SLAC-R-742

SLAC-ABC-64

Linear Coupling for B-Factory Tilted Solenoid

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Work supported by Department of Energy contract DE–AC03–76SF00515.

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ASYMMETRIC B-FACTORY COLLIDER NOTE ABC-64

LINEAR COUPLING FOR B – FACTORY TILTED SOLENOID[1]

Thesis by

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In Fulfillment of the Requirements
for the Degree of
Master of Science

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Stanford Linear Accelerator Center
California

1992

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1 Introduction

Let us give first an overview of the idea for the Asymmetric B Factory and then summarize the goals of this thesis.

The main physics motivation for the B Factory is a full and exhaustive study of CP violation (a small deviation in Nature's otherwise symmetric order that has been clearly observed but whose origin remain a mystery), using the rich spectrum of B meson decays. The rich resonance structure above the b quark threshold is called the upsilon (Υ) system. The first three prominent resonances are the lowest-lying S states of a bound $b\bar{b}$ quark system. The narrowness of the resonances reflects their stability against strong decays; the states have insufficient energy to decompose into a pair of mesons, each carrying a b quark. The fourth state, Υ (4S) [3], has just sufficient energy to decay to a pair of B mesons (B and $\bar{B}$); this decay totally dominates the disintegration of the Υ(4S). The Υ(4S) is thus an ideal laboratory for the study of B decays. In order to measure CP Asymmetries, the experiment involves measuring the time difference between the decay points of the two B mesons produced in the decay of the Υ(4S). It is the need to measure this difference that is responsible for the energy asymmetry of the accelerator. With modern detectors it is possible to measure the distance between the decay points of two B mesons with a resolution of about $50\mu m$. If a B meson facility is run with equal beam energies, the Υ(4S) is produced at rest in the laboratory and the two mesons do not propogate very far before they decay. The typical distance between the B meson decay points in this equal-beam energy geometry would be about $30\mu m$, a distance too small to discern with today's detectors. The solution to this dilemma is to boost the Υ(4S) in the laboratory frame by running the storage ring with unequal beam energies, hence the name Asymmetric B Factory. The asymmetry denotes the difference in energy between the electron and positron beams. For example, if one chooses 9 and 3.1 GeV for two beam energies ($E^2_{c.m.} = 4E_{low}E_{high}$) the center-of- mass energy is thus that of the Υ(4S) with the average distance between the two B meson decays - $180\mu m$. The justification for an asymmetry in the beam energy is now clear: it is required to give the Υ(4S) system a sufficient Lorentz boost to provide a measurable $t_1 - t_2$ distribution. But how large does the asymmetry need to be? A precipitous dependence on the asymmetry for energy choices below 8 GeV was found. To remain safely above this region the energy of the high- energy beam is set at 9 GeV. So B Factory is a high-luminosity electron- positron colliding-beam accelerator that will operate in the 10-GeV

[3] $E_{c.m.} = 10.58\ GeV$

center-of-mass energy regime (the electron and positron beam energies were chosen to be 9 and 3.1 GeV, respectively.

High luminosity asymmetric B-Factories with separate beam trajectories require final focusing and bending magnets to be located inside the magnetic field of the detector solenoid. It is not possible to align the solenoid axis with both of this separate beam trajectories. So one of the trajectories must pass through significant radial fringe field components of the detector solenoid (recent discussions show that for decreasing significant radial fringe field it is better to have both beam trajectories misaling the solenoid axis). The total magnetic field acting on the two separate beams [4] is a complex superposition of three dimensional fields from solenoid, quadrupole and bending magnet, which in our case are connected with different axis. Such uncompensated solenoid field at the intersection point couple the horizontal and vertical betatron oscillations, dispersions and emittances, that lowers peak luminosity attainable. In order to account the influence of solenoid fringing fields it is necessary to expand the magnetic field through high order. The goal of this thesis is to understand the effect of linear coupling for the B Factory tilted solenoid with the expansion of the magnetic field . The new code for the computation of the transfer map is given. The theory of this thesis is based on Hamiltonian formalism. It is useful to describe the idea of integration of Hamiltonian equations, to discuss what happens with a phase space portrait in the cases of symplectic and not symplectic maps. Here is also shown that use of not symplectic integration gives a good model in special cases. The actual compensation of coupling is done by means of 4 tilted quadrupoles on each side of the IP. Some applications of transfer matrix (including blow-up of emmitance) are also discussed.

2 Theory of linear coupling

Inside of solenoid the longitudinal magnetic field component $B_s = const$. At the solenoid end the field is radial:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) = -\frac{\partial B_s}{\partial s}$$

So near solenoid axis the end field is given by:

$$B_x = -(\frac{1}{2}B_s')x$$

[4]For simplicity the next discussion will be done only for single (HE) beam

$$B_y = -(\frac{1}{2}B_s')y$$

And for force we have:

$$F_x = \frac{e}{c}(v_y B_s - v_s B_y)$$

$$F_y = \frac{e}{c}(-v_x B_s + v_s B_x)$$

This yields the following equations of motion:

$$x'' = \frac{e}{c}v_y B_s + \frac{e}{c}v_s \frac{\partial B_s}{2\partial s}y$$

$$y'' = -\frac{e}{c}v_x B_s - \frac{e}{c}v_s \frac{\partial B_s}{2\partial s}x$$

Using $\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_s}{\partial s} = 0$, we obtain

$$x'' = \frac{e}{c}v_y B_s - \frac{e}{c}v_s \frac{\partial B_y}{\partial y}y$$

$$y'' = -\frac{e}{c}v_x B_s + \frac{e}{c}v_s \frac{\partial B_x}{\partial x}x$$

Let us denote the radius of curvature of the equilibrium orbit by ρ and the y-component of the ideal magnetic field at the equilibrium orbit by B, so that our equations become:

$$x'' = \frac{1}{|B\rho|}\frac{\partial B_y}{\partial y}y - \frac{1}{|B\rho|}B_s y'$$

$$y'' = \frac{1}{|B\rho|}\frac{\partial B_x}{\partial x}x - \frac{1}{|B\rho|}B_s x'$$

We can write the equations of motion in the presence of other external magnetic focussing fields, represented by k_1 and k_2 as

$$x'' + k_1 x = \frac{1}{|B\rho|}\frac{\partial B_y}{\partial y}y - \frac{1}{|B\rho|}B_s y' \quad (1)$$

$$y'' + k_2 y = \frac{1}{|B\rho|}\frac{\partial B_x}{\partial x}x - \frac{1}{|B\rho|}B_s x' \quad (2)$$

where B_x, B_y, B_s are the three field components and $B\rho$ is the magnetic rigidity. One can rewrite (1) and (2) in another form [1]:

$$x'' + k_1 x = -(K + \frac{1}{2}M')y - My' \quad (3)$$

$$y'' + k_2 y = -(K - \frac{1}{2}M')x + Mx' \quad (4)$$

$K(s)$ and $M(s)$ are introduced like

$$K(s) = \frac{1}{|B\rho|}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}\right)$$

$$M(s) = \frac{1}{|B\rho|}B_s \qquad M' = \frac{1}{|B\rho|}\frac{\partial B_s}{\partial s}$$

The associated Hamiltonian might be found in the following way:

$$(x' + \frac{1}{2}My)' = -k_1 x - k_y - \frac{1}{2}My' \ ; \ p_x = x' + \frac{1}{2}My$$

$$(y' - \frac{1}{2}Mx)' = -k_2 y - k_x + \frac{1}{2}Mx' \ ; \ p_y = y' - \frac{1}{2}Mx$$

$$p_x' = -k_1 x - Ky - \frac{1}{2}M(p_y + \frac{1}{2}Mx) = -\frac{\partial H}{\partial x}$$

$$p_y' = -k_2 y - Kx + \frac{1}{2}M(p_x - \frac{1}{2}My) = -\frac{\partial H}{\partial y}$$

$$H = \frac{1}{2}(k_1 x^2 + k_2 y^2 + 2Kxy + (p_x - \frac{1}{2}My)^2 + (p_y + \frac{1}{2}Mx)^2) \quad (5)$$

The general solution of Hill's equation, i.e., linear equations with periodic coefficients and without first derivative terms, are given by well known Courant-Snyder theory [9]. Representing by z either x or y we have:

$$z = au(s)cos(Q\phi + \delta) = Ae^{iQ\phi} + c.c. \quad (6)$$

where $u(s) = \sqrt{\beta}$, $\phi = \int \frac{ds}{Q\beta}$, Q denotes the number of betatron oscillations per revolution. The term $c.c.$ denotes the complex conjugate. The arbitrary constants a, δ or the complex constant $A = \frac{a}{2}e^{i\delta}$ are determined from initial conditions. From the canonical [5] equations $\frac{dx}{ds} = \frac{\partial H}{\partial p_x}$, $\frac{dy}{ds} = \frac{\partial H}{\partial p_y}$ we can also get the expressions for p_x and p_y. In order to get the expression of H as function of (A_j, s) we have to put the expressions for x, y and p_x, p_y into the Hamiltonian, Eq.(5). Knowing the expression for $H(A_j, \bar{A}_j, s)$ it is possible to find the explicit form of A_j. That was done by Guignard [1]. The proportional coefficient in the expressions of A_j is given in the general form (using $n = \pm 1$ for difference and sum resonances respectively) by Guignard theory [1]:

$$\begin{aligned} k &= \frac{1}{4\pi\rho} \oint \sqrt{\beta_x\beta_y} \left[\frac{K}{\rho} + \frac{1}{2} M (\frac{\alpha_x}{\beta_x} - \frac{\alpha_y}{\beta_y}) - \frac{i}{2} M (\frac{1}{\beta_x} - \frac{n}{\beta_y}) \right] \\ & exp \left(i \left[- \oint \frac{ds}{\beta_x} + n \oint \frac{ds}{\beta_y} + \Delta \frac{s}{\rho} \right] \right) ds \end{aligned} \tag{7}$$

Where Δ-distanse from resonance.

In such expression for k the real term represents the skew quadrupole field effects and the image one represents the longitudinal field effects. $\oint \frac{ds}{\beta_x}$, $\oint \frac{ds}{\beta_y}$ are phases μ_x , μ_y, respectively. Therefore it is possible to compensate the effects of a longitudinal field with skew fields by virtue of the phase terms. In other words, a set of skew quadrupoles can compensate both the random tilts of the main magnets and the solenoids contribution. So, as $n = \pm 1$, we see from such Hamilton pertubation theory that in order to decouple our equations it is necessary to use *4* decoupling skew quadrupoles. To show this fact was the goal of the passage above.

In the case of B Factory tilted solenoid we should write the Hamiltonian with expansion of the magnetic field. This is done in the next section.

3 Hamiltonian of particle motion to arbitrary order

The Lagrangian for the relativistic motion of a particle with charge e and rest mass m in a magnetic field described by vector potential A is given by the familiar relation:

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} (\dot{\vec{r}} \vec{A}) \tag{8}$$

The position vector $\vec{r}$ in Eq. (8) refers to a fixed coordinate system. It is useful to introduce the natural coordinates x, y, s. We assume that an ideal closed orbit (design

[5] The definition of canonical transformation is given in Appendix 5

orbit) exists describing the path of a particle of constant energy E_0.[6] We also assume that the close orbit comprises piecewise flat curves which lie either in the horizontal or vertical plane so that it has no torsion.[7] The design orbit which will be used as the reference system will be described by the vector $\vec{r_0}(s)$, where s is the length along the design orbit. An arbitrary particle orbit $\vec{r}(s)$ is then described by the deviation $\delta\vec{r}$ of the particle orbit $\vec{r}(s)$ from the design orbit $\vec{r_0}(s)$: $\vec{r}(s) = \vec{r_0}(s) + \delta\vec{r}(s)$ The vector $\delta\vec{r}$ can usually be described using an orthogonal coordinate system accompanying the particles:

$$\vec{n}(s) \; - \; \textit{a unit normal vector}$$

$$\vec{\tau}(s) \; - \; \textit{a unit tangent vector}$$

$$\vec{b}(s) = \vec{\tau}(s) \times \vec{n} \; - \; \textit{a unit binormal vector}$$

we require that the vector $\vec{n}(s)$ is directed outwards if the motion takes place in the horizontal plane and upwards if the motion takes place in the vertical plane. So that this vectors and their derivatives are connected by Frenet relations;

$$\frac{d\vec{\tau}}{ds} = -K(s)\vec{n}(s) \quad \frac{d\vec{n}}{ds} = K(s)\vec{\tau}(s) \quad \frac{d\vec{b}}{ds} = 0$$

such a representation has the disadvantage that the direction of the normal vector $\vec{n}(s)$ changes discontinuously while the particle trajectory is going over from the vertical plane to the horizontal plane and vice versa. So is more convenient to introduce new unit vectors $\vec{\tau}, \vec{e_x}, \vec{e_y}$, which change their directions continuously [4].

$$\vec{e_x}(s) = \begin{cases} \vec{n}(s) & \textit{if the orbit lies in the horizontal plane} \\ -\vec{b}(s) & \textit{if the orbit lies in the vertical plane} \end{cases}$$

$$\vec{e_y}(s) = \begin{cases} \vec{b}(s) & \textit{if the orbit lies in the horizontal plane} \\ \vec{n}(s) & \textit{if the orbit lies in the vertical plane} \end{cases}$$

the orbit-vector $\vec{r}(s)$ can be written as:

$$\vec{r}(s,x,y) = \vec{r_0}(s) + x(s)\vec{e_x}(s) + y(s)\vec{e_y}(s) \tag{9}$$

in such definitions we have following Frenet relations:

$$\frac{d\vec{e_x}}{ds} = K_x\vec{\tau}$$

[6]Neglecting of course energy variations due to radiation loss
[7]is discussed in Appendix 1

$$\frac{d\vec{e_y}}{ds} = K_y\vec{\tau} \tag{10}$$

$$\frac{d\vec{\tau}}{ds} = -K_x\vec{e_x} - K_y\vec{e_y}$$

using (9) and (10) we get:

$$\dot{\vec{r}} = \dot{s}\left(\frac{d\vec{r_0}}{ds} + x\frac{d\vec{e_x}}{ds} + y\frac{d\vec{e_y}}{ds}\right) + \dot{x}\vec{e_x} + \dot{y}\vec{e_y} \tag{11}$$

or

$$\dot{\vec{r}} = \vec{\tau}\dot{s}(1 + K_x x + K_y y) + \dot{x}\vec{e_x} + \dot{y}\vec{e_y}$$

Thus in our coordinate system x, y, s the Lagrangian becomes:

$$\begin{aligned} L &= -mc^2\{1 - \frac{1}{c^2}[\dot{x}^2 + \dot{y}^2 + (1 + K_x x + K_y y)^2\dot{s}^2]\}^{\frac{1}{2}} \\ &+\frac{e}{c}[\dot{x}A_x + \dot{y}A_y + (1 + K_x x + K_y y)\dot{s}A_s] \end{aligned} \tag{12}$$

with the corresponding Hamiltonian:

$$H = p_x\dot{x} + p_y\dot{y} + p_s\dot{s} - L \tag{13}$$

We defined the conjugate momenta as:

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = \frac{e}{c}A_x + \frac{m\dot{x}}{\sqrt{1 - \frac{v^2}{c^2}}} \\ p_y &= \frac{\partial L}{\partial \dot{y}} = \frac{e}{c}A_y + \frac{m\dot{y}}{\sqrt{1 - \frac{v^2}{c^2}}} \\ p_s &= \frac{\partial L}{\partial \dot{s}} = \frac{m\dot{s}}{\sqrt{1 - \frac{v^2}{c^2}}}(1 + K_x x + K_y y)^2 + \frac{e}{c}(1 + K_x x + K_y y)A_s \end{aligned} \tag{14}$$

Putting Eq.(14) and Eq.(12) into (13) one can get for the Hamiltonian:

$$H = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{15}$$

And using the relation;

$$\frac{m^2v^2}{1-\frac{v^2}{c^2}} = \frac{m^2(\dot{x}^2+\dot{y}^2+\dot{s}^2(1+K_x x+K_y y)^2)}{1-\frac{v^2}{c^2}} =$$

$$\frac{m^2c^2}{1-\frac{v^2}{c^2}} - m^2c^2 = (p_x - \frac{e}{c}A_x)^2 + (p_y - \frac{e}{c}A_y)^2 + \left(\frac{p_s}{1+K_x x+K_y y} - \frac{e}{c}A_s\right)^2$$

we have:

$$H = c\sqrt{(p_x - \frac{e}{c}A_x)^2 + (p_y - \frac{e}{c}A_y)^2 + \left(\frac{p_s}{1+K_x x+K_y y} - \frac{e}{c}A_s\right)^2 + m^2c^2} \qquad (16)$$

Let us denote $K_x = \frac{1}{\rho}$ (we assume $K_y = 0$). The coresponding canonical equations are given by:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} & \dot{p_x} &= -\frac{\partial H}{\partial x} \\ \dot{y} &= \frac{\partial H}{\partial p_y} & \dot{p_y} &= -\frac{\partial H}{\partial y} \\ \dot{s} &= \frac{\partial H}{\partial p_s} & \dot{p_s} &= -\frac{\partial H}{\partial s} \end{aligned} \qquad (17)$$

In Eq.(17) the time t appeared as independent variable, it is more useful to introduce the arc length s of the design orbit as independent variable and define a new Hamiltonian as $K = -p_s$.

$$K = -p_s = -(1+\frac{x}{\rho})\sqrt{\frac{H^2}{c^2} - m^2c^2 - (p_x - \frac{e}{c}A_x)^2 - (p_y - \frac{e}{c}A_y)^2} - (1+\frac{x}{\rho})\frac{e}{c}A_s$$

Let us denote new variable $\delta = \frac{E-E_0}{E_0}$ thus

$$\tilde{K} = -\frac{c}{E_0}p_s = -(1+\frac{x}{\rho})\sqrt{(1+\delta)^2 - (\hat{p_x} - \frac{e}{E_0}A_x)^2 - (\hat{p_y} - \frac{e}{E_0}A_y)^2} - (1+\frac{x}{\rho})\frac{e}{E_0}A_s$$

as the term $(\frac{mc^2}{E_0})^2 << 1$ we dropped it here. [8] Since the variable $t(s)$ increases without limit, it is more useful to introduce the new variable $t^* = s - ct(s)$, which describes the

[8]The general form of Hamiltonian H, including the term (m^2c^2) is given at the Appendix 3.

delay in arrival time at position s of a particle travelling at the speed of light c. The further change of variables can be achieved using the generating function [4]:

$$F(p,\hat{q},s) = F(p_x,p_y,x,y,\delta,t^*,s) = -p_x\hat{x} - p_y\hat{y} - t^*\delta + s\delta + s$$

corresponding transformation equations:

$$(-ct) = -\frac{\partial F}{\partial \delta} \qquad t^* = s - ct$$

$$\hat{\delta} = -\frac{\partial F}{\partial t^*} \qquad \hat{\delta} = \delta$$

So that $\frac{\partial F}{\partial s}$ gives additional term:

$$\hat{K} = \tilde{K} + \frac{\partial F}{\partial s} = \tilde{K} + (1+\delta)$$

and the Hamiltonian becomes: [9]

$$H = -(1+\frac{x}{\rho})\sqrt{(1+\delta)^2 - (\hat{p}_x - \frac{e}{E_0}A_x)^2 - (\hat{p}_y - \frac{e}{E_0}A_y)^2} + (1+\delta) - (1+\frac{x}{\rho})\frac{e}{E_0}A_s \quad (18)$$

Since we have $|(p_y - \frac{e}{E_0}A_y)| = |\frac{e}{E_0}mv_y| << 1$ (the same for x and z) the square root can be expanded in a series. For B Factory solenoid we should also account contribution to this Hamiltonian from bending magnet and quadrupoles. Expanding the form of our Hamiltonian and accounting bending magnet and quadrupoles terms, we have:

$$\begin{aligned} H(x,p_x,y,p_y,t,\delta;s) &= \frac{1}{2(1+\delta)}\left[(p_x - \frac{e}{c}A_x)^2 + (p_y - \frac{e}{c}A_y)^2\right] + \\ &\quad +\delta\frac{x}{\rho_x} + \frac{x^2}{2\rho_x^2} - \frac{K_1}{2}(x^2 - y^2) \end{aligned} \quad (19)$$

where $K_I = \frac{\partial B_y}{\partial x}|_{x=y=0}$

4 Expansion of Vector Potential

In this section we are going to remind the expansion of solenoid fields, which is given by Ripken [4].

[9] Later $\hat{K}$ will be $\equiv H$ and $\hat{p} \equiv p$

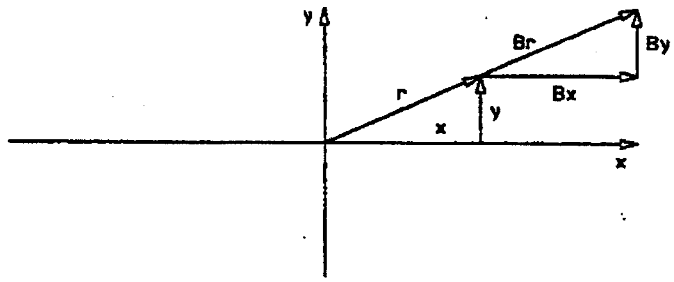

Figure 1: *Radial field.*

$$B = rotA \Longrightarrow$$

$$B_x = -\frac{\partial}{\partial s}A_y$$

$$B_y = \frac{\partial}{\partial s}A_x$$

$$B_s = \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x$$

$$B_r = \sqrt{B_x^2 + B_y^2}$$

In the current free region (Fig.1) the radial field B_r and the longitudinal field B_s can be written as power series [3]:

$$B_r(x, y, s) = \sum_{\nu=0}^{\infty} b_{2\nu+1}(s) r^{2\nu+1} \tag{20}$$

$$B_s(x, y, s) = \sum_{\nu=0}^{\infty} b_{2\nu}(s) r^{2\nu} \tag{21}$$

Putting (20) , (21) into the Maxwell equations:

$$div\vec{B} = 0 \ ; \ rot\vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) = -\frac{\partial}{\partial s}B_s \ ; \ \frac{\partial}{\partial s}B_r = \frac{\partial}{\partial r}B_s$$

one obtains [4]:

$$\sum_{\nu=0}^{\infty} b_{2\nu+1}(s)\cdot(2\nu+2)r^{2\nu} = -\sum_{\nu=0}^{\infty} r^{2\nu}\cdot\frac{d}{ds}b_{2\nu}(s) \tag{22}$$

$$\sum_{\nu=0}^{\infty} r^{2\nu+1}\cdot\frac{d}{ds}b_{2\nu+1}(s) = \sum_{\nu=0}^{\infty} b_{2\nu+2}(s)\cdot(2\nu+2)r^{2\nu+1} \tag{23}$$

By equating coefficients of each power one then obtains:

$$b_{2\nu+1}(s) = -\frac{1}{(2\nu+2)}b'_{2\nu}(s)$$

$$b_{2\nu+2}(s) = \frac{1}{(2\nu+2)}b'_{2\nu+1}(s)$$

If we now define the longitudinal field on the $s-$axis $B_s(0,0,s) = b_0(s)$, the coefficients b_1, b_2, b_3.... can be calculated. Though the field components in the field free region are given by:

$$B_x(x,y,s) = \frac{x}{r}B_r = x\sum_{\nu=0}^{\infty} b_{2\nu+1}(s)\cdot(x^2+y^2)^{\nu}$$

$$B_y(x,y,s) = \frac{y}{r}B_r = y\sum_{\nu=0}^{\infty} b_{2\nu+1}(s)\cdot(x^2+y^2)^{\nu}$$

$$B_s(x,y,s) = \sum_{\nu=0}^{\infty} b_{2\nu}(s)\cdot(x^2+y^2)^{2\nu}$$

So the vector potential:

$$A_x = -\sum_{\nu}\frac{1}{(2\nu+2)}b_{2\nu}(s)\cdot r^{2\nu}y$$

$$A_y = \sum_{\nu}\frac{1}{(2\nu+2)}b_{2\nu}(s)\cdot r^{2\nu}x$$

$$A_s = 0$$

Now we can write Hamiltonian equations of motion.

There are many different ways of integration such differential equations. This question will be discussed in the next part.

5 Integration of Hamiltonian equations

As it was mentioned above, many different ways of integration differential equations numerically are exist. The methods are usually described by the accuracy of single step in time. We have deal with differential equations, which are derivable from a Hamiltonian. The exact solution of such a system of differential equation leads to a symplectic map from the initial conditions to the present state of the system.

5.1 Symplectic conditions

Using the definition of canonical transformation [10] for our transformation from the initial conditions to the values at time t we have:

$$[q_i, q_j] = 0 \qquad [p_i, p_j] = 0 \qquad [q_i, p_j] = \delta_{ij}$$

or

$$\begin{aligned}
\sum_{j0}\left(\frac{\partial q_i}{\partial q_{j0}}\frac{\partial q_k}{\partial p_{j0}} - \frac{\partial q_i}{\partial p_{j0}}\frac{\partial q_k}{\partial q_{j0}}\right) &= 0 \\
\sum_{j0}\left(\frac{\partial p_i}{\partial q_{j0}}\frac{\partial p_k}{\partial p_{j0}} - \frac{\partial p_i}{\partial p_{j0}}\frac{\partial p_k}{\partial q_{j0}}\right) &= 0 \\
\sum_{j0}\left(\frac{\partial q_i}{\partial q_{j0}}\frac{\partial p_k}{\partial p_{j0}} - \frac{\partial q_i}{\partial p_{j0}}\frac{\partial p_k}{\partial q_{j0}}\right) &= 0
\end{aligned} \qquad (24)$$

If we consider the six dimensional vector $x = (q_1, p_1, q_2, p_2, q_3, p_3)$ and define the Jacobian of our trasformation:

$$M = \begin{pmatrix} \frac{\partial q_1}{\partial q_{10}} & \cdot & \frac{\partial q_1}{\partial p_{30}} \\ \cdot & \cdot & \cdot \\ \frac{\partial p_3}{\partial q_{10}} & \cdot & \frac{\partial p_3}{\partial p_{30}} \end{pmatrix}$$

Then the condition (24) can be expressed in terms of matrix M as following:

$$M^T \cdot S \cdot M = S \qquad (25)$$

[10]Appendix 5

where

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Eq.(25)-symplectic condition. All solutions to Hamiltonian equations must satisfy this condition.

5.2 Symplectic integrator

Most of the high-order ($k >> 2$) integration methods are not exactly symplectic. The determinant of the Jacobian of the transformation for one time step differs slightly from unity and thus the system will be damped or excited artificially.

In addition there is another way of viewing this approach [2]: If we iterate any explicit integration step whether canonical or not, eventually the absolute error in the coordinates and the momenta gets large. For not symplectic integration step, where spurious damping or antidamping occurs, q and p either settle onto some fixed point or diverge roughly exponentially. In the case of the symplectic map it does not happen. The symplectic integration step with a sufficiently small step size generates a phase space portrait which is close to that of the original system. So in the symplectic case it is possible and sometimes attractive to replace the differential equation by a symplectic map. The development of such symplectic map was done by E.Forest and R.D.Ruth [2].

Unfortunately, their technique of successive canonical transformations works for the Hamiltonian, which is written in form $H = A(p) + V(x)$. In our case we have $H = \frac{[p-A(x,t)]^2}{2}$ or the troublesome term: $p \cdot A$. This leads to matrix inversion even in the first order case. It is possible to write down a second order map, but in assumption for first order in $\vec{A}$. We are interested in high-order expansion of fields and even a third order map is not enough. It is seemed possible to use for this purpose the one-map integrator, based on Lie groups [2] (Use of higher map integrator needs to split H into two pieces which can be solved exactly. For our Hamiltonian the achievement of such condition is difficult) such symplectic integrator will be used later if it be necessary to make a modal for our symplectic problem. But now we use another way: In many applications the salient features of the solution appear only after long time or large number of iterations.

We are not interested in "tracking" so for our purposes and for simplicity we can use not symplectic integrator. Here is used Runge Kutta fourth order method (that means that in a small time step h the integration is accurate through order h^4).

5.3 Fourth order Runge Kutta formula

$$k_1 = hf'(x_n, y_n)$$

$$k_2 = hf'(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$

$$k_3 = hf'(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$

$$k_4 = hf'(x_n + h, y_n + k_3)$$

In each step the derivative is evaluated four times: once at the initial point, twice at trial midpoints and once at a trial endpoint. From this derivatives the final function value is calculated:

$$y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + o(h^5)$$

In our case such procedure is made for five parameters x, p_x, y, p_y, p_t (no direct dependence from time t).

Using such technique the transfer matrix of solenoid with field expansion was achieved. The next step was to write the coordinate transformation due to tilted solenoid. This is shown in the next section.

6 Coordinate transformation due to tilted solenoid

A number of features suggested to write a new code rather than using present accelerators codes. As it was mentioned before, the solenoid axis does not lie on top of the reference orbit, since the beam have to be separated to minimize parasitic crossings. But quadrupoles and horizotal bending magnets, which are located inside the detector, must lie on the reference orbit. The code which is presented here overcomes this troubles.

We use a Hamiltonian, which is written in the form of Eq.(19), where the canonical variables and the independent variable refer to a charged particle in the frame of the reference orbit, that coincides neither with reference frame at the collision point nor with frame of the solenoid axis.

We use the trajectory of the beam center with solenoid off as a reference orbit. This reference orbit is defined solely by the horizontal and vertical bending fields and the quadrupoles. Note that a solenoid tilted horizontal by an angle ϕ will produce a vertical bending field of the strength $k_s sin(\phi)$ that acts on top of the horizontal bending fields.[11]

Thus we have to introduce two coordinate transformations T_1 relates the frame of the reference orbit to the frame of the collision axis and T_2 relates the coordinates of the collision axis to the coordinate inside the solenoid frame. We introduce three sets of unit vectors corresponding to each of the three frames: e_x^b, e_y^b, e_z^b for the reference orbit, e_x^i, e_y^i, e_z^i for the collision point and e_x^s, e_y^s, e_z^s for the frame of the solenoid.

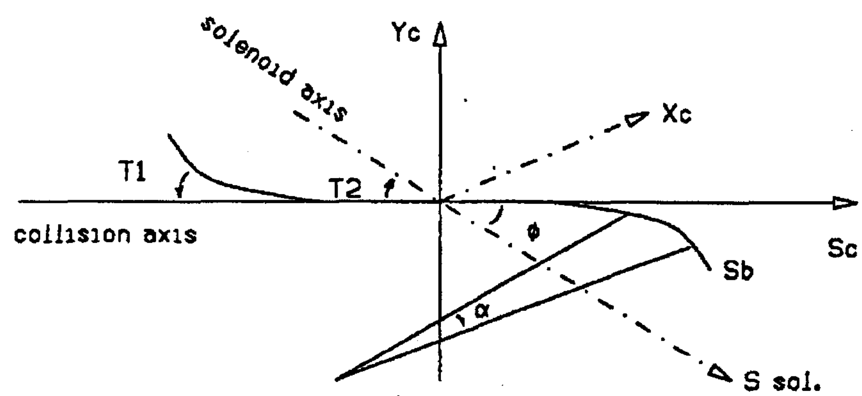

Figure 2: *The axis of the reference orbit, the solenoid axis and the longitudinal axis of the beam at the collision point.*

[11]The vertical difference between the trajectory with and without solenoid is only about 1 mm for both rings.

$$T_2 = \begin{cases} x_s &= x_c \cdot cos(\phi) + z_c \cdot sin(\phi) \\ z_s &= x_c \cdot sin(\phi) + z_c \cdot cos(\phi) \end{cases}$$

The origin of the reference orbit frame: O_b is moving along the reference orbit which is determined by the horisontal bending angle of the dipoles α. Let us denote the entrance of the magnet by s_b^i. The origin of the reference orbit frame in the collision frame may now be expressed by:

$$O_b = e_x^c[x_c(s_b^i)+x_c'(s_b^i)p_x\alpha - p_x(cos(\alpha)-1)] + e_z^c[z_c(s_b^i)+z_c'(s_b^i)p_x\alpha - p_x(\alpha - sin(\alpha))] \quad (26)$$

where we have used:

$$O_b(s_b^i) = e_x^c \cdot x_c(s_b^i) + e_z^c \cdot z_c(s_b^i)$$

and

$$\frac{dO_b(s_b^i)}{ds_b} = e_x^c \cdot x_c'(s_b^i) + e_z^c \cdot z_c'(s_b^i)$$

We assume that a positive bending angle bends the particle towards the positive horizontal axis. For small bending angle we may approximate Eq.(26) by:

$$O_b = e_x^c[x_c(s_b^i) + x_c'(s_b^i)p_x\alpha + p_x\alpha^2/2] + e_z^c[z_c(s_b^i) + z_c'(s_b^i)p_x\alpha - p_x\alpha^2/6]$$

And the coordinate transform valid on the reference orbit lying in the field region at the bending magnet is:

$$T_1 = \begin{cases} x_c &= x_c(s_b^i) + x_c'(s_b^i)p_x\alpha + p_x\frac{\alpha^2}{2} + x_b cos(\alpha) + z_b sin(\alpha) \\ \\ z_c &= z_c(s_b^i) + z_c'(s_b^i)p_x\alpha - p_x\frac{\alpha^3}{6} - x_b sin(\alpha) + z_b cos(\alpha) \end{cases}$$

Using this code the transfer matrix for tilted solenoid (without effect of torsion[12]) was calculated. To prove our use of not symplectic integrator the transfer matrix was checked with symplectic condition (25). The deviation from a symplectic transfer matrix is smaller than 10^{-5}, that allow us to make the statement that use of not symplectic integrator in our case is possible through very good accuracy.

[12]this effect is discussed in Appendix 1

7 Transfer matrix for B Factory solenoid

In the particular case of B Factory tilted solenoid with the extension of the magnetic field up to the fifth order we got the following numerical matrices.

$$T_{uncouple} = \begin{bmatrix} 1.589 & 3.623 & 0.000 & 0.000 & 0.000 & 0.03229 \\ 0.462 & 1.683 & 0.000 & 0.000 & 0.000 & 0.01646 \\ 0.000 & 0.000 & 0.482 & 2.063 & 0.000 & 0.00000 \\ 0.000 & 0.000 & -0.390 & 0.404 & 0.000 & 0.00000 \\ -0.011 & -0.005 & 0.000 & 0.000 & 1.000 & 0.00000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 1.00000 \end{bmatrix}$$

$$T_{couple} = \begin{bmatrix} 1.588 & 3.620 & 0.0400 & 0.1028 & 0.0000 & 0.03235 \\ 0.462 & 1.682 & 0.0074 & 0.0445 & 0.0000 & 0.01648 \\ -0.025 & -0.078 & 0.4807 & 2.0609 & -0.0002 & 0.00103 \\ 0.006 & 0.020 & -0.3907 & 0.4031 & 0.0001 & -0.00042 \\ -0.011 & -0.005 & 0.0000 & 0.0007 & 1.0000 & 0.00000 \\ 0.000 & 0.000 & 0.0000 & 0.0003 & 0.0000 & 1.00000 \end{bmatrix}$$

where:

$$X_0 = \begin{pmatrix} x_0 \\ p_{x0} \\ y_0 \\ p_{y0} \\ t_0 \\ p_{t0} \end{pmatrix}$$

$$X = T \cdot X_0$$

Length from I.P. to the end of solenoid - *2m*. Uncoupled and coupled matrices are given for the exit of the quadrupole (*2.1m*). The following list contains the location and the strength of the magnets inside the experimental solenoid.

8 Decoupling

Our purpose is to use this transfer matrix to decouple the normal modes at the interaction point.

Type	exit [m]	$B_0[kG]$	$B_1[kG/m]$
DRIFT	0.2	0.000	0.000
HBEND	0.25	3.100	0.000
HBEND	0.30	5.400	0.000
HBEND	0.35	6.500	0.000
HBEND	0.70	7.500	0.000
DRIFT	0.90	0.000	0.000
QUAD	1.20	0.000	-0.355
QUAD	1.50	0.000	-0.355
QUAD	1.80	0.000	-0.355
QUAD	2.10	0.000	-0.355

Table 1: *List of elements.*

A variaty of compensation schemes exist which have been used in order to decouple the equations of motion. For example, 1. A half strength anti-solenoid on either side of the experemental solenoid. 2. With two or three pairs of skew quadrupoles.

When we discussed Hamilton pertubation theory we saw that for this purpose we need four skew quadrupoles. The necessety of four decouplers can be also shown, using matrix formalism. [13] Our transfer matrix has been put into program MAD. We located four skew quadrupoles according to independent phases $\mu_x \pm \mu_y$, which is used at formula (7). The results for the integrated strength of the tilted quadrupoles are shown at table 2.

Name	Type	integrated strength [1/m]	arc length [m]
QT1	tilted QUAD	8.99E-04	7.45
QT2	tilted QUAD	-5.75E-03	35.976
QT3	tilted QUAD	1.02E-02	45.040
QT4	tilted QUAD	8.08E-03	58.954

Table 2: *Strength of skew quad. necessary for decoupling.*

These values were obtained by matching the off diagonal elements of the transfer matrix $R_{13}, R_{14}, R_{23}, R_{24}$. Note that the same skew quadrupoles must be also placed on the other side to compensate the second part of solenoid. But as we used four

[13]Such way is discussed at the Appendix 4.

additional elements, the Twiss parameters at the IP shifted slightly: β_x from 0.7[m] to 0.74[m] and β_y from 0.03[m] to 0.0297[m]. In order to return back the values of β-functions IP, we used *6* regular quadrupoles (QD4 to QDS4). So we matched the transfer matrix elements $R_{11}, R_{12}, R_{13}, R_{14}, R_{22}, R_{23}, R_{24}, R_{33}, R_{34}, R_{44}$ to their values in the condition solenoid off. For complete compensation we should also compensate the dispersive elements. In the case of tilted solenoid we have horizontal dispersion which produces the vertical dispersion. But the main influence on dispersion gives the vertical corrector, which is placed immediately after the solenoid. So the compensation of the dispersion lies out off this paper as we discuss here only the effect of solenoid.

9 Some application of transfer matrix

9.1 Beam shape, change in beam-size

Let us introduce the beam ellipse, using the definition of sigma matrices used at TRANSPORT [10] (Fig.3)

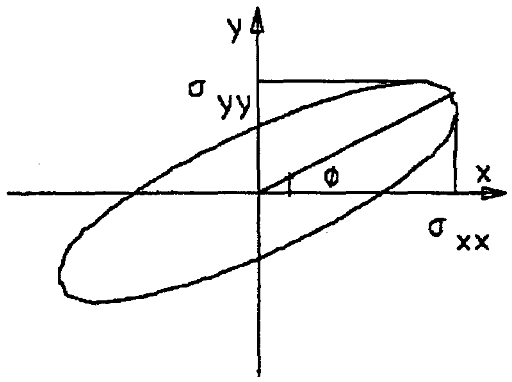

Figure 3: *Beam ellipse in definitions of sigma matrices.*

$\sqrt{\sigma_{xx}}$=x_{max}=the maximum (half)-width of the beam envelope in the x (bend)-plane.
$\sqrt{\sigma_{yy}}$=y_{max}
In such definitions one can write the equation of ellipse in the couple case:

$$\sigma_{yy}x^2 - 2\sigma_{xy}xy + \sigma_{xx}y^2 = \epsilon \tag{27}$$

If now define new axis through main axis of the tilted ellipse and write the relations between new and old coordinates:

$$x' = x \cdot cos(\phi) + y \cdot sin(\phi) \quad y' = y \cdot cos(\phi) - x \cdot sin(\phi)$$

One can easily get the relation for angle, putting this relations into equation (27):

$$\tan(2\phi) = \frac{2\sigma_{xy}}{\sigma_{xy} - \sigma_{xy}} \tag{28}$$

The transverse beam area (for luminosity calculation) is given by:

$$A = \pi(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)^{\frac{1}{2}}$$

The maximum angle for the coupled beam shape was 1.4 degree. The dependence of angle ϕ from longitudinal coordinates during the interection region is shown on Fig.4

The numbers for ϕ and s are given at table 3.

It is interesting to note that the low energy beam feels an about three times stronger solenoid field compared with the high energy beam. Fig.(5) shows that the tilt of the normal modes is about three times larger compared to the high energy beam. The numbers for ϕ and s for LE beam are given at table 4.

element	exit	$\phi[deg]$
DRIFT	0.2	0.190
HBEND	0.25	0.238
HBEND	0.30	0.286
HBEND	0.35	0.333
HBEND	0.70	0.665
DRIFT	0.90	0.851
QUAD	2.10	1.424
DRIFT	2.80	1.275

Table 3: *Dependence of angle ϕ, which corresponds to the tilted beam ellipse in coupling case, from longitudinal coordinate*

9.2 Equilibrium transverse emittance

It is interesting now to get information about emittance change due to the coupling. Let us first remind some general ideas from theory.

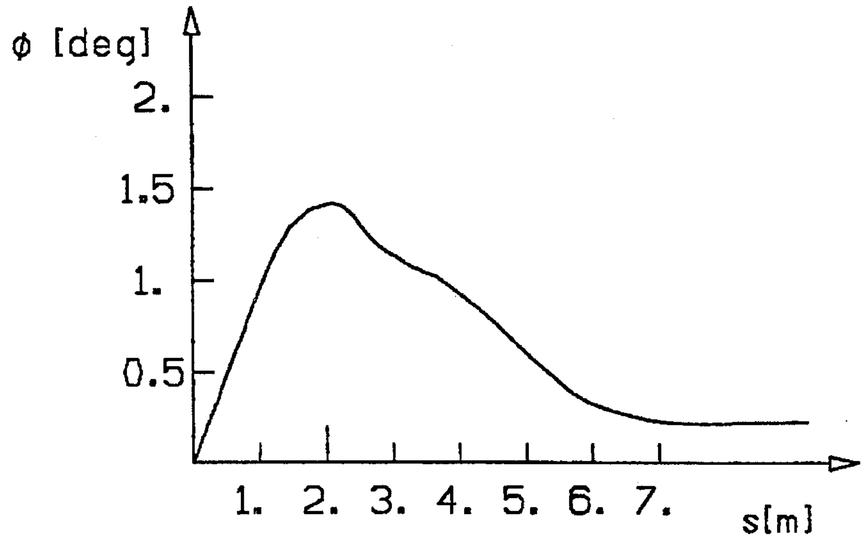

Figure 4: *Dependence of the tilt of horizontal mode w.r.t. the vertical axis from longitudinal coordinate.*

The equilibrium transverse emittance of the beam in a storage ring is the result of two competing processes: 1) the quantum fluctuations due to the emission of synchrotron radiation blow up the beam 2) the process of radiation damping tends to reduce the transverse beam size. The balance stuck between these two processes determines the value for the equilibrium emittance.

9.2.1 General introduction

The Courant-Snyder invariant of the betatron motion or the square of the invariant amplitude is given by:

$$a^2 = \gamma x_\beta^2 + 2\alpha x_\beta x'_\beta + \beta x'^2_\beta$$

this formula describes an ellipse in the (x_β, x'_β) phase space and the square of the invariant amplitude is just the area of the ellipse divided by π or the emittance. The presence of dispersion in the ring causes a particle, with an energy different from the design energy

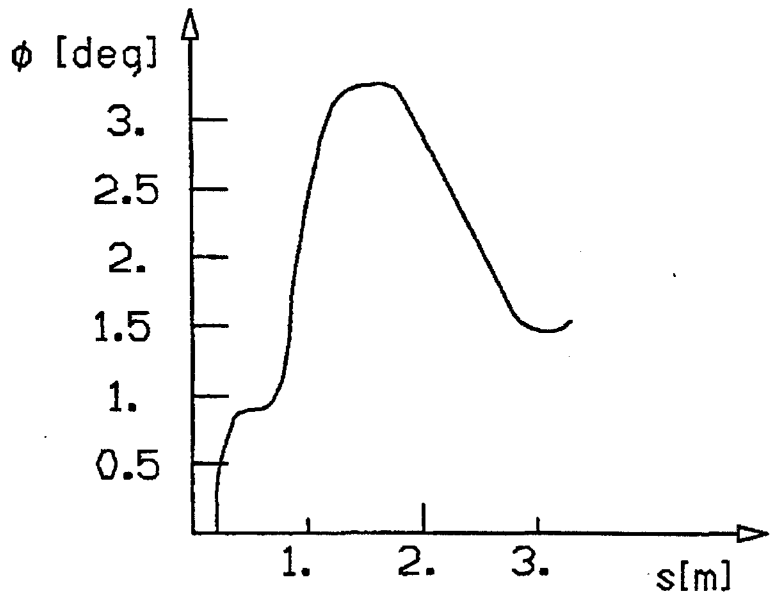

Figure 5: *Dependence of the tilt of horizontal mode w.r.t. the vertical axis from longitudinal coordinate for LE beam. Length from I.P. to the end of solenoid - 2m.*

to execute betatron oscillations around a new closed orbit, different from the design orbit. So that we can separate the radial motion into two parts:

$$x(s) = x_\beta(s) + x_\epsilon$$

If we use the standard form of the equations of motion, we can write for the displacement x_ϵ

$$x_\epsilon'' = k(s)x_\epsilon + K(s)\delta$$

where $K(s)$- curvature function, $k(s)$- periodic function with a period of at most the circumference of the ring, δ-fractional energy deviation. The displacement x_ϵ is proportional to the energy deviation: $x_\epsilon(s) = \eta(s)\delta$ If we put such relation in the equation above, we see that the dispersion function η satisfies the following equation: $\eta'' = k(s)\eta + K(s)$

element	exit	$\phi[deg]$
DRIFT	0.2	0.00
HBEND	0.25	0.55
HBEND	0.30	0.69
HBEND	0.35	0.82
HBEND	0.70	0.96
DRIFT	0.90	1.93
QUAD	1.20	3.05
QUAD	1.50	3.25
QUAD	1.80	3.20
QUAD	2.10	2.74
DRIFT	2.80	1.60
QUAD	3.30	1.54

Table 4: *Dependence of angle ϕ, which corresponds to the tilted beam ellipse in coupling case, from longitudinal coordinate for LE beam.*

9.2.2 Quantum fluctuation

If account the radiation, the invariant amplitude of the electron betatron oscillations will no longer remain constant. Due to the emission of photon the particle's energy is reduced, but its displacement and the slope of the trajectory do not change, so we can write:

$$\Delta x = 0 = \Delta x_\beta - \eta \frac{U}{E} \quad ; \quad \Delta x' = 0 = \Delta x'_\beta - \eta' \frac{U}{E}$$

If our particle was following the design trajectory before radiating , it now starts performing betatron oscillations with its square of invariant amplitude given by:

$$a^2 = \gamma \Delta x_\beta^2 + 2\alpha \Delta x_\beta \Delta x'_\beta + \beta \Delta x'^2_\beta = (\frac{U}{E})^2 H$$

where $H = \gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2$ this function describes the growth of the invariant betatron amplitudes due to the radiation-induced quantum fluctuations.

One can write the full expression for the growth of the square of the invariant betatron amplitude due to the quantum excitation as

$$\frac{da^2}{dt} = Q_x = \frac{2C_q E U_0}{T_0 \oint K^2 ds} \oint |K^3| H ds \qquad (29)$$

where

$$C_q \ (quantum\ constant) \ = \frac{55\hbar c}{32\sqrt{3}(mc^2)^3} == 1.468 \cdot 10^{-6} \frac{m}{Gev^2}$$

$$E - design\ energy$$

$$U_0 \ (radiation\ loss\ per\ turn) \ = \frac{2r_e E^4}{3(mc^2)^3} \oint K^2 ds$$

9.2.3 Radiation damping

The betatron oscillations decay exponentially with the damping time. For example, the damping time for horizontal betatron oscillation is given by $\tau_x = \frac{1}{J_x} \frac{2ET_0}{U_0}$, where damping partition number J_x is introduced by $\tau_x = \frac{\tau_0}{J_x}$, $\tau_0 = \frac{2ET_0}{U_0}$

9.2.4 Variety of technique for emittance calculation

The total rate of change of the square of the invariant betatron amplitudes can be summarized then as:

$$\frac{d < a^2 >}{dt} = Q_x - \frac{2 < a^2 >}{\tau_x} \tag{30}$$

A stationary distribution of the horizontal betatron oscillations of many particles is then characterized by the mean square horizontal spread at the beam:

$$\sigma_{x\beta}^2 = \frac{1}{2} < a^2 > \beta_x(s) = \frac{1}{4} \tau_x Q_x \beta_x(s) \tag{31}$$

The ratio $\epsilon_x = \frac{\sigma_{x\beta}^2(s)}{\beta_x(s)} = \frac{1}{4}\tau_x Q_x$ is exactly the equilibrium beam emittance. Note that it is independent of the asimuth s.

$$\epsilon_x = \frac{C_q E^2}{J_x} \frac{\oint |K^2| H ds}{\oint K^2 ds} \tag{32}$$

For emittance calculation we used computer optics program MAD, which uses two different way to find the emittance:

1) Chao's technique [6] of the transport matrices. At that technique coupling is included in the evaluation of the distribution parameters, assuming that the coupling effects can be approximately described by a set of coupling coefficients which specify the coupling strength averaged over one revolution of the storage ring. In this method, each

linear element in the storage ring lattice is represented by 6×6 TRANSPORT matrix, which transforms the state vector X as an electron passes through the element. Knowing the TRANSPORT matrix transformations around the storage ring, the distribution parameters can be obtained from the eigenvalues and eigenvectors of some matrices, which are described in his paper.

2) Another technique uses the definition of emittance with synchrotron radiation integrals.

$$I_1 = \oint \eta K ds \;\; ; \;\; I_2 = \oint K^2 ds \;\; ; \;\; I_3 = \oint |K^3| ds$$

$$I_4 = \oint \eta K(2k + K^2) ds \;\; ; \;\; I_5 = \oint |K^3| H ds$$

So that $J_x = 1 - \frac{I_4}{I_2}$

$$\epsilon_x = \frac{C_q E^2}{J_x} \cdot \frac{I_5}{I_2} = C_q \cdot E^2 \frac{I_5}{I_4 - I_2} \tag{33}$$

Note also that one of the present day technique for coupled emittance is based on the projections of β functions on old axis, so that you have deal with four new β functions and new emittances ϵ_1 and ϵ_3 [7]. This technique gives:

$$< x^2(s) >= \epsilon_1 \beta_{x1}(s) + \epsilon_3 \beta_{x3}$$

$$< y^2(s) >= \epsilon_1 \beta_{y1}(s) + \epsilon_3 \beta_{y3}$$

$$< xy >= \epsilon_1 \sqrt{\beta_{x1}\beta_{y1}} cos(\phi_{x1} - \phi_{y1}) + \epsilon_3 \sqrt{\beta_{x3}\beta_{y3}} cos(\phi_{x3} - \phi_{y3})$$

In the case of small coupling ϵ_1 and ϵ_3 are very close to ordinary ϵ_x and ϵ_y. We compared ϵ_x using both techniques 1) and 2), that gave us in the uncoupled case the same result $\epsilon_x[\pi \; micro \; m] = 0.047$

For coupled case the first technique was used and we got the following blow up of emittance:

$$Emittance \; ratio[vert./hor.] = 0.123$$

10 Conclusion

In this thesis we have presented the transfer matrix for B Factory tilted solenoid with the expansion of magnetic field up to the fifth order. Starting with the general theory of linear coupling, we got the Hamiltonian for solenoid with the bending magnet and quadrupole inside. The solenoid axis is tilted by 20 mrad horizontally w.r.t. the collision axis and at the entrance and the exit of the solenoid the beam will sense transverse and longitudinal non-linear fields. To account both this effects the expansion of the magnetic field was done. The code of coordinate transformation, which relates the frame of the reference orbit to the frame of the collision axis and to the solenoid frame, has been introduced. We tried to show that not symplectic fourth-order Runge Kutta integration method, which had been used for integration of our Hamiltonian equations, might be used as a model for "not tracking" problems. The deviation from a symplectic transfer matrix is smaller than 10^{-5}. Using the transfer matrix, the change in beam shape and blow up of emittance, due to the solenoid coupling, was discussed. In order to compensate this effect we used 4 tilted quadrupoles on each side of the IP. Our method based on the Hamiltonian in Eq.19 integrates along a reference orbit which is defined only by the horizontal and vertical bending fields and not by the tilted solenoid. In order to get the Hamiltonian, which is associated with a non-planar curvature of the reference orbit, it is necessary to account the effect of torsion. [14] In that case the transformation between the three different coordinate systems will become more complicated.

11 Appendix 1

If torsion $\equiv 0$, we have reference orbit (the trajectory of the beam center with solenoid off) in horizontal plane. When we switch on solenoid the trajectory of central particle goes in vertical plane, but the reference orbit lies at the same horizontal plane. The effect of torsion gives us the change of the reference orbit, it no longer lies in horizontal plane. In other words we must account torsion when we go from plane problem to the motion of particle in space. Now curves are placed in space. Let us consider the natural coordinate system (Fig. 6):

$$n(s) - normal\ vector \ \ ; \ \ b(s) - binormal\ vector$$

$$\tau(s) - tangent\ vector \ \ ; \ \ K(s) - curvature \ \ ; \ \ \Omega - torsion$$

For our curvilinear system of coordinates the reference curve is given by $r = r_0(s)$, tangent vector to the curve at s is given by $\tau(s) = \frac{dr_0(s)}{ds}$. $\tau(s), b(s), n(s)$ and its

[14]This effect is discussed in Appendix 1

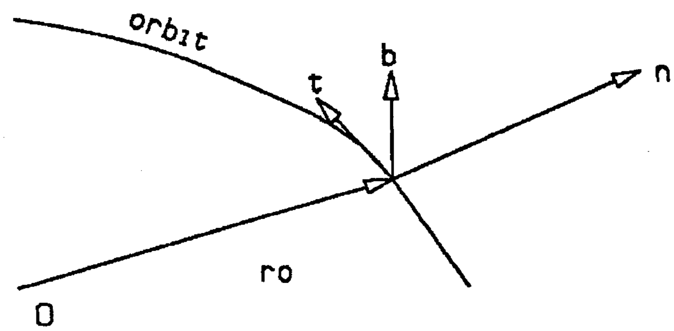

Figure 6: *Moving tripod of the orbit.*

derivatives are connected by Frenet- Serret formulas:

$$\vec{\tau} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \; ; \; \vec{n} = \frac{\ddot{\vec{r}}}{|\ddot{\vec{r}}|} \; ; \; \vec{b} = \vec{\tau} \times \vec{n} \tag{34}$$

$$\frac{d\vec{\tau}}{dt} = K\vec{n} \; ; \; \frac{d\vec{n}}{dt} = K\vec{\tau} + \Omega\vec{b} \; ; \; \frac{d\vec{b}}{dt} = -\Omega\vec{n}$$

Let us explain the definition of torsion: The plane containing the point $r_0(s)$ and parallel to $\vec{\tau}$ and $\vec{n}$ is usually called the osculating plane. Unless the curve be plane, the osculating plane varies as the point moves along the curve. The change in the direction depends evidently upon the form of the curve. The ratio of the angle $\Delta\theta$, between the binormals at two points of the curve and their curvilinear distance Δs expresses our idea of the mean change in the direction of the osculating plane. If we take the limit of this ratio, as one point approaches the other as the measure of the rate of this change at the latter point. This limit is called the *second curvature* or *torsion* and its inverse the radius of torsion.

In curvilinear system of coordinates one can write magnetic field like $B = B_s\tau + B_x n + B_y b$, where $s, x, y-$ curvilinear coordinates of point P, which is represented by radius-vector $r = r_0(s) + xn(s) + yb(s)$. For solving Maxwell equations we must know the relations between our field components and contravariant and covariant components of B field [8].

$$B^1 = B_x \quad B^2 = \frac{1}{1+Kx}B_s \quad B^3 = B_y - contravariant\ components \tag{35}$$

$$B_1 = B_x \quad B_2 = (1+Kx)B_s \quad B_3 = B_y - covariant\ components$$

here we use $K = \frac{l}{\rho}$ and denote torsion by Ω. Putting (35) into Maxwell equations one can get equations of motion [8]:

$$\left[\frac{ds}{dt}(x'+\Omega y)\right]' = \left\{(1+Kx)\left(\frac{ecB_y}{E}+K\frac{ds}{dt}\right)-\left(\frac{ecB_s}{E}+\Omega\frac{ds}{dt}\right)(y'-\Omega x)\right\} \tag{36}$$

$$\left[\frac{ds}{dt}(y'-\Omega x)\right]' = \left\{\left(\frac{ecB_s}{E}+\Omega\frac{ds}{dt}\right)(x'+\Omega y)-\frac{ecB_x}{E}(1+Kx)\right\}$$

where $\frac{ds}{dt} = \frac{pc^2}{E}[(1+Kx)^2+(x'+\Omega y)^2+(y'-\Omega x)^2]^{\frac{1}{2}}$ or in linearized form:

$$x''+g_x x = [\frac{e}{pc}\frac{\partial B_y}{\partial y}-\Omega']y-[\frac{eB_s}{pc}+2\Omega]y' \tag{37}$$

$$y''+g_y y = [-\frac{e}{pc}\frac{\partial B_x}{\partial x}+\Omega']x+[\frac{eB_s}{pc}+2\Omega]x'$$

where

$$g_x = K^2(1-n)-\Omega^2+\frac{eB_s\Omega}{pc}$$

$$g_y = K^2 n-\Omega^2-\frac{eB_s\Omega}{pc}$$

In that case it is easy to get Hamiltonian: Let us rewrite Eq.(37) in form similar to (3) and (4):

$$x''+k_1^* x = -[K+\frac{1}{2}M']y-\Omega' y-My'-2\Omega y' \tag{38}$$

$$y''+k_2^* y = -[K-\frac{1}{2}M']x+\Omega' x+Mx'+2\Omega x' \tag{39}$$

$$p_x = x'+\frac{1}{2}My+\Omega y \qquad x' = p_x-\frac{1}{2}My-\Omega y$$

$$p_y = y'-\frac{1}{2}Mx-\Omega x \qquad y' = p_y+\frac{1}{2}Mx+\Omega x$$

$$p'_x = -k_1^* x - Ky - (\frac{1}{2}M + \Omega) \cdot (p_y + \frac{1}{2}Mx + \Omega x) = -\frac{\partial H}{\partial x}$$
$$p'_y = -k_2^* y - Kx + (\frac{1}{2}M + \Omega) \cdot (p_x - \frac{1}{2}My - \Omega y) = -\frac{\partial H}{\partial y}$$

and we have Hamiltonian:

$$H = \frac{1}{2}[k_1^* x^2 + k_2^* y^2 + 2Kxy(p_x - \frac{1}{2}My)^2 + (p_y + \frac{1}{2}Mx)^2 + (p_x - \Omega y)^2 + (p_y + \Omega x)^2 + \frac{1}{2}M\Omega y^2 + \frac{1}{2}M\Omega x^2] \quad (40)$$

$k_1^* = k_1 - \Omega^2 + M\Omega \qquad k_2^* = k_2 - \Omega^2 - M\Omega$ If compare Eq.(40) with Eq.(5), we see that effect of torsion gives us additional terms for our Hamiltonian:

$$\frac{1}{2}[(p_x - \Omega y)^2 + (p_y + \Omega x)^2 + \frac{M\Omega}{2}y^2 + \frac{M\Omega}{2}x^2]$$

For not linear case one can get [9]:

$$H = c[m^2c^2 + \frac{1}{(1+Kx)^2}(p_t - eA_t + \Omega y(p_x - eA_x) - \Omega x(p_y - eA_y))^2 + (p_x - eA_x)^2 + (p_y - eA_y)^2]^{\frac{1}{2}} \quad (41)$$

12 Appendix 2

This section will be devote to the definition of torsion and curvature in real variables.

For partical motion in constant magnetic field we have:

$$\frac{d^2}{d\tau^2} = \omega_1 \frac{dy}{d\tau} \qquad \frac{d^2y}{d\tau^2} = -\omega_1 \frac{dx}{d\tau} \quad (42)$$

$$\frac{d^2z}{d\tau^2} = 0 \qquad \frac{d^2t}{d\tau^2} = 0\ , \quad z||\vec{B}\ , \quad \omega_1 = \frac{eB}{mc} \quad (43)$$

From Eq.(43) $\Rightarrow ct = \frac{E_0\tau}{mc}$, $E_0 = c\sqrt{p_0^2 + m^2c^2}$, $E = mc^2\frac{dt}{d\tau} = E_0$ and from Eq.(42) we have:

$$r(t) = \begin{cases} x = Rcos(\omega_2 t + \alpha) + \frac{cp_{0y}}{eB} + x_0 \\ y = -Rsin(\omega_2 t + \alpha) + \frac{cp_{0x}}{eB} + y_0 \\ z = v_{0z}t \end{cases}$$

where $R = \frac{p_{0\perp}c}{eB}$ $p_{0\perp} = \sqrt{p_{0x}^2 + p_{0y}^2}$ $\omega_2 = \frac{eBc}{E}$ $sin\alpha = -\frac{p_{0x}}{p_{0\perp}}$ $cos\alpha = -\frac{p_{0y}}{p_{0\perp}}$
Using Frenet-Serret relations (34) we have:

$$\vec{\tau} = \frac{1}{\sqrt{v_{0z}^2 + R^2\omega^2}} \begin{pmatrix} -R\omega sin(\omega t + \alpha) \\ -R\omega cos(\omega t + \alpha) \\ v_{0z} \end{pmatrix}$$

$$\vec{n} = \frac{1}{\sqrt{v_{0z}^2 + R^2\omega^2}(R\omega^2)} \begin{pmatrix} -R\omega^2 cos(\omega t + \alpha) \\ R\omega^2 sin(\omega t + \alpha) \\ 0 \end{pmatrix}$$

$$\vec{n} = \frac{1}{\sqrt{v_{0z}^2 + R^2\omega^2}} \begin{pmatrix} -cos(\omega t + \alpha) \\ sin(\omega t + \alpha) \\ 0 \end{pmatrix}$$

$$\vec{b} = \vec{\tau} \times \vec{n} = \frac{1}{(v_{0z}^2 + R^2\omega^2)} \begin{pmatrix} -R\omega sin(\omega t + \alpha) \\ -R\omega cos(\omega t + \alpha) \\ v_{0z} \end{pmatrix} \times \begin{pmatrix} -cos(\omega t + \alpha) \\ sin(\omega t + \alpha) \\ 0 \end{pmatrix}$$

$$\frac{d\vec{b}}{dt} = \frac{v_{0z}\omega}{(v_{0z}^2 + R^2\omega^2)} \begin{pmatrix} -cos(\omega t + \alpha) \\ sin(\omega t + \alpha) \\ 0 \end{pmatrix} = -\Omega\vec{n}$$

this relation gives us torsion:

$$\Omega = \frac{v_{0z}\omega}{\sqrt{v_{0z}^2 + R^2\omega^2}}$$

and from $\frac{d\vec{\tau}}{dt} = K\vec{n}$ we have:

$$K = R\omega^2$$

here $v_{0z} = \frac{p_{0z}c^2}{E}$ $R = \frac{\sqrt{p_{0x}^2 + p_{0y}^2}}{eB}$ $\omega = \frac{eBc}{E}$

13 Appendix 3

At that section we introduce the Hamiltonian with small term $\frac{m^2c^2}{E}$:

$$H = -p_s = -(1 + \frac{x}{\rho})\sqrt{\frac{H^2}{c^2} - m^2c^2 - (p_x - \frac{e}{c}A_x)^2 - (p_y - \frac{e}{c}A_y)^2}$$

$$\frac{(E_0+\Delta E)^2}{c^2} - m^2c^2 = \frac{E_0^2}{c^2} + \frac{2E_0\Delta E}{c^2} + \frac{\Delta E^2}{c^2} - m^2c^2 =$$

$$= \frac{E_0^2}{c^2}(1-\frac{1}{\gamma^2}) + \frac{2E_0\Delta E}{c^2} + \frac{\Delta E^2}{c^2}$$

or using $p_0 = \frac{\beta_0 E_0}{c}$ and defining $p_t = -\frac{\Delta E}{p_0 c}$, we have:

$$\frac{(E_0+\Delta E)^2}{c^2} - m^2c^2 = p_0^2(1-\frac{2p_t}{\beta_0}+p_t^2)$$

so our Hamiltonian becomes:

$$H = -(1+\frac{x}{\rho})\sqrt{(1+\eta)^2 - (p_x - \frac{e}{c}A_x)^2 - (p_y - \frac{e}{c}A_y)^2}$$

where $(1+\eta)^2 = (1-\frac{2p_t}{\beta_0}+p_t^2)$

14 Appendix 4

The insertion as a whole does not couple horizontal and vertical motion if the four by four transfer matrix across it, T_{BA} , is block diagonal in form. When all the coupling fields are turned off, the linear motion from any point i to any point j is represented by the block diagonal matrix M_{ij}. Motion across the k-th coupler is given by $N_{2k,k-1}$. So the insertion is exactly decoupled if

$$T_{BA} = \begin{bmatrix} T_{xBA} & 0 \\ 0 & T_{zBA} \end{bmatrix}$$

$$T_{BA} = M_{B,2n}N_{2n,2n-1}M_{2n-1,2n-2}\cdot\cdot NMN\cdot\cdot M_{1,A} \tag{44}$$

that is, if eight simultaneous equations containing the n coupler strengths $k_1,\cdots,k_n$ are satisfied. This description is simplified if define the projection matrix P_i [5]. Note that I.P. is now called like C.

$$P_i = M_{c,2i}N_{2i,2i-1}M_{2i-1,c} \tag{45}$$

$$T_{BA} = M_{BC}(P_n\cdots P_2, P_1)M_{CA}$$

and decoupling conditions become:

$$P_n\cdot\cdot P_2.P_1 = \begin{bmatrix} P_x & 0 \\ 0 & P_z \end{bmatrix} \tag{46}$$

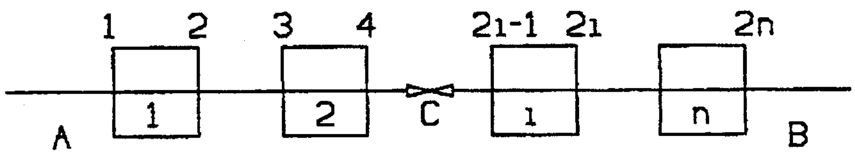

Figure 7: *Symbolic representation of I.R.*

In practice P_i is often very close to the identity matrix and can be expanded as a polynomial in k_i:

$$P_i = I + k_i \cdot K_i + k_i^2 \cdots \tag{47}$$

where K_i is a block anti-diagonal matrix

Putting (47) into (46) we have the general first order decoupling conditions:

$$\sum_{i=1}^{n} k_i K_i = 0 \tag{48}$$

For solenoid we have linear matrix:

$$N_s = F(l, \theta^2) R(\theta) \tag{49}$$

where

$$l - solenoid\ length$$

$$\theta - angle\ of\ rotation\ about\ a\ longitudinal\ axis.$$

$$R(\theta) - \textit{coupled matrix}$$

$$F(l, \theta^2) - \textit{an uncoupled matrix}$$

If M is the uncoupled matrix from C to the entrance plane of the solenoid, then:

$$P_s = I + \theta K_s + \theta^2 \cdots = M^{-1} L^{-1} F R M$$

so that using first order approximation for F and R, we have:

$$K_s = \begin{bmatrix} 0 & S \\ -S^+ & 0 \end{bmatrix} \tag{50}$$

where $S = M_x^{-1} M_z$ and $S^+ = M_z^{-1} M_x$

For quadrupole length l and gradient g, which has been rotated about the beam axis by an angle ϕ away from midplane symmetry, we have coupling matrix:

$$N_q(\phi, l, g) = R(-\phi) N_q(0, l, g) R(\phi) \tag{51}$$

where $N_q(0, l, g)$-uncoupled matrix

Any rotated quadrupole field can be decomposed into a superposition of a regular quadrupole ($\phi = 0$) and a "skew" quadrupole ($\phi = 45deg.$). A thin skew quadrupole has the coupling matrix:

$$N_{4s} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{f} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{f} & 0 & 0 & 1 \end{bmatrix} \tag{52}$$

$$P_q = I + q K_q = M^{-1} N_{4s} M$$

dimensionless strength of a skew quadrupole

$$q = \frac{\sqrt{\beta_x \beta_z}}{f}$$

then

$$K_q = \begin{bmatrix} 0 & Q \\ -Q^+ & 0 \end{bmatrix} \tag{53}$$

where, in terms of Twiss parameters at C and at the skew quadrupole:

$$Q = \frac{1}{\sqrt{\beta_x \beta_z}} M_x^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} M_z \longrightarrow$$

$$Q = \begin{bmatrix} -S_x C_z (\frac{\beta_x^*}{\beta_z^*})^{1/2} & -S_x S_z (\beta_x^* \beta_z^*)^{1/2} \\ \frac{C_x C_z}{(\beta_x^* \beta_z^*)^{1/2}} & C_x S_z (\frac{\beta_z^*}{\beta_x^*})^{1/2} \end{bmatrix}$$

here $S_x = sin(\phi_x)$ and $C_z = cos(\phi_z)$ are trigonometric functions of the betatron phase. Note that x, z- transverse coordinates.

Comparing (50) and (53) the first order decoupling conditions have become the four independent simultaneous equations:

$$\sum_i \theta_i S_i + \sum_j q_i Q_i = 0 \tag{54}$$

So, in general, to compensate solenoid we need four couplers.

15 Appendix 5

The purpose of this section is to remind the definition of canonical transformation. We speak about motion in three dimensional space. Let us denote the coordinate by $q = (q_1, q_2, q_3)$. The motion of particles can be described by the Hamiltonian equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{55}$$

where p_i are the conjugate moments of the variables q_i, H-is the Hamiltonian, describing our system.

The transformation $Q = Q(q,p,t), P = P(q,p,t)$ is canonical when there exists a function $\tilde{H}(Q,P,t)$ such that the Q and P satisfy the equations: $\dot{Q}_i = \frac{\partial \tilde{H}}{\partial P_i}$; $\dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q_i}$ From Hamiltonian mechanics follows that the function Q and P satisfy the following conditions:

$$[Q_i, Q_j] = 0 \quad [P_i, P_j] = 0 \quad [Q_i, P_j] = \delta_{ij} \tag{56}$$

where $[A, B]$ is the Poison bracket of A, B defined as:

$$[A, B] = \sum_j \left(\frac{\partial A}{\partial q_j} \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j} \frac{\partial B}{\partial q_j} \right)$$

If q_{i0} and p_{i0}-initial conditions for the variables q_i and the moments p_i, we can write the solution in such form:

$$q_i = q_i(q_{i0}, p_{i0}, t) \qquad p_i = p_i(q_{i0}, p_{i0}, t)$$

16 Acknowledgements

I want to thank my advisor Herbert Moshammer for all I learned from him. And I am very indepted to Volker Ziemann and Jeff Corbett for many challengging discussions.

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