematics

Today, we’ll mainly study a kinematical limit, the origin, where we can (conjecturally) compute the amplitude nonperturbatively in terms of a “tilted BES kernel”

Loops
3
4, 5
6, 7
LD, Drummond, Henn, 1108.4461, 1111.1704;
Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington,
1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;
Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou,
1903.10890, 1906.07116; LD, Dulat, 200m.nnnnn (NMHV 7 loop)
Quantum Symmetries

- Massless QCD has classical scale + conformal symmetry: $\text{SO}(3,1) \rightarrow \text{SO}(4,2)$
- Spoiled at quantum level by nonvanishing $\beta$ function (asymptotic freedom).
- N=4 SYM has $\beta = 0$ $\rightarrow$ full (position space) $\text{SO}(4,2)$, actually full N=4 superconformal algebra, $\text{PSU}(2,2|4)$
- Planar N=4 SYM also has momentum-space version of $\text{SO}(4,2)$ [PSU(2,2|4)] $\rightarrow$ dual N=4 superconformal invariance
Dual conformal invariance is geometric: from AdS/CFT + T-duality

Alday, Maldacena, 0705.0303

SO(4,2) isometry of space-time
T-duality symmetry of string theory

- Exchanges string world-sheet variables $\sigma, \tau$
- $X^\mu(\tau, \sigma) = x^\mu + k^\mu \tau + \text{oscillators}$
  $\rightarrow X^\mu(\tau, \sigma) = x^\mu + k^\mu \sigma + \text{oscillators}$
- Strong coupling limit of planar gauge theory is semi-classical limit of string theory: world-sheet stretches tight around minimal area surface in AdS.
- Boundary determined by momenta of external states: light-like polygon with null edges = momenta $k^\mu$
Amplitudes = Wilson loops

- Polygon vertices $x_i$ are not positions but dual momenta, $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Duality verified to hold at weak coupling too!
The [Dual] Conformal Group

\[ \text{SO}(4,2) \supset \text{SO}(3,1) \text{ [rotations+boosts]} + \text{translations+dilatations} + \text{special-conformal} \]

\[ 15 = 3 + 3 + 4 + 1 + 4 \]

- Nontrivial generators are special conformal \( K^\mu \)
- Correspond to inversion \cdot translation \cdot inversion
- \( \rightarrow f(x_{ij}^2) \) is [dual] conformally invariant if it’s invariant under inversion,
  \[
  x_i^\mu \rightarrow x_i^\mu / x_i^2
  \]
Dual conformal invariance

- Wilson $n$-gon invariant under inversion:
  
  \[ x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\ldots,j-1} \]

- Fixed, up to functions of invariant cross ratios:

  \[
  u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}
  \]

- $x_{i,i+1}^2 = k_i^2 = 0 \rightarrow$ no such variables for $n = 4,5$

- $n = 6 \rightarrow$ precisely 3 ratios:

\[
\begin{align*}
  u_1 &= u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}} \\
  u_2 &= v = \frac{s_{23} s_{56}}{s_{234} s_{123}} \\
  u_3 &= w = \frac{s_{34} s_{61}}{s_{345} s_{234}}
\end{align*}
\]
Solving Planar N=4 SYM Scattering

\[ \lambda = N_c \, g_{YM}^2 \]

\[ A_n \sim \exp[-\sqrt{\lambda} S_{cl}^E] \]

Images: A. Sever, N. Arkani-Hamed

\( \text{collinear limit} \)  

\( \text{perturbative gluons} \)  

\( \text{minimal surface} \)  

\( \text{flux tube} \)  

\( \text{Kinematical variables} \)  

L. Dixon  

Hexagon at the Origin  

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(Near) collinear (OPE) limit
Flux tubes at finite coupling

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045 BSV+Caetano+Cordova, 1412.1132, 1508.02987

- Tile $n$-gon with pentagon transitions.
- Quantum integrability $\rightarrow$ compute pentagons exactly in 't Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit
Kinematical playground

Multi-particle factorization $u, w \to \infty$

(near) collinear
$v = 0, \ u + w = 1$

multi-Regge
$(1,0,0)$

self-crossing

spurious pole $u = 1$
Removing Divergences

- On-shell amplitudes **IR divergent** due to long-range gluons

- Polygonal Wilson loops **UV divergent** at cusps, anomalous dimension $\Gamma_{\text{cusp}}$
  - known to all orders in planar N=4 SYM:
    Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by **BDS-like ansatz**
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708

- Normalized [MHV] amplitude is finite, dual conformal invariant.
- BDS-like also maintains important relation due to causality (Steinmann).

$$\mathcal{E}(u_i) = \lim_{\epsilon \to 0} \frac{\mathcal{A}_6(s_{i,i+1}, \epsilon)}{\mathcal{A}^{\text{BDS-like}}_6(s_{i,i+1}, \epsilon)} = \exp[\mathcal{R}_6 + \frac{\Gamma_{\text{cusp}}}{4} \mathcal{E}^{(1)}]$$
Properties of Amplitudes

• Having determined 6-point amplitudes to 7 loops, study their properties:
  • Analytic behavior in various factorization limits.
  • Simple “bulk” lines like \((u,u,1), (u,1,1), (u,u,u)\).
  • Singular line \((u,0,0)\), then take \(u \to 0\) to approach origin.
  • Planar N=4 SYM has finite radius of convergence of perturbative expansion (unlike QCD, QED, whose perturbative series are asymptotic).

• For BES solution to cusp anomalous dimension, using coupling \(g^2 = \frac{\lambda}{16 \pi^2}\), radius is \(\frac{1}{16}\).

• Ratio of successive terms \(\frac{\Gamma^{(L)}_{\text{cusp}}}{\Gamma^{(L-1)}_{\text{cusp}}} \to -16\)
NMHV Amplitude on $(u,u,1)$

Finite radius of convergence equal to that of cusp anomalous dimension?

\[
\frac{\Gamma_{cusp}^{(L)}}{\Gamma_{cusp}^{(L-1)}} \to -16 \quad \text{as} \quad L \to \infty
\]
Remainder function on \((u,u,u)\)

- **Amazing proportionality** of each perturbative coefficient at small \(u\), also with the strong coupling result.
- Suggests we should take all \(u_i \to 0\)

Alday, Gaiotto, Maldacena, 0911.4708
Weak coupling at origin

• Remarkably, $\ln \mathcal{E}$ is quadratic in logarithms through 7 loops
  CDDvHMP, 1903.10890
• Previously observed through 2 loops, and at strong coupling, on the diagonal $(u,u,u)$ AGM, 0911.4708

$$
\ln \mathcal{E}(u_i) \approx -\frac{\Gamma_{\text{oct}}}{24} \ln^2(u_1 u_2 u_3) - \frac{\Gamma_{\text{hex}}}{24} \sum_{i=1}^{3} \ln^2 \frac{u_i}{u_{i+1}} + C_0
$$

<table>
<thead>
<tr>
<th>$L = 1$</th>
<th>$L = 2$</th>
<th>$L = 3$</th>
<th>$L = 4$</th>
<th>$L = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{\text{oct}}$</td>
<td>4</td>
<td>$-16 \zeta_2$</td>
<td>256$\zeta_4$</td>
<td>$-3264 \zeta_6$</td>
</tr>
<tr>
<td>$\Gamma_{\text{cusp}}$</td>
<td>4</td>
<td>$-8 \zeta_2$</td>
<td>$88 \zeta_4$</td>
<td>$-876 \zeta_6 - 32 \zeta_3^2$</td>
</tr>
<tr>
<td>$\Gamma_{\text{hex}}$</td>
<td>4</td>
<td>$-4 \zeta_2$</td>
<td>$34 \zeta_4$</td>
<td>$-603 \zeta_6 - 24 \zeta_3^2$</td>
</tr>
<tr>
<td>$C_0$</td>
<td>$-3 \zeta_2$</td>
<td>$\frac{77}{4} \zeta_4$</td>
<td>$-\frac{4463}{24} \zeta_6 + 2 \zeta_3^2$</td>
<td>$\frac{67645}{32} \zeta_8 + 6 \zeta_2 \zeta_3^2 - 40 \zeta_3 \zeta_5$</td>
</tr>
</tbody>
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Mysterious octagon connection

- Remarkably, \[ \Gamma_{\text{oct}} = \frac{2}{\pi^2} \ln \cosh(2\pi g) \]

recently appeared in light-like limit of correlator of 4 large \( R \)-charge operators, dubbed the octagon

Coronado, 1811.00467, 1811.03282; Kostov, Petkova, Serban, 1903.05038; Belitsky, Korchemsky, 1907.13131; Bargheer, Coronado, Vieira, 1904.00965, 1909.04077
BES Equations

Beisert, Eden, Staudacher, hep-th/0610251

- Integral equation for spin fluctuation density $\sigma(t)$ with magic kernel $K(t, t')$

$$\frac{e^t - 1}{t} \sigma(t) = K(2gt, 0) - 4g^2 \int_0^\infty dt' K(2gt, 2gt') \sigma(t')$$

- Solution provides $\Gamma_{\text{cusp}}(g^2) = 8g^2\sigma(0)$

- Expanding in Bessel functions, equivalent to inverting a semi-infinite matrix,

$$\Gamma_{\text{cusp}}(g^2) = 4g^2\left[\frac{1}{1 + KK}\right]_{11}$$

$$K_{ij} = 2j(-1)^{i+j} + \int_0^\infty \frac{dt}{t} \frac{J_i(2gt)J_j(2gt)}{e^t - 1}$$

Benna, Benvenuti, Klebanov and Scardicchio, hep-th/0611135
Our Tilted BES Proposal

• Write $K_{ij}$ in 2 x 2 block form, according to whether $i, j$ are odd/even:

$$K = \begin{bmatrix} K_{oo} & K_{o*} \\ K_{*o} & K_{**} \end{bmatrix}$$

• Introduce “tilt angle” $\alpha = 0, \, \frac{\pi}{4}, \, \frac{\pi}{3}$ for oct, cusp, hex

$$K(\alpha) = 2\cos \alpha \begin{bmatrix} \cos \alpha & K_{oo} & \sin \alpha & K_{o*} \\ \sin \alpha & K_{*o} & \cos \alpha & K_{**} \end{bmatrix}$$

• Then

$$\Gamma(\alpha(g^2)) = 4g^2 \left[ \frac{1}{1 + K(\alpha)} \right]_{11}$$
We also find that

\[ C_0 = -D \left( \frac{\pi}{3} \right) - \frac{1}{2} D(0) + D \left( \frac{\pi}{4} \right) - \frac{\zeta_2}{2} \Gamma_{\text{cusp}} \]

where

\[ D(\alpha) = \ln \det[1 + K(\alpha)] \]

A number-theoretic “coaction principle” Schnetz, 1302.6445; Panzer, Schnetz, 1603.04289; Brown, 1512.06409 suggests a best (“cosmic”) normalization for amplitude:

\[ \ln \mathcal{E} \to \ln \mathcal{E} - \ln \rho, \] and through 7 loops [CDDvHMP, 1906.07116]

\[ \ln \rho^{[\text{new}]} = D \left( \frac{\pi}{4} \right) - \frac{\zeta_2}{2} \Gamma_{\text{cusp}} \]

In this normalization, only \( \alpha = 0, \frac{\pi}{3} \) enter hexagon!
Log of Determinants $\frac{D(\alpha)}{2g}$

- oct
- cusp
- hex

$g$ values range from 0 to 7.
Origin of the results

- To approach origin via pentagon OPE, must sum over large number $N$ of large helicity $a_k$ gluonic bound state flux tube excitations.
- Framed Wilson loop:
  \[ \mathcal{W}_6 = \mathcal{E} \times \exp \left[ \frac{\Gamma_{\text{cusp}}}{2} (\sigma^2 + \tau^2 + \zeta_2) \right] \]

Gluconic contribution:

\[
\mathcal{W}_6 = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_a e^{i\phi} \sum_{k=1}^N a_k \int \frac{du}{(2\pi)^N} \frac{e^{-\tau E + i\sigma P}}{\prod_{k<l} P_{kl} P_{lk}} \prod_k \mu_k
\]
Weak coupling

- Expand $\mathbf{u}, E, p, \mu_k, P_{kl}$ in $g$.
- $N$ excitation contribution only starts at $N$ loops, so can get to 8 loops (9 loops if log) with only 2 excitations.
- Large $a_k \rightarrow$ Sommerfeld-Watson transform

$$\sum_{a \geq 1} (-1)^a f(a) \rightarrow \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{if(a)da}{2\sin(\pi a)}$$

- Deform $a$ integral to $a = 0$ residue (after doing $u$ integrals)
- Agrees with full amplitude limit, goes to 8 (9) loops
Finite coupling

• To simplify $E, p, \mu_k, P_{kl}$, analytically continue $u$ to “Goldstone sheet” Basso, Sever, Vieira, 1407.1736

... →

$$\mathcal{E} = N \int \prod_{i=1}^{\infty} d\xi^+ d\xi^- F_\varphi(\bar{\xi}) e^{-\bar{\xi} \cdot M \cdot \xi} \quad M \sim 1 + K$$

• $\bar{\xi}$ is conjugate to charges $\bar{Q} = \sum_{k=1}^{N} \bar{q} (u_k, a_k)$

• $F_\varphi$ is Fredholm determinant,

$$\ln F_\varphi = - \sum_{N \geq 1} \frac{1}{N} \sum a \oint \frac{du}{(2\pi)^N} \prod_{k=1}^{N} \frac{\hat{\mu}_k e^{\varphi a_k}}{x_k^+ - x_{k+1}^-} e^{2i \bar{Q} \cdot \bar{\xi}}$$

• Agrees with full amplitude limit, goes to 8 (9) loops
A Secretly Gaussian Integral

- At weak coupling, $Q_i \sim g^i$, and can expand

$$\ln F_\varphi(\vec{\xi}) = \langle 1 \rangle + 2i \langle Q_i^m \rangle \xi_i^m - 2 \langle Q_i^m Q_j^n \rangle \xi_i^m \xi_j^n + \ldots$$

- All moments $> 2$ vanish as $\varphi \to \infty$ (!):

$$\lim_{\varphi \to \infty} \langle Q_i^m Q_j^n \ldots \rangle = 0$$

- Also compute $\langle 1 \rangle$, $\langle \vec{Q} \rangle$, $\langle \vec{Q} \vec{Q} \rangle$ explicitly through 4 loops, extrapolate by writing in terms of $\mathbb{K}(\alpha)$

- Leads to our finite-coupling proposals.
Validation

• All formulas agree with weak coupling expansions through 8 or 9 loops

• Strong coupling limit tested against string theory: area of minimal surface Alday, Maldacena, 0705.0303 plus constant from determinant of scalars for $S^5$ in $\text{AdS}_5 \times S^5$ Basso, Sever, Vieira, 1405.6350

• On diagonal, TBA can be done analytically →

$$\frac{\ln \mathcal{E}(u, u, u)}{\Gamma_{\text{cusp}}} \bigg|_{g \to \infty} = -\frac{3}{4\pi} \ln^2 u - \frac{\pi^2}{12} - \frac{\pi}{6} + \frac{\pi}{72}$$

• Agrees perfectly with strong coupling limit of $C_0$
Summary & Outlook

- Planar N=4 SYM scattering amplitudes/Wilson Loops determined to high loop order by writing linear combination of right functions and imposing boundary constraints.
- Rich information about many different kinematic limits.
- Along with pentagon OPE, leads to proposal for finite-coupling behavior at origin, $u,v,w \sim 0$, in terms of tilted BES equations.
- Three anomalous dimensions and three determinants, all with similar analytic structure and behavior.
- Next challenges:
  - NMHV at origin
  - analogous kinematics for $> 6$ gluons (new $\alpha$ values?)
  - interpolation between origin and near-collinear limits
Extra Slides
BDS-like ansatz

\[
\frac{A_{6}^{\text{BDS-like}}}{A_{6}^{\text{MHV}(0)}} = \exp \left[ \sum_{L=1}^{\infty} a^{L} \left( f^{(L)}(\epsilon) \frac{1}{2} \hat{M}_{6}(L\epsilon) + C^{(L)} \right) \right]
\]

where

\[
 f^{(L)}(\epsilon) = \frac{1}{4} \gamma_{K}^{(L)} + \epsilon \frac{L}{2} G_{0}^{(L)} + \epsilon^{2} f_{2}^{(L)}
\]

are constants, and

\[
\hat{M}_{6}(\epsilon) = M_{6}^{1-\text{loop}}(\epsilon) + Y(\mathit{u}, \mathit{v}, \mathit{w})
\]

\[
= \sum_{i=1}^{6} \left\{ \frac{1}{\epsilon^{2}} (1 - \epsilon \ln s_{i,i+1}) - \ln s_{i,i+1} \ln s_{i+1,i+2} + \frac{1}{2} \ln s_{i,i+1} \ln s_{i+3,i+4} \right\} + 6 \zeta_{2}
\]

- \(Y\) is dual conformally invariant part of one-loop amplitude \(M_{6}^{1-\text{loop}}\) containing all 3-particle invariants:

\[
Y(\mathit{u}, \mathit{v}, \mathit{w}) = -\mathcal{E}^{(1)} = -\text{Li}_{2} \left( 1 - \frac{1}{\mathit{u}} \right) - \text{Li}_{2} \left( 1 - \frac{1}{\mathit{v}} \right) - \text{Li}_{2} \left( 1 - \frac{1}{\mathit{w}} \right)
\]

- More minimal BDS-like ansatz contains all IR poles, but no 3-particle invariants.
Cosmic normalization

- To fit amplitudes into the minimal space of functions requires, starting at 3 loops, redefining the BDS-like ansatz, by a multi-loop constant $\rho$:

$$A_{6}^{\text{BDS-like}'} = A_{6}^{\text{BDS-like}} \times \rho$$

$$\rho(g^2) = 1 + 8(\zeta_3)^2 g^6 - 160\zeta_3\zeta_5 g^8 + \left[1680\zeta_3\zeta_7 + 912(\zeta_5)^2 - 32\zeta_4(\zeta_3)^2\right] g^{10}$$

$$- \left[18816\zeta_3\zeta_9 + 20832\zeta_5\zeta_7 - 448\zeta_4\zeta_3\zeta_5 - 400\zeta_6(\zeta_3)^2\right] g^{12}$$

$$+ \left[221760\zeta_3\zeta_{11} + 247296\zeta_5\zeta_9 + 126240(\zeta_7)^2 - 3360\zeta_4\zeta_3\zeta_7 - 1824\zeta_4(\zeta_5)^2$$

$$- 5440\zeta_6\zeta_3\zeta_5 - 4480\zeta_8(\zeta_3)^2\right] g^{14} + \mathcal{O}(g^{16}).$$

- Now we have a flux tube interpretation for $\rho$!
A little number theory

- Classical polylogs evaluate to Riemann zeta values
  \[ \text{Li}_n(u) = \int_0^u \frac{dt}{t} \text{Li}_{n-1}(t) = \sum_{k=1}^\infty \frac{u^k}{k^n} \]
  \[ \text{Li}_n(1) = \sum_{k=1}^\infty \frac{1}{k^n} = \zeta(n) \equiv \zeta_n \]

- HPL’s evaluate to nested sums called multiple zeta values (MZVs):
  \[ \zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0} \frac{1}{k_1^{n_1}k_2^{n_2}\cdots k_m^{n_m}} \]
  Weight \( n = n_1 + n_1 + \ldots + n_m \)

- MZV’s obey many identities, e.g. stuffle
  \[ \zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2} \]

- All reducible to Riemann zeta values until weight 8.
  Irreducible MZVs: \( \zeta_{5,3}, \zeta_{7,3}, \zeta_{5,3,3}, \zeta_{9,3}, \zeta_{6,4,1,1}, \ldots \)

- At the origin, no MZV’s
At \((u,v,w) = (1,1,1)\), amplitude \(\to \text{MZVs}\)

\[
\begin{align*}
\mathcal{E}^{(1)}(1,1,1) &= 0, \\
\mathcal{E}^{(2)}(1,1,1) &= -10 \zeta_4, \\
\mathcal{E}^{(3)}(1,1,1) &= \frac{413}{3} \zeta_6, \\
\mathcal{E}^{(4)}(1,1,1) &= -\frac{5477}{3} \zeta_8 + 24 \left[ \zeta_{5,3} + 5 \zeta_3 \zeta_5 - \zeta_2 (\zeta_3)^2 \right], \\
\mathcal{E}^{(5)}(1,1,1) &= \frac{379957}{15} \zeta_{10} - 12 \left[ 4 \zeta_2 \zeta_{5,3} + 25 (\zeta_5)^2 \right] \\
&\quad - 96 \left[ 2 \zeta_{7,3} + 28 \zeta_3 \zeta_7 + 11 (\zeta_5)^2 - 4 \zeta_2 \zeta_3 \zeta_5 - 6 \zeta_4 (\zeta_3)^2 \right].
\end{align*}
\]

\[E^{(1)}(1,1,1) = -2 \zeta_2,\]
\[E^{(2)}(1,1,1) = 26 \zeta_4,\]
\[E^{(3)}(1,1,1) = -\frac{940}{3} \zeta_6,\]

\[E^{(4)}(1,1,1) = -\frac{36271}{9} \zeta_8 - 24 \left[ \zeta_{5,3} + 5 \zeta_3 \zeta_5 - \zeta_2 (\zeta_3)^2 \right],\]
\[E^{(5)}(1,1,1) = -\frac{1666501}{30} \zeta_{10} + 12 \left[ 4 \zeta_2 \zeta_{5,3} + 25 (\zeta_5)^2 \right] \\
&\quad + 132 \left[ 2 \zeta_{7,3} + 28 \zeta_3 \zeta_7 + 11 (\zeta_5)^2 - 4 \zeta_2 \zeta_3 \zeta_5 - 6 \zeta_4 (\zeta_3)^2 \right].\]

Allowed MZV’s obey a Galois “co-action” principle, restricting the combinations that can appear

Brown, Panzer, Schnetz
Branch cut condition

• All massless particles $\rightarrow$ all branch cuts start at origin in

$$s_{i,i+1}, \; s_{i,i+1,i+2}$$

$\rightarrow$ Branch cuts all start from 0 or $\infty$ in

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \quad \text{or} \quad v \quad \text{or} \quad w$$

$\rightarrow$ Only 3 weight 1 functions, not 9: $\{ \ln u, \ln v, \ln w \}$

• Discontinuities commute with branch cuts
• Require derivatives of higher weight functions to obey branch-cut condition too.
• Powerful constraint: At weight 8 (four loops) would have 1,675,553 functions without it; exactly 6,916 with it.
• But almost all of the 6,916 functions are still unphysical.
Steinmann relations


- Amplitudes should not have overlapping branch cuts:

\[
\text{Disc}_{s_{234}} \left[ \text{Disc}_{s_{123}} E(u, v, w) \right] = 0
\]

Not Allowed

Allowed

Violated by ABDK and BDS ansatz!

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Steinmann relations

S. Caron-Huot, LD, M. von Hippel, A. McLeod, 1609.00669

\[
\text{Disc}_{s_{234}} \left[ \text{Disc}_{s_{123}} \mathcal{E}(u, v, w) \right] = 0 + \text{cyclic conditions}
\]

\[
u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \quad v = \frac{s_{23}s_{56}}{s_{234}s_{123}} \quad w = \frac{s_{61}s_{34}}{s_{345}s_{234}}
\]

\[
\ln^2 u \quad \ln^2 \frac{uv}{w} \quad \frac{uv}{w} = \frac{s_{12}s_{23}s_{45}s_{56}}{s_{34}s_{61}s_{123}^2}
\]

Weight 2 functions restricted to 6 out of 9:

\[
\text{Li}_2(1 - 1/u) \quad \text{Li}_2(1 - 1/v) \quad \text{Li}_2(1 - 1/w)
\]

\[
\ln^2 \frac{uv}{w} \quad \ln^2 \frac{vw}{u} \quad \ln^2 \frac{wu}{v}
\]

Analogous constraints for \( n=7 \)

LD, J. Drummond, T. Harrington, A. McLeod, G. Papathanasiou, M. Spradlin, 1612.08976