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RARE K MESON DECAYS IN THE CASE OF A HEAVY TOP QUARK*

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IN THE CASE OF A HEAVY TOP QUARK

By

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The rare decays $K_L \to \pi^0 \ell^+ \ell^-$, $K_L \to \pi^0 \nu_\ell \overline{\nu}_\ell$ and $K^+ \to \pi^+ \nu_\ell \overline{\nu}_\ell$ are studied. The main contributions to these decays do not appear at tree level, but only at one-loop level in electroweak interactions. For this reason, they are highly suppressed within the Standard Model and, at the same time, very sensitive to heavy quark masses.

The effect of a top quark mass of the order of the W boson mass or higher is considered, with the inclusion of strong interaction corrections in the form of perturbative Quantum Chromodynamics (QCD). The calculation is done by building a short-distance, effective electroweak Hamiltonian –with leading QCD corrections included– which is scaled down from the electroweak scale M_W to an appropriate hadronic scale μ.

The short-distance contributions to these decays are sensitive to quark masses and Kobayashi-Maskawa matrix elements. These contributions are estimated here for top quark masses from 50 to 200 GeV. The range of values of the Kobayashi-Maskawa elements is determined by constraints coming from experimental measurements of the $B - \overline{B}$ meson mixing and the CP violating parameter ϵ in the

neutral K system.

Of particular interest are the decays $K_L \to \pi^0 \ell^+ \ell^-$ and $K_L \to \pi^0 \nu_\ell \overline{\nu}_\ell$, which contain important CP violating contributions. The observed decay $K_L \to \pi\pi$, which is CP violating, is dominated by contributions coming from the mixing of CP eigenstates in the K mass matrix. Instead, for the processes under consideration, CP violation may arise mainly from the decay amplitudes themselves. Therefore, experimental measurement of these processes would not only provide further evidence of the occurence of CP violation in Nature, but also help determine whether the origin of this phenomenon is within the Standard Model.

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The Superior Beings that I ignore but must be behind all this.

For my parents Omar and Marta, and my sister Cinthia.

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1. Introduction

One of the paths for searching for physics beyond the Standard Model is by probing low energy processes which are sensitive to the effects of high mass, virtual particles. This inspires much of the present round of rare K decay experiments.[1] Of particular interest are processes that are forbidden in lowest order, such as neutral current processes which change quark flavors, but which can occur through one-loop Feynman diagrams.

Decays such as $K_L \to \pi^0 \ell^+ \ell^-$ and $K \to \pi\nu\overline{\nu}$ are of this type. The change in quark flavor, from an s (in the K) to a d (in the π), occurs in the Standard Model through diagrams involving one or more loops. There is a large window between the present experimental limits on the branching ratios for these processes and the corresponding standard model predictions. Within that window there is the possibility of a branching ratio arising due to new high mass particles in the loop. Even if these processes are finally observed at roughly the expected level, they provide information on the parameters of the Standard Model and, in the case of $K \to \pi\nu\overline{\nu}$, a rate which depends on the number of light neutrino species. In the case of $K_L \to \pi\ell^+\ell^-$, on the other hand, an additional interest arises from the fact that this decay may provide a new place to observe CP violation.

It is almost 25 years since the original observation of CP violation in long-lived neutral K decays.[2] Until very recently, all experiments were consistent with this phenomenon originating in a "superweak" interaction,[3] whose one measurable manifestation was in the mass matrix of the neutral K system. As a result, the long-lived neutral K meson, $K_L \approx K_2 + \epsilon K_1$, is dominantly the CP odd state

K_2, but contains a small admixture ($\propto \epsilon$) of the CP even state K_1.

A different, more definite origin of CP violation occurs in the three generation standard model where CP violating effects arise through the presence of a single, non-trivial phase in the matrix which expresses the mixing of quark flavors under the weak interactions.[4] For the K^0 mass matrix, the CP violating phase enters through "box" diagrams that involve heavy quarks and can connect the quarks in a K^0 ($d\bar{s}$) to those in a $\overline{K}^0$ ($s\bar{d}$), mimicking in this regard a "superweak" theory.

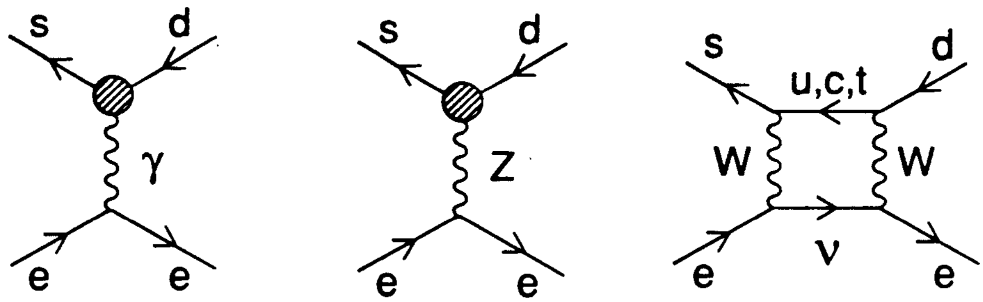
In the past year the NA31 collaboration has presented statistically significant evidence[5] for a non-zero value of the parameter ϵ', which is a measure of CP violation in the $K \to \pi\pi$ decay amplitude itself. Experiments at Fermilab[6] and at CERN[5] are continuing with the aim of reducing the statistical and systematic errors to a level where, if the central value of the CERN experiment holds, a non-zero value of ϵ' will be firmly established and a "superweak" explanation made untenable.

Such a value of ϵ' is consistent,[7–9] within rather large uncertainties of the relevant hadronic matrix element, with the three generation standard model. Indeed, it was suggested[10] 10 years ago that if CP violation originated in a phase of the three generation quark mixing matrix and if one-loop "penguin" diagrams give an important part of the $K \to \pi\pi$ decay amplitude, then a non-zero and measurable ϵ' would result.

While the three generation standard model plausibly explains CP violation as it is observed up to now in Nature, we would like to obtain additional evidence that points in this direction. If we could find several experimental processes which exhibit measurable CP violating effects and all could be fit by a single value of the *ab initio* free phase in the mixing matrix, then we will have gone a long way toward establishing this as the correct explanation. If along the way the standard model cannot account for the results of these experiments, so much the better – we would have evidence for physics beyond the standard model.

There are several avenues toward accomplishing this; none of them is easy. One is to look for CP violating effects in the B meson system. Here, the CP violating asymmetries can potentially be very large – of order 10^{-1} or more in some rare modes, rather than the order 10^{-3} effects in the neutral K mass matrix. The sheer numbers of B mesons estimated to be necessary to get a statistically significant effect, however, put this exciting possibility many years in the future.[11] Another avenue is to consider other K decays where CP violating effects, although very small, may occur with a different weighting (from that in $K \to \pi\pi$) between effects originating in the mass matrix and in the decay amplitude. Although these experiments are also very difficult, there is the advantage of high intensity beams and sophisticated detectors already in existence to perform the measurements of ϵ' and search for rare K decays.

An example of such a process is $K_L \to \pi^0 \ell^+ \ell^-$. If CP were conserved, the long-lived eigenstate would be the CP odd state, K_2. It would not decay to $\pi^0 \gamma_{\text{virtual}} \to \pi^0 \ell^+ \ell^-$, this being forbidden by CP invariance.[12] Since Nature has chosen to break CP invariance, the decay can proceed through: (1) the small part, $\approx \epsilon K_1$, of the K_L wave function that is CP even (we call this "indirect" CP violation); and (2) CP violating effects in the $K_2 \to \pi^0 \ell^+ \ell^-$ decay amplitude itself (we call this "direct" CP violation). In addition to these two CP violating amplitudes, the decay can



(a) (b) (c)

Figure 1.1. Feynman diagrams for the process $\bar{s}d \rightarrow e^+e^-$. a) "electromagnetic penguin"; b) "Z penguin"; c) "box" diagram.

proceed in a CP conserving manner *via* the decay chain $K_2 \rightarrow \pi^0\gamma\gamma \rightarrow \pi^0\ell^+\ell^-$, where the photons are either real or virtual. Although higher order in α, this latter amplitude is not necessarily negligible in comparison to either the "indirect" or "direct" CP violating amplitudes which are also suppressed precisely because they contain factors that are related to CP violation.

Naturally, we are most interested in the question of whether one can see the "direct" CP violation effects and especially to investigate if they can be the dominant amplitude contributing to the decay. This amplitude comes from "penguin" diagrams with a photon or Z boson and also from "box" diagrams, as shown in Figure 1.1. For values of $m_t^2 << M_W^2$, it is the "electromagnetic penguin" that gives the dominant short-distance contribution to the amplitude. This was discussed, with estimates of the CP violating effects,[13] before evidence for the b quark was found. A full analysis, including QCD corrections, was carried out in the case of six quarks,[14] building upon work done with four quarks.[15,16] A principal conclusion of that study was that the "direct" CP violation could be comparable to the "indirect" effects.

Why do we reconsider this process now? First, the possible mass range for the t quark has been pushed upward considerably since Ref. 14. The QCD corrections, which turned out to be quite important, need to be redone when m_t^2/M_W^2 cannot be considered to be a small number. The successive steps of removing heavy particles from the theory and developing an effective Hamiltonian involving only the light quarks can no longer be carried out by first removing the W and then the t quark. Rather, they must be removed together. Second, the "Z penguin" and "W box" diagrams, which are "suppressed" by factors of m_t^2/M_W^2 and were neglected in old calculations, are important for large m_t. We need to consider the QCD corrections to them as well. Third, experiments at the required level of sensitivity are beginning to be considered.[17]

The other process of interest here is $K \to \pi\nu\overline{\nu}$, which is particularly attractive from a theoretical point of view due to the absence of long distance contributions.[18,19] In Standard Model predictions for this process, QCD corrections are often neglected or, to the same end result, stated to be small.[20] When included, they are sometimes treated as an overall multiplicative factor for the whole amplitude, even though it arises from a sum of pieces due to c and t quarks in the loop. An exception is the work of Ellis and Hagelin[21] where QCD corrected top-quark contributions are given in the case where the mass of the top quark in the loop is much smaller or comparable to that of the W.

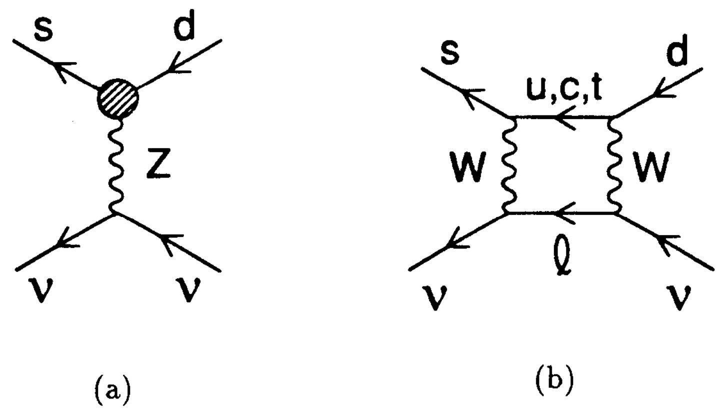


Figure 1.2. Feynman diagrams for the process $\bar{s}d \to \nu\bar{\nu}$: a) "$Z$ penguin"; b) "box" diagram.

In this work, we re-evaluate the QCD corrections to the short-distance amplitude for $K \to \pi\nu\bar{\nu}$ when $m_t \sim M_W$. We give an analytic form for the corrections to the leading logarithmic pieces and discuss the ambiguities in non-leading terms. Quantitatively, the rate for $K^+ \to \pi^+\nu\bar{\nu}$ is decreased by 15 to 30%, a result which is in fact numerically similar to that[22] of applying Ref. 21, even though the detailed expressions are different.

In the next two chapters we introduce the formalism we use in the calculations: Chapter 2 is dedicated to the calculation of the free quark effective Hamiltonian for strangeness-changing processes, while Chapter 3 introduces leading QCD effects into this short-distance effective Hamiltonian. By no means is this the only place where strong interactions affect these decays. The main effect of strong interactions is the formation of hadronic wave functions, which in our case is reflected in the matrix elements of quark currents between hadron states. The calculation of these matrix elements, which is non perturbative, can be avoided by using experimental results in K_{e3} decays, as we discuss in the Appendix. The other potential effect of (non perturbative) strong interactions occurs in the formation of light hadrons in some intermediate states. The possible appearance of this long distance phenomenon will be discussed independently for each one of the decays. In Chapter 4, we briefly describe the range of Kobayashi-Maskawa matrix elements that enter in the amplitudes of the different processes in question. We include here bounds on these parameters coming from various measurements, like the mixing of neutral B mesons and the CP violating parameter ϵ from the neutral K mesons. In the last two chapters we turn to our numerical estimates of the rare decay rates. In Chapter 5 we treat the decay $K_L \to \pi^0 \ell^+ \ell^-$, examining the CP violating amplitudes as well as the different estimates for the CP conserving components. Finally, in Chapter 6 we present our results for the decays $K^+ \to \pi^+ \nu \overline{\nu}$ and $K_L \to \pi^0 \nu \overline{\nu}$. We have included an Appendix with the calculation of the decay rates, starting from the effective Hamiltonians and explicitly showing the normalization and phase conventions used.

2. Free Quark Lagrangian for $\Delta S = 1$ Rare Decays

We calculate the electroweak processes $s \to de^+e^-$ and $s \to d\nu\overline{\nu}$ in the Standard Model to one loop order. Since these decays do not appear at tree level in the absence of flavor-changing neutral currents, the leading contribution occurs at one loop in electroweak interactions.[23]

There are three kinds of diagrams that contribute to the first process, as shown in Fig. 1.1: the "electromagnetic penguin", "Z penguin" and "box" diagram. In the second process (Fig. 1.2), only the last two kinds of diagram are present.

We were able to reproduce the results of Ref. 23 using the Feynman-'tHooft gauge to treat the boson propagators, and dimensional regularization to treat the divergent integrals.

Since the integrals are, in most cases, dominated by virtual momenta larger than the hadron masses (it is not so when the internal quark is the *up*-quark, but in that case the contribution turns out to be irrelevant, as will be shown later), it is a good approximation to neglect all external momenta and masses in the calculation of the sZd vertex and the "box" diagram.

The one-loop diagrams for the sZd vertex are shown in Fig. 2.1, from which we obtain their individual contributions:

$$i\Gamma^{(d)}_{Z,\mu} = i\Gamma^{(s)}_{Z,\mu} = -2i(N.F.)_q\, L'(2+x_q)\,\{\delta + A(x_q)\}\,\overline{s}\gamma_\mu(1-\gamma_5)d$$

$$i\Gamma^{(W)}_{Z,\mu} = 4i(N.F.)_q\;\{L(\delta + B(x_q) - \ln x_q - 1) - 2Rx_qB(x_q)\}\,\overline{s}\gamma_\mu(1-\gamma_5)d$$

$$i\Gamma^{(\phi)}_{Z,\mu} = 2i(N.F.)_q\;\{R(\delta + B(x_q) - \ln x_q - 1) - 2Lx_qB(x_q)\}\,x_q\;\overline{s}\gamma_\mu(1-\gamma_5)d$$

$$i\Gamma^{(WW)}_{Z,\mu} = -8i(N.F.)_q\ \cos^2\theta_W\,\{3\delta + 3A(x_q) - 1\}\,\bar{s}\gamma_\mu(1-\gamma_5)d$$
$$i\Gamma^{(W\phi)}_{Z,\mu} = i\Gamma^{(\phi W)}_{Z,\mu} = -8i(N.F.)_q\ \sin^2\theta_W A(x_q)\bar{s}\gamma_\mu(1-\gamma_5)d \tag{2.1}$$
$$i\Gamma^{(\phi\phi)}_{Z,\mu} = -2i(N.F.)_q\ \cos 2\theta_W\ x_q\,\{\delta + A(x_q)\}\,\bar{s}\gamma_\mu(1-\gamma_5)d$$

where $(N.F.)_q$ is a common normalization factor, L and R the coupling of Z to left-handed and right-handed charge-$\frac{2}{3}e$ internal quarks, L' the coupling of Z to left-handed external quarks, δ an infinite constant, and $A(x_q)$ and $B(x_q)$ finite functions of $x_q \equiv (m_q/M_W)^2$:

$$\begin{aligned}
(N.F.)_q &\equiv \frac{1}{16\pi^2}\left(\frac{g}{2\sqrt{2}}\right)^2 V^*_{qs}V_{qd}\left(\frac{g}{4\cos\theta_W}\right),\\
L &\equiv 1 - \frac{4}{3}\sin^2\theta_W,\\
R &\equiv -\frac{4}{3}\sin^2\theta_W,\\
L' &\equiv -1 + \frac{2}{3}\sin^2\theta_W,\\
\delta &\equiv \frac{2}{4-d} - \gamma_E + \ln\left(\frac{4\pi\nu^2}{M_W^2}\right) + \frac{1}{2},\\
A(x_q) &\equiv \frac{x_q}{x_q-1} - \frac{x_q^2}{(x_q-1)^2}\ln x_q,\\
B(x_q) &\equiv -\frac{1}{x_q-1} + \frac{1}{(x_q-1)^2}\ln x_q.
\end{aligned} \tag{2.2}$$

In the expression for δ, d is the number of spacetime dimensions and ν an arbitrary mass parameter.

The sZd vertex, henceforth denoted as $i\Gamma_{Z,\mu}$, is obtained by adding all the

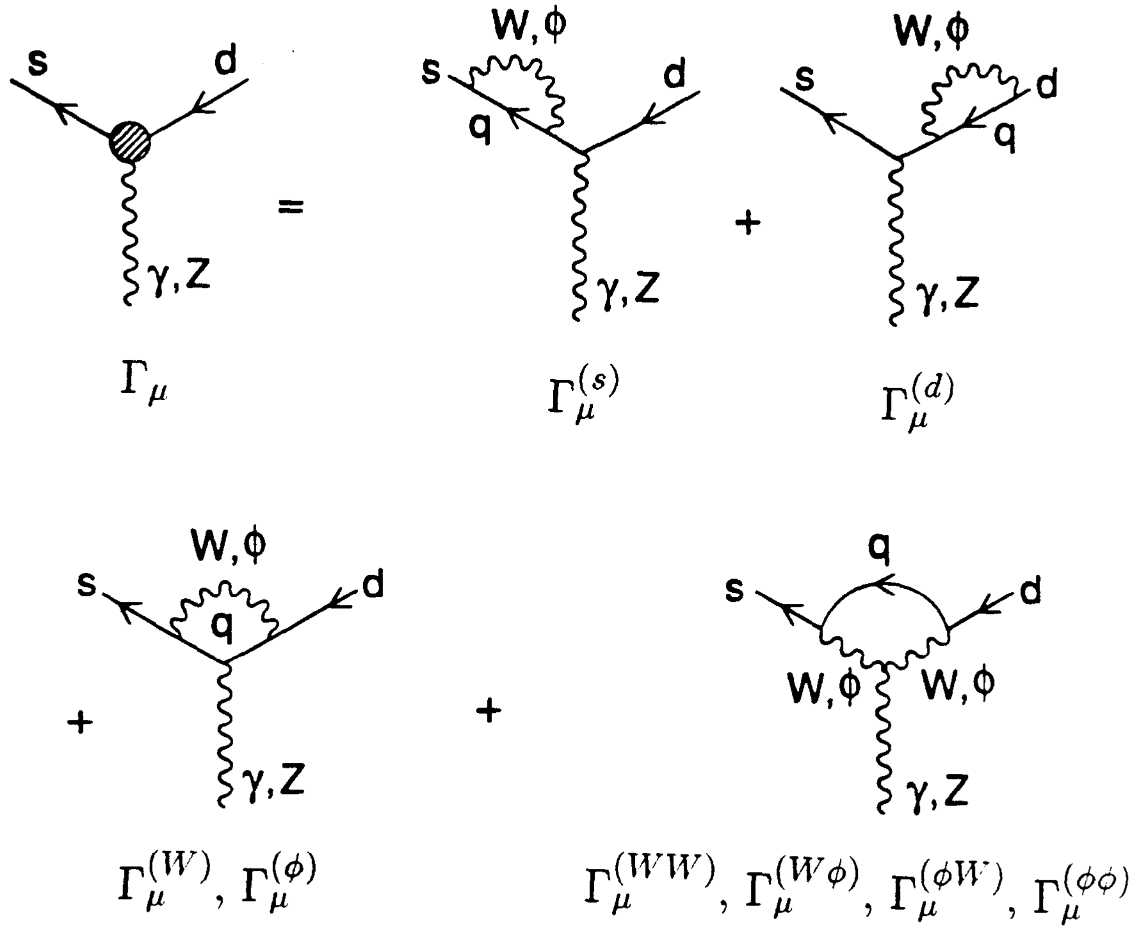


Figure 2.1. Contributions to the electromagnetic and Z flavor-changing vertices.

individual contributions, with the particular exception of $i\Gamma_{Z,\mu}^{(s)}$ and $i\Gamma_{Z,\mu}^{(d)}$, which contribute with only half of their value to the vertex (to this order of approximation, half of their value contribute to the normalization of the external quark wave functions):

$$
\begin{aligned}
i\Gamma_{Z,\mu} = i\sum_q \Big(& \frac{\Gamma^{(s)}_{Z,\mu}}{2} + \frac{\Gamma^{(s)}_{Z,\mu}}{2} + \\
& \Gamma^{(W)}_{Z,\mu} + \Gamma^{(\phi)}_{Z,\mu} + \Gamma^{(WW)}_{Z,\mu} + \Gamma^{(W\phi)}_{Z,\mu} + \Gamma^{(\phi W)}_{Z,\mu} + \Gamma^{(\phi\phi)}_{Z,\mu} \Big) \\
= & \frac{ie}{\pi \sin\theta_W \cos\theta_W} \left(\frac{g}{2\sqrt{2}}\right)^2 \sum_q V^*_{qs} V_{qd} \; F_Z(x_q) \; \bar{s}\gamma_\mu(1-\gamma_5)d.
\end{aligned} \tag{2.3}
$$

where we define the Z coupling form factor:

$$
F_Z(x_q) \equiv \frac{x_q}{16\pi} \left[\frac{(x_q-6)(x_q-1) + (3x_q+2)\ln x_q}{(x_q-1)^2} \right]. \tag{2.4}
$$

Notice that the infinite term δ cancels in the sum: due to the Glashow-Iliopoulos-Maiani (G.I.M.) mechanism -*i.e.* unitarity of the Kobayashi-Maskawa matrix-, there are no flavor changing neutral currents at tree level, so divergent terms at higher order must cancel to assure renormalizability. In addition, terms proportional to $\sin^2\theta_W$ also cancel since these terms couple like the electromagnetic current; the electromagnetic current is conserved, so the vertex vanishes as the gauge boson momentum q_μ goes to zero (for further explanation see below, in the paragraph on the electromagnetic vertex).

Also notice that all the meaningful terms in Γ_Z are x_q-dependent; again, due to G.I.M., physical amplitudes depend only on differences of the sort $\Gamma(x_i) - \Gamma(x_j)$, so that any constant term cancels. Moreover, in the hypothetical case where all quark masses are equal, flavor changing neutral currents vanish exactly to all orders; indeed, in that hypothetical case, the Kobayashi-Maskawa matrix is trivially the identity and, as a consequence, any quark generation mixing vanishes identically.

Using the effective vertex (2.3) and the Feyman rules for the diagram in Fig. 1.1.a, the amplitude for the process $sd \to Z^* \to e^+e^-$ can be easily calculated:

$$i\mathcal{L}_{ee}^{(Z)} = i\frac{G_F}{\sqrt{2}}\sum_q V_{qs}^* V_{qd}\, F_Z(x_q)\left\{\left(\frac{1}{\sin^2\theta_W} - 4\right)Q_V - \frac{1}{\sin^2\theta_W}Q_A\right\} \;+\; h.c., \tag{2.5}$$

where we have denoted the semileptonic effective operators by Q_V and Q_A:

$$\begin{aligned} Q_V &= \frac{e^2}{4\pi}[\bar{s}\gamma_\mu(1-\gamma_5)d][\bar{e}\gamma^\mu e], \\ Q_A &= \frac{e^2}{4\pi}[\bar{s}\gamma_\mu(1-\gamma_5)d][\bar{e}\gamma^\mu\gamma_5 e]. \end{aligned} \tag{2.6}$$

In much the same way, the amplitude for $sd \to Z^* \to \nu\bar{\nu}$ is obtained, with the corresponding changes in the couplings and the substitution of the lepton fields $e \to \nu$ in (2.6):

$$i\mathcal{L}_{\nu\nu}^{(Z)} = -i\frac{G_F}{\sqrt{2}}\sum_q V_{qs}^* V_{qd}\, F_Z(x_q)\,\frac{1}{\sin^2\theta_W}\left\{Q_V - Q_A\right\} \;+\; h.c. \tag{2.7}$$

The other vertex we need to consider is the one with a photon instead of a Z. Here the calculation is slightly more complicated. The $s\gamma d$ vertex vanishes as $q \to 0$ (q is the momentum of the photon), due to electromagnetic current conservation. This did not happen for the Z vertex: the Z current is not conserved, because the Z boson couples differently to left and right handed fermions. While the vector part of the Z current is still conserved, the axial vector part is not; instead, its divergence is proportional to fermion masses.

Since the $s\gamma d$ vertex vanishes as $q \to 0$, it can not be calculated neglecting the external momenta; the calculation must be carried out to order q^2.

Apart from this detail, the rest of the calculation follows that of sZd. In particular, the divergent and x_q-independent terms cancel here as well. The final result can be expressed in terms of two form factors, $F_E(x_q)$ and $F_M(x_q)$:

$$\begin{aligned} i\Gamma_{\gamma,\mu} = \frac{ie}{4\pi}\frac{G_F}{\sqrt{2}}\sum_q V_{qs}^* V_{qd}\Big[&F_E(x_q)\overline{s}(q^2\gamma_\mu - \not{q} q_\mu)(1-\gamma_5)d \\ &+ F_M(x_q)\overline{s}\big(im_s\sigma_{\mu\nu}q^\nu(1-\gamma_5) + im_d\sigma_{\mu\nu}q^\nu(1+\gamma_5)\big)d\Big], \end{aligned} \tag{2.8}$$

where the form factors are

$$\begin{aligned} F_E(x_q) &\equiv \frac{(25-19x_q)x_q^2}{72\pi(x_q-1)^3} - \frac{(3x_q^4 - 30x_q^3 + 54x_q^2 - 32x_q + 8)}{36\pi(x_q-1)^4}\ln x_q, \\ F_M(x_q) &\equiv \frac{(8x_q^2 + 5x_q - 7)x_q}{24\pi(x_q-1)^3} - \frac{(3x_q-2)x_q^2}{4\pi(x_q-1)^4}\ln x_q. \end{aligned} \tag{2.9}$$

The individual contributions from each diagram of Fig. 2.1 to this vertex are rather complicated and do not provide any further insight than what was mentioned above, so we will not reproduce them here.

As we see, there are two form factors for the electromagnetic vertex. The first one, $F_E(x_q)$, which has a logarithmic dependence for small x_q, will be the main contribution to the process $sd \to \gamma^* \to e^+e^-$. The second form factor corresponds to an effective magnetic moment and becomes the only term that survives when the photon is on mass shell; however, for the processes of interest here, its contribution turns out to be negligible.

The amplitude for $sd \to \gamma^* \to e^+e^-$ is then calculated using this effective vertex together with the usual Feynman rules for the diagram in Fig. 1.1.b:

$$i\mathcal{L}_{ee}^{(\gamma)} = -i\frac{G_F}{\sqrt{2}}\sum_q V_{qs}^* V_{qd}\Big\{F_E(x_q)Q_V \;+\; F_M(x_q)Q_M\Big\} \;+\; h.c. \tag{2.10}$$

The operator Q_V is defined in (2.6), and the "magnetic" operator Q_M is given by:

$$Q_M(x_q) \equiv \frac{e^2}{4\pi}\Big[\bar{s}\big(im_s\sigma_{\mu\nu}q^\nu(1-\gamma_5) + im_d\sigma_{\mu\nu}q^\nu(1+\gamma_5)\big)d\Big]\frac{1}{q^2}\Big[\bar{e}\gamma^\mu e\Big]. \tag{2.11}$$

Q_M is not explicitly a local operator (it contains powers of momenta); however, this is not a problem, since this momentum dependence cancels after taking its matrix element.

The last contribution to be considered comes from the "box" diagrams of Fig. 2.2. In a general gauge, there are four different diagrams, due to the presence of would-be Goldstone bosons. However, if we neglect the external masses and momenta, as well as the internal lepton masses, only the diagram with two W bosons gives a non vanishing result. The "box" diagrams are all convergent and their computation is straightforward.

The amplitude for $sd \to e^+e^-$ given by the "box" diagram in Fig. 2.2.a is

$$\begin{aligned} i\mathcal{L}_{ee}^{(Box)} &= -i\frac{G_F}{\sqrt{2}}\sum_q V_{qs}^* V_{qd}\; F_B(x_q)\times \\ &\frac{e^2}{16\pi\sin^2\theta_W}\Big[\bar{s}\gamma_\alpha\gamma_\mu\gamma_\beta(1-\gamma_5)d\Big]\Big[\bar{e}\gamma^\beta\gamma^\mu\gamma^\alpha(1-\gamma_5)e\Big] \;+\; h.c.\,, \end{aligned} \tag{2.12}$$

where

$$F_B(x_q) \equiv \frac{x_q}{8\pi}\left[\frac{1 - x_q + \ln x_q}{(x_q-1)^2}\right]. \tag{2.13}$$

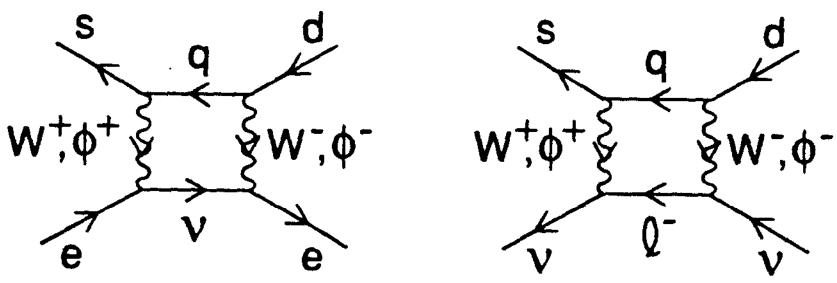


Figure 2.2. Semileptonic "box" diagrams. a) for the process $\bar{s}d \to e^+e^-$; b) for the process $\bar{s}d \to \bar{\nu}\nu$.

After using the identity:

$$\left[\gamma_\alpha\gamma_\mu\gamma_\beta(1-\gamma_5)\right]_{ij}\left[\gamma^\beta\gamma^\mu\gamma^\alpha(1-\gamma_5)\right]_{kl} \equiv 4\left[\gamma_\mu(1-\gamma_5)\right]_{ij}\left[\gamma^\mu(1-\gamma_5)\right]_{kl}, \quad (2.14)$$

the amplitude (2.12) can be expressed as

$$i\mathcal{L}_{ee}^{(Box)} = -i\frac{G_F}{\sqrt{2}}\sum_q V_{qs}^* V_{qd}\ F_B(x_q)\frac{1}{\sin^2\theta_W}\Big\{Q_V - Q_A\Big\} \ + \ h.c. \quad (2.15)$$

There is a slight difference between the "box" contribution to this process and the one to $sd \to \nu\bar{\nu}$: in the latter, the lepton line runs in the opposite direction, which gives a relative minus sign coming from the internal lepton propagator (we neglect lepton masses) and a different order of the γ matrices in the identity (2.14).

The appropriate identity to be used now is:

$$\Big[\gamma_\alpha\gamma_\mu\gamma_\beta(1-\gamma_5)\Big]_{ij}\Big[\gamma^\alpha\gamma^\mu\gamma^\beta(1-\gamma_5)\Big]_{kl} \equiv 16\Big[\gamma_\mu(1-\gamma_5)\Big]_{ij}\Big[\gamma^\mu(1-\gamma_5)\Big]_{kl}, \quad (2.16)$$

which contributes with an extra factor of 4 to the amplitude:

$$i\mathcal{L}_{\nu\nu}^{(Box)} = i\frac{G_F}{\sqrt{2}}\sum_q V_{qs}^* V_{qd}\ F_B(x_q)\frac{4}{\sin^2\theta_W}\Big\{Q_V - Q_A\Big\}\ +\ h.c. \quad (2.17)$$

We are now ready to write the full effective lagrangian for $sd \to e^+e^-$ and $sd \to \nu\overline{\nu}$, calculated to one loop order in electroweak interactions:

$$\begin{aligned} i\mathcal{L}_{ee} =& i\frac{G_F}{\sqrt{2}}\sum_q V_{qs}^* V_{qd}\Bigg[\Bigg\{\left(\frac{1}{\sin^2\theta_W}-4\right)F_Z(x_q) - \frac{1}{\sin^2\theta_W}F_B(x_q) - F_E(x_q)\Bigg\}Q_V \\ &- \frac{1}{\sin^2\theta_W}\Big\{F_Z(x_q) - F_B(x_q)\Big\}Q_A - F_M(x_q)Q_M\Bigg]\ +\ h.c.\,, \quad (2.18) \\ i\mathcal{L}_{\nu\nu} =& -i\frac{G_F}{\sqrt{2}}\sum_q V_{qs}^* V_{qd}\Big[F_Z(x_q) - 4F_B(x_q)\Big]\frac{1}{\sin^2\theta_W}\Big\{Q_V - Q_A\Big\}\ +\ h.c. \quad (2.19) \end{aligned}$$

Before ending this chapter, there are three important points we would like to address.

First, we want to comment on the G.I.M. mechanism, or equivalently the unitarity of the Kobayashi-Maskawa matrix. The unitarity condition implies:

$$\sum_{q=u,c,t} V_{qs}^* V_{qd} = 0.$$

As a consequence, any x_q-independent term in the form factors previously calculated is of no physical significance since it cancels automatically in the amplitudes. Indeed, in the amplitudes, the form factors appear only as differences between their value at two different quark masses. This can be easily seen by replacing, *e.g.*

$$V_{us}^* V_{ud} \;\to\; -V_{cs}^* V_{cd} - V_{ts}^* V_{td} \tag{2.20}$$

in the effective lagrangians; the form factors then appear only in the combinations $F(x_t) - F(x_u)$ and $F(x_c) - F(x_u)$, and the sum over $q = u, c, t$ is replaced by a reduced sum over $q = c, t$. In this manner, G.I.M. is explicitly introduced into the lagrangian and no redundant terms are left. However, the symmetry between fermion generations becomes obscure. Here we prefer to leave the generation symmetry explicit, and introduce the G.I.M. mechanism only at the very end, so we will not do the replacement (2.20) until we reach the final step of calculating decay rates.

The second point we want to address is related to the small-x_q limit. In fact, it is only for the *top* quark that we need to consider values of x_q close to unity; for *charm* and *up* quarks, x_q is of the order of 10^{-3} or less, so that to very good accuracy we can use the small-x_q limits:

$$F_Z(x_q) \;\to\; \frac{x_q}{8\pi}\{\ln x_q + 3\} \;+\; \mathcal{O}(x_q^2), \tag{2.21.a}$$

$$F_B(x_q) \;\to\; \frac{x_q}{8\pi}\{\ln x_q + 1\} \;+\; \mathcal{O}(x_q^2), \tag{2.21.b}$$

$$F_E(x_q) \;\to\; -\frac{2}{9\pi}\ln x_q \;+\; \mathcal{O}(x_q^2), \tag{2.21.c}$$

$$F_M(x_q) \;\to\; \frac{7}{24\pi}x_q \;+\; \mathcal{O}(x_q^2). \tag{2.21.d}$$

We notice from the above expressions that, in the small-x_q limit, the largest form factor is, by far, F_E. All the others are suppressed by at least one power of x_q.

However, we can not conclude much from this, since what has physical significance is not the value of these functions at a given mass, but the difference between the functions evaluated at two different masses. Still, even after considering the differences $F(x_t) - F(x_c)$ and $F(x_c) - F(x_u)$, the electromagnetic form factor is the largest.

In $K \to \pi ee$, therefore, the "electromagnetic penguin" is the most important contribution; in fact, it was the only term considered in past literature, when the *top* quark was believed to be much lighter than the W boson.

In other decays, like $K_L \to \mu\mu$ or $K \to \pi\nu\nu$, the electromagnetic term is absent, so that the "Z penguin" and "box" contributions play a major role. Moreover, in $K_L \to \mu\mu$ the form factors appear in the combination $F_Z - F_B$; this causes the logarithmic terms to cancel, adding an extra suppression to the decay rate. However, strong interactions affect F_Z and F_B differently, so that this cancellation does not actually occur. We should also mention that in these form factors, there are two comparable terms, one of which is logarithmic and the other which is not. Strong interactions have a different effect on these terms as well.

Third and last, we wish to briefly comment on the gauge invariance of the form factors and effective lagrangians. The expressions for F_Z, F_E and F_B given in (2.4), (2.9) and (2.13) are gauge dependent: they were calculated in the Feynman-'tHooft gauge. In a general R_ξ gauge (the Feynman-'tHooft gauge corresponds to $\xi = 1$) the form factors become:[23]

$$
\begin{aligned}
F_Z(x_q) &\rightarrow F_Z(x_q) + g(x_q,\xi) - g(x_q,1),\\
F_E(x_q) &\rightarrow F_E(x_q) - 4g(x_q,\xi) + 4g(x_q,1),\\
F_B(x_q) &\rightarrow F_B(x_q) + g(x_q,\xi) - g(x_q,1) \qquad \text{for } e^+e^-,\\
4F_B(x_q) &\rightarrow 4F_B(x_q) + g(x_q,\xi) - g(x_q,1) \qquad \text{for } \nu\overline{\nu},
\end{aligned}
\tag{2.22}
$$

with the gauge dependent term:

$$
\begin{aligned}
g(x_q,\xi) \equiv \frac{1}{32\pi}\bigg[&\frac{(6\xi+1)x_q-7}{(\xi x_q-1)^2(x_q-1)}x_q \ln x_q -\\
&\frac{x_q}{\xi x_q-1}\Big\{1 + \frac{5\xi+1}{\xi-1}\ln\xi\Big\} + \frac{(2-\xi x_q)}{(\xi x_q-1)^2}x_q\ln\xi\bigg].
\end{aligned}
\tag{2.23}
$$

Although the form factors are gauge dependent, the effective lagrangians (2.18) and (2.19) are clearly gauge independent, since the form factors appear only in combinations such that the term $g(x_q,\xi)$ cancels.

The important issue here is to know what will happen with this cancellation after strong interaction (QCD) corrections are applied. In general, each form factor receives a different correction from QCD, which could spoil the cancellation of gauge dependent terms in the effective lagrangians. Of course, if the calculations were done exactly to all orders, gauge invariance would be automatically satisfied; however, here we are only dealing with an effective theory calculated to a particular order in masses and couplings. Therefore, we need to check for consistency up to that order.

QCD corrections are calculated to the leading logarithm approximation. Since logarithmic terms generated by loop integrals depend on all scales inside the integration region, and the QCD coupling α_s varies with scale, the most important

QCD effects appear in the logarithmic terms of the form factors, provided the range of scales is large. Subleading terms may also get corrected, but in this case a scheme dependence may arise. In order to check the effect of our QCD corrections on the gauge invariance of the effective lagrangians, we need to look into the leading terms of (2.23) when x_q is small (when x_q is large *-i.e.* of order unity-, there are no large logarithms, so QCD effects are negligible).

The expansion of $g(x_q, \xi)$ for small x_q gives:

$$g(x_q, \xi) \; \rightarrow \; \frac{1}{32\pi}\Big[7x_q \ln x_q \; + \; \Big(1 \; + \; \frac{7\xi - 1}{\xi - 1}\ln\xi\Big)x_q\Big]. \tag{2.24}$$

The leading logarithm in (2.24) does not depend on ξ, which assures gauge invariance for the leading logarithmic corrections of the lagrangian. However, the non-logarithmic terms still show a ξ-dependence; gauge invariance for the non-logarithmic terms can be maintained if either the same QCD corrections are applied to all of these terms or, at most, only a part of these terms that do not depend on ξ receive different corrections. Neither of these two procedures is completely systematic, because the QCD coupling $\alpha_s(q^2)$ is calculated up to the leading logarithmic term only. Moreover, there is an infinite number of ways of extracting a ξ-independent piece from the non-logarithmic terms; this is where the scheme dependence enters. The problem of gauge invariance for these terms will be analyzed in the next chapter, once we have chosen our scheme.

3. Leading Logarithmic QCD Corrections and the Effective Hamiltonian

In Chapter 2 we calculated a $\Delta S = 1$ Hamiltonian at the quark level, which contains all short distance electroweak effects in the coefficients of effective low energy operators. The next step in the calculation is to consider the effect of strong (*i.e.* QCD) interactions. We can classify the QCD effects into three categories: first, there is the calculation of matrix elements of quark currents between hadronic wave functions; we will treat this problem in the Appendix. Second, strong interactions can cause hadronization at intermediate stages, where these intermediate hadrons later decay into the external hadrons; these are the so called long distance effects and will be briefly discussed in Chapter 5. Finally, the short distance electroweak vertices also receive corrections through the presence of virtual gluons; this last effect can be calculated perturbatively to a good degree of approximation and will be the main objective of this chapter.

We will calculate the QCD effects to the short distance interaction by building an effective Hamiltonian at a high scale M_W, where QCD can be treated perturbatively. QCD corrections are thus calculated by the inclusion of one gluon at the high scale, and then by running the Hamiltonian down to a hadronic scale with the use of Renormalization Group equations.[14] The one-gluon calculation is kept to the leading logarithm at the scale M_W. Then, using the Renormalization Group equations, the logarithmic terms can be summed to all orders of α_s.

The effective Hamiltonian is an expansion of the standard model Hamiltonian in terms of effective operators, where the heavy fields are removed from the theory in succesive steps, and the coefficients of the operators are determined by means of renormalization group equations. QCD effects are then included in the leading logarithmic terms for scales below the W boson mass.

As we saw in the previous chapter, there are three contributions to the $\Delta S = 1$ Hamiltonian, namely the "electromagnetic penguin", the "Z penguin" and the "W box". These contributions do not mix under renormalization, so they can be treated separately; moreover, they obey a similar scale running pattern, so they can all be treated using the same general method. We first describe this general method to calculate the effective Hamiltonian and then give numerical results for the "electromagnetic penguin", the "Z penguin" and the "W box".

At the scale of M_W or above, the terms in the Hamiltonian are taken to be those in a free (no strong interactions), six quark theory. Below M_W, the effects of QCD are included through the mixing of the effective operators using the machinery of the renormalization group. We first assume a succession of scales characterized by the "old" hierarchy of scales: M_W, m_t, m_b, m_c and finally μ. At the end, we remove the W boson and the top quark together in order to treat the case $m_t \gtrsim M_W$.

At each stage of the calculation, we will be left with an effective Hamiltonian in the form of a sum of Wilson coefficients times operators Q_i:

$$
\begin{aligned}
Q_1 &= (\overline{s}_\alpha \gamma_\mu (1-\gamma_5) d_\alpha)(\overline{u}_\beta \gamma^\mu (1-\gamma_5) u_\beta) \\
Q_2 &= (\overline{s}_\alpha \gamma_\mu (1-\gamma_5) d_\beta)(\overline{u}_\beta \gamma^\mu (1-\gamma_5) u_\alpha) \\
Q_3 &= (\overline{s}_\alpha \gamma_\mu (1-\gamma_5) d_\alpha)[(\overline{u}_\beta \gamma^\mu (1-\gamma_5) u_\beta) \ + \ (\overline{d}_\beta \gamma^\mu (1-\gamma_5) d_\beta) \ + \ \cdots] \\
Q_4 &= (\overline{s}_\alpha \gamma_\mu (1-\gamma_5) d_\beta)[(\overline{u}_\beta \gamma^\mu (1-\gamma_5) u_\alpha) \ + \ (\overline{d}_\beta \gamma^\mu (1-\gamma_5) d_\alpha) \ + \ \cdots]
\end{aligned}
$$

$$Q_5 = (\bar{s}_\alpha \gamma_\mu (1-\gamma_5) d_\alpha)[(\bar{u}_\beta \gamma^\mu (1+\gamma_5) u_\beta) \ + \ (\bar{d}_\beta \gamma^\mu (1+\gamma_5) d_\beta) \ + \ \cdots]$$
$$Q_6 = (\bar{s}_\alpha \gamma_\mu (1-\gamma_5) d_\beta)[(\bar{u}_\beta \gamma^\mu (1+\gamma_5) u_\alpha) \ + \ (\bar{d}_\beta \gamma^\mu (1+\gamma_5) d_\alpha) \ + \ \cdots]$$
$$Q_V = \frac{e^2}{4\pi}(\bar{s}_\alpha \gamma_\mu (1-\gamma_5) d_\alpha)(\bar{e}\gamma^\mu e)$$
$$Q_A = \frac{e^2}{4\pi}(\bar{s}_\alpha \gamma_\mu (1-\gamma_5) d_\alpha)(\bar{e}\gamma^\mu \gamma_5 e) \ . \tag{3.1}$$

The color indices α and β are summed over the three colors. The quark fields appearing in the second factor in the definition of Q_3, Q_4, Q_5, and Q_6 generally include all those which have not yet been removed from the theory. At the last stage, where this includes only the u, d and s quarks, one of the operators in (3.1) is linearly dependent (this is usually taken to be Q_4). We have chosen the same operators as in Ref. 14, with the addition of Q_A, whose presence is required now that we include the contributions from the "Z penguin" and "W box" graphs of Figures 1.1 and 1.2. We have neglected operators of the form $m_s \ \bar{s}\sigma_{\mu\nu} \ F^{\mu\nu} d$ as giving a very small contribution to the net amplitude.

Although in principle we should use the whole set of operators, we neglect the mixing of strong interaction "penguin" operators (Q_3 to Q_6) since we know their coefficients and matrix elements are at least one order of magnitude smaller than those of Q_V and Q_A.

After the W is treated as heavy and removed from the theory, the Hamiltonian can be expressed as:

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \sum_{q=u,c,t} V_{qs}^* V_{qd} \sum_{i=+,-,7} A_i{}^{(q)} O_i{}^{(q)} \ + \ h.\ c.\ , \tag{3.2}$$

where

$$O_\pm^{(q)} \equiv \frac{1}{2}(Q_2 \pm Q_1) \tag{3.3}$$

and O_7 is a linear combination of the semileptonic operators Q_V and Q_A, slightly different for the "electromagnetic penguin", the "Z penguin" and the "W box". The precise normalization of O_7 will be specified later on. The operators $O_\pm^{(q)}$ appear only at scales above m_q where the quark q is still extant in the theory and where they mix with $O_7^{(q)}$ through one-loop electroweak corrections. The effective operator O_7 appears at all scales, and its coefficient contains leading logarithmic QCD corrections as well as non leading terms coming from the free quark theory. These operators satisfy a renormalization group equation of the form:

$$[\mathcal{D}\delta_{ij} \; + \; \gamma_{ij}]\, O_j \; = \; 0, \tag{3.4}$$

with

$$\mathcal{D} \; \equiv \; \mu\frac{\partial}{\partial\mu} \; + \; \beta(g_s)\frac{\partial}{\partial g_s} \; + \; \gamma_q m_q \frac{\partial}{\partial m_q}. \tag{3.5}$$

Since the Hamiltonian is μ-independent, the coefficients A_i satisfy the equation:

$$[\mathcal{D}\delta_{ij} \; - \; \gamma^T{}_{ij}]\, A_j\left(\frac{M_W}{\mu}\right) \; = \; 0, \tag{3.6}$$

with the boundary conditions at $\mu = M_W$ given by $A_+(1) = A_-(1) = 1$. The value of $A_7(1)$ must correspond to the coefficient of O_7 in an effective free quark theory at the scale of M_W. In a case where all quarks are much lighter than the W boson, the coefficient of O_7 at M_W is negligibly small and can be taken to be zero, as was done in Ref. 14. However, for the cases where $m_t \gtrsim M_W$, $A_7(1)$ receives important nonleading-logarithmic contributions, which should not be neglected.

If all the elements of the anomalous dimension matrix γ are of the same order in the strong coupling g and the quark masses m_q, the solution to (3.6) can be easily found by first transforming the components to a basis where γ is diagonal, then solving in this diagonal basis a set of uncoupled differential equations, and finally transforming the solution back to the original basis. It is important that all elements of γ are of the same order in g and m_q, otherwise the transformation matrix will be μ-dependent and the equations will not decouple. To order g^2, it is always possible to define γ in this form by choosing an appropriate normalization of the operators O_i.

Denoting the original basis by latin indices and the diagonal basis by greek indices, the transformation matrix T is defined as:

$$T^{-1}{}_{\alpha i}\, \gamma^T{}_{ij}\, T_{j\beta} \;=\; \gamma(\alpha)\delta_{\alpha\beta}, \qquad\qquad A_i \;=\; T_{i\alpha}\, A_\alpha, \tag{3.7}$$

where $\gamma(\alpha)$ are the eigenvalues of γ, and sum over repeated indices is understood. The solution for $A_i(\frac{M_W}{\mu})$ then becomes:

$$A_i\left(\frac{M_W}{\mu}\right) \;=\; T_{i\alpha} K_{W/\mu}^{r(\alpha)} T^{-1}{}_{\alpha j}\, A_j(1), \tag{3.8}$$

where

$$\begin{aligned} K_{W/\mu} &\equiv \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right], \\ r(\alpha) &\equiv 24\pi^2\gamma(\alpha)/(33-2N_f), \end{aligned} \tag{3.9}$$

and N_f is the number of quark flavors with masses below the scale μ. For scales above the *top* quark mass, $N_f = 6$.

Going below m_t, $N_f = 5$; we then need to expand the operators O_i in terms of operators $O_{i'}$ of an effective theory where the *top* quark has been removed (primed indices will denote a theory with $N_f = 5$).[#1] Using the expansion:

$$O_i \;=\; B_{ik'}\, O_{k'}, \tag{3.10}$$

the coefficients $B_{ij'}$ satisfy a renormalization group equation of the form:

$$[\mathcal{D}\delta_{ij}\delta_{k'l'} \;+\; \gamma_{ij}\delta_{k'l'} \;-\; \delta_{ij}\gamma^T{}_{k'l'}]\, B_{jl'}\left(\frac{m_t}{\mu}\right) \;=\; 0, \tag{3.11}$$

with the boundary conditions at $\mu = m_t$ given by $B_{ik'}(1) = \delta_{ik'}$.

Notice that while γ_{ij} is the anomalous dimension matrix for a theory with six flavors, $\gamma_{k'l'}$ is the anomalous dimension matrix for a theory with five flavors.

Again, the equation for $B_{lk'}$ can be solved by going to a diagonal basis:

$$T^{-1}{}_{\alpha' k'}\; \gamma^T{}_{k'l'}\; T_{l'\beta'} \;=\; \gamma(\alpha')\delta_{\alpha'\beta'}, \qquad S^{-1}{}_{\alpha i}\; \gamma_{ij}\; S_{j\beta} \;=\; \gamma(\alpha)\delta_{\alpha\beta},$$

$$B_{ik'} \;=\; S_{i\alpha}\; T_{k'\alpha'}\; B_{\alpha\alpha'}, \qquad S \;\equiv\; T^{-1^T}, \tag{3.12}$$

and then reexpressing the solution in the original basis:

$$B_{ik'}\left(\frac{m_t}{\mu}\right) \;=\; S_{i\alpha}K_{t/\mu}^{-r(\alpha)}S^{-1}{}_{\alpha j}\;\; T_{k'\alpha'}K_{t/\mu}^{r(\alpha')}T^{-1}{}_{\alpha' l'}\;\; B_{jl'}(1), \tag{3.13}$$

where $K_{t/\mu} \equiv \left[\frac{\alpha_s(m_t)}{\alpha_s(\mu)}\right]$.

#1 We use primes on the indices, not on the operators; this notation is particularly useful when dealing with transformation matrices: if it is the indices that are primed, there is no ambiguity in telling from which to which basis the matrix transforms.

For scales below m_b, the b quark is treated as heavy and removed from the theory. Therefore, the operators from the theory with $N_f = 5$ need to be expanded in terms of operators from a theory with $N_f = 4$. The expansion goes as follows:

$$O_{i'} \;=\; C_{i'k''}\, O_{k''}, \tag{3.14}$$

where the coefficients $C_{i'k''}$ satisfy the boundary conditions $C_{i'k''}(1) = \delta_{i'k''}$ and a renormalization group equation analogous to (3.11):

$$[\mathcal{D}\delta_{i'j'}\delta_{k''l''} \;+\; \gamma_{i'j'}\delta_{k''l''} \;-\; \delta_{i'j'}\gamma^T{}_{k''l''}]\, C_{j'l''}\left(\frac{m_b}{\mu}\right) \;=\; 0, \tag{3.15}$$

with its solution given by:

$$C_{i'k''}\left(\frac{m_b}{\mu}\right) \;=\; S_{i'\alpha'}K_{b/\mu}^{-r(\alpha')}S^{-1}{}_{\alpha'j'}\;\; T_{k''\alpha''}K_{b/\mu}^{r(\alpha'')}T^{-1}{}_{\alpha''l''}\;\; C_{j'l''}(1), \tag{3.16}$$

where $K_{b/\mu} \equiv \left[\frac{\alpha_s(m_b)}{\alpha_s(\mu)}\right]$.

Finally, below m_c, the effective theory has $N_f = 3$ and the operators from the previous theory are expanded as:

$$O_{i''} \;=\; D_{i''k'''}\, O_{k'''}. \tag{3.17}$$

The coefficients satisfy the boundary conditions $D_{i''k'''}(1) = \delta_{i''k'''}$ and the equation:

$$[\mathcal{D}\delta_{i''j''}\delta_{k'''l'''} \;+\; \gamma_{i''j''}\delta_{k''l''} \;-\; \delta_{i''j''}\gamma^T{}_{k'''l'''}]\, D_{j''l'''}\left(\frac{m_c}{\mu}\right) \;=\; 0; \tag{3.18}$$

the solution for $D_{j''l'''}$ is given by:

$$D_{i''k'''}\left(\frac{m_c}{\mu}\right) \;=\; S_{i''\alpha''}K_{c/\mu}^{-r(\alpha'')}S^{-1}{}_{\alpha''j''}\;\; T_{k'''\alpha'''}K_{c/\mu}^{r(\alpha''')}T^{-1}{}_{\alpha'''l'''}\;\; D_{j''l'''}(1), \tag{3.19}$$

where $K_{c/\mu} = \left[\frac{\alpha_s(m_c)}{\alpha_s(\mu)}\right]$.

In consequence, the effective Hamiltonian at a scale μ below m_c is given by the general expression:

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \sum_{q=u,c,t} \Bigg[V_{qs}^* V_{qd} \; A_i^{(q)}\left(\frac{M_W}{\mu}\right) \; B_{ij'}^{(q)}\left(\frac{m_t}{\mu}\right) \times \\ C_{j'k''}^{(q)}\left(\frac{m_b}{\mu}\right) \; D_{k''l'''}^{(q)}\left(\frac{m_c}{\mu}\right) \; O_{l'''} \Bigg] + h.c. \tag{3.20}$$

Although numerical values change from one region of scales to another, the anomalous dimension matrices, in the basis of $O_+^{(q)}$, $O_-^{(q)}$ and O_7 and above m_q, have the general form:

$$\gamma = g^2 \begin{pmatrix} \gamma_+ & 0 & \gamma_{+7} \\ 0 & \gamma_- & \gamma_{-7} \\ 0 & 0 & \gamma_7 \end{pmatrix}. \tag{3.21}$$

Below m_q, all entries are zero except γ_7. The transformation matrices that diagonalize the matrix γ in (3.21) are of the form:

$$T = \begin{pmatrix} \gamma_7 - \gamma_+ & 0 & 0 \\ 0 & \gamma_7 - \gamma_- & 0 \\ -\gamma_{+7} & -\gamma_{-7} & 1 \end{pmatrix}, \qquad S = T^{-1^T}. \tag{3.22}$$

Replacing (3.12), (3.16) and (3.19) in (3.20), we obtain an explicit expression for the QCD corrected effective Hamiltonian. Since O_+ and O_- do not mix, the effective Hamiltonian separates in terms involving these two operators plus terms coming from their mixing to O_7 and subleading terms coming from the free quark theory:

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}}\Bigg[V_{us}^* V_{ud}\Big\{c_+(\mu)Q_+ \;+\; c_-(\mu)Q_-\Big\}$$
$$+ \sum_{q=u,c,t} V_{qs}^* V_{qd}\Big\{c_{7,q}^{(+)}(\mu) + c_{7,q}^{(-)}(\mu) + A_{7,q}(\mu)\Big\}O_7\Bigg] \;+\; h.c.\,, \tag{3.23}$$

where

$$\begin{aligned} c_{7,u}^{(\pm)}(\mu) = &\left[\frac{\gamma_{\pm 7}}{\gamma_7 - \gamma_\pm}\right]_{(u)} \left[K_{W/\mu}^{r(7)} - K_{W/\mu}^{r(\pm)}\right] K_{t/\mu}^{r'(7)} K_{b/\mu}^{r''(7)} K_{c/\mu}^{r'''(7)} \\ &+ K_{W/\mu}^{r(\pm)} \left[\frac{\gamma_{\pm 7}}{\gamma_7 - \gamma_\pm}\right]'_{(u)} \left[K_{t/\mu}^{r'(7)} - K_{t/\mu}^{r'(\pm)}\right] K_{b/\mu}^{r''(7)} K_{c/\mu}^{r'''(7)} \\ &+ K_{W/\mu}^{r(\pm)} K_{t/\mu}^{(r'(\pm)} \left[\frac{\gamma_{\pm 7}}{\gamma_7 - \gamma_\pm}\right]''_{(u)} \left[K_{b/\mu}^{r''(7)} - K_{b/\mu}^{r''(\pm)}\right] K_{c/\mu}^{r'''(7)} \\ &+ K_{W/\mu}^{r(\pm)} K_{t/\mu}^{(r'(\pm)} K_{b/\mu}^{r''(\pm)} \left[\frac{\gamma_{\pm 7}}{\gamma_7 - \gamma_\pm}\right]'''_{(u)} \left[K_{c/\mu}^{r'''(7)} - K_{c/\mu}^{r'''(\pm)}\right], \end{aligned} \tag{3.24.a}$$

$$\begin{aligned} c_{7,c}^{(\pm)}(\mu) = &\left[\frac{\gamma_{\pm 7}}{\gamma_7 - \gamma_\pm}\right]_{(c)} \left[K_{W/\mu}^{r(7)} - K_{W/\mu}^{r(\pm)}\right] K_{t/\mu}^{r'(7)} K_{b/\mu}^{r''(7)} K_{c/\mu}^{r'''(7)} \\ &+ K_{W/\mu}^{r(\pm)} \left[\frac{\gamma_{\pm 7}}{\gamma_7 - \gamma_\pm}\right]'_{(c)} \left[K_{t/\mu}^{r'(7)} - K_{t/\mu}^{r'(\pm)}\right] K_{b/\mu}^{r''(7)} K_{c/\mu}^{r'''(7)} \\ &+ K_{W/\mu}^{r(\pm)} K_{t/\mu}^{(r'(\pm)} \left[\frac{\gamma_{\pm 7}}{\gamma_7 - \gamma_\pm}\right]''_{(c)} \left[K_{b/\mu}^{r''(7)} - K_{b/\mu}^{r''(\pm)}\right] K_{c/\mu}^{r'''(7)}, \end{aligned} \tag{3.24.b}$$

$$c_{7,t}^{(\pm)}(\mu) = \left[\frac{\gamma_{\pm 7}}{\gamma_7 - \gamma_\pm}\right]_{(t)} \left[K_{W/\mu}^{r(7)} - K_{W/\mu}^{r(\pm)}\right] K_{t/\mu}^{r'(7)} K_{b/\mu}^{r''(7)} K_{c/\mu}^{r'''(7)}, \tag{3.24.c}$$

$$A_{7,q}(\mu) = A_7^{(q)}(1)\; K_{W/\mu}^{r(7)} K_{t/\mu}^{r'(7)} K_{b/\mu}^{r''(7)} K_{c/\mu}^{r'''(7)}, \tag{3.24.d}$$

$$c_\pm(\mu) = K_{W/\mu}^{r(\pm)} K_{t/\mu}^{r'(\pm)} K_{b/\mu}^{r''(\pm)} K_{c/\mu}^{r'''(\pm)}. \tag{3.24.d}$$

The coefficients $c_{7,q}^{(\pm)}(\mu)$ come from the mixing of $O_\pm$ into O_7 and correspond to the leading logarithmic pieces of the form factors (2.4), (2,9) and (2.13), with inclusion of QCD corrections; $A_{7,q}(\mu)$ contains the nonleading terms.

In order to fix the boundary conditions $A_7^{(q)}(1)$ at the scale of M_W, we require that the Hamiltonian of the free electroweak theory coincide with the $\alpha_s \to 0$ limit of the effective Hamiltonian:

$$\begin{aligned}\mathcal{H}_{eff} \to & \frac{G_F}{\sqrt{2}}\Big[V_{us}^* V_{ud}\Big(O_+^{(u)} + O_-^{(u)} + \Big\{A_7^{(u)}(1) - 2\pi(\gamma_{+7}+\gamma_{-7})\alpha_s \log(\frac{M_W^2}{\mu^2})\Big\}O_7\Big) \\ & + \ V_{cs}^* V_{cd}\Big(A_7^{(c)}(1) - 2\pi(\gamma_{+7}+\gamma_{-7})\alpha_s \log(\frac{M_W^2}{m_c^2})\Big)O_7 \\ & + \ V_{ts}^* V_{td}\Big(A_7^{(t)}(1) - 2\pi(\gamma_{+7}+\gamma_{-7})\alpha_s \log(\frac{M_W^2}{m_t^2})\Big)O_7\Big] \ + \ h.c. \end{aligned} \quad (3.25)$$

In addition, we must consider the matrix element of $O_\pm^{(u)}$ to one loop order, which is:

$$< O_+^{(u)} + O_-^{(u)} > = \ 2\pi(\gamma_{+7}+\gamma_{-7}) \ \alpha_s \log(\frac{m_u^2}{\mu^2}) < O_7 > . \quad (3.26)$$

Equations (3.25) and (3.26), while written for the $\alpha_s \to 0$ limit, are illustrative of general properties with respect to μ dependence, renormalization-scheme-dependent matrix elements, and subleading terms in $\mathcal{H}_{eff}$. First, the μ dependence explicitly cancels between Eqs. (3.25) and (3.26), as it should. Second, there are possible subleading terms on the right-hand-side of Eq. (3.26) which depend on the renormalization scheme, as do subleading terms in $\mathcal{H}_{eff}$. Since we use the anomalous dimensions and beta function calculated in leading order we do not consistently predict subleading terms in the expansion of $\mathcal{H}_{eff}$; consequently, only the leading logarithmic terms in (3.26) are meaningful. The subleading terms are introduced only as boundary conditions in $A_7^{(q)}(1)$, which are obtained by comparing the free Hamiltonian with the limit of the effective Hamiltonian in Eq. (3.25).

Now we use the method just described to treat the QCD corrections to the "electromagnetic penguin", the "Z penguin" and the "box".

For the "electromagnetic penguin", we recognize that $O_{\pm}^{(u)} \equiv Q_{\pm}$, and that the appropriate operators $O_7^{(q)}$ for $q = u, c, t$ are

$$O_7^{(q)} = O_7 \equiv \frac{1}{\alpha_s} Q_V \; . \tag{3.27}$$

A factor of $1/\alpha_s$ is absorbed in the normalization of O_7 to make all the elements of the anomalous dimension matrix be of the same order in α_s. At the end of the calculation the effective Hamiltonian is expressed in terms of the operators $Q_{\pm}$ and Q_V, and the factor $1/\alpha_s$ put back into the coefficient of the latter operator.

The anomalous dimension matrix for the "electromagnetic penguin", calculated to order g^2 is:

$$\gamma^{(q)} = \begin{pmatrix} g^2/4\pi^2 & 0 & -2g^2/9\pi^2 \\ 0 & -g^2/2\pi^2 & g^2/9\pi^2 \\ 0 & 0 & -(33-2N_f)g^2/24\pi^2 \end{pmatrix} , \tag{3.27}$$

Substituting the operator Q_V and numerical values into (3.23-24), an expression similar to (2.10) is obtained:

$$\begin{aligned} \mathcal{H}_{eff}^{(\gamma)} = \frac{G_F}{\sqrt{2}} \Big[V_{us}^* V_{ud} \Big\{ c_+(\mu) Q_+ \; + \; c_-(\mu) Q_- \Big\} \\ + \sum_{q=u,c,t} V_{qs}^* V_{qd} F_E^{QCD}(x_q, \mu) Q_V \Big] \; + \; h.c. \; , \end{aligned} \tag{3.28}$$

where

$$F_E^{QCD}(x_q, \mu) \equiv \Big\{ c_{7,q}^{(+)}(\mu) + c_{7,q}^{(-)}(\mu) + A_{7,q}(\mu) \Big\} \frac{1}{\alpha_s(\mu)}$$

corresponds to the QCD corrected form factor F_E. The terms $c_{7,q}^{(\pm)}$ and $A_{7,q}$ contain the QCD-corrected leading logarithmic pieces and the nonleading terms, respectively. The explicit expressions for the form factors are:

$$
\begin{aligned}
c_+(\mu) =& K_{W/t}^{6/21}\ K_{t/b}^{6/23}\ K_{b/c}^{6/25}\ K_{c/\mu}^{6/27}, \\
c_-(\mu) =& K_{W/t}^{-12/21}\ K_{t/b}^{-12/23}\ K_{b/c}^{-12/25}\ K_{c/\mu}^{-12/27},
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
F_E^{QCD}(x_u,\mu) =& F_E\big(x_u(M_W)\big) + \frac{2}{9\pi}\ln x_u(M_W) \\
& + \left\{ \left(1 - K_{W/t}^{27/21}\right)\left(\frac{16}{81}\right) + \left(1 - K_{W/t}^{9/21}\right)\left(-\frac{8}{27}\right) \right\} \frac{1}{\alpha_s(M_W)} \\
& + \left\{ K_{W/t}^{6/21}\left(1 - K_{t/b}^{29/23}\right)\left(\frac{16}{87}\right) + K_{W/t}^{-12/21}\left(1 - K_{t/b}^{11/23}\right)\left(-\frac{8}{33}\right) \right\} \frac{1}{\alpha_s(m_t)} \\
& + \Bigg\{ K_{W/t}^{6/21} K_{t/b}^{6/23}\left(1 - K_{b/c}^{31/25}\right)\left(\frac{16}{93}\right) \\
& \quad + K_{W/t}^{-12/21}\ K_{t/b}^{-12/23}\left(1 - K_{b/c}^{13/25}\right)\left(-\frac{8}{39}\right) \Bigg\} \frac{1}{\alpha_s(m_b)} \\
& + \Bigg\{ K_{W/t}^{6/21} K_{t/b}^{6/23}\ K_{b/c}^{6/25}\left(1 - K_{c/\mu}^{33/27}\right)\left(\frac{16}{99}\right) \\
& \quad + K_{W/t}^{-12/21} K_{t/b}^{-12/23}\ K_{b/c}^{-12/25}\left(1 - K_{c/\mu}^{15/27}\right)\left(-\frac{8}{45}\right) \Bigg\} \frac{1}{\alpha_s(m_c)},
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
F_E^{QCD}(x_c,\mu) =& F_E\big(x_c(M_W)\big) + \frac{2}{9\pi}\ln x_c(M_W) \\
& + \left\{ \left(1 - K_{W/t}^{27/21}\right)\left(\frac{16}{81}\right) + \left(1 - K_{W/t}^{9/21}\right)\left(-\frac{8}{27}\right) \right\} \frac{1}{\alpha_s(M_W)} \\
& + \left\{ K_{W/t}^{6/21}\left(1 - K_{t/b}^{29/23}\right)\left(\frac{16}{87}\right) + K_{W/t}^{-12/21}\left(1 - K_{t/b}^{11/23}\right)\left(-\frac{8}{33}\right) \right\} \frac{1}{\alpha_s(m_t)} \\
& + \Bigg\{ K_{W/t}^{6/21} K_{t/b}^{6/23}\left(1 - K_{b/c}^{31/25}\right)\left(\frac{16}{93}\right) \\
& \quad + K_{W/t}^{-12/21}\ K_{t/b}^{-12/23}\left(1 - K_{b/c}^{13/25}\right)\left(-\frac{8}{39}\right) \Bigg\} \frac{1}{\alpha_s(m_b)},
\end{aligned}
\tag{3.31}
$$

$$F_E^{QCD}(x_t,\mu) = F_E(x_t(M_W)) + \frac{2}{9\pi}\ln x_t + \left\{ \left(1 - K_{W/t}^{27/21}\right)\left(\frac{16}{81}\right) + \left(1 - K_{W/t}^{9/21}\right)\left(-\frac{8}{27}\right) \right\} \frac{1}{\alpha_s(M_W)}. \tag{3.32}$$

Following the same steps, the QCD corrections to the "Z penguin" can be found. In this case we use the process $sd \to Z^* \to e^+e^-$, so that the appropriate normalization of $O_7^{(q)}$ is:

$$O_7^{(q))} = \frac{x_q}{\alpha_s}\left[\left(4 - \frac{1}{\sin^2\theta_W}\right) Q_V \; + \; \frac{1}{\sin^2\theta_W} Q_A \right], \tag{3.33}$$

and the anomalous dimension matrices $\gamma^{(q)}$ become:

$$\gamma^{(q)} = \begin{pmatrix} g^2/4\pi^2 & 0 & g^2/8\pi^2 \\ 0 & -g^2/2\pi^2 & -g^2/16\pi^2 \\ 0 & 0 & (-9+2N_f)g^2/24\pi^2 \end{pmatrix} \tag{3.34}$$

for $\mu > m_q$.

The effective Hamiltonian can then be expressed as in (3.23):

$$\mathcal{H}_{eff}^{(Z)} = \frac{G_F}{\sqrt{2}}\Big[V_{us}^* V_{ud} \Big\{ c_+(\mu) Q_+ \; + \; c_-(\mu) Q_- \Big\} + \sum_{q=u,c,t} V_{qs}^* V_{qd} F_Z^{QCD}(x_q,\mu) \Big[\left(4 - \frac{1}{\sin^2\theta_W}\right) Q_V \; + \; \frac{1}{\sin^2\theta_W} Q_A \Big] \; + \; h.c. \tag{3.35}$$

We should mention that the normalization factor

$$\left(4 - \frac{1}{\sin^2\theta_W}\right) Q_V \; + \; \frac{1}{\sin^2\theta_W} Q_A$$

is only due to the Z coupling to electrons, and is irrelevant in the calculation of

QCD corrections. Indeed, this normalization always factors out in the calculations; what actually becomes corrected by QCD is the sZd vertex, *i.e.* the form factor $F_Z(x_q)$ of the hadronic Z current. In this manner, the same QCD corrections apply to $sd \to Z^* \to \nu\bar{\nu}$ and other semileptonic processes mediated by the Z current.

The corrected form factors are thus found:

$$
\begin{aligned}
F_Z^{QCD}(x_u,\mu) = {} & F_Z\big(x_u(M_W)\big) - \frac{1}{8\pi} x_u(M_W)\ln x_u(M_W) \\
& + \left\{ \left(1 - K_{W/t}^{3/21}\right)\frac{3}{3} + \left(1 - K_{W/t}^{-15/21}\right)\frac{3}{30} \right\} \frac{x_u(W)}{\alpha_s(W)} \\
& + \left\{ K_{W/t}^{6/21}\left(1 - K_{t/b}^{5/23}\right)\frac{3}{5} + K_{W/t}^{-12/21}\left(1 - K_{t/b}^{-13/23}\right)\frac{3}{26} \right\} \frac{x_u(t)}{\alpha_s(t)} \\
& + \left\{ K_{W/t}^{6/21} K_{t/b}^{6/23}\left(1 - K_{b/c}^{7/25}\right)\frac{3}{7} \right. \\
& \qquad \left. + K_{W/t}^{-12/21} K_{t/b}^{-12/23}\left(1 - K_{b/c}^{-11/25}\right)\frac{3}{22} \right\} \frac{x_u(b)}{\alpha_s(b)} \\
& + \left\{ K_{W/t}^{6/21} K_{t/b}^{6/23} K_{b/c}^{6/25}\left(1 - K_{c/\mu}^{9/27}\right)\frac{3}{9} \right. \\
& \qquad \left. + K_{W/t}^{-12/21} K_{t/b}^{-12/23} K_{b/c}^{-12/25}\left(1 - K_{c/\mu}^{-9/27}\right)\frac{3}{18} \right\} \frac{x_u(c)}{\alpha_s(c)},
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
F_Z^{QCD}(x_c,\mu) = {} & F_Z\big(x_c(M_W)\big) - \frac{1}{8\pi} x_c(M_W)\ln x_c(M_W) \\
& + \left\{ \left(1 - K_{W/t}^{3/21}\right)\frac{3}{3} + \left(1 - K_{W/t}^{-15/21}\right)\frac{3}{30} \right\} \frac{x_c(W)}{\alpha_s(W)} \\
& + \left\{ K_{W/t}^{6/21}\left(1 - K_{t/b}^{5/23}\right)\frac{3}{5} + K_{W/t}^{-12/21}\left(1 - K_{t/b}^{-13/23}\right)\frac{3}{26} \right\} \frac{x_c(t)}{\alpha_s(t)} \\
& + \left\{ K_{W/t}^{6/21} K_{t/b}^{6/23}\left(1 - K_{b/c}^{7/25}\right)\frac{3}{7} \right.
\end{aligned}
$$

$$+ \; K_{W/t}^{-12/21} \; K_{t/b}^{-12/23} \left(1 - K_{b/c}^{-11/25}\right) \frac{3}{22} \Big\} \frac{x_c(b)}{\alpha_s(b)}, \tag{3.37}$$

$$\begin{aligned} F_Z^{QCD}(x_t, \mu) &= F_Z(x_t(M_W)) - \frac{1}{8\pi} x_t(M_W) \ln x_t(M_W) \\ &+ \left\{ \left(1 - K_{W/t}^{3/21}\right) \frac{3}{3} + \left(1 - K_{W/t}^{-15/21}\right) \frac{3}{30} \right\} \frac{x_t(W)}{\alpha_s(W)}. \end{aligned} \tag{3.38}$$

Finally, QCD corrections to the "box" diagram can be treated in a similar way, now using a semileptonic operator $O_B^{(q)}$ instead of $O_\pm^{(q)}$ and a different normalization for $O_7^{(q)}$:

$$\begin{aligned} O_B^{(q)} &= -i \frac{G_F}{\sqrt{2}} [\bar{s}\gamma_\mu(1-\gamma_5)q][\bar{\nu}\gamma^\mu(1-\gamma_5)e] \times [\bar{q}\gamma_\nu(1-\gamma_5)d][\bar{e}\gamma^\nu(1-\gamma_5)\nu] \\ O_7^{(q)} &= \frac{x_q}{\alpha_s} \frac{1}{\sin^2\theta_W} \Big\{ Q_{7V} - Q_{7A} \Big\}. \end{aligned} \tag{3.38}$$

The anomalous dimension matrices, for $\mu > m_q$, are now only 2×2:

$$\gamma^{(q)} = \begin{pmatrix} 0 & g^2/16\pi^2 \\ 0 & (-9 + 2N_f)g^2/24\pi^2 \end{pmatrix}, \tag{3.39}$$

The effective Hamiltonian can again be written in the form of (3.23):

$$\begin{aligned} \mathcal{H}_{eff}^{(box)} = \frac{G_F}{\sqrt{2}} \Bigg[& V_{us}^* V_{ud} c_B(\mu) O_B^{(u)} \\ & + \sum_{q=u,c,t} V_{qs}^* V_{qd} F_B^{QCD}(x_q, \mu) \frac{1}{\sin^2\theta_W} \Big\{ Q_V - Q_A \Big\} \Bigg] \; + \; h.c. \end{aligned} \tag{3.40}$$

Just like in the previous case, the normalization factor $1/\sin^2\theta_W$ comes from the fact that we chose electrons as external leptons. For the case of $sd \to \nu\bar{\nu}$, there is an additional factor of -4, but in neither case does this affect the QCD corrections to the form factors $F_B(x_q)$.

The corrected form factors are given by:

$$\begin{aligned}
F_B^{QCD}(x_u,\mu) = {} & F_B\big(x_u(M_W)\big) - \frac{1}{8\pi} x_u(M_W)\ln x_u(M_W) \\
& + \frac{1}{2}\left(1 - K_{W/t}^{-3/21}\right)\frac{x_u(W)}{\alpha_s(W)} + \frac{3}{2}\left(1 - K_{t/b}^{-1/23}\right)\frac{x_u(t)}{\alpha_s(t)} \\
& - \frac{3}{2}\left(1 - K_{b/c}^{1/25}\right)\frac{x_u(b)}{\alpha_s(b)} - \frac{1}{2}\left(1 - K_{c/\mu}^{3/27}\right)\frac{x_u(c)}{\alpha_s(c)},
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
F_B^{QCD}(x_c,\mu) = {} & F_B\big(x_c(M_W)\big) - \frac{1}{8\pi} x_c(M_W)\ln x_c(M_W) \\
& + \frac{1}{2}\left(1 - K_{W/t}^{-3/21}\right)\frac{x_c(W)}{\alpha_s(W)} + \frac{3}{2}\left(1 - K_{t/b}^{-1/23}\right)\frac{x_c(t)}{\alpha_s(t)} \\
& - \frac{3}{2}\left(1 - K_{b/c}^{1/25}\right)\frac{x_c(b)}{\alpha_s(b)},
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
F_B^{QCD}(x_t,\mu) = {} & F_B\big(x_t(M_W)\big) - \frac{1}{8\pi} x_t(M_W)\ln x_t(M_W) \\
& + \frac{1}{2}\left(1 - K_{W/t}^{-3/21}\right)\frac{x_t(W)}{\alpha_s(W)}.
\end{aligned} \tag{3.43}$$

We have included a coefficient $c_B(\mu)$ in the effective Hamiltonian (3.40) in analogy with $c_\pm(\mu)$ of the electromagnetic and Z Hamiltonians. However, unlike $c_\pm(\mu)$ which do depend on μ, $c_B(\mu)$ is actually equal to 1 at all scales. The reason for this is clearly that the operator O_B does not run, as can be seen in (3.39): the upper left element of the anomalous dimension matrix is zero.

There is a simple and practical way to obtain the QCD corrections to the leading logarithm of the form factors above. The main idea is to identify the region of the loop momentum that contributes to the logarithmic term and then include the momentum dependence of the effective four-quark operators coming from gluon effects.[24]

The leading term of the electromagnetic form factor $F_E(x_q)$, which is given in (2.21.c), can be generated as:

$$-\frac{2}{9\pi}\ln x_q = \frac{2}{9\pi}\int_{m_q^2}^{M_W^2}\frac{dq^2}{q^2}. \tag{3.44}$$

QCD effects are then taken into account by simply including the factor $\left(2c_+(q^2) - c_-(q^2)\right)$ in the integral, which reflects the running of the four-quark vertices Q_+ and Q_- from the scale M_W^2 down to q^2:

$$\frac{2}{9\pi}\int_{m_q^2}^{M_W^2}\frac{dq^2}{q^2} \rightarrow \frac{2}{9\pi}\int_{m_q^2}^{M_W^2}\frac{dq^2}{q^2}\left(2c_+(q^2) - c_-(q^2)\right), \tag{3.45}$$

with $c_+(q^2)$ and $c_-(q^2)$ given in (3.29).

The corrections to the leading term of the Z form factor $F_Z(x_q)$ can be found in a similar way, with one further complexity: not only the running of $Q_\pm$ from M_W^2 down to q^2 needs to be to be included in the integrand, but also the running of m_q^2 from q^2 down to m_q^2, which reflects the running of Q_7 in the region where this operator is active. Consequently, the leading term of $F_Z(x_q)$ in (2.21.a) can be corrected as:

$$
\begin{aligned}
\frac{1}{8\pi} x_q \ln x_q &= -\frac{1}{8\pi} \int\limits_{m_q^2}^{M_W^2} \frac{dq^2}{q^2} x_q \\
&\to -\frac{1}{8\pi} \int\limits_{m_q^2}^{M_W^2} \frac{dq^2}{q^2} x_q(q^2)\big(2c_+(q^2) - c_-(q^2)\big),
\end{aligned}
\tag{3.46}
$$

with

$$
x_q(q^2) = x_q \left(\frac{\alpha_s(q^2)}{\alpha_s(m_q^2)} \right)^{24/(33-2N_f)} . \tag{3.47}
$$

In much the same way, the corrections to the "box" factor $F_B(x_q)$ are calculated. In this case, however, there are no $Q_\pm$ operators in the process; instead, there is only the effective operator Q_B, which is not affected by QCD to this order of approximation [*i.e.* $c_B(q^2) \equiv 1$]. Therefore, the corrections to $F_B(x_q)$ come from the running of the quark mass alone:

$$
\begin{aligned}
\frac{1}{8\pi} x_q \ln x_q &= -\frac{1}{8\pi} \int\limits_{m_q^2}^{M_W^2} \frac{dq^2}{q^2} x_q \\
&\to -\frac{1}{8\pi} \int\limits_{m_q^2}^{M_W^2} \frac{dq^2}{q^2} x_q(q^2).
\end{aligned}
\tag{3.48}
$$

To conclude this chapter, we wish to comment on our treatment of the non logarithmic terms. In the treatment presented above, all terms except the leading logarithms are introduced as matching conditions at the scale M_W. In order to do this, we choose a scheme in which all one-loop matrix elements of effective operators, such as $< O_\pm >$ in (3.26), are kept only to its leading logarithm, subtracting off all other terms that arise in the regularization procedure. In this manner, all nonleading terms appear as coefficients evaluated at the same scale M_W and, therefore, gauge dependent terms cancel in the physical amplitudes, just like in the free quark theory of Chapter 2.

If we choose instead a scheme that leaves some non-logarithmic terms in the evaluation of one-loop matrix elements, like:

$$< O_{\pm}^{(q)} > \sim \left\{ \log(\frac{m_q^2}{\mu^2}) \; + \; 1 \right\} \tag{3.49}$$

then not all non-logarithmic terms are introduced at the scale M_W. There are terms that appear at the scale m_q, which is where the operators $O_{\pm}^{(q)}$ drop out of the effective theory. Consequently, the Hamiltonian (3.23) has an additional set of terms:

$$\begin{aligned} \mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \Bigg[& V_{us}^* V_{ud} \Big\{ c_+(\mu) Q_+ \; + \; c_-(\mu) Q_- \Big\} \qquad (3.50) \\ & + \sum_{q=u,c,t} V_{qs}^* V_{qd} \Big\{ c_{7,q}^{(+)}(\mu) + c_{7,q}^{(-)}(\mu) + B_{7,q}^{(+)}(\mu) + B_{7,q}^{(-)}(\mu) + A_{7,q}(\mu) \Big\} O_7 \Bigg] \; + \; h.c., \end{aligned}$$

with

$$B_{7,q}^{(\pm)}(\mu) \; = \; K_{W/q}^{r(\pm)} \; B_{7,q}^{(\pm)} \; K_{q/\mu}^{r(7)}. \tag{3.51}$$

The non-logarithmic terms, now included in $A_{7,q}$ and $B_{7,q}$, are affected differently by QCD.

For example, for the "Z penguin", the non-logarithmic terms on the first row of Eqs. (3.37) and (3.38) respectively become:

$$\begin{aligned}&F_Z(x_c(M_W)) - \frac{1}{8\pi}x_c(M_W)\{\ln x_c(M_W) + 1\}\\ &+ \frac{1}{8\pi}\left\{2K_{W/t}^{6/21}K_{t/b}^{6/23}K_{b/c}^{6/25} - K_{W/t}^{-12/21}K_{t/b}^{-12/23}K_{b/c}^{-12/25}\right\}x_c(m_c),\end{aligned}$$

and

$$\begin{aligned}&F_Z(x_t(M_W)) - \frac{1}{8\pi}x_t(M_W)\{\ln x_t(M_W) + 1\}\\ &+ \frac{1}{8\pi}\left\{2K_{W/t}^{6/21} - K_{W/t}^{-12/21}\right\}x_t(m_t).\end{aligned}$$

Although this scheme dependence is an undesirable ambiguity that arises from the one-loop leading logarithm approximation, the numerical differences are fairly small in most cases, so that it is still possible to make reasonably good physical predictions.

With respect to possible gauge dependences, there is still cancellation of the gauge dependent terms in the physical amplitudes. Even when some non leading terms do get different corrections from QCD, these terms are gauge independent. All of the gauge dependent terms still appear only at the scale M_W, where the effect of QCD reduces to simply replacing all quark masses by their values at that scale.

4. The Kobayashi-Maskawa Matrix Elements

The analytical expressions for the decay rates under consideration can be easily obtained with the use of the effective Hamiltonians of the previous chapter. However, in order to give numerical estimates, we need to take into account the uncertainties in the parameters that enter these expressions. Once the top quark mass is fixed, there are still uncertainties coming from the unknown Kobayashi-Maskawa elements. Since the top quark has not yet been observed, there are no direct measurements of either m_t or the K.M. elements that involve a top quark. In addition, some of the other K.M. elements are not well known either. In order to find estimates for the unknown K.M. elements, we need to use constraints coming from other experimental results that indirectly depend on these quantities.

The Kobayashi-Maskawa matrix for three generations of quarks can be parametrized by three real angles θ_{12}, θ_{23}, θ_{13} and one phase δ_{13} [25]

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}, \qquad (4.1)$$

where the indexes i and j run over u, c, t and d, s, b respectively, and c_{kl} and s_{kl} are abbreviations for $\cos\theta_{kl}$ and $\sin\theta_{kl}$.

Most of the elements that mix the first two generations are well known: [26]

$$|V_{ud}| = 0.9747 \pm 0.0011,$$

$$|V_{us}| = 0.2197 \pm 0.0023,$$

$$|V_{cd}| = 0.21 \pm 0.03 \ .$$

For $|V_{cs}|$, there is an experimental lower bound of 0.66, which is rather weak. A much tighter bound is found by imposing unitarity of the K.M. matrix in three generations:

$$|V_{cs}| = 0.9744 \pm 0.0011,$$

which clearly does not hold if there exists a fourth generation.

The elements involving the third generation are less certain. From the measurement of B decays into charmed mesons, a value of $|V_{cb}| = 0.046 \pm 0.010$ is extracted, and from the lack of observation of decays into non-charmed mesons, an upper limit for $|V_{ub}|$ can be found:

$$q \equiv \left|\frac{V_{ub}}{V_{cb}}\right| < 0.2 \quad . \tag{4.2}$$

There is no direct measurement of the elements involving the top quark. We find upper and lower bounds for these elements using unitarity relations of the K.M. matrix and constraints coming from the measurement of $B^0 - \overline{B}^0$ mixing[27] and the CP violating parameter ϵ of the $K^0 - \overline{K}^0$ system.[28] The last two quantities provide an indirect measurement of the K.M. elements, which appear in loops containing a top quark. The bounds so obtained are rather loose and, in addition, strongly dependent on the top quark mass. While $B^0 - \overline{B}^0$ mixing restricts the value of $|V_{td}|$, pushing it to smaller values as m_t is taken to be larger, the result on ϵ forces the phase δ_{13} to be far from the extreme values 0 or π.

These constraints seem to be sufficient to consistently determine bounds for the K.M. elements. However, the observation of $K_L \to \mu^+\mu^-$, which is now in progress,[29] may provide a stronger bound than that of $B - \overline{B}$ mixing if this decay rate turns out to be smaller than the present value. Although we will not include this constraint here, it may have to be considered in future analyses.

The $B^0 - \overline{B}^0$ mixing parameter x_d is defined as

$$x_d = \frac{2|M_{12}|}{\Gamma}, \tag{4.3}$$

where M_{12} is the non diagonal element of the $B^0 - \overline{B}^0$ mass matrix and Γ is the B meson width. In the standard model, M_{12} is calculated from "box" diagrams.[30] Assuming only three generations of quarks, x_d is given by:

$$x_d = \frac{G_F^2}{6\pi^2}\, f_B^2\, B_B\, \tau_B\, m_B\, M_W^2\, \eta_t\, S(x_t)\, \left|V_{td}V_{tb}^*\right|^2, \tag{4.4}$$

where all contributions coming from the charm quark have been neglected. In the above expression, f_B is the B meson decay constant, B_B the bag parameter coming from the hadronic matrix element of the four-quark operator, τ_B the B meson lifetime and m_B its mass. η_t is a QCD correction to the four-quark operator[31] and $S(x_t)$ a function of the top quark mass coming from the loop integral:

$$S(x) = \frac{x}{4(1-x)^2}\left(4 - 11x + x^2\right) - \frac{3x^3}{2(1-x)^3}\ln x. \tag{4.5}$$

with $x_t = m_t^2/M_W^2$.

In order to determine the constraints coming from Eq. (4.4), we fix the values of the quantities that are better known:

$$m_B = 5.28 \text{ GeV},$$

$$M_W = 81.\text{ GeV},$$

$$\eta_t = 0.85\ .$$

and allow the rest to vary inside their accepted ranges:[32]

$$f_B^2 B_B = (0.1 \rightarrow 0.2 \text{ GeV})^2,$$

$$x_d = 0.6 \rightarrow 0.85,$$

$$\tau_B = (1.04 \rightarrow 1.32) \cdot 10^{-12} \text{ sec.}$$

Unitarity relations force $|V_{tb}| \approx 1$, so that (4.4) practically becomes a bound for $|V_{td}|$ as a function of m_t.

The constraint imposed by the CP violating parameter ϵ can be found in much the same way. The expression for ϵ in the standard model, calculated from the $\Delta S = 2$ "box" diagram, becomes:

$$\begin{aligned} \epsilon = e^{i\pi/4} \frac{G_F^2\ f_K^2\ m_K\ M_W^2\ B_K}{12\pi^2\ \sqrt{2}\ \Delta m_K}\ Im\Big\{&(V_{cs}^* V_{cd})^2\ \eta_1\ S(x_c) \\ &+ 2\ (V_{cs}^* V_{cd} V_{ts}^* V_{td})\ \eta_3\ S(x_c, x_t)\ +\ (V_{ts}^* V_{td})^2\ \eta_2\ S(x_t)\Big\}, \end{aligned} \tag{4.6}$$

where all the quantities are defined in analogy to Eq. (4.4). The values of the QCD corrections in this case are $\eta_1 = 0.7$, $\eta_2 = 0.6$ and $\eta_3 = 0.4$,[33] and the function $S(x_c, x_t)$, for $x_c << x_t$, is given by:

$$S(x_c, x_t) = x_c \Big\{ \ln \frac{x_t}{x_c} - \frac{3x_t}{4(1 - x_t)} \big(1 + \frac{x_t}{1 - x_t} \ln x_t\big) \Big\}. \tag{4.7}$$

Eq. (4.6) can be simplified in the phase convention of Eq. (4.1) if we neglect $Re[V_{ts}^* V_{td}]$ compared to $Re[V_{cs}^* V_{cd}]$ and use

$$Im[V_{ts}^* V_{td}] = -Im[V_{cs}^* V_{cd}] = s_{23}\ s_{13}\ s_{\delta_{13}},$$

which is the factor that appears in all CP violating quantities within the standard model (in the original parametrization of Kobayashi and Maskawa,[4] this factor corresponds to $-s_1 s_2 s_3 s_\delta$). In this manner, (4.6) can be expressed as:

$$\epsilon = e^{i\pi/4} \frac{G_F^2\ f_K^2\ m_K\ M_W^2\ B_K}{6\pi^2\ \sqrt{2}\ \Delta m_K}\ Im[V_{ts}^* V_{td}] \Big\{ Re[V_{cs}^* V_{cd}] (\ \eta_3\ S(x_c, x_t) - \eta_1\ S(x_c)\ ) + Re[V_{ts}^* V_{td}]\ \eta_2\ S(x_t) \Big\}. \tag{4.8}$$

Here we use $\epsilon = 2.28 \cdot 10^{-3}$. The large uncertainties in (4.8) come from B_K and m_c, which are taken to be:

$$B_K = 0.3 \rightarrow 1.5$$

$$m_c = 1.3 \rightarrow 1.7\ \text{GeV}.$$

The main effect of the constraint (4.8) is to keep the phase δ_{13} away from 0 or π, and to provide a lower limit for $|V_{ub}| = s_{13}$. Both restrictions, which depend on m_t, become weaker as m_t is taken to be larger.

The combination of constraints coming from unitarity, $B - \overline{B}$ mixing and ϵ defines a complicated region in the space of K.M. parameters which, in addition, depends on m_t.[34] Unitarity constraints fix $s_{12} = 0.22$, and determine the bounds $0.036 < s_{23} < 0.056$ and $q = s_{13}/s_{23} < 0.2$, while $B - \overline{B}$ and ϵ define further constraints on s_{23}, s_{13} and δ_{13} as a function of m_t.

We search for the extreme values of the decays $K_L \rightarrow \pi^0 \ell^+ \ell^-$, $K_L \rightarrow \pi^0 \nu\overline{\nu}$ and $K^+ \rightarrow \pi^+ \nu\overline{\nu}$ as a function of m_t, by scanning the region of K.M. parameters allowed by the constraints above mentioned. In the first two decays, which are CP violating, the quantity that enters is $Im[V_{ts}^* V_{td}]$, shown in Table (4.1). Instead, what enters in the third decay is a combination of $Re[V_{ts}^* V_{td}]$ and $Im[V_{ts}^* V_{td}]$, weighted by the coefficients $C_{\nu,c}$ and $C_{\nu,t}$, as shown in the Appendix. The values

m_t (GeV)	$Im[V_{ts}^* V_{td}]$ $min.(max.) \times 10^{-4}$	s_{23}	δ_{13}	q	m_c (GeV)
50.	0.86 (6.2)	.055 (.056)	172 (102)	.2 (.2)	1.7 (1.3)
100.	0.37 (5.1)	.055 (.056)	176 (53)	.18 (.2)	1.7 (1.3)
150.	0.24 (4.2)	.055 (.056)	176 (41)	.12 (.2)	1.7 (1.3)
200.	0.21 (3.5)	.053 (.055)	174 (35)	.07 (.2)	1.7 (1.3)

Table 4.1. Parameters that minimize (maximize) $Im[V_{ts}^* V_{td}]$ as a function of m_t.

of these parameters can be found in Table (4.2). It is interesting to notice that, while the CP violating decays are bound by $Im[V_{ts}^* V_{td}]$, the extreme values of the charged K decay are determined mainly by $Re[V_{ts}^* V_{td}]$.

m_t (GeV)	$B(K^+ \to \pi^+ \nu\overline{\nu})$ $min.(max.) \times 10^{-10}$	$Re[V_{ts}^* V_{td}]$ $\times 10^{-4}$	$Im[V_{ts}^* V_{td}]$ $\times 10^{-4}$
50.	0.15 (0.4)	−7.9 (−13.)	1.4 (.87)
100.	0.127 (1.16)	−2.6 (−13.)	2.6 (.44)
150.	0.135 (1.75)	−1.5 (−11.)	2.2 (1.1)
200.	0.151 (2.10)	−1.0 (−8.9)	1.9 (.86)

Table 4.2. Values of $V_{ts}^* V_{td}$ that minimize (maximize) $B(K^+ \to \pi^+ \nu\overline{\nu})$ as a function of m_t.

5. Branching Ratios for $K_L \to \pi^0 \ell^+ \ell^-$

In previous chapters we set the general formalism and physical constraints in order to treat rare decays of K mesons. We now proceed to apply our formalism to the calculation of the decay $K_L \to \pi^0 \ell^+ \ell^-$. This decay is expected to be mainly CP violating, although it may contain important CP conserving contributions. The CP violating amplitudes originates from two sources, namely the small admixture of a CP even state in K_L measured by the parameter ϵ, and a direct CP violating decay amplitude.

In section 5.1 we discuss the possible magnitudes of the CP conserving amplitudes. The CP violating contribution coming from ϵ —which we call "indirect"— with inclusion of long distance effects is treated in section 5.2. The main part of our work is dedicated to the calculation of the direct CP violating amplitude, which is shown in section 5.3. We close this chapter in section 5.4, where we compare the different contributions and draw our conclusions.

5.1. The CP Conserving Amplitude

A CP conserving contribution to the process $K_2 \to \pi^0 \ell^+ \ell^-$ is induced through the chain $K_2 \to \pi^0 \gamma\gamma \to \pi^0 \ell^+ \ell^-$, which is shown in Figure 5.1. We give here a brief review of the checkered history of this amplitude, partly because it is of interest in and of itself, but mainly to see whether the CP conserving contribution to $\Gamma(K_2 \to \pi^0 \ell^+ \ell^-)$ is comparable to the CP violating contribution or might even "drown out" the latter.

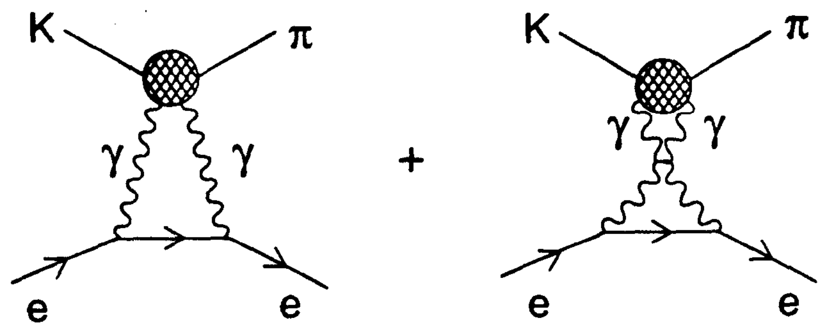


Figure 5.1. CP conserving contribution to $K_L \to \pi^0 e^+ e^-$ mediated by two photons.

The absorptive part of Figure 5.1 can be calculated with the two intermediate photons on shell. For the first part of this process, $K_2 \to \pi^0\gamma\gamma$, there are two invariant amplitudes.[35] If we take the momenta as p, p', q_1 and q_2, respectively, the photon field tensors as $F^{(1,2)}_{\mu\nu}$, and define $x_{1,2} = p \cdot q_{1,2}/p^2$, then they may be expressed in a gauge invariant way as:

$$< \pi\gamma\gamma||K_2 > \; = \; \frac{1}{2}A(x_1,x_2)F^{(1)}_{\mu\nu}F^{\mu\nu}_{(2)} + B(x_1,x_2)\frac{p_\rho p_\sigma}{p^2}F^{\rho\nu}_{(1)}F^{\sigma}_{(2)_\nu}. \qquad (5.1.1)$$

To this order, any other gauge invariant term is a combination of these two. Expressing the fields in terms of the polarization vectors of the photons $\epsilon_{1,2}$, the amplitude becomes:

$$\begin{aligned} < \pi\gamma\gamma||K_2 > \; = \; & A(x_1,x_2)\big[q_2\cdot\epsilon_1\ q_1\cdot\epsilon_2 - q_1\cdot q_2\ \epsilon_1\cdot\epsilon_2\big] \; + \\ & B(x_1,x_2)\big[p^2\ x_1x_2\epsilon_1\cdot\epsilon_2 + q_1\cdot q_2\ p\cdot\epsilon_1\ p\cdot\epsilon_2/p^2 \qquad (5.1.2) \\ & - x_1\ q_2\cdot\epsilon_1\ p\cdot\epsilon_2 - x_2\ q_1\cdot\epsilon_2\ p\cdot\epsilon_1\big] \end{aligned}$$

When joined with the QED amplitude for $\gamma\gamma \to \ell^+\ell^-$ to form the amplitude for $K_2 \to \pi^0\ell^+\ell^-$, the contribution from the A amplitude gets a factor of m_ℓ in front of it. This is not hard to understand, as the total angular momentum of the $\gamma\gamma$ system that pertains to the A amplitude is zero; the same is then true of the final $\ell^+\ell^-$ system. However, the interactions, being electroweak, always match (massless) left-handed leptons to right-handed antileptons and *viceversa*, causing the decay of a J = 0 system to massless leptons and antileptons to be forbidden. Hence the factor of m_ℓ in the overall amplitude for $K_2 \to \pi^0\ \ell^+\ell^-$, which causes the A amplitude to give a negligible contribution for $K_2 \to \pi^0 e^+e^-$. A corollary of this theorem applies when the $K_2 \to \pi^0\ \gamma\gamma$ amplitude is calculated using traditional current algebra techniques in the limit of vanishing pion four-momentum. Only a non-vanishing A-type amplitude is predicted. In order to see this, we may simply take the limit of (5.1.2) as the pion four-momentum vanishes, in which case,

$$p = q_1 + q_2 \qquad \text{and} \qquad x_1 = x_2 = \frac{q_1 \cdot q_2}{p^2} = \frac{1}{2}. \tag{5.1.3}$$

The amplitude (6.3.1) then turns into a term of the A-type:

$$< \pi\gamma\gamma || K_2 > \quad \to \quad \Big(A(x_1, x_2) + \frac{1}{2}B(x_1, x_2)\Big)\big[q_2 \cdot \epsilon_1\ q_1 \cdot \epsilon_2 - q_1 \cdot q_2\ \epsilon_1 \cdot \epsilon_2\big]. \tag{5.1.4}$$

The factor of m_e then found[36] to be produced in the absorptive part of the amplitude for $K_2 \to \pi^0\ e^+e^-$ merely reflects the presence of a single amplitude of the kind of (5.1.4) in the current algebra calculation. If this were the end of the story, the CP conserving contribution to $K_2 \to \pi^0 e^+e^-$ would produce negligible branching ratios at the 10^{-13} level[36] or smaller.[35]

On the other hand, the contraction of the amplitude for $\gamma\gamma \rightarrow e^+e^-$ with the B amplitude produces terms with no such factor of m_e. B does however contain a coefficient with two more powers of momentum, and one might hope for its contribution to be suppressed by angular momentum barrier factors. Because of the extra powers of momentum, in chiral perturbation theory this amplitude is put in by hand and its coefficient not predicted. An order of magnitude estimate may be obtained by pulling out the known dimensionful factors in terms of powers of f_π, and asserting that the remaining coupling strength should be of order one.[35] The branching ratio for $K_2 \rightarrow \pi^0\ e^+e^-$ is then of order 10^{-14}. Again, the CP conserving amplitude would make a negligible contribution to the decay rate.

However, an old fashioned vector dominance, pole model predicts[37] comparable A and B amplitudes in $K_2 \rightarrow \pi^0\gamma\gamma$ and a branching ratio for $K_2 \rightarrow \pi^0 e^+e^-$ of order 10^{-11}, roughly at the level of that arising from the CP violating amplitudes (see below). The B amplitude is far bigger[38] than would be estimated[35] in chiral perturbation theory. The applicability of such a model, however, can be challenged on the grounds that the low energy theorems and Ward identities of chiral perturbation theory are not being satisfied.[39] The consistent implementation of vector dominance with the chiral and other constraints may lead to an extra suppression factor, and to a smaller prediction than in the old fashioned model.

At this point the burden is still on the theorists to show that the CP conserving contribution is truly negligible in $K_L \rightarrow \pi^0 e^+e^-$. After a short period when factors of m_e^2 seemed to assure this, we are presently not able to claim it. In the longer run, it will be in the hands of experimentalists to measure $K_L \rightarrow \pi^0\gamma\gamma$ and eventually to separate the A and B amplitudes by measuring the Dalitz plot distributions, particularly the invariant mass of the two photons.[40]

5.2. The CP Violating Amplitude from the Mass Matrix

As already noted in the Introduction, the presence of CP violation in the mass matrix of the neutral K system results in a small admixture of the CP even K_1 state being found in the long-lived eigenstate:

$$K_L = \frac{K_2 + \epsilon\, K_1}{\left[1 \;+\; |\epsilon|^2\right]^{1/2}} \;, \tag{5.2.1}$$

where the denominator is unity to an excellent approximation, as[28] $|\epsilon| \approx 2.28 \cdot 10^{-3}$. We define "indirect" CP violation as arising from the part of the K_L eigenstate which is proportional to ϵ in (5.2.1): CP is violated within the mass matrix, producing the K_1 admixture in the K_L, while the decay $K_1 \to \pi^0 \ell^+ \ell^-$ itself proceeds in a CP conserving manner.

So defined, the magnitude of "indirect" CP violation is dependent upon the choice of phase convention for the K^0 and $\overline{K}^0$ states, as the value of ϵ depends on this choice. We choose the commonly used convention where the weak interaction amplitude for $K^0 \to \pi\pi$ is real when the $\pi\pi$ system has isospin zero. As we do most calculations in a quark basis where this is not true (precisely because of CP violation in the decay amplitude for $K \to \pi\pi$), we will have to do a transformation

$$\begin{aligned} |K^0> &\to e^{-i\xi}|K^0> \\ |\overline{K}^0> &\to e^{+i\xi}|\overline{K}^0> \;, \end{aligned} \tag{5.2.2}$$

with $15.6|\xi| \;=\; |\epsilon'/\epsilon|$ from strong interaction "penguin" effects,[10] to get to the commonly used phase convention. This induces a term in the $K_L \to \pi^0 \ell^+ \ell^-$

amplitude proportional to $i \sin \xi \approx i\xi$ (which is about an order of magnitude smaller than that which is proportional to ϵ); we shall take due account of this term later when we consider the total CP violating amplitude that includes both "indirect" and "direct" pieces. This net amplitude, being a physical quantity, is independent of phase convention.

With the above definition of "indirect" CP violation we may estimate its contribution to the decay rate from the identity:

$$\begin{aligned} B(K_L \to \pi^0 e^+ e^-)_{\text{indirect}} \equiv\; & B(K^+ \to \pi^+ e^+ e^-) \times \\ \frac{\tau_{K_L}}{\tau_{K^+}} \times \frac{\Gamma(K_1 \to \pi^0 e^+ e^-)}{\Gamma(K^+ \to \pi^+ e^+ e^-)} & \times \frac{\Gamma(K_L \to \pi^0 e^+ e^-)_{\text{indirect}}}{\Gamma(K_1 \to \pi^0 e^+ e^-)} \quad . \end{aligned} \tag{5.2.3}$$

This allows us to relate the desired quantity to the known branching ratio for the CP conserving decay $K^+ \to \pi^+ e^+ e^-$. Experimental values[26] of 2.7×10^{-7} and 4.2 may be inserted for the first two factors on the right hand side, while the last factor is $|\epsilon|^2$ by the definition above of what we mean by "indirect" CP violation. The third factor can be measured directly one day. For the moment it is the subject of model dependent theoretical calculations, with a value of 1 if the transition between the K and the π is $\Delta I = 1/2$. This is the case for the short-distance amplitude which involves a transition from a strange to a down quark. For $\Delta I = 3/2$, the corresponding value is 4. With both isospin amplitudes present and interfering, any value is possible.[41] Using a value of unity for this factor makes

$$B(K_L \to \pi^0 e^+ e^-)_{\text{indirect}} = 0.58 \times 10^{-11} \quad .$$

This is quite close to the previous estimate in Ref. 14, although the discussion is phrased in a different manner. Instead of relating the branching ratio back to

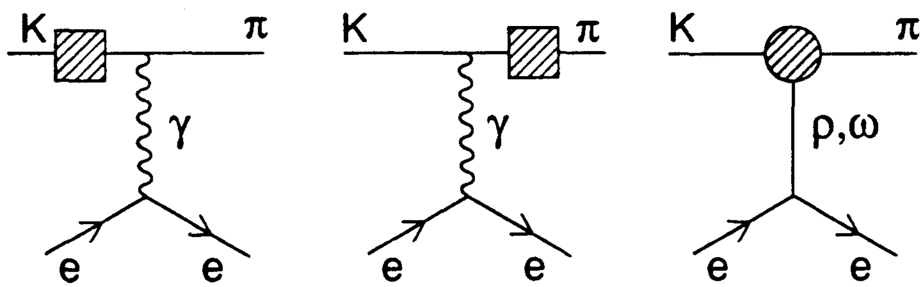


Figure 5.2. Long distance contributions to $K_L \to \pi^0 e^+ e^-$.

that for $K^+ \to \pi^+ e^+ e^-$, one could proceed directly from the amplitude for $K_1 \to \pi^0 e^+ e^-$ using the theoretical, QCD corrected, short-distance contribution to the real part of this amplitude. This is dangerous; the QCD corrections to the real part of the short-distance contribution are so large as to change its sign, as pointed out in Ref. 14, and discussed in the next Section. As a result, its magnitude cannot be calculated reliably. It is too small to explain $\Gamma(K^+ \to \pi^+ e^+ e^-)$ and there is a high likelihood that long-distance contributions are important. Ultimately all of this discussion can be bypassed by an experimental measurement of $\Gamma(K_S \to \pi^0 e^+ e^-)$. This will provide a direct determination of the third factor on the right-hand side of Eq. (5.2.3), removing all the present uncertainty that stems from our theoretical inability to supply a precise prediction for this decay rate.

5.3. CP Violation from the Decay Amplitude

We now turn to the calculation of the CP violating contributions to the $K_2 \to$

$\pi^0\ell^+\ell^-$ amplitude. After QCD effects are introduced in the theory, we are left with an effective Hamiltonian for the process $K_L \to \pi^0 e^+e^-$ in the form of a sum of Wilson coefficients times local operators which, following the notation of Ref. 42, is expressed as:

$$\begin{aligned}\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}}\Bigg[V_{us}^* V_{ud}\Big\{c_+(\mu)Q_+ \;+\; c_-(\mu)Q_- \;+\; c_B(\mu)Q_B\Big\} \\ +\sum_{q=u,c,t} V_{qs}^* V_{qd}\Big\{\widetilde{C}_{7V,q}Q_V \;+\; \widetilde{C}_{7A,q}Q_A\Big\}\Bigg] \;+\; h.c.\ ,\end{aligned} \tag{5.3.1}$$

where the coefficients

$$\begin{aligned}\widetilde{C}_{7V,q} &\equiv \Big\{F_E^{QCD}(x_q,\mu) + \frac{1}{\sin^2\theta_W}F_B^{QCD}(x_q,\mu) + \left(4 - \frac{1}{\sin^2\theta_W}\right)F_Z^{QCD}(x_q,\mu)\Big\},\\ \widetilde{C}_{7A,q} &\equiv \frac{1}{\sin^2\theta_W}\Big\{F_Z^{QCD}(x_q,\mu) - F_B^{QCD}(x_q,\mu)\Big\},\end{aligned} \tag{5.3.2}$$

are given in terms of the QCD corrected form factors defined in Chapter 3.

We have neglected operators of the form $m_s\, \bar{s}\sigma_{\mu\nu}\, F^{\mu\nu}d$ as giving a very small contribution to the net amplitude, after their coefficients and matrix elements are taken into account. In addition, some important features of the QCD effects are worth mentioning. First, to order e^0, non-zero coefficients are generated for the first six operators as we move successively down from the weak scale to one quark mass and then another. The operators Q_3, Q_4, Q_5, and Q_6 arise from "penguin" diagrams involving gluons. The operators $Q_\pm = \frac{1}{2}[Q_2 \pm Q_1]$ are multiplicatively renormalized:

$$c_\pm(\mu^2) = \left[\frac{\alpha_s(M_W^2)}{\alpha_s(\mu^2)}\right]^{r(\pm)} c_\pm(M_W^2)\ , \tag{5.3.3}$$

with $c_\pm(M_W^2) = 1$, and where $r(+) = 6/(33-2N_f)$ and $r(-) = -12/(33-2N_f)$ for

N_f quark flavors in leading logarithmic approximation between the scale M_W and the scale μ. At the same time, to order e^2 the coefficients of the operators Q_V and Q_A are generated from their values at M_W plus mixing effects of the operators Q_1 and Q_2 with Q_V or Q_A. The "penguin" operators, Q_3, Q_4, Q_5, and Q_6, which arise only through QCD effects, have coefficients which start out at zero at the weak scale. They typically never grow to be more than an order of magnitude smaller than the coefficients for $Q_\pm$. So, while we in principle consider the whole 8×8 anomalous dimension matrix[43] which describes the mixing among all the operators in Eq. (3.1) as we go from one scale to another, it is an excellent approximation to consider the mixing only of $Q_\pm$ with Q_V and Q_A and the renormalization of $Q_\pm$ as in (5.3.3). In the same spirit we neglect the effect of taking order e^2 matrix elements of the "penguin operators," which is also known to give a small effect.[44]

The derivation of the QCD corrected contributions when $m_t \sim M_W$ proceeds in a straightforward manner, if one follows the general method given in Chapter 3. The QCD corrections to $\widetilde{C}^{(\gamma)}_{7V,t}$ are negligible and those to $\widetilde{C}^{(\gamma)}_{7V,c}$ are large. However, the corrections to $\widetilde{C}^{(\gamma)}_{7V,u}$ are enormous, for they can easily change not only the magnitude but the sign of this coefficient. As pointed out in Ref. 14, this is readily understandable by considering the correction to $F_E(x_c) - F_E(x_u)$ rewritten as:

$$-\frac{2}{9\pi}\int_{\mu^2}^{m_c^2} \frac{dq^2}{q^2}\left[2\ c_+(q^2) - c_-(q^2)\right] .$$

Before QCD effects are considered, the integrand is $[2 \times 1 - 1] = 1$. When QCD is included, the coefficient $c_+(q^2)$ decreases and $c_-(q^2)$ increases so that the cancellation between the terms in the integrand becomes more complete. In fact, over most or all of the region of integration from μ^2 to m_c^2 the second term overwhelms the first and the integrand is negative.

For the real (CP conserving) part of the short-distance generated amplitude, the contribution from the top quark is negligible because of the Kobayashi - Maskawa factor. It is $\widetilde{C}_{7V,c}^{(\gamma)} - \widetilde{C}_{7V,u}^{(\gamma)}$ which gives the important short-distance contribution to the real part of the amplitude for $K \to \pi\ell^+\ell^-$, and the possibilities for making a precise theoretical prediction are nil because of the situation we have just described: The QCD corrections typically change not just the magnitude but even the sign of the coefficient of Q_V. Aside from this explicit indication of danger from delicate cancellations in the calculation, a comparison of the magnitude of the resulting amplitude with that required from the measured rate for $K^+ \to \pi^+e^+e^-$ shows that the theoretical calculation gives a result that is much too small to explain the data. Long-distance contributions, not unexpectedly, are necessary to understand the magnitude of the real part of the amplitude.

This is entirely different than the situation with regard to the imaginary (CP violating) part of the amplitude. The Kobayashi - Maskawa factors for charm and top are the same, up to a sign:

$$Im\ \frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} = -Im\ \frac{V_{cs}^* V_{cd}}{V_{us}^* V_{ud}} = \frac{Im\ [V_{ts}^* V_{td} V_{us} V_{ud}^*]}{|V_{us} V_{ud}|^2}\ . \tag{5.3.4}$$

These quantities are all invariant under a (quark field) rephasing,[45,46] and in (5.3.4) have been kept in a form to exhibit that fact. The numerator on the right-hand-side is just a form of the invariant measure of CP non-conservation proposed by Jarlskog[45] for three generations. In the original parametrization of Ref. 4, the quantities in (5.3.4) are expressible as $\sin\theta_2 \sin\theta_3 \sin\delta = s_2 s_3 s_\delta$, with cosines of small angles set equal to unity. We shall use this shorthand to refer to the rephase invariant quantity in (5.3.4) in what follows.

Because of the Kobayashi-Maskawa factors, it is momentum scales from m_c^2 to m_t^2 that contribute to the imaginary part. This can be seen, for example, by combining the charm and top quark leading logarithmic contributions to the imaginary part of $C_{7V}^{(\gamma)} \equiv C_{7V,t}^{(\gamma)} - C_{7V,c}^{(\gamma)}$ in the absence of QCD:

$$\begin{aligned} ImC_{7V}^{(\gamma)} &= s_2 s_3 s_\delta \left[\widetilde{C}_{7V,t}^{(\gamma)}(\mu^2) - \widetilde{C}_{7V,c}^{(\gamma)}(\mu^2)\right] \\ &= s_2 s_3 s_\delta \left[\frac{2}{9\pi} \int_{m_t^2}^{M_W^2} \frac{dq^2}{q^2} - \frac{2}{9\pi} \int_{m_c^2}^{M_W^2} \frac{dq^2}{q^2}\right] \\ &= -\, s_2 s_3 s_\delta \left[\frac{2}{9\pi} \int_{m_c^2}^{m_t^2} \frac{dq^2}{q^2}\right] . \end{aligned} \tag{5.3.5}$$

Thus, no dependence on the scale μ appears in this expression. There is every reason to expect the short-distance contributions to give the dominant part of the "direct" CP violating amplitude.[47]

Once QCD corrections are applied, the integrand is reduced, but over most or all of the range of integration it does not change sign (from that for free quarks). Thus, while the QCD corrections are non-negligible, they are fairly insensitive to changes in parameters and reliably calculable for the imaginary part. This is shown in Figure 5.3, where the QCD corrected $\widetilde{C}_{7V}^{(\gamma)} = \widetilde{C}_{7V,t}^{(\gamma)} - \widetilde{C}_{7V,c}^{(\gamma)}$ is indicated with solid curves for $\Lambda_{QCD} = 100$ and 250 MeV as a function of the top quark mass. The result is independent of μ^2. While about a factor of two smaller than the result without QCD (dashed curve), the result does not depend strongly on Λ_{QCD} or *top* quark mass.

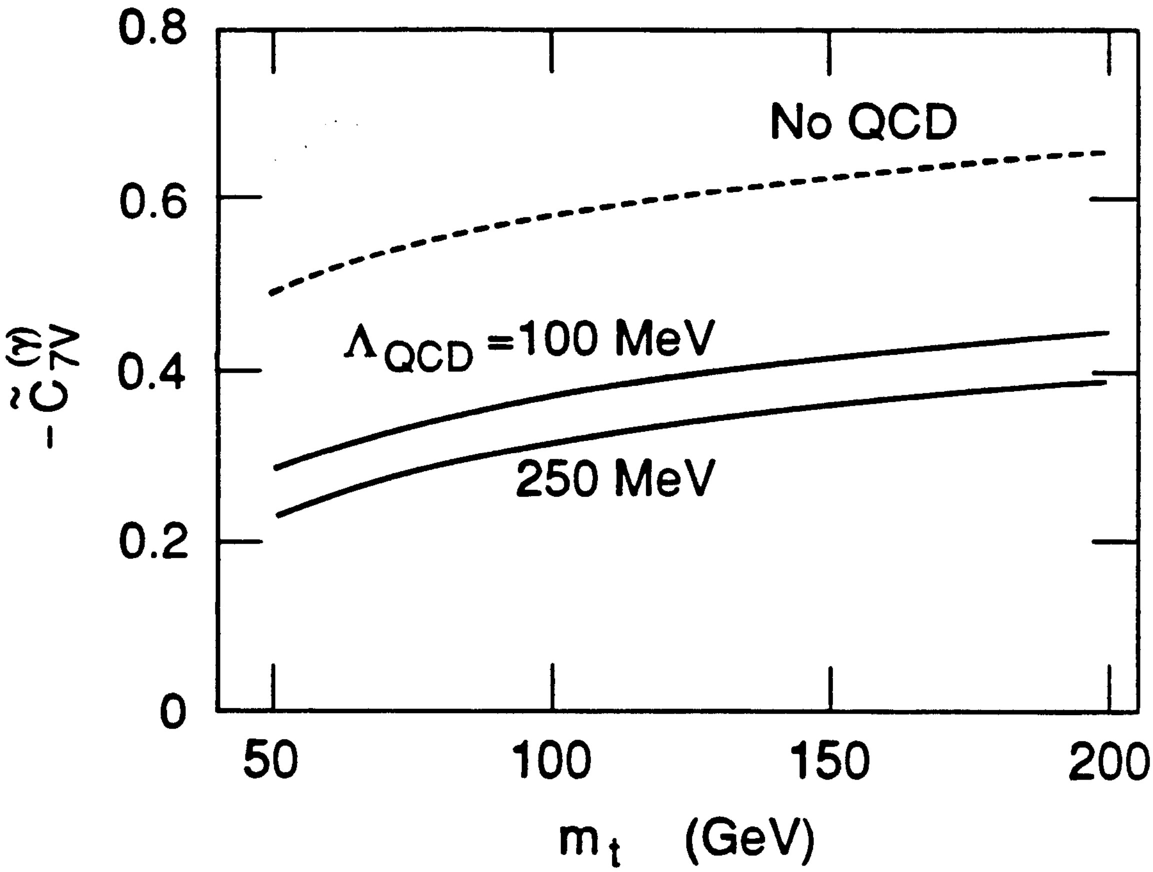


Figure 5.3. "Electromagnetic penguin" contribution $\widetilde{C}_{7V}^{(\gamma)} = \widetilde{C}_{7V,t}^{(\gamma)} - \widetilde{C}_{7V,c}^{(\gamma)}$ as a function of m_t without (dashed curve) and with (solid curves) QCD corrections for Λ_{QCD} = 100 and 250 MeV.

To assemble the full coefficient, C_{7V}, we need to add the "Z penguin" and "W box" contributions. For those involving the t quark, they may be taken directly from their value in the free quark theory. When $m_t \sim M_W$ there are no QCD corrections to be applied, as these contributions are generated at momentum scales from m_t to M_W where there are no large logarithms.[48] For those contributions involving the c quark, there are important QCD corrections., However,

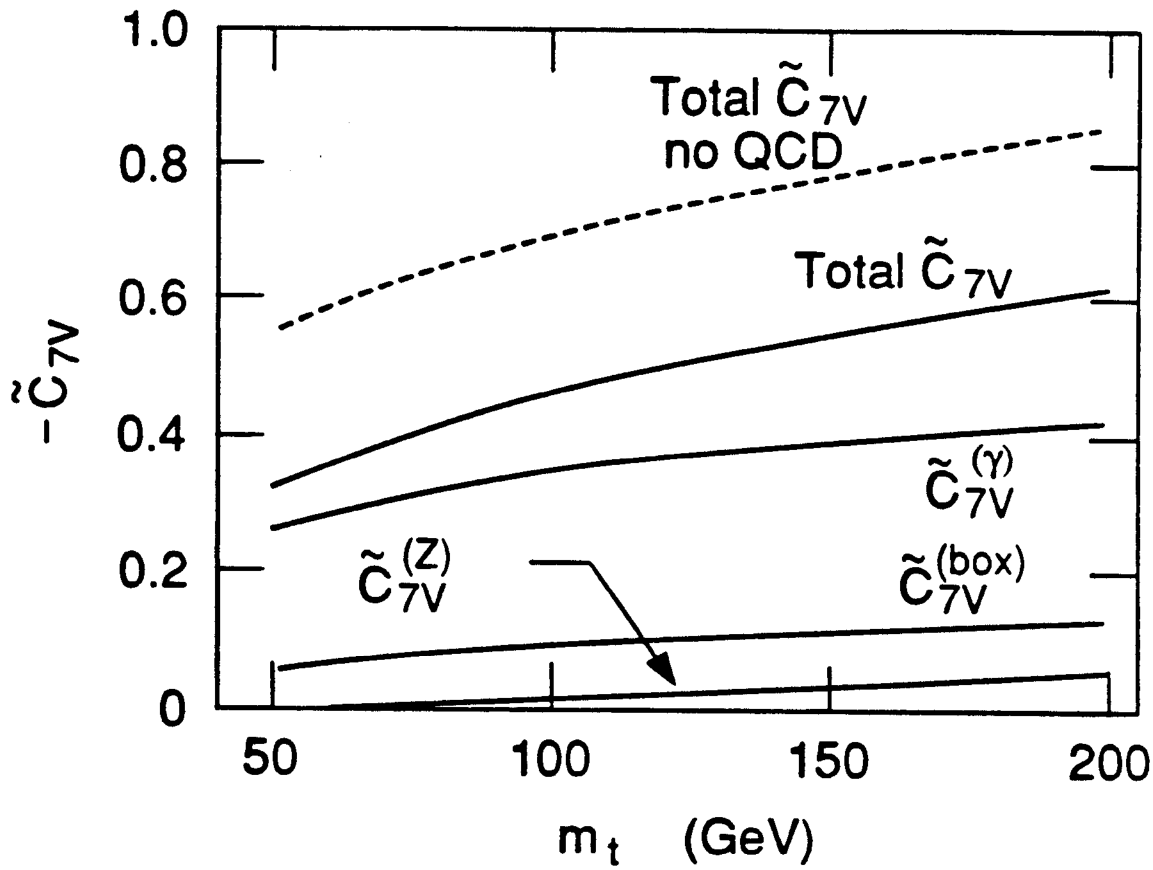


Figure 5.4. Contributions to the coefficient $\widetilde{C}_{7V}$ from each of its components, the "electromagnetic penguin", the "Z penguin" and the "box" diagrams and the total $\widetilde{C}_{7V}$ with QCD corrections (solid curves) for $\Lambda_{QCD} = 150$ MeV, and the total coefficient without QCD corrections (dashed curve) as a function of m_t.

these contributions, being proportional to $x_c = m_c^2/M_W^2$, are themselves negligible compared to those coming from the "electromagnetic penguin".

The total coefficient $\widetilde{C}_{7V} \equiv \widetilde{C}_{7V,t} - \widetilde{C}_{7V,c}$ and the contributions from each of its components is shown in Figure 5.4. Even after being reduced by QCD corrections, the contribution from the "electromagnetic penguin," $\widetilde{C}_{7V}^{(\gamma)}$, is the largest of the three. This is in good part due to the smallness of the vector coupling of the Z to charged leptons (which is proportional to $1 - 4\sin^2\theta_W$). Otherwise, the contribution of the "Z penguin" would dominate for large values of m_t.

The dominance of the "Z penguin" contributions at large m_t can be seen in Figure 5.5, where the total and component parts of the coefficient $\widetilde{C}_{7A} \equiv \widetilde{C}_{7A,t} - \widetilde{C}_{7A,c}$ are shown. As $m_t \to \infty$, $\widetilde{C}^{(Z)}_{7A,t}$ grows as m_t^2, while $\widetilde{C}^{(box)}_{7A,t}$ goes to a constant. In $\widetilde{C}_{7A}$, the "box" contribution is less than that from the "Z penguin" for $m_t \gtrsim M_W$.

Note that in the opposite situation where $m_t^2 << M_W^2$, both these contributions behave as $x_t \cdot \ln x_t$ and are non-leading when compared to the "electromagnetic penguin" contribution (to $\widetilde{C}_{7V}$), which behaves in the same limit as $\ln x_t$. QCD provides corrections to such large logarithms, which can arise when there is a large ratio of momentum scales. Our philosophy here, with $m_t \sim M_W$, has been to keep the leading and non-leading contributions at the scale M_W, and to also carry out the QCD corrections to the large logarithms that arise from integration over scales with a big ratio. In fact, the nonleading contribution $\widetilde{C}^{(\gamma)}_{7V,t}(M_W^2)$ is a small part of the full $\widetilde{C}^{(\gamma)}_{7V}$. Some of the ambiguities at the low scale μ also cancel out in the imaginary part of the amplitude, where the different sign in Kobayashi-Maskawa factors for charm and top makes the resulting amplitude arise from scales larger than m_c.

To proceed to actual branching ratios or decay rates, we may avoid some arithmetic by relating the hadronic matrix element of the operator, $(\overline{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha)$, which occurs in Q_V and Q_A, to that of the corresponding charged current operator, $(\overline{s}_\alpha\gamma_\mu(1-\gamma_5)u_\alpha)$, which occurs in $K_{\ell 3}$ decay. Then the form factors and phase

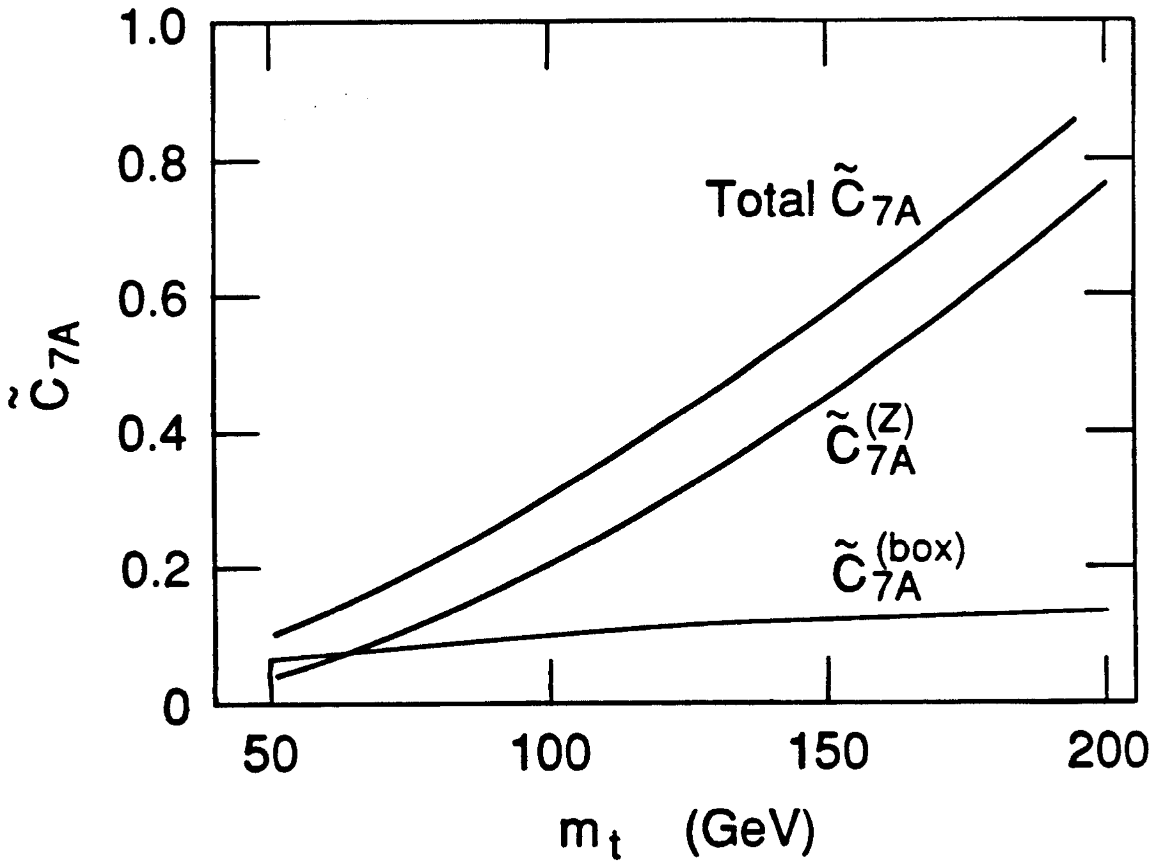


Figure 5.5. Contributions to the coefficient $\widetilde{C}_{7A}$ from the "Z penguin" and the "box" diagrams as a function of m_t.

space involved in this latter decay are automatically entered by Nature into the measured branching ratio for that mode. Using this, we find from the measured[26] branching ratio for K_{e3} decay that

$$B(K_2 \rightarrow \pi^0 e^+ e^-) = 1.0 \times 10^{-5}\ (s_2 s_3 s_6)^2\ \left[(\widetilde{C}_{7V})^2 + (\widetilde{C}_{7A})^2\right] . \tag{5.3.6}$$

The factor in square brackets is shown in Figure 5.6. With QCD corrections, and with m_t between 50 and 200 GeV, it ranges between about 0.1 and 1.0. While

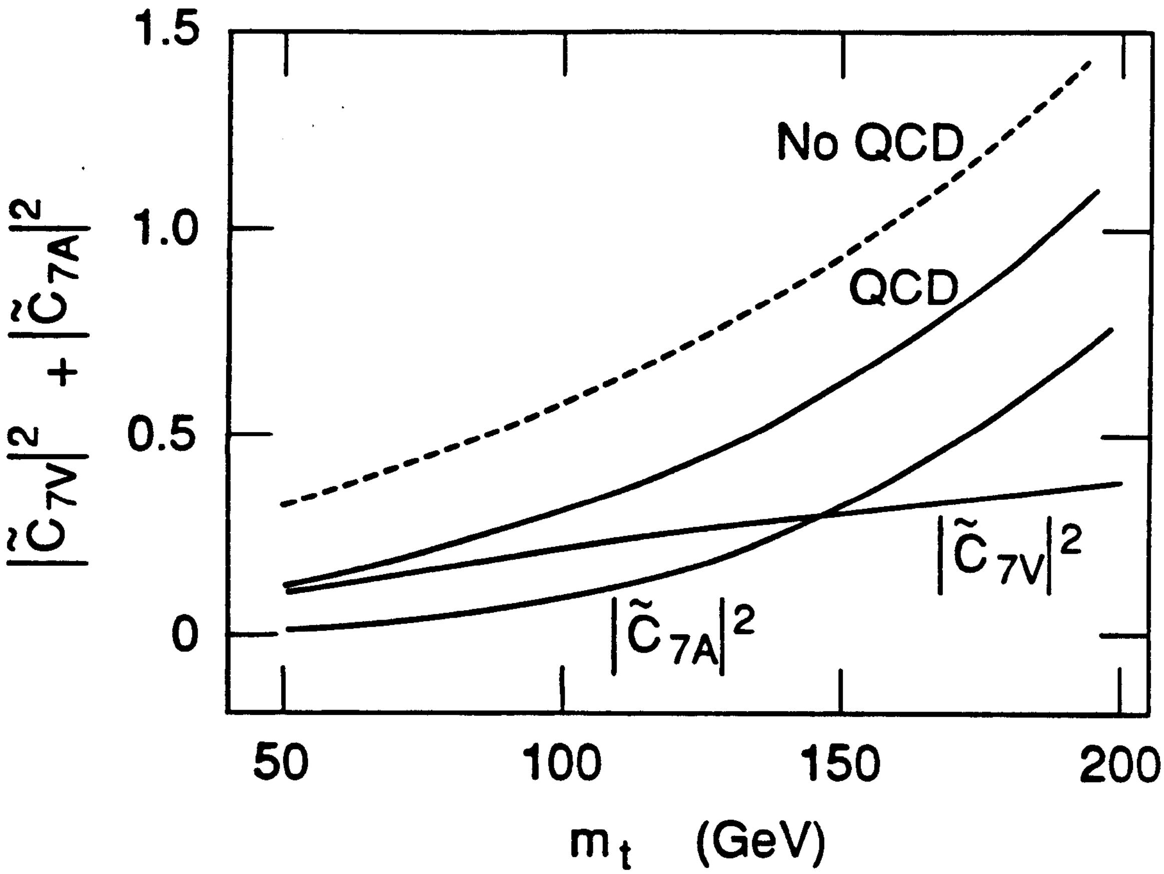


Figure 5.6. The quantities $(\tilde{C}_{7V})^2$ and $(\tilde{C}_{7A})^2$ as a function of m_t, and their sum, $(\tilde{C}_{7V})^2 + (\tilde{C}_{7A})^2$, with (solid curve, $\Lambda_{QCD} = 150$ MeV) and without (dashed curve) QCD corrections.

the combination $s_2 s_3 s_\delta$ enters other CP violating quantities such as ϵ and ϵ', imprecisely known hadronic matrix elements and m_t presently allow a broad range of values of this combination. From measurement of Kobayashi-Maskawa matrix elements, $s_2 s_3 s_\delta \leq 2.5 \times 10^{-3}$. For m_t at the low end of the acceptable range (as constrained by $B^0 - \overline{B}^0$ mixing), the allowed region of Kobayashi-Maskawa parameters contracts and $s_2 s_3 s_\delta$ must be quite close[8] to 10^{-3}. More generally, a typical value is in this neighborhood. Putting this information into (5.3.6) we see that the branching ratio for $K_L \to \pi^0 e^+ e^-$ from CP violation in the decay amplitude alone is around 10^{-11}.

5.4. Conclusions for $K_L \to \pi^0 \ell^+ \ell^-$

From the results of the previous three sections, it appears that from our present knowledge, the three contributions to the process $K_L \to \pi^0 \ell^+ \ell^-$ could each give rise to a branching ratio in the 10^{-11} range. With further theoretical and/or experimental work, it is possible that the CP conserving contribution might yet be shown to be well below this level.

This is not the case for the effects of CP violation in the mass matrix and in the decay amplitude. Their contributions are comparable, roughly at the 10^{-11} level in branching ratio, and in general will interfere in the expression for the total decay rate.

Some care must be exercised about phase conventions in calculating this interference. We have been calculating the CP violation in the decay amplitude in terms of what happens at the quark level, where strong interaction "penguin" diagrams induce a $\Delta I = 1/2$ $K \to \pi\pi$ transition which has a CP violating phase. The standard convention, on the other hand, where $(\mathrm{CP})K^0 = \overline{K}^0$, starts from making the amplitude for $K \to \pi\pi$ real when the final state has $I = 0$ (as it would from a $\Delta I = 1/2$ transition). To get to the standard convention from the quark basis requires absorbing a phase ξ proportional to ϵ' into the neutral K field, as described in the Appendix. As a result, in the amplitude for "indirect" CP violation, $\epsilon \to \epsilon - i\xi$, if $|\xi|$ is small. A somewhat abbreviated expression for the branching ratio from all CP violating effects is then,

$$B(K_L \to \pi^0 e^+ e^-) \approx \left[\left| 0.76 \left(e^{i\pi/4} - i\frac{\xi}{|\epsilon|} \right) \left(\frac{\Gamma(K_1 \to \pi^0 e^+ e^-)}{\Gamma(K^+ \to \pi^+ e^+ e^-)} \right)^{1/2} \right. \right.$$
$$\left. \left. + i\left(\frac{s_2 s_3 s_\delta}{10^{-3}} \right) \widetilde{C}_{7V} \right|^2 + \left| \left(\frac{s_2 s_3 s_\delta}{10^{-3}} \right) \widetilde{C}_{7A} \right|^2 \right] \cdot 10^{-11} , \qquad (5.4.1)$$

where we have taken into account the phase conventions mentioned above. In the last term of Eq. (5.4.1) we have neglected the contribution from ϵ times the real part of C_{7A}, which is only a few percent of the imaginary part of C_{7A} (see Appendix). Eq. (5.4.1) indicates the interference of amplitudes coming from "indirect" and "direct" CP violation. Neglected is the fact that the two interfering amplitudes (those which involve vector coupling to the lepton pair) can have a different dependence on the pair invariant-mass and the interference can then vary with this quantity. If both amplitudes came from short-distance effects (which we have indicated is very unlikely for the "indirect" CP violation), then $(\Gamma(K_1 \to \pi^0 e^+ e^-)/\Gamma(K^+ \to \pi^+ e^+ e^-))^{1/2}$ is negative, the interference is the same for all values of the pair invariant-mass, and (5.4.1) stands as written.

Since $\epsilon'/\epsilon = -15.6\xi \approx 3 \times 10^{-3}$, the extra piece from the change of basis is small, but interferes constructively with that from ϵ. The terms coming from "direct" CP violation are comparable to those from the mass matrix ("indirect" CP violation), and we cannot give a definitive conclusion as to their relative magnitudes without further knowledge of $A(K_1 \to \pi^0 \ell^+ \ell^-)$, $s_2 s_3 s_\delta$, and m_t. Nor can we give a statement as to constructive or destructive interference without a model for the long-distance effects which we suspect are inherent in the "indirect" CP violation amplitude. As m_t becomes larger, more of the "direct" CP violation comes through

Q_A (see Figures 5.4, 5.5, and 5.6). As a result, the theoretical predictions become more definitive, as the QCD corrections to C_{7A} are very small and this contribution does not interfere in the expression for the decay rate with that from "indirect" CP violation. Even for large m_t, however, it is hard to get a branching ratio that is more than a few times 10^{-11}.

We have a major advantage over calculations of other CP violating effects in the K^0 system in that the hadronic matrix element of the relevant operators (Q_V and Q_A) from the short-distance physics is given to us from $K_{\ell 3}$ decay. There is no uncertainty here. Nevertheless, we would assign an uncertainty from the QCD corrections, the neglect of non-leading QCD terms, and possible "direct" CP violating contributions from order e^2 matrix elements of Q_1 to Q_6, of 10 to 20% for $\widetilde{C}_{7V}$, even if we knew m_t precisely along with all the Kobayashi-Maskawa parameters. Conversely, if there were both a precise measurement of m_t and of the $K_L \to \pi^0 \ell^+ \ell^-$ branching ratio that resulted in an isolation of the amplitude for "direct" CP violation, there would be an uncertainty of this magnitude in the extracted value of $s_2 s_3 s_\delta$. While not as precise as one might like, this would be far better than the determination from ϵ and ϵ', where non-trivial hadronic matrix elements enter.

There are a number of experimental observations which would help to sort out various contributions and their magnitudes. We conclude by briefly discussing some of them:

- The short-distance generated amplitudes have a dependence on the kinematic variables of the final state which is identical to that in $K_{\ell 3}$ decay, with obvious substitutions of particle names. This allows an easy calculation of decay rates with cuts on final state kinematic variables, *e.g.*, restrictions on $m_{\ell\bar{\ell}}$. Comparison with observations of $K^+ \to \pi^+\ell^+\ell^-$, $K_S \to \pi^0\ell^+\ell^-$, and $K_L \to \pi^0\ell^+\ell^-$, would help to sort out long-distance contributions from short-distance ones.

- The relative rates for $K_L \to \pi^0 e^+ e^-$ and $K_L \to \pi^0\mu^+\mu^-$ are sensitive as well to the CP conserving two-photon contribution, with the factor of m_ℓ that accompanies the A amplitude (see Section 5.1) acting to enhance its contribution in the latter reaction in comparison to the former.

- The direct measurement of $K_L \to \pi^0\gamma\gamma$ can be used as an input to calculations of the two-photon, CP conserving contribution to $K_L \to \pi^0\ell^+\ell^-$. In particular, one could separate the A and B amplitudes by measuring the Dalitz plot distributions, such as the invariant mass distribution of the two photons.[40]

- If both CP conserving and CP violating amplitudes are present with even roughly comparable strengths, they will in general interfere on the Dalitz plot, giving rise to a large lepton - antilepton energy asymmetry.[37]

- The "indirect" CP violating amplitude can be obtained from a measurement of $K_S \to \pi^0\ell^+\ell^-$. Any deviation in the then measured rate for $K_L \to \pi^0\ell^+\ell^-$ from the straightforward prediction involving multiplication of the former rate by $|\epsilon - i\xi|^2$ is then evidence for "direct" CP violation in the decay amplitude (assuming the CP conserving contribution has been shown experimentally or theoretically to be small).

- One can imagine a full interference pattern being measured, as was done for the $\pi\pi$ mode, where one sees both the regime of $K_S \to \pi^0 \ell^+ \ell^-$ decay followed by that for $K_L \to \pi^0 \ell^+ \ell^-$, with an interference region between the two regimes of exponential decay. This would permit not only the measurement of the two rates, but the phase between the "indirect" and "direct" amplitudes whose interference is indicated in Eq. (5.4.1).

As of now, we have a long way to go experimentally. While recent upper limits[49,50] are around 4×10^{-8}, and are improvements by orders of magnitude on earlier limits,[51] we have about three orders of magnitude further improvement in sensitivity needed to see the standard model signal.

Finally we note that in the large m_t regime, all the decays $K^+ \to \pi^+ \nu\overline{\nu}$, $K_L \to \pi^0 \ell^+ \ell^-$, and $K_L \to \pi^0 \nu\overline{\nu}$ have amplitudes which are dominated by contributions from the "Z penguin" and "W box" graphs, and the latter two, which are CP violating, have comparable rates in this regime.

6. Branching Ratios for $K \to \pi\nu\overline{\nu}$

We just studied the decay of K_L into π and charged leptons, where the presence of different kinds of contributions at the same order of magnitude, like CP conserving and CP violating amplitudes or long distance and short distance terms, make the analysis rather complex and theoretical predictions hard to disentangle.

These difficulties are less likely to occur in rare decays of K mesons into neutrinos.[18] The reason for this is the absence of electromagnetic coupling to neutrinos, suppressing the major long distance contributions found above.[19] In addition, and for the same reason, there is no "electromagnetic penguin" in $K \to \pi\nu\overline{\nu}$, but only "$Z$ penguin" and "box" diagrams. As was shown in Chapter 5, it is the "electromagnetic penguin" which gives the most important QCD corrections to the decay $K_L \to \pi^0\ell^+\ell^-$; in the present case, however, QCD corrections have to come from the "Z penguin" and "box", which are quite different from those coming from the "electromagnetic penguin". We therefore expect to find a new effect of QCD in these decays.

As additional features, the charged meson decay $K^+ \to \pi^+\nu\overline{\nu}$ contains a different combination of Kobayashi - Maskawa elements, and the neutral meson decay $K_L \to \pi^0\nu\overline{\nu}$, with long distance and CP conserving amplitudes highly suppressed, could provide a clear signal of direct CP violation in the decay amplitude.

6.1. The decay $K^+ \to \pi^+\nu\overline{\nu}$

Feynman diagrams for the process $K \to \pi\nu_\ell\overline{\nu}_\ell$ are shown in Figure 1.2. They are similar to the diagrams for the short-distance contributions to the process

$K \to \pi\ell^+\ell^-$, except for the absence of the "electromagnetic penguin" and the appearance of different lepton lines. The latter induces a dependence on the mass of the charged lepton in the loop and, as shown in Chapter 2, a different weighting of the diagrams: the "box" appears enhanced by a factor of 4 with respect to the appropriate "box" for the decay into charged leptons.

At a hadronic scale μ below the charm mass and appropriate for K decays, we write the effective Hamiltonian for $\Delta S = 1$ processes as:

$$\mathcal{H}_{\nu\nu} = \frac{G_F}{\sqrt{2}}\left[V_{us}^* V_{ud} c_i(\mu) Q_i \; + \sum_{q=u,c,t} V_{qs}^* V_{qd} \widetilde{C}_{\nu,q}\left\{Q_V - Q_A\right\}\right] \; + \; h.c., \qquad (6.1.1)$$

where the effective four-quark operators Q_1 to Q_6 are the same as in Eq. (3.1). The $V - A$ character of the gauge boson coupling to neutrinos allows only the operator

$$Q_V - Q_A = \frac{e^2}{4\pi}(\overline{s}_\alpha \gamma_\mu (1-\gamma_5) d_\alpha)(\overline{\nu}_\ell \gamma^\mu (1-\gamma_5)\nu_\ell) \qquad (6.1.2)$$

to appear to lowest order in electroweak interactions to represent the short-distance contributions to $K \to \pi\nu\overline{\nu}$ in the summation.

After comparing the effective Hamiltonian (6.1.1) with the short distance calculation (2.19), we match the coefficients $\widetilde{C}_{\nu,q}$ of the semileptonic operator, which receives contributions involving quarks $q = u, c, t$ from both the "Z penguin" and "box" diagrams:[52]

$$\widetilde{C}_{\nu,q} = \frac{1}{\sin^2\theta_W}\{F_Z^{QCD}(x_q,\mu) - 4F_B^{QCD}(x_q,\mu)\}, \qquad (6.1.3)$$

where $x_q = m_q^2/M_W^2$. For small x_q, the leading terms in (6.1.3) come from the logarithm [see Eqs. (2.21)], with its coefficient in $\widetilde{C}_{\nu,q}^{(Box)}$ being four times larger and of opposite sign to that in $\widetilde{C}^{(Z)}_{\nu,q}$. The contribution of the u quark is neglected (because of an overall factor of x_u), leaving c and t quark contributions which are comparable in the amplitude for $K^\pm \to \pi^\pm \nu\overline{\nu}$ when the respective Kobayashi-Maskawa factors are included.

The leading logarithmic QCD corrections are applied to the term proportional to $\ln x_q$, which arises here from an integration from the scale m_q^2 to that of the weak interaction, M_W^2. With the introduction of QCD, the integrand gets an additional scale dependence reflecting those of the four-fermion interaction and the running of masses, so that[24]

$$\int_{m_q^2}^{M_W^2} \frac{dq^2}{q^2} \xrightarrow[QCD]{} \int_{m_q^2}^{M_W^2} \frac{dq^2}{q^2} C(q^2) = \eta_q \int_{m_q^2}^{M_W^2} \frac{dq^2}{q^2} .$$

Since by assumption the t quark has a mass comparable to the W, its contribution has no large logarithms and in addition comes from a region where α_s is small. Therefore the QCD corrections to $\widetilde{C}_{\nu,t}$ are neglected. Consequently, the only significant QCD corrections of interest here are those to the charm contribution. From the results of Chapter 3, we find for the "box"

$$\eta_c^{(Box)} = \left(\frac{12\pi}{ln(M_W^2/m_c^2)}\right)\left(\frac{(K_{c/b}^{1/25} - 1)}{\alpha_s(m_c^2)} + \frac{K_{c/b}^{-24/25}(1 - K_{b/W}^{-1/23})}{\alpha_s(m_b^2)}\right) , \quad (6.1.4)$$

with $K_{b/W} = \alpha_s(m_b^2)/\alpha_s(M_W^2)$ and $K_{c/b} = \alpha_s(m_c^2)/\alpha_s(m_b^2)$ in effective five and four quark theories, respectively.

For the "Z penguin" the corresponding QCD correction factor is:

$$
\begin{aligned}
\eta_c^{(Z)} &= \left(\frac{12\pi}{ln(M_W^2/m_c^2)}\right) \times \\
&\left(\left[\frac{2}{7}K_{b/W}^{-6/23}K_{c/b}^{-6/25}\frac{(K_{c/b}^{7/25}-1)}{\alpha_s(m_c^2)} - \frac{1}{11}K_{b/W}^{12/23}K_{c/b}^{12/25}\frac{(1-K_{c/b}^{-11/25})}{\alpha_s(m_c^2)}\right]\right. \\
&\left.+\ K_{c/b}^{-24/25}\left[\frac{2}{5}K_{b/W}^{-6/23}\frac{(K_{b/W}^{5/23}-1)}{\alpha_s(m_b^2)} - \frac{1}{13}K_{b/W}^{12/23}\frac{(1-K_{b/W}^{-13/23})}{\alpha_s(m_b^2)}\right]\right) .
\end{aligned}
\tag{6.1.5}
$$

Numerical values of the charm contribution, before and after QCD corrections, can be found for various values of Λ_{QCD} in Table 6.1. The values there correspond to $\eta_c^{(Box)} = 0.61$ and $\eta_c^{(Z)} = 0.31$ when $\Lambda_{QCD} = 150$ MeV. Especially for the "Z penguin," the QCD corrections are large (by "large" meaning a value of η far from 1). However, since the leading logarithm, $ln(M_W^2/m_c^2)$, enters the amplitude for $K \to \pi\nu\overline{\nu}$ in the ratio of 4 to -1 (of "box" to "Z penguin"), the effective QCD correction factor to the leading logarithm in the overall charm contribution is $[4(0.61) - 1(0.31)]/[4 - 1] = 0.71$.

There remains the question of how to treat the QCD corrections to the non-leading terms, which appear to be more important here than in the case of $K_L \to \pi^0\ell^+\ell_-$. In general, the coefficients $\widetilde{C}_{\nu,q}$ may contain different (non-leading) renormalization-scheme dependent terms of the form: $(constant) \times x_q$. Being a physical quantity, the net amplitude can not change in going from one scheme to another, as there are compensating changes in the matrix elements of the other operators. Without a higher order QCD calculation of the anomalous dimensions and the matrix elements, a scheme dependence remains in the QCD corrections to the non-leading terms in the coefficients.

	$\widetilde{C}^{(Z)}_{\nu,c}$	$\widetilde{C}^{(Box)}_{\nu,c}$	$\widetilde{C}_{\nu,c}$
Leading Log Only			
No QCD	-4.7	18.9	14.2
$\Lambda_{QCD} = 100$ MeV	-1.7	12.1	10.4
$\Lambda_{QCD} = 150$ MeV	-1.5	11.5	10.0
$\Lambda_{QCD} = 250$ MeV	-1.1	10.5	9.4
Full Contribution			
No QCD	-3.0	16.6	13.6
QCD applied to leading log only ($\Lambda_{QCD} = 150$ MeV)	0.3	9.1	9.4
QCD applied to leading and non-leading terms ($\Lambda_{QCD} = 150$ MeV)	-1.0	10.6	9.6

Table 6.1. The coefficient $\widetilde{C}_{\nu,c}$ for $m_c = 1.5$ GeV (Units of 10^{-4}).

If we take the non-leading terms of the charm contribution from Eq. (6.1.3), then it does not make much difference what is done as far as QCD corrections to them. The next-to-leading terms are in the ratio of -4 to $+3$ and cancel against each other, as can be seen from Eqs. (2.21) and (6.1.1), or by comparing the (no QCD) leading logarithm portion with the full contribution of charm in Table 6.1. As QCD corrections reduce the coefficient of the leading logarithm, the non-leading terms become relatively more important if no correction is applied to them. Even in this case, there is only a 10% difference in the total charm contribution (compare the sixth and third row of Table 6.1) if the non-leading terms are included, although the effects are very much bigger in the component pieces, especially $\widetilde{C}^{(Z)}_{\nu,c}$. The application of QCD corrections characteristic of the scale m_c to these next-to-leading terms reduces their magnitude and makes them even less significant (row seven of Table 6.1). The lesson is that there is a sizable difference in the charm contribution due to QCD corrections to the leading logarithm, but only small differences induced from changing the value of Λ_{QCD} or from handling the QCD corrections to the non-leading terms in different ways.

With the QCD corrections in hand, we can apply them to the amplitudes for the processes of interest. The branching ratio (per neutrino flavor) for $K^{\pm} \to \pi^{\pm}\nu_\ell\overline{\nu}_\ell$ can be related to that for K_{e3} decay, as shown in the Appendix, to yield

$$\begin{aligned} B(K^{\pm} \to \pi^{\pm}\nu_\ell\overline{\nu}_\ell) &= 2\ |V_{ud}|^2\ \alpha^2\ |C_\nu|^2\ B(K^+ \to \pi^0 e^+ \nu_e) \\ &= 5.1 \times 10^{-6}\ |V_{ud}|^2\ |C_\nu|^2 \end{aligned} , \tag{6.1.6}$$

where

$$C_\nu \approx -\widetilde{C}_{\nu,c} + \frac{V^*_{ts}V_{td}}{V^*_{us}V_{ud}}\ \widetilde{C}_{\nu,t}\ . \tag{6.1.7}$$

We have performed a full numerical search over Kobayashi-Maskawa parameter space to obtain maximum and minimum values of the branching ratio as a function of m_t, which is shown in Figure 6.1. Our results without QCD corrections (dashed curve) are very close to those of Nir.[53]

An upper limit[53] on the rate occurs when m_c is as large as allowed (1.7 GeV here), and we replace $V^*_{ts}V_{td}/V^*_{us}V_{ud}$ in Eq. (6.17) by minus its maximum magnitude, allowing complete constructive interference between the charm and top contributions. Unitarity of the KM matrix gives $|V_{td}| < 0.024$, while for $m_t \geq 120$ GeV,

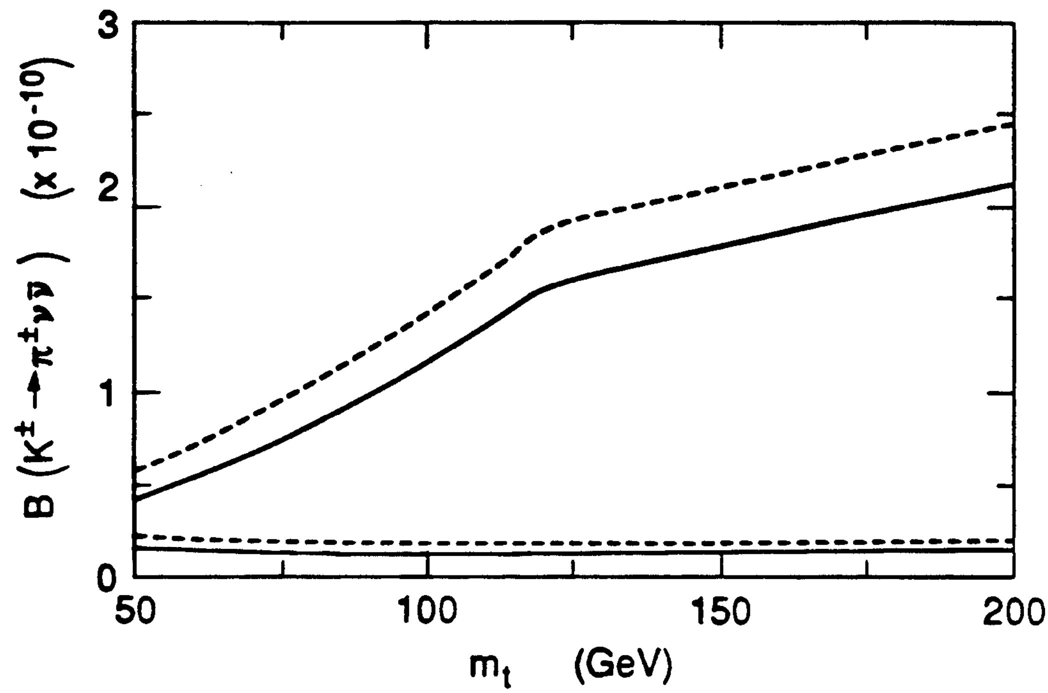


Figure 6.1. Maximum and minimum of the branching ratio (per neutrino flavor) for $K^+ \to \pi^+\nu\bar{\nu}$ without (dashed curve) and with (solid curve) QCD corrections (Λ_{QCD} = 150 MeV).

a more stringent upper limit on $|V_{td}|$ occurs from $B_d - \overline{B}_d$ mixing and possibly the branching ratio for $K_L \to \mu^+\mu^-$. The upper bound on the rate so derived holds for three generations of quarks irrespective of whether CP violation arises from the Kobayashi-Maskawa matrix. In fact, adding the ϵ constraint lowers the maximum rate by at most a few percent. On the other hand the minimum of $B(K^{\pm} \to \pi^{\pm}\nu_{\ell}\overline{\nu}_{\ell})$ for a given m_t occurs both when m_c is as small as allowed (1.3 GeV here) and the potentially constructive interference between the charm and top contributions tends to be as small as possible.

When one compares to the branching ratio with QCD corrections (solid curve), there is a decrease in the minimum by $\approx 30\%$. The maximum, on the other hand, decreases by $\approx 25\%$ for smaller m_t and $\approx 15\%$ for larger values. Although the detailed formulas are different, this is numerically similar to the results of Refs. 21 and 22. From the preceding discussion this is to be expected in that, even though we take account of the change in operative quark flavors at the b-scale and m_t being comparable to M_W, the basic physics is the same and the magnitude of the QCD corrections is not sensitive to the details of the running of α_s.[54]

6.2. The CP violating decay $K_L \to \pi^0 \nu \overline{\nu}$

The process $K_L^0 \to \pi^0 \nu_\ell \overline{\nu}_\ell$ is determined by the same diagrams and effective Hamiltonian as above. The only difference comes from the external hadrons: since these are CP eigenstates, the decay becomes particularly interesting.[18] Being the coupling of the hadrons to the neutrinos primarily mediated by a vector current, the decay turns out to be CP violating.

It is clear that the coupling of hadrons to neutrinos is a current-current type of interaction for the "Z penguin", since the Z boson itself provides the coupling between the currents. This is less obvious for the "box" diagram, where there are two W bosons mediating the quark and lepton sectors. However, in the short distance expansion of the interaction, the leading term of the "box" is also of the current-current type, as can be seen from the identity (2.16). Once the interaction is factorized as:

$$< \pi^0 \nu \overline{\nu} | \mathcal{H} | K_L > = < \pi^0 | J_\mu^{Had} | K_L >< \nu \overline{\nu} | J_{lep}^\mu | 0 >,$$

we can examine the CP properties of the hadronic matrix element. From the fact that the hadrons are (pseudo-)scalars, the only vectors available to reproduce the Lorentz structure of the current are the particles' momenta. Consequently, the axial component vanishes and the current becomes pure vector-like. It is then clear that the hadronic transition from the K_L to the π^0 and vector current follows a J^{PC} structure:

$$0^{-+} \rightarrow 0^{-+} + 1^{--}.$$

So far, CP seems to be conserved. Nevertheless, in order to conserve angular momentum, the final state must be in an orbital p—wave, adding an extra factor of -1 to the parity transformation, and making this mode to be CP forbidden.

The branching ratio per neutrino flavor is calculated following the lines of the Appendix, where hadronic form factors are determined by relating this amplitude to that for K_{e3} decay:

$$B(K_L^0 \rightarrow \pi^0 \nu_\ell \overline{\nu}_\ell) = 2.1 \times 10^{-5} \; |V_{ud}|^2 \; |(\epsilon - i\xi) Re \; C_\nu + i Im \; C_\nu|^2 \; . \tag{6.2.1}$$

The term proportional to $Re\ C_\nu$ gives a negligible contribution and

$$Im \; C_\nu = Im \left(\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} \right) \left(\widetilde{C}_{\nu,t} - \widetilde{C}_{\nu,c} \right) \; , \tag{6.2.2}$$

with the rephase invariant quantity $Im\ \left(V_{ts}^* V_{td} / V_{us}^* V_{ud}\right) \approx s_2 s_3 s_\delta$ in the original parametrization of Kobayashi and Maskawa.[4] Therefore

$$B(K_L^0 \rightarrow \pi^0 \nu_\ell \overline{\nu}_\ell) \approx 2.1 \times 10^{-5} \left(s_2 s_3 s_\delta\right)^2 \; |\widetilde{C}_{\nu,t} - \widetilde{C}_{\nu,c}|^2 \; . \tag{6.2.3}$$

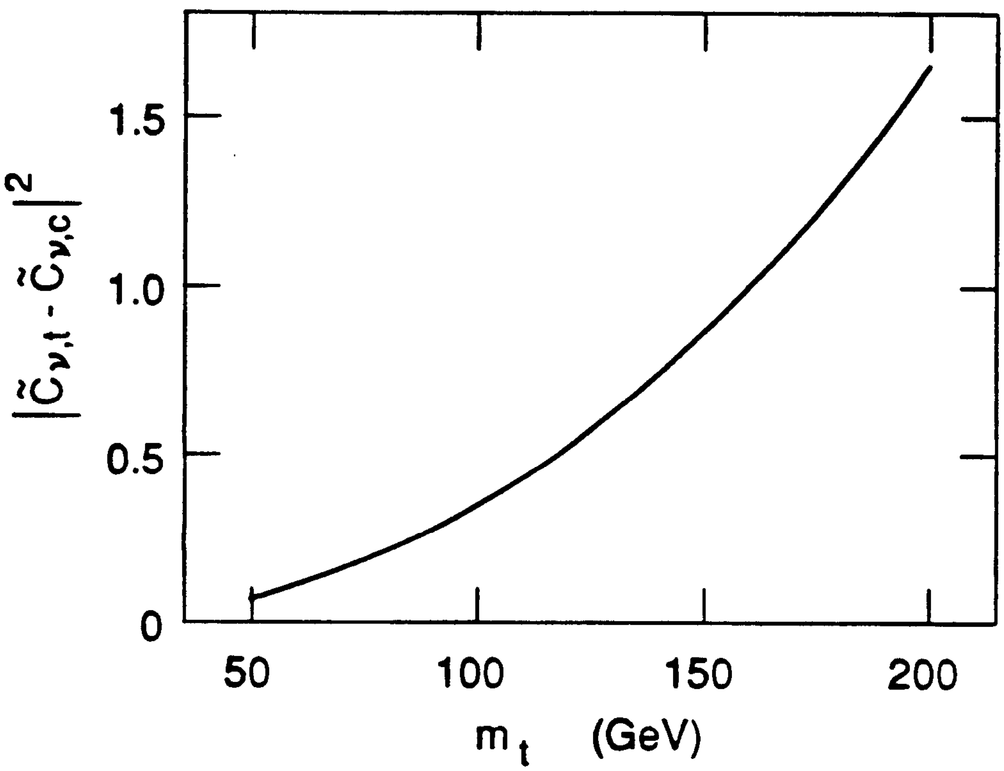


Figure 6.2. The quantity $|\widetilde{C}_{\nu,t} - \widetilde{C}_{\nu,c}|^2$, which enters the branching ratio for the CP violating decay $K_L \to \pi^0 \nu_\ell \overline{\nu}_\ell$, as a function of m_t.

Theoretical predictions are much cleaner here than in the case of $K_L \to \pi^0 \ell^+ \ell^-$ due to the absence of electromagnetic coupling to neutrinos. First of all, we do not expect large CP conserving amplitudes here, which in the previous case were mediated by two photons. A CP conserving amplitude mediated by two Z is certainly negligible. Second, recent estimates of long distance contributions show that these are negligible as well.[19] Finally, the quantity $|\widetilde{C}_{\nu,t} - \widetilde{C}_{\nu,c}|^2$, which is shown in Figure 6.2, is completely dominated by the top contribution, where QCD corrections should be very small. As $s_2 s_3 s_\delta$ is of order 10^{-3}, the branching ratio with three generations of neutrinos is of order 10^{-11}. The major contribution to

$K_L \rightarrow \pi^0 \nu\overline{\nu}$ has to come directly from a CP violating decay amplitude, which is calculable with very small theoretical uncertainties. This is clearly an ideal place to test the validity of the Kobayashi-Maskawa phase as the origin of CP violation in the three generation Standard Model. Unfortunately, the experimental difficulties to measure a process of this kind are very formidable.[18]

Appendix

In this appendix we want to show some details involved in the calculation of the decay rates, such as phase conventions, notation and the like. Starting from the effective Hamiltonian (2.18) for $K \to \pi e^+e^-$, expressed as:

$$\mathcal{H}_{ee} = \frac{G_F}{\sqrt{2}} \sum_q V_{qs}^* V_{qd} \Big\{ \widetilde{C}_{7V,q} Q_V + \widetilde{C}_{7A,q} Q_A + \widetilde{C}_M Q_M \Big\} + h.c. \qquad (A.1)$$

we calculate the amplitude $< e^+e^-\pi^0|\mathcal{H}|K_L >$. In order to define all phases consistently, we choose the states K^0 and $\overline{K}^0$ such that:

$$\begin{aligned} (\mathrm{CP})K^0 &= \overline{K}^0 \\ (\mathrm{CP})\overline{K}^0 &= K^0 \end{aligned} \qquad (A.2)$$

We choose this basis to be the one where the amplitude for $K^0 \to \pi\pi(I=0)$ is real, after taking out the phase shift coming from $\pi\pi$ strong interactions. In this basis, we can express K_L as the combination:

$$K_L \;=\; p\, K^0 \;-\; q\, \overline{K}^0. \qquad (A.3)$$

The coefficients p and q are slightly different, to take into account CP violation in the mixing of K^0 and $\overline{K}^0$:

$$p = \frac{1}{\sqrt{2}}(1+\epsilon) \qquad q = \frac{1}{\sqrt{2}}(1-\epsilon) \qquad (A.4)$$

If we choose another phase for the K^0, $\overline{K}^0$, as it is the case when working in a quark basis, then the CP transformation on the K's become:

$$\begin{aligned}(\mathrm{CP})K^0 &= e^{2i\xi}\,\overline{K}^0 \\ (\mathrm{CP})\overline{K}^0 &= e^{-2i\xi}\,K^0\end{aligned} \tag{A.5}$$

and a corresponding phase appears in p and q. For ξ small:

$$p = \frac{1}{\sqrt{2}}(1+\epsilon-i\xi) \qquad q = \frac{1}{\sqrt{2}}(1-\epsilon+i\xi) \tag{A.6}$$

Now we turn our attention to the amplitude $< e^+e^-\pi^0|\mathcal{H}|K_L >$. In the phase convention where the pion and K wave functions are expressed as:

$$\begin{aligned}\pi^+ &= \overline{d}u \\ \pi^0 &= \frac{1}{\sqrt{2}}(\overline{u}u-\overline{d}d) \\ \pi^- &= \overline{u}d\end{aligned} \qquad \begin{aligned}K^0 &= \overline{s}d \\ \overline{K}^0 &= s\overline{d}\end{aligned} \tag{A.7}$$

we define the hadronic charged current form factors f_+ and f_-:

$$\begin{aligned}< \pi^+(p')|\overline{u}\gamma_\mu s|\overline{K}^0(p) > = < \pi^-(p')|\overline{s}\gamma_\mu u|K^0(p) > \equiv \\ f_+(q^2)(p+p')_\mu \; + \; f_-(q^2)q_\mu\end{aligned} \tag{A.8}$$

with $q = p - p'$. Doing an isospin rotation, we can also obtain similar relations for the neutral currents:

$$\begin{aligned}< \pi^0(p')|\overline{d}\gamma_\mu s|\overline{K}^0(p) > = < \pi^0(p')|\overline{s}\gamma_\mu d|K^0(p) > \equiv \\ -\frac{1}{\sqrt{2}}\Big(f_+(q^2)(p+p')_\mu \; + \; f_-(q^2)q_\mu\Big).\end{aligned} \tag{A.9}$$

Now we are in position to calculate the amplitude $< e^+e^-\pi^0|\mathcal{H}|K_L >$ from the effective Hamiltonian (A.1). The only non-vanishing matrix elements of the operators Q_V and Q_A that contribute to this amplitude are:

$$\begin{aligned}
<e^+e^-\pi^0|Q_V|K^0> &= \alpha <\pi^0|\bar{s}\gamma_\mu(1-\gamma_5)d|K^0>(\bar{e}\gamma_\mu e) \\
<e^+e^-\pi^0|Q_V^\dagger|\overline{K}^0> &= \alpha <\pi^0|\bar{d}\gamma_\mu(1-\gamma_5)s|\overline{K}^0>(\bar{e}\gamma_\mu e) \\
<e^+e^-\pi^0|Q_A|K^0> &= \alpha <\pi^0|\bar{s}\gamma_\mu(1-\gamma_5)d|K^0>(\bar{e}\gamma_\mu\gamma_5 e) \\
<e^+e^-\pi^0|Q_A^\dagger|\overline{K}^0> &= \alpha <\pi^0|\bar{d}\gamma_\mu(1-\gamma_5)s|\overline{K}^0>(\bar{e}\gamma_\mu\gamma_5 e)
\end{aligned} \tag{A.10}$$

Due to parity, only the vector part of the hadronic currents survives, which can be expressed in terms of f_+ and f_-. Moreover, the term proportional to q_μ vanishes when contracted with the electron current, so that only f_+ is relevant in our calculation.

Since the vector and axial vector lepton currents do not interfere, the square of the amplitude separates into two terms:

$$\begin{aligned}
|<\pi^0e^+e^-|\mathcal{H}|K_L>|^2 = \Big[\frac{G_F}{\sqrt{2}}\,\alpha\,\frac{f_+(q^2)}{\sqrt{2}}\Big]^2(K.F.)\times \\
\Big\{\Big|\sum_q \widetilde{C}_{7V,q}(p\ V_{qs}^*V_{qd}-q\ V_{qs}V_{qd}^*)\Big|^2 + \Big|\sum_q \widetilde{C}_{7A,q}(p\ V_{qs}^*V_{qd}-q\ V_{qs}V_{qd}^*)\Big|^2\Big\}.
\end{aligned} \tag{A.11}$$

The kinematic factor $(K.F.)$ comes from the square of the lepton currents contracted with the meson momenta, and is the same for both vector and axial vector pieces.

The form factor $f_+(q^2)$ cannot be calculated exactly. Fortunately, we do not need to, since it can be determined from the experimental value of the charged K

decay $K^+ \to \pi^0 e^+ \nu$:

$$| < \pi^0 e^+ \nu | \mathcal{H} | K^+ > |^2 \; = \; 2\Big[\frac{G_F}{\sqrt{2}}\frac{f_+(q^2)}{\sqrt{2}}\Big]^2 (K.F.)\; |V_{us}|^2.$$

The important point here is that the kinematics of this process is identical to that of $K_L \to \pi^0 e^+ e^-$, since the electron mass is negligible. This means that the invariant mass of the lepton pair, q^2, is the same for both decays, so there is no need for a model to extrapolate the value of the form factor at two different momentum scales.

After using (A.6), we can express (A.11) in a form that separates the direct (*i.e.* $ImV^*_{qs}V_{qd}$) from the indirect (*i.e.* $\sim \epsilon$) CP violating contributions:

$$\begin{aligned} \left|\frac{< \pi^0 e^+ e^- |\mathcal{H}| K_L >}{< \pi^0 e^+ \nu |\mathcal{H}| K^+ >}\right|^2 \; &= \frac{\alpha^2}{|V_{us}|^2}\Big\{\Big|\sum_q \widetilde{C}_{7V,q}\{(\epsilon - i\xi)Re[V^*_{qs}V_{qd}] \; + \; iIm[V^*_{qs}V_{qd}]\}\Big|^2 \\ &+ \Big|\sum_q \widetilde{C}_{7A,q}\{(\epsilon - i\xi)Re[V^*_{qs}V_{qd}] \; + \; iIm[V^*_{qs}V_{qd}]\}\Big|^2\Big\}. \qquad (A.12) \end{aligned}$$

An important remark about this expression is that the unitarity relations of the Kobayashi-Maskawa elements for three generations of quarks allows us to write the direct CP violating pieces in terms of a single factor:

$$\frac{Im[V^*_{cs}V_{cd}]}{|V_{us}|} \; = \; -\frac{Im[V^*_{ts}V_{td}]}{|V_{us}|} \; = \; s_2 s_3 s_\delta .$$

Also, the term proportional to $(\epsilon - i\xi)$ is negligible compared to $Im[V^*_{qs}V_{qd}]$. Consequently, we neglect this term in the sum involving $\widetilde{C}_{7A,q}$. However, we do not neglect it in the sum that contains $\widetilde{C}_{7V,q}$. It is true that in this last case the term is also negligible but, as it is, it only takes into account the short distance contributions to the decay $K_1 \to \pi^0 e^+ e^-$. Since we know the decay $K^+ \to \pi^+ e^+ e^-$ contains important long distance contributions, we expect the same to happen here. Therefore, instead of neglecting the term under discussion, we reexpress it in terms of the decay rate of the charged K. After appropriate normalizations, we obtain for the branching ratio:

$$
\begin{aligned}
B(K_L \to \pi^0 e^+ e^-) = B(K^+ \to \pi^0 e^+ \nu) \frac{\tau(K_L)}{\tau(K^+)} \Bigg\{ \Bigg| & i\alpha(s_2 s_3 s_\delta)(\widetilde{C}_{7V,t} - \widetilde{C}_{7V,c}) \\
+ (\epsilon - i\xi) \Bigg[\left(\frac{B(K^+ \to \pi^+ e^+ e^-)}{B(K^+ \to \pi^0 e^+ \nu)} \right) & \left(\frac{\Gamma(K_1 \to \pi^0 e^+ e^-)}{\Gamma(K^+ \to \pi^+ e^+ e^-)} \right) \Bigg]^{\frac{1}{2}} \Bigg|^2 \\
+ \Big| \alpha(s_2 s_3 s_\delta)(\widetilde{C}_{7A,t} - \widetilde{C}_{7A,c}) \Big|^2 \Bigg\}. & \qquad (A.13)
\end{aligned}
$$

In this last expression, all branching ratios and lifetimes on the right hand side, except $\Gamma(K_1 \to \pi^0 e^+ e^-)$, are experimentally known quantities. Our predictions are for the direct CP violating terms, which involve the factor $s_2 s_3 s_\delta$; the unknown quantity $\Gamma(K_1 \to \pi^0 e^+ e^-)/\Gamma(K^+ \to \pi^+ e^+ e^-)$ may hopefully be determined in future experiments through the measurement of $\Gamma(K_S \to \pi^0 e^+ e^-)$, pinning down in this manner the long distance contributions to this rare decay. An estimated value of 1 is obtained for this ratio if only $\Delta I = 1/2$ transitions are assumed, but in the presence of comparable $\Delta I = 3/2$ transitions, any value for this ratio is possible.

The derivation of the branching ratio for $K^+ \to \pi^+ \nu \overline{\nu}$ goes along the same lines as above, but with the use of the effective Hamiltonian (2.19) expressed as:

$$
\mathcal{H}_{\nu\nu} = \frac{G_F}{\sqrt{2}} \sum_q V_{qs}^* V_{qd} \widetilde{C}_{\nu,q} \{ Q_V - Q_A \} + h.c. \qquad (A.14)
$$

and the hadronic matrix elements:

$$
\begin{aligned}
< \pi^+(p')|\bar{s}\gamma_\mu d|K^+(p) > = < \pi^-(p')|\bar{d}\gamma_\mu s|K^-(p) > \equiv \\
f_+(q^2)(p+p')_\mu \ + \ f_-(q^2)q_\mu
\end{aligned}
\tag{A.15}
$$

obtained from (A.8) also by means of an isospin transformation.

Normalizing the amplitude with that for $K^+ \to \pi^0 e^+ \nu$ our result takes the form:

$$
\left|\frac{< \pi^+\nu\bar{\nu}|\mathcal{H}|K^+ >}{< \pi^0 e^+\nu|\mathcal{H}|K^+ >}\right|^2 = \frac{2\alpha^2}{|V_{us}|^2}\Big|\sum_q \widetilde{C}_{\nu,q}V_{qs}^* V_{qd}\Big|^2 . \tag{A.16}
$$

After reducing the sum over $q = c, t$ with the use of the unitarity relation (2.20), the only coefficients that enter the expression (A.16) are $\widetilde{C}_{\nu,c} - \widetilde{C}_{\nu,u}$ and $\widetilde{C}_{\nu,t} - \widetilde{C}_{\nu,u}$. Moreover, the u quark contribution can be neglected in both cases, so that actually the only important coefficients are $\widetilde{C}_{\nu,c}$ and $\widetilde{C}_{\nu,t}$. Even though the first of these coefficients is much smaller than the second, it cannot be neglected, since they become comparable after taking the Kobayashi - Maskawa factors into account. The branching ratio is then expressed within this order of approximation as:

$$
B(K^+ \to \pi^+\nu\bar{\nu}) = B(K^+ \to \pi^0 e^+\nu) 2\alpha^2 \left|V_{ud}\right|^2 \left|\widetilde{C}_{\nu,t}\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} - \widetilde{C}_{\nu,c}\right|^2 . \tag{A.17}
$$

Now, let us turn to the CP violating decay $K_L \to \pi^0\nu\bar{\nu}$. The expression for its branching ratio has a strong resemblance to that for the decay into charged leptons. Proceeding in a similar way, after the corresponding change in the coefficients $\widetilde{C}$, we can express the branching ratio as:

$$B(K_L \to \pi^0 \nu\overline{\nu}) = B(K^+ \to \pi^0 e^+ \nu)\frac{\tau(K_L)}{\tau(K^+)}\frac{2\alpha^2}{|V_{us}|^2} \times \Big| \sum_q \widetilde{C}_{\nu,q}\{(\epsilon - i\xi)Re[V^*_{qs}V_{qd}] \;+\; iIm[V^*_{qs}V_{qd}]\}\Big|^2. \tag{A.18}$$

From a theoretical perspective, however, we are here in a more fortunate situation, since the indirect CP violating terms (those proportional to ϵ) do not receive important long-distance contributions and so can be actually neglected. The main contribution to the decay comes directly from a CP violating amplitude, where the only uncertainty besides the top quark mass is the CP violating parameter (A.13) of the Kobayashi - Maskawa matrix. QCD corrections are also negligible in this case, since the coefficients $\widetilde{C}_{\nu,c}$ and $\widetilde{C}_{\nu,t}$ appear now as a simple difference:

$$B(K_L \to \pi^0 \nu\overline{\nu}) = B(K^+ \to \pi^0 e^+ \nu)\frac{\tau(K_L)}{\tau(K^+)}2\alpha^2\frac{(ImV^*_{ts}V_{td})^2}{|V_{us}|^2}(\widetilde{C}_{\nu,t} - \widetilde{C}_{\nu,c})^2.$$

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