SELF-CONSISTENT MODEL FOR THE BEAMS IN ACCELERATORS

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Abstract

It is proposed to use Ensembles of particles instead of ”macro” particles for modeling the beams in accelerators. Each Ensemble describes the dynamics of the real bunch in the 6 dimensional phase space, taking into account all coupling effects, coming from the relativistic relation between momentum projections and energy. Ensemble parameters include average coordinate and momentum, bunch sizes, momentum spread and all second order correlation parameters of the bunch in the phase space. Self-consistent equations for the Ensemble parameters are derived from Vlasov equation. Examples of application of this model for the TESLA Linear Collider are presented.

1 INTRODUCTION

Usually for the beam dynamics calculations, the beam is described by a set of ”macro” particles. ”Macro” particle is an ensemble of particles for the bunch field calculation. However for the trajectory calculation, ”macro” particle becomes a single particle. The motion of the particles inside ”macro” particle is not considered. Therefore, for the emittance calculations of the beam with non zero energy spread, relatively large number of ’macro” particles is needed.

2 PHASE DISTRIBUTION FUNCTION

There is another possibility to describe the beam by the phase distribution function f of particle density in the phase space of coordinates and momentum: $(\overrightarrow{r}, \overrightarrow{p})$

$$f = f(t, x, y, z, p_x, p_y, p_z) \qquad \int f(t, \overrightarrow{r}, \overrightarrow{p}) d\overrightarrow{r} d\overrightarrow{p} = 1$$

Normalized momentum ($\overrightarrow{p}$) and energy (γ) are used

$$\overrightarrow{p} = \frac{\overrightarrow{P}}{mc} \qquad \gamma = \frac{E}{mc^2} = \sqrt{1 + \overrightarrow{p} \bullet \overrightarrow{p}}$$

Phase distribution function f satisfies the Vlasov equation

$$\frac{d}{dt} f = \frac{\partial f}{\partial t} + \overrightarrow{grad_r}(f) \bullet \frac{\overrightarrow{p}}{\gamma} c + \overrightarrow{grad_p}(f) \bullet \frac{\overrightarrow{F}}{mc} = 0$$

The examples of of the distribution function on the two dimensional phase plane (Energy- Coordinate), as the solution of Vlasov equation with noise and radiation damping (Fokker-Plank equation) are shown on the Fig.1 and Fig.2. Results of the simulation of the longitudinal instability in the damping ring are presented.

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Our estimations of the amplitude and frequency of the surface roughness wake field [1] show that this field can be responsible for ”saw-tooth” instability in the damping ring. On the pictures the shape of the distribution function is shown at the moment, when energy spread and bunch size get extreme values. Left graphic shows the energy distribution and right - the particle distribution together with cavity RF and wake voltage. The distributions of the bunch with small number of particles are shown by dotted lines. Upper curves show energy spread and bunch size in time, measured in the period of synchrotron oscillations.

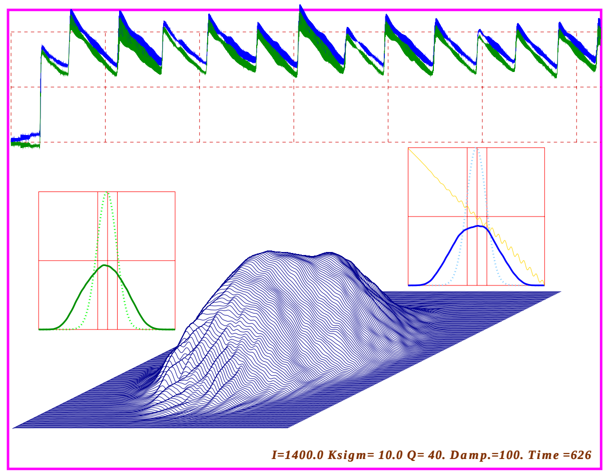

Figure 1: The shape of the distribution function at the moment, when energy spread and bunch size have maximum value.

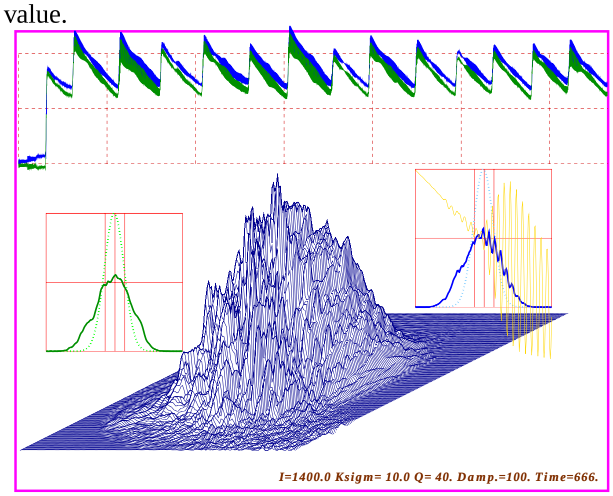

Figure 2: The shape of the distribution function at the moment, when energy spread and bunch size have minimum value.

Unfortunately, the direct numerical solution of the Vlasov equation in 6 dimensional case needs a great amount of the computer memory. If, for finite - difference approximation, we take only 50 mesh points in each direction, then the total number of mesh points will be $50^6 = 1.56\ 10^{10}$, that is equal to the number of real particles in the bunches of Linear Colliders.

3 SELF-CONSISTENT MODEL OF THE BEAM

However, the Vlasov equation can be numerically solved more easily for the bunch of particles, interacting with linear forces only. In this case the distribution function is described by a small number of parameters, only by the first and second order moments. Let us define such a bunch as an "Ensemble". If we can have full description of the dynamics of an Ensembles, then by a set of Ensembles we can describe the dynamics of any beam and all kinds of forces. Nevertheless, even one Ensemble can give a lot of information about beam dynamics in the Linear Collider.

The self-consistent equations for the Ensemble parameters are derived from Vlasov equation. Opposite to the usual model for the second order moments (nice description can be find in the book of A.Chao[2]), the model of Ensembles includes longitudinal motion of the particles, and all corresponding correlation with transverse motion. Let us define the Ensemble parameters.

3.1 Ensemble Parameters

First order moments

$$\langle\xi\rangle = \int f(t,\vec{r},\vec{p})\xi d\vec{r}d\vec{p}$$

give the Average Position in the Phase Space

$$\langle x\rangle \quad \langle y\rangle \quad \langle z\rangle \quad \langle p_x\rangle \quad \langle p_y\rangle \quad \langle p_z\rangle$$

Second order moments

$$M_{\xi\nu} = \langle\xi\nu\rangle = \int f(t,\vec{r},\vec{p})(\xi - \langle\xi\rangle)(\nu - \langle\nu\rangle)d\vec{r}d\vec{p}$$

give Effective Beam Sizes (squared)

$$M_{xx} = \sigma_x^2 = \langle(x - \langle x\rangle)^2\rangle$$
$$M_{yy} = \sigma_y^2$$
$$M_{zz} = \sigma_z^2$$

Correlated Sizes

$$M_{xy} = \langle xy\rangle_{avc} = \langle(x - \langle x\rangle)(y - \langle y\rangle)\rangle$$
$$M_{xz} = \langle xz\rangle_{avc}$$
$$M_{yz} = \langle yz\rangle_{avc}$$

Correlated Momentum Spread

$$M_{xp_x} = \langle xp_x\rangle_{avc} = \langle(x - \langle x\rangle)(p_x - \langle p_x\rangle)\rangle$$
$$M_{yp_y} = \langle yp_y\rangle_{avc}$$
$$M_{zp_z} = \langle zp_z\rangle_{avc}$$
$$M_{xp_y} = \langle xp_y\rangle_{avc}$$
$$M_{yp_x} = \langle yp_x\rangle_{avc}$$
$$M_{xp_z} = \langle xp_z\rangle_{avc}$$
$$M_{zp_x} = \langle zp_x\rangle_{avc}$$
$$M_{yp_z} = \langle yp_z\rangle_{avc}$$
$$M_{zp_y} = \langle zp_y\rangle_{avc}$$

Uncorrelated Momentum Spread (squared)

$$M_{p_xp_x} = \sigma_{p_x}^2$$
$$M_{p_yp_y} = \sigma_{p_y}^2$$
$$M_{p_zp_z} = \sigma_{p_z}^2$$

Momentum Correlations

$$M_{p_xp_y} = \langle p_xp_y\rangle_{avc}$$
$$M_{p_xp_z} = \langle p_xp_z\rangle_{avc}$$
$$M_{p_yp_z} = \langle p_yp_z\rangle_{avc}$$

3.2 Presentation by a Matrix

At the same time, the second order moments describe the 6 dimensional ellipse in phase space. If we combine the second order moments in a matrix M

$$\begin{pmatrix} M_{xx} & M_{p_xx} & M_{yx} & M_{p_yx} & M_{zx} & M_{p_zx} \\ M_{xp_x} & M_{p_xp_x} & M_{yz} & M_{p_yp_x} & M_{zp_x} & M_{p_z,p_x} \\ M_{xy} & M_{p_xy} & M_{yy} & M_{p_yy} & M_{zy} & M_{p_zy} \\ M_{xp_y} & M_{p_xp_y} & M_{yp_y} & M_{p_yp_y} & M_{zp_y} & M_{p_zp_y} \\ M_{xz} & M_{p_xz} & M_{yz} & M_{p_yz} & M_{zz} & M_{p_zz} \\ M_{xp_z} & M_{p_xp_z} & M_{yp_z} & M_{p_yp_z} & M_{zp_z} & M_{p_zp_z} \end{pmatrix}$$

then, the determinant of this matrix will give the volume (squared) of the ellipse

$$V^2 = det\{M\}$$

As this determinant is invariant for translations and rotation of the ellipse, then the volume V is the 6 dimensional normalized emittance of the Ensemble. In the case of uncoupling motion the matrix takes the form

$$\begin{pmatrix} M_{xx} & M_{p_xx} & 0 & 0 & 0 & 0 \\ M_{xp_x} & M_{p_xp_x} & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{yy} & M_{p_yy} & 0 & 0 \\ 0 & 0 & M_{yp_y} & M_{p_yp_y} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{zz} & M_{p_zz} \\ 0 & 0 & 0 & 0 & M_{zp_z} & M_{p_zp_z} \end{pmatrix}$$

Each, non zero 2×2 matrix, represents the projection of the emittance on the coordinates plane $\{xp_x\}$, or $\{yp_y\}$, or $\{zp_z\}$ and its determinant gives squared value of the emittance, like

$$\epsilon_x = \sqrt{\sigma_x{}^2\sigma_{p_x}{}^2 - \langle xp_x\rangle_{avc}{}^2} = \sqrt{M_{xx}M_{p_xp_x} - (M_{xp_x})^2}$$

In this case the full emittance is the multiplication of projection emittances

$$V^2 = \epsilon_x^2\epsilon_y^2\epsilon_z^2$$

However, when the correlated parameters are excited, the projection emittances have to be changed. For example, when we excite M_{xp_z}, the full emittance is the difference of positive values

$$V^2 = \epsilon_y^2(\epsilon_x^2\epsilon_z^2 - M_{xp_z}^2\sigma_{p_x}^2\sigma_x{}^2)$$

So, to keep it invariant, the x- and z- emittance projections have to be increased.

3.3 Time Equations

Time equations for Ensemble parameters can be derived from the Vlasov equation under two assumptions:

1) The full emittance is invariant only for the forces, that satisfy the condition

$$\langle\mu \bullet \overrightarrow{grad}_p \frac{\vec{F}}{mc^2}\rangle = 0$$

for any Ensemble parameter μ.

2) The energy spread in the beam is not very large and the energy can be presented in the following expanded way

$$\frac{1}{\gamma} = \frac{1}{\gamma_m} - \frac{1}{\gamma_m^3}\sum_n[\langle p_n\rangle(p_n - \langle p_n\rangle) + \frac{1}{2}((p_n - \langle p_n\rangle)^2 - M_{p_np_n})]$$

where γ_m is the average beam energy

$$\gamma_m = \sqrt{1 + \sum_n [\langle p_n \rangle^2 + M_{p_n p_n}]}$$

Under these conditions the average velocity $\vec{v}$ comtains additionally the momentum correlations parts

$$\mathbf{v}_n = \langle \frac{p_n}{\gamma} \rangle = \frac{\langle p_n \rangle}{\gamma_m} - \frac{1}{\gamma_m^3} \sum_k \langle p_k \rangle M_{p_n p_k}$$

Now we can use average value relation

$$\frac{\partial}{c\partial t}\langle \mu \rangle = \langle \overrightarrow{grad_r} \mu \bullet \frac{\vec{p}}{\gamma} \rangle + \langle \overrightarrow{grad_p} \mu \bullet \frac{\vec{F}}{mc^2} \rangle$$

for deriving time equations:

Average Trajectory

$$\frac{\partial}{c\partial t}\langle l \rangle = \frac{\langle p_l \rangle}{\gamma_m} - \frac{1}{\gamma_m^3} \sum_n \langle p_n \rangle M_{p_l p_n}$$

$$\frac{\partial}{c\partial t}\langle p_l \rangle = \langle \frac{F_l}{mc^2} \rangle$$

Coordinate Coordinate Relations

$$\frac{\partial}{c\partial t} M_{lk} = \frac{1}{\gamma_m}(M_{kp_l} + M_{lp_k}) -$$

$$-\frac{1}{\gamma_m^3}(\langle p_l \rangle \sum_n \langle p_n \rangle M_{kp_n} + \langle p_k \rangle \sum_n \langle p_n \rangle M_{lp_n})$$

Coordinate Momentum Relations

$$\frac{\partial}{c\partial t} M_{lp_k} = \frac{1}{\gamma_m} M_{p_l p_k} - \frac{1}{\gamma_m^3}\langle p_l \rangle \sum_n \langle p_n \rangle M_{p_k p_n} +$$

$$+\langle (l - \langle l \rangle) \frac{F_k}{mc^2} \rangle$$

Momentum Momentum Relations

$$\frac{\partial}{c\partial t} M_{p_l p_k} = \langle (p_l - \langle p_l \rangle) \frac{F_k}{mc^2} \rangle + \langle (p_k - \langle p_k \rangle) \frac{F_l}{mc^2} \rangle$$

4 PROPERTIES OF ENSEMBLE

4.1 Fundamental Conservation Laws

We can check the model for realization of the dynamics laws for average values of the bunch. It is easy to show, that from presented equations one obtains:

$$\frac{\partial}{c\partial t}\langle \vec{p} \rangle = \langle \frac{\vec{F}}{mc^2} \rangle$$

$$\frac{\partial}{c\partial t}\langle \gamma \rangle^2 = \frac{\partial}{c\partial t}\gamma_m^2 = 2\langle \vec{p} \bullet \overrightarrow{\frac{F}{mc^2}} \rangle$$

$$\frac{\partial}{c\partial t}\vec{\mathcal{M}} = \frac{\partial}{c\partial t}\langle \vec{r} \times \vec{p} \rangle = \langle \vec{r} \times \overrightarrow{\frac{F}{mc^2}} \rangle$$

So, the model fulfills the fundamental conservation laws for momentum, energy and angular momentum.

4.2 Emittance Equation

We can also estimate the modification of the full emittance for the case, when the presented assumptions are not fulfilled. When we have only longitudinal force and transverse average momentums are zero, the model gives the following equation for longitudinal emittance

$$\frac{\partial}{c\partial t}\epsilon_z^2 = 2M_{zz}\langle (p_z - \langle p_z \rangle) \frac{F_z}{mc^2} \rangle - 2M_{zp_z}\langle (z - \langle x \rangle) \frac{F_z}{mc^2} \rangle$$

When the force is proportional to the momentum

$$F_z = \alpha mc^2 p_z$$

the equation takes the following form

$$\frac{\partial}{c\partial t}\epsilon_z^2 - 2\alpha\epsilon_z^2 = 0$$

This equation is solved very easily. In agreement with the sign of α, the full emittance will exponentially increase, or decrease (friction force) with the time

$$\epsilon_z = \epsilon_z^0 exp(\alpha ct)$$

and $1/\alpha$ is the effective distance of the effect.

At the same time the transverse emittances are invariant

$$\frac{\partial}{c\partial t}\epsilon_x^2 = 0 \qquad \frac{\partial}{c\partial t}\epsilon_y^2 = 0$$

4.3 Bunch Compression in Free Space

Here we check the modification of the bunch in free space, when the bunch has negative correlated momentum spread $M^0_{xp_x} < 0$, or $M^0_{yp_y} < 0$, or $M^0_{zp_z} < 0$

In case, when $\langle p_z \rangle \gg \langle p_x \rangle$

$$M_{xx} = M^0_{xx} + 2\frac{c\Delta t}{\gamma_m} M^0_{xp_x} + (\frac{c\Delta t}{\gamma_m})^2 M^0_{p_x p_x}$$

$$M_{zz} = M^0_{zz} + 2\frac{c\Delta t}{\gamma_m^3} M^0_{zp_z} + (\frac{c\Delta t}{\gamma_m^3})^2 M^0_{p_z p_z}$$

Minimum size of the bunch is determined by the emittance and uncorrelated momentum spread, as it usually is

$$min(\sigma_x^2) = M^0_{xx} - \frac{(M^0_{xp_x})^2}{M^0_{p_x p_x}} = \frac{\epsilon_x^2}{M^0_{p_x p_x}} = \frac{\epsilon_x^2}{\sigma_{p_x}^2}$$

5 APPLICATION OF THE MODEL

The derived equations for Ensemble parameters are nonlinear. For some simple cases we can find analytical solution. However, to use these equations for computer simulations correct numerical algorithm is needed. We present examples of analytical and numerical solution.

5.1 Ensemble in an Accelerating Structure

From this model we can have the estimation for the frequency ϖ of the longitudinal oscillations of a bunch in an accelerating field. Let the average center of the bunch move with the speed, equal to the phase velocity of the accelerating wave, having the phase displacement φ_0. The linear part of the force is

$$\frac{F_z}{mc^2} = E + \delta E(z - \langle z\rangle)$$

$$E = \frac{e}{mc^2} E_{acc} \cos\varphi_0 \quad \delta E = \frac{e}{mc^2} E_{acc} \frac{2\pi}{\lambda} \sin\varphi_0$$

For the longitudinal bunch size (squared) the model gives the equations

$$\frac{\partial}{c\partial t} M_{zz} = \frac{2}{\gamma_m^3} M_{zp_z}$$

$$\frac{\partial}{c\partial t}(\gamma_m^3 \frac{\partial}{c\partial t} M_{zp_z}) + 4\delta E M_{zp_z} = \delta E M_{zz} \frac{\partial \gamma_m^3}{c\partial t}$$

From which we can estimate ϖ

$$\varpi / c = \sqrt{\frac{\delta E}{\gamma^3}} = \sqrt{\frac{2\pi}{\lambda} \frac{eE_{acc}}{\gamma^3 mc^2} \sin\varphi_0}$$

If the sign of $\sin\varphi_0$ is negative, we have ”exponential” growth of the longitudinal beam size.

5.2 *Space Charge Effect*

The force, acting on particles, moving together is

$$\vec{F} = \frac{e}{\gamma} * \frac{\vec{p}(\vec{p}\vec{R}) + \vec{R}}{((\vec{p}\vec{R})^2 + R^2)^{3/2}}$$

Where $\vec{p}$ is the momentum, $\vec{R} = \vec{r_0} - \vec{r}$ is the vector between coordinates of the particles. To find the space charge force in the bunch with homogenous charge density, one have to calculate the force from ”uncompensated” charge only. This charge ”appears”, when the point of observation is shifted from the average center (Fig.3). This approach

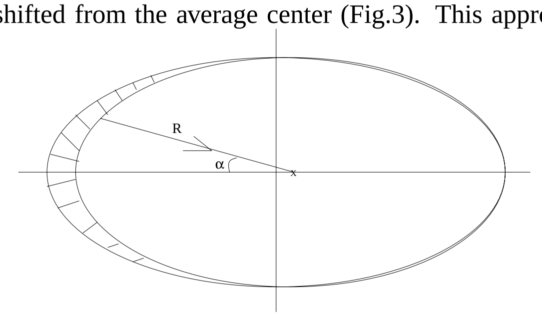

Figure 3: Effective charged area, that forms the force in the point X near the average center.

gives estimation for the space charge force

$$\vec{F} = \frac{eQ}{\gamma^2} \frac{\vec{R}}{V_g}$$

The force is linear with the distance from the average center and inversely proportional to geometrical volume

$$V_g = det \begin{pmatrix} M_{xx} & M_{xy} & M_{xz} \\ M_{yx} & M_{yy} & M_{yz} \\ M_{zx} & M_{zy} & M_{zz} \end{pmatrix}$$

This formula and the model give the equations for the charged bunch in free space. For the transverse beam size and correlated momentum spread

$$\sigma_x = \sqrt{M_{xx}} \quad \sigma_{p_x} = \sqrt{M_{p_x x}}$$

we have

$$\frac{\partial}{c\partial t} M_{xx} = \frac{2}{\gamma_m} M_{xp_x} \quad æ = \frac{r_0 N}{\gamma_m^2 V_g}$$

$$\frac{\partial^2}{c^2 \partial t^2} M_{xp_x} - \frac{4æ}{\gamma_m} M_{xp_x} = 0$$

For longitudinal beam size and correlated momentum spread we have the same equation, but with the

$$æ_l = æ/\gamma_m^2$$

The equations give growing in time solutions. Effective length $L_{eff}^{\perp}$, where the space charge really enlarges the transverse beam size is

$$L_{eff}^{\perp} = \sqrt{\frac{\gamma_m}{æ}} = \sqrt{\frac{\gamma_m^3 \sigma_x \sigma_y \sigma_z}{r_0 N}}$$

The effective parameter for the longitudinal size is γ_m larger. However this size does not increase too much, as the transverse enlargement decrease the charge density very soon.

5.3 *Ensemble in the Homogeneous Magnetic Field*

Imagine, that relativistic bunch has initial energy spread $\delta\gamma$ before injection into the magnetic field. Then the particles with different energy will have different radiuses of rotation and therefore different time for one turn, as they are moving with the speed close to the speed of light

$$T = \langle T\rangle \frac{\gamma}{\langle\gamma\rangle}$$

where $\langle T\rangle$ and $\langle\gamma\rangle$ are the average period and energy. The bunch size σ is increasing in time t as

$$\sigma = \langle R\rangle \sin(2\pi \frac{t}{\langle T\rangle} \frac{\delta\gamma}{\langle\gamma\rangle})$$

where $\langle R\rangle$ is the average radius. After the time

$$T = \frac{\langle T\rangle}{4} \frac{\langle\gamma\rangle}{\delta\gamma}$$

the bunch takes the circumference of the circle.

Now we use numerical calculations to study this phenomena in the frame of our model. We carry simulations for the bunch with initial energy spread of ± 1 %. Comparison of the numerical and analytical results for the bunch size is given in Fig.4. Results for the bunch momentum and energy spread are shown on Fig.5. After 25 turns the center of bunch comes to center of rotation and the beam size reaches the value of rotation radius. This position is not stable, but it is repeated with the period of 50 turns. After 25 turns average momentum and its projections become zero, but energy spread reaches maximum value.

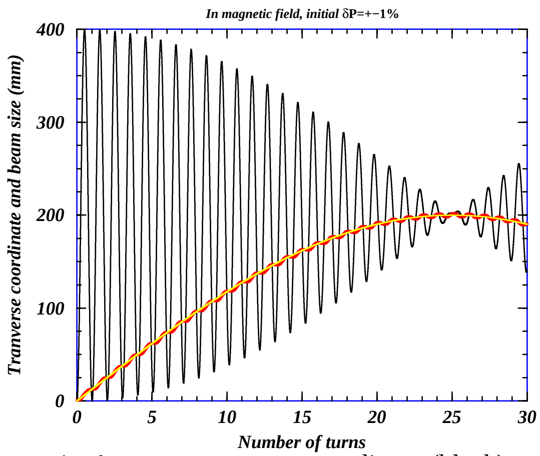

Figure 4: Average transverse coordinate (black) and the bunch size (red). Yellow line presents the analytical formula.

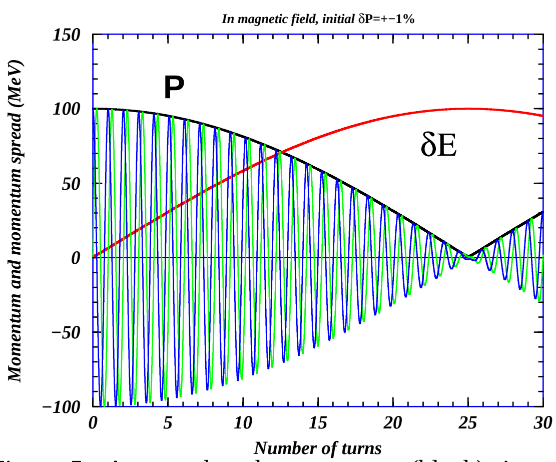

Figure 5: Average bunch momentum (black), its projections (blue and green) and energy spread (red).

5.4 *Chicane Bunch Compression*

The TESLA bunch compressor consists of four rectangular dipole magnets, where bunch at first is deflecting in transverse direction and then is forwarding back. When the bunch gets the transverse deflection, it also gets correlated moments M_{xp_z},M_{zp_x} and $M_{p_xp_z}$, M_{xz}, that change the emittance projections. In symmetrical chicane the emittance projections come back to the initial values. If the second pair of magnets have another magnetic strength, then the bunch do not come to the initial transverse position (non symmetric chicane). The transverse beam size increases and the transverse emittance grows up. Emittance growth can be estimated by formula

$$\Delta\epsilon_x^2 = \frac{X^2}{\gamma_m^2} M^0_{p_zp_z} M^0_{p_xp_x}$$

Where X is the difference of initial and final transverse coordinate. Another effect, that can change the compression parameters and increase the emittance projections, is the action of the space charge forces. Results of computer simulation for the bunch with the charge of 1 nC are presented on Fig.6 and Fig.7.

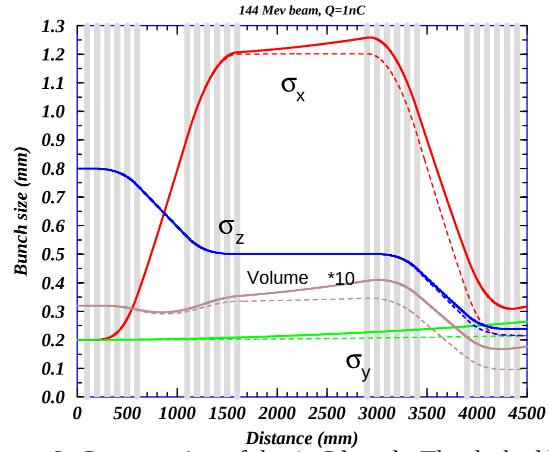

Figure 6: Compression of the 1nC bunch. The dashed lines show results without consideration of the space charge effect.

Longitudinal bunch size is compressed from 0.8 mm to 0.22 mm. Bunch volume is changed mainly in the second half of the chicane. Space charge effect do not greatly change the ratio of the longitudinal compression, but increase transverse bunch size. Space charge forces destroy the symmetry of the compression system. Transverse emittance does not come back to initial value. Emittance increase approximately five times.

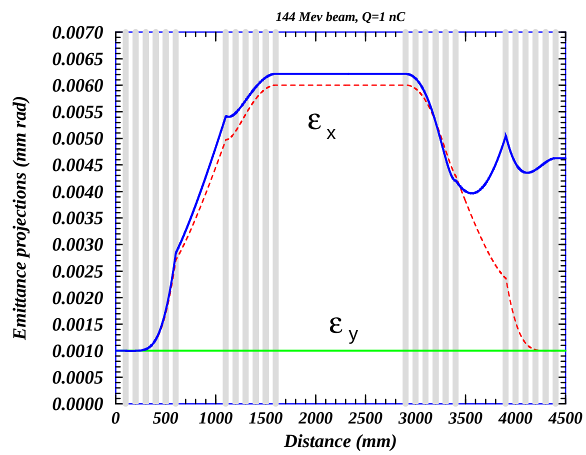

Figure 7: Emittance projections. The dashed lines show results without consideration of the space charge effect.

6 ACKNOWLEDGEMENTS

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