WAKE OF A ROUGH BEAM WALL SURFACE

K. L. F. Bane and G. V. Stupakov
SLAC, 2575 Sand Hill Road, Menlo Park, CA. 95014 USA

Abstract

We review and compare two models recently developed for the impedance calculation of a rough surface.

1 INTRODUCTION

Future linear accelerators and FELs tend to use short, intense bunches with small emittances and small energy spreads. For example, the design of the Linac Coherent Light Source (LCLS) at SLAC requires a bunch with a peak current of 3.4 kA, an rms bunch length of 30 μm, a normalized emittance of 1 mm-mr, and an rms energy spread of 0.1%[1]. One concern in such machines is that induced wakefields may significantly increase the beam emittance or energy spread. It has been pointed out in [2, 3] that one major source of wakefields in machines with short bunches might be the roughness of the beam tube surface.

The model developed in Ref. [2] assumes that a rough surface can be represented as a collection of bumps of relatively simple shapes (hemisheres, half cubes, etc.), and the total impedance can be approximated as the sum of the impedances of the individual bumps. Recently, another approach has been developed [4], one using a small-angle approximation in the wall surface profile. It assumes that the wall surface discontinuities are gradual in the direction along the wall surface. In this approach the impedance of the rough surface is expressed in terms of the spectral function of the surface profile. The result represents the contribution of different scales, and can be used to estimate the impedance based on the statistical properties of the surface.

In this paper we review and compare the two approaches. Note that in both models we assume that the depth of the surface perturbations are large compared to the skin depth at the frequencies of interest, and that we can therefore ignore the effect of the resistance of the wall material. Note further that both models yield a total impedance that is inductive in character. Another model, one that says that the effect of a rough surface is similar to that of a thin dielectric layer, and that yields a resonator type of impedance [3], will not be discussed here. Finally, note that, for brevity, we consider here only the longitudinal impedance. In the case of the LCLS undulator beam tube, for example, it appears that this is the dominant wakefield effect. Once the longitudinal impedance is known, however, the transverse impedance of a rough surface on a cylindrical beam tube can be easily obtained, as is shown, for example, in Ref. [2].

2 SIMPLE MODEL

In the model developed in Ref. [2], it was assumed that a rough surface can be represented as a random distribution of small bumps and cavities of a certain size — the granularity size — on a smooth surface. Since the impedance of a small bump tends to be significantly larger than that of a cavity of similar size, the effect of cavity-like features was neglected. Then a rough surface can be represented as a collection of bumps, as sketched in Fig. 1. The longitudinal impedance of a single hemisphere of radius r on the surface of a tube of radius b for $\omega \ll r/b$ is given by [5]

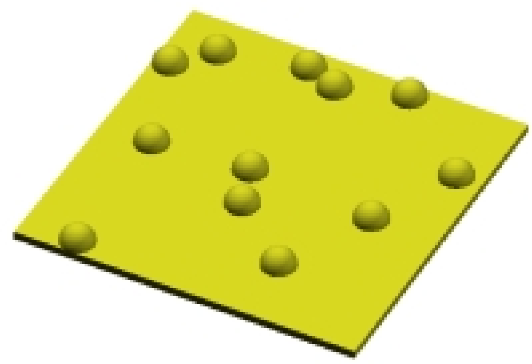


Figure 1: Rough surface is represented as collection of bumps of a given shape randomly distributed on the surface.

$$Z_1(\omega) = -i\omega\mathcal{L} = -i\omega\frac{Z_0}{4\pi c}\frac{r^3}{b^2}, \tag{1}$$

where $\mathcal{L}$ is the inductance, ω the frequency, $Z_0 = 377\ \Omega$, and c the speed of light. For a small object of a different shape the above formula needs to be multiplied by a form factor f. Numerically obtained form factors for some simple shapes are given in Table 1 [2]. By comparing the result for a cube and a half cube note a strong, roughly quadratic dependence of f on bump height.

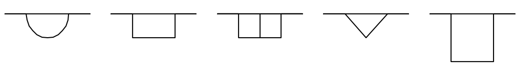
For many bumps, assuming they are separated by at least their size, the total impedance is approximated by the sum of the impedances of the individual bumps. The total impedance *per unit length* of the beam tube then becomes

$$Z(\omega) = -\alpha f\frac{iZ_0\omega}{2\pi c}\frac{r}{b}, \tag{2}$$

with α a packing factor equal to the relative area on the surface occupied by the bumps. As we see, the longitudinal impedance in this model is purely imaginary (inductive).

Table 1: Form factors for 5 selected objects with the same base area. The figure at the bottom of the table shows the shapes of the respective objects, counted from left to right.

Case	f
Hemisphere	1
Half Cube	2.6
Rotated Half Cube	0.6
Wedge	1.1
Cube	10.8



In applications such as the LCLS undulator, an important parameter is the energy spread of the bunch, which can be increased due to the roughness impedance. For a Gaussian bunch with rms length $\sigma_z \gg r$ the total rms energy spread induced by the roughness of the beam tube is given by [2]

$$\delta E_{rms} = Ne^2 L W_{rms}, \tag{3}$$

where N is the number of particles in the bunch, L is the beam tube length, and

$$W_{rms} = -\alpha f \frac{cZ_0}{3^{1/4}2^{3/2}\pi^{3/2}} \frac{r}{b\sigma_z^2}. \tag{4}$$

Using the above expression for the impedance, we can now estimate the effect of the roughness wake in the LCLS undulator using the following parameters: undulator length – $L = 100\,\mathrm{m}$, beam charge – $Ne = 1\,\mathrm{nC}$, $f = 1$, $\alpha = 0.5$, $\sigma_z = 30\,\mu\mathrm{m}$, $b = 3\,\mathrm{mm}$, beam energy – $E = 15\,\mathrm{GeV}$. For the energy spread increase due to the wake $\sigma_\delta < 0.05\%$, the height of the bumps should be

$$r < 50\,\mathrm{nm}. \tag{5}$$

If these parameters are accurate, then the requirement on the smoothness of the beam tube surface are severe.

Small-Angle Approximation

The detailed derivation of the impedance in the small-angle approximation can be found elsewhere [4]. Here we outline the main assumptions and present the final result of this model.

The approach is based on the assumption that the angle between the normal to the rough surface and the radial direction is small compared to unity. If we assume that the rough surface is given by the equation $y = h(x, z)$, where x, y and z are the cartesian coordinates, and h is the local height of the surface, then the small-angle approximation means that

$$|\nabla h| \ll 1\,. \tag{6}$$

This assumption allows us to develop a rather general theory of the impedance, which gives good accuracy even when $|\nabla h| \sim 1$.

In addition to Eq. (6), we also require that the height of the bumps and their characteristic width g be small compared to the radius of the pipe b,

$$g,\, |h| \ll b\,. \tag{7}$$

Evidently, this inequality is easily satisfied for realistic values of g, h and b. Finally, because typically the size of the surface bumps g is on the order of microns, and the bunch length σ_z is on the order of at least tens of microns, we also assume that the characteristic frequency of interest $\omega \sim c/\sigma_z$ is small compared to c/g,

$$\omega \ll c/g\,. \tag{8}$$

Using approximations (6) – (8), one can show that for a single bump of arbitrary shape $h_0(x, z)$ sitting on the surface of a round beam pipe, the impedance is

$$Z_1(\omega) = -\frac{ikZ_0}{b^2} \int_{-\infty}^{\infty} \frac{\kappa_z^2 |\hat{h}_0(\kappa_z, \kappa_x)|^2}{\sqrt{\kappa_x^2 + \kappa_z^2}} d\kappa_z d\kappa_x, \tag{9}$$

where $\hat{h}_0$ is a two dimensional Fourier transform of the bunch shape:

$$\hat{h}_0(\kappa_z, \kappa_x) = \frac{1}{(2\pi)^2} \int_{\infty}^{\infty} h_0(x, z) e^{-i\kappa_z z - i\kappa_x x}\, dz dx, \tag{10}$$

where the z-axis is directed along the pipe axes, and the x axis is locally directed along the azimuthal coordinate θ. We note, that due to assumed smallness of the surface structures, we can use the local Cartesian coordinate system x, y and z in Eqs. (9) and (10) instead of the global cylindrical coordinate system θ, r and z.

To describe a rough surface with a random profile, we assume that $h(x, y)$ is a random function with zero average, $\langle h(x, z)\rangle = 0$. Statistical properties of such a surface are characterized by the correlation function $K(x, y)$,

$$K(x - x', z - z') = \langle h(x', z') h(x, z)\rangle\,, \tag{11}$$

where the angular brackets denote averaging over possible realizations of $h(x, z)$. Eq. (11) implies that statistical properties of $h(x, z)$ do not depend on the position of the surface. An important statistical characteristic of the roughness is the *spectral density* (or *spectrum*) $R(\kappa_z, \kappa_x)$, defined as a Fourier transform of the correlation function,

$$R(\kappa_x, \kappa_z) = \frac{1}{(2\pi)^2} \int dx\, dz\, K(x, z) e^{-i\kappa_x x - i\kappa_z z}\,. \tag{12}$$

If the surface is statistically isotropic (all direction in the $x - y$ plane are statistically equivalent), the spectrum R depends only on the absolute value κ of the vector (κ_x, κ_z), $\kappa = \sqrt{\kappa_x^2 + \kappa_z^2}$, $R = R(\kappa)$.

The main result of Ref. [4] is that the longitudinal impedance of a circular pipe of radius b_0 with a rough perfectly conducting surface characterized by the spectral function $R(\kappa_x, \kappa_z)$ in the frequency range limited by the condition (8) is given by the following equation:

$$Z(\omega) = -\frac{ikZ_0 L}{2\pi b} \int d\kappa_z \, d\kappa_x \, R(\kappa_x, \kappa_z) \frac{\kappa_z^2}{\kappa}, \qquad (13)$$

where L is the length of the pipe.

The presence of the factor κ_z^2 in the integrand of Eq. (13) means that the contribution to Z of roughness in longitudinal (z) and azimuthal (x) directions are different. For example, bellow-type variations on the surface have spectral components with $\kappa_z \neq 0$ and $\kappa_x = 0$, and result in non-vanishing $Z(\omega)$. On the other hand, ridges going in the longitudinal direction generate a spectrum with $\kappa_x \neq 0$ and $\kappa_z = 0$, and according to Eq. (13) do not contribute to $Z(\omega)$.

As an application of Eq. (13), we can calculate the impedance of a rough surface with a Gaussian spectrum

$$S(\kappa) = \frac{l_c^2 d^2}{2\pi} e^{-\kappa^2 l_c^2/2}, \qquad (14)$$

where d is the rms height of the roughness and l_c is the correlation length in the spectrum. Performing the integration, one finds

$$\frac{Z(\omega)}{L} = -\frac{\sqrt{\pi}}{4\sqrt{2}} \frac{ikZ_0 d^2}{l_c b}. \qquad (15)$$

It is seen, that the impedance not only depends on the rms height of the bumps, but also on the correlation length l_c. Increasing this lengths makes the impedance smaller for a given rms height of the roughness. Qualitatively, l_c can be considered as a typical transverse size of the bumps in the statistical distribution.

3 FRACTAL SURFACE

Another model of a rough surface is given by a power spectrum, limited at small wavelengths,

$$\begin{aligned} R(\kappa) &= A\kappa^{-q}, \text{ for } \kappa > \kappa_0, \\ R(\kappa) &= 0, \qquad \text{for } \kappa < \kappa_0, \end{aligned} \qquad (16)$$

where κ_0 is the minimal value of the spectrum, $q > 0$ is a power index, and A defines the amplitude of the roughness. For spatial scales much smaller than κ_0^{-1}, this surface gives an example of a fractal landscape with a fractal dimension q. The parameter κ_0 can be related to the characteristic correlation length, l_c, of the random profile, $\kappa_0 \sim \pi/l_c$. We can also relate the factor A to the rms height d of the roughness,

$$d^2 = 2\pi \int_0^\infty \kappa \, d\kappa \, R(\kappa) = \frac{2\pi A}{q-2} \kappa_0^{2-q}. \qquad (17)$$

For convergence of the integral it is required that $q > 2$. The shape of the surface for two different values of q obtained with a help of computer code described in [6] is shown in Fig. 2. It turns out, that increasing the value

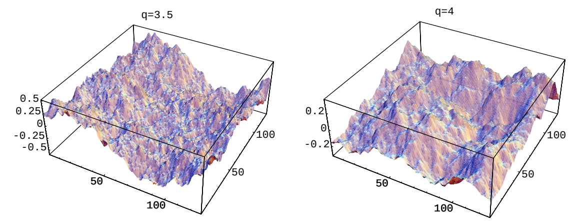


Figure 2: Fractal surfaces for $q = 3.5$ and $q = 4$. Smaller values of q give more "spiky" profiles.

of q makes the surface smoother. Using Eq. (13) we can calculated the impedance of such a surface,

$$\frac{Z(\omega)}{L} = -\frac{ikZ_0}{4\pi b} \frac{q-2}{q-3} d^2 \kappa_0. \qquad (18)$$

Again, for convergence, we require that $q > 3$, otherwise the integral diverges as $\kappa \to \infty$. This requirement is stronger than the convergence condition for Eq. (17), and is due to a relatively slow decay of the spectrum at large κ.

4 COMPARISON OF THE TWO MODELS

To compare the two models, we will calculate the impedance of a surface covered by bumps of a given shape, as illustrated by Fig. 1, using the small-angle approximation, and compare it with Eq. (2) . For the sake of generality, we will assume an arbitrary shape of the bump given by the function $h_0(x, z)$. The bumps are randomly scattered over the surface, with the average number of bumps per unit area equal to ν. We will assume that the average distance between the bumps, $\nu^{-1/2}$, is much larger than the transverse size of the bump g; then we can neglect the events when bumps overlap.

Let us consider a square on the surface of size $\mathcal{L} \times \mathcal{L}$, where $\mathcal{L}$ is large in comparison with the bump width g, but small relative to the pipe radius b, so that the effects of curvature are negligible. If this area contains N bumps, located at positions (x_n, z_n), $n = 1, 2, \ldots, N$, then the surface profile $h(x, z)$ is a superposition of all N bumps,

$$h(x, z) = \sum_{n=1}^{N} h_0(x - x_n, z - z_n). \qquad (19)$$

To calculate the spectrum $R(\kappa_x, \kappa_z)$ needed in the small-angle approximation model, we will first find the correlation function K,

$$\begin{aligned} K(\xi, \zeta) &= \langle h(x, z) h(x + \xi, z + \zeta) \rangle \qquad (20) \\ &= \sum_{n,k=1}^{N} \langle h_0(x - x_n, z - z_n) \\ &\times \quad h_0(x + \xi - x_k, z + \zeta - z_k) \rangle. \end{aligned}$$

Since we neglect overlapping of the bumps, only terms with $n = k$ contribute to the sum of Eq. (20)

$$K(\xi,\zeta) \approx \sum_{n=1}^{N} \langle h_0(x-x_n, z-z_n) \times h_0(x+\xi-x_n, z+\zeta-z_n)\rangle. \quad (21)$$

To perform averaging in Eq. (21), we will assume that the probability $p(x_n, z_n)$ for the bump to be located at the point (x_n, z_n) within dx_n and dz_n does not depend on the position, and is equal $p = \mathcal{L}^{-2}$. This assumption corresponds to a uniform distribution of bumps on the surface. Then averaging means integration over the square,

$$\langle f(x,z)\rangle = \mathcal{L}^{-2} \int_{\mathcal{L}\times\mathcal{L}} dx\, dz f(x,z), \quad (22)$$

and it reduces Eq. (21) to

$$K(\xi,\zeta) = \frac{N}{\mathcal{L}^2} \int_{\mathcal{L}\times\mathcal{L}} dx\, dz h_0(x,z) h_0(x+\xi, z+\zeta). \quad (23)$$

From Eq. (21) it follows that the correlations function for the randomly distribute bumps is equal to the correlation for a single bump multiplied by the bump density $\nu = N/\mathcal{L}^2$. Correspondingly, the spectral function R is

$$R(\kappa_x, \kappa_z) = (2\pi)^2 \nu |\hat{h}_0(\kappa_x, \kappa_z)|^2, \quad (24)$$

where $\hat{h}_0$ is given by Eq. (10). Putting this correlation function into Eq.(13) gives

$$Z(\omega) = 2\pi b L \nu Z_1(\omega), \quad (25)$$

where $Z_1(\omega)$ is given by Eq. (9). This equation tells us that the impedance of a rough surface consisting of a collection of identical bumps randomly scattered over the surface is equal to the impedance of a single bump multiplied by the number of bumps on the surface area. This result agrees with the approach used in the first model. Hence, the only difference between the two models in this limit is due to the calculation of the single bump impedance Z_1. Indeed, as shown in Ref. [4], for hemispheres, the small-angle approximation theory gives the result that is about two times smaller than the exact solution Eq. (1). Hence, for the rough surface, we will find that the two models agree within the factor of 2, with the small-angle theory giving a smaller impedance.

5 CONCLUSIONS

We have shown that the two models of roughness impedance investigated in this report have some similarities and some differences, and can be thought of as being complementary. They both are applicable only when the frequencies of interest are low compared to c/r, with r the typical size of the surface discontinuities, and both yield an approximation to the impedance that is purely inductive. The first model approximates a rough surface by a random collection of non-interacting bumps. It finds the impedance of a single, small bump on a beam tube surface, and then uses averaging to estimate the impedance of a rough surface. The second model, through use of the spectral function of the rough surface analytically finds the impedance, though it is limited to surfaces with slowly varying discontinuities. In the specific case of a surface with non-interacting, smooth bumps the two models will give the same result.

The micro-geometry of a metallic surface—for example, the beam tube in the LCLS undulator— depends on the manufacturing and polishing process that had been applied to that surface. For either of the models discussed in this report to accurately estimate the impedance of a surface requires a specific characterization of the micro-geometry. Once such a characterization is performed, through measurement, one can begin to apply these models to obtain a realistic estimate of the surface impedance and derive conclusions about the effect of the impedance on beam dynamics.

6 ACKNOWLEDGMENT

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