Derivation of solid-angle formula for a right rectangular pyramid

We want to know the solid angle subtended by the base of a right rectangular pyramid when viewed from the apex. Note that this pyramid consists of four planes that pass through the origin and intersect the unit sphere on great-circle arcs. Therefore, it will not be correct to simply integrate the differential area element \( \sin \theta d\phi d\theta \) in spherical coordinates using constant limits of integration. This procedure would give us the solid angle between two lines of latitude and two lines of longitude; however, lines of latitude are not great circles, so this is clearly incorrect.

Instead, to find the proper limits of integration, it will be necessary to write down the equations for the four planes making up the faces of the pyramid and integrate the area between them on the unit sphere. It is convenient to work symmetrically about the \((\phi = 0, \theta = \pi/2)\) axis—that is, the \(x\) axis in a Cartesian system. Then two of the planes of interest will pass through the \(y\) axis and two will pass through the \(z\) axis.

Let our pyramid have apex angles (measured between opposite faces of the pyramid) \(\alpha\) and \(\beta\), both in the range \((0, \pi)\). Then one of our faces is defined by a plane passing through the \(y\) axis at an angle \(\alpha/2\) from the \(x\) axis. Its equation is \(z = x \tan \alpha/2\), or, converting to spherical coordinates,
\[
\theta = \cot^{-1} \left[ \tan \frac{\alpha}{2} \cos \phi \right].
\]
Similarly, a plane passing through the \(y\) axis at an angle \(\beta/2\) from the \(x\) axis is given simply by
\[
\phi = \beta/2.
\]

These equations define our limits of integration, and the solid angle becomes
\[
\Omega = \int_{-\beta/2}^{\beta/2} d\phi \int_{\theta_-}^{\theta_+} \sin \theta d\theta,
\]
where \(\theta_\pm = \cot^{-1}[\tan(\pm \alpha/2) \cos \phi] = \pm \cot^{-1}[\tan(\alpha/2) \cos \phi]\). Performing the \(\theta\) integral gives
\[
\Omega = 2 \int_{-\beta/2}^{\beta/2} d\phi \cos |\theta_\pm|.
\]

We now apply four trigonometric identities to the integrand:
\[
cot^{-1} x = \pi/2 - \tan^{-1} x \\
\cos(\pi/2 - x) = -\sin(x)
\]
\[
\sin[\tan^{-1} x] = \frac{x}{\sqrt{1 + x^2}}
\]
\[
\cos^2 x = 1 - \sin^2 x.
\]

From this we obtain

\[
\Omega = 2 \int_{-\beta/2}^{\beta/2} \frac{\cos \phi d\phi}{\sqrt{\cot^2(\alpha/2) + 1 - \sin^2 \phi}}
\]
\[
= 2 \sin^{-1} \left( \frac{\sin \phi}{\sqrt{\cot^2(\alpha/2) + 1}} \right) \bigg|_{-\beta/2}^{\beta/2},
\]

where we have changed variables to \( u = \sin \phi \) in performing the integral. Applying the identity \( 1 + \cot^2 x = 1 / \sin^2 x \), we get

\[
\Omega = 2 \sin^{-1} \left[ \sin \phi \sin(\alpha/2) \right] \bigg|_{-\beta/2}^{\beta/2},
\]

and making use of the antisymmetry of the sine and arcsine functions, we arrive at the final formula:

\[
\Omega = 4 \sin^{-1} \left[ \sin(\alpha/2) \sin(\beta/2) \right].
\]