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QUARKS AND BUBBLES: THE DYNAMICS OF A FIELD THEORY MODEL OF HADRON STRUCTURE*

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ABSTRACT

This report analyzes the semi-classical dynamics of the field theory model of hadron structure proposed by Bardeen, Chanowitz, Drell, Weinstein, and Yan (BCDWY). The BCDWY model is based on a field theory of quarks interacting via the Yukawa coupling with a quartically self-coupled scalar field which acquires a non-zero vacuum expectation value. BCDWY have shown that in a strong coupling limit, though the quark acquires a large dynamically generated bare mass, low mass particle-like bound states containing quarks ("bubbles") can form.

We show that, in the infinitely strong coupling limit, the semi-classical field equations of the BCDWY theory admit finite energy bound state solutions which correspond, in general, to bubbles whose surfaces are time dependent and upon which quarks are trapped. The dynamics of such states can be completely characterized in terms of the geometric variables which define the bubble surface and a set of quark fields localized on this surface. There remains but a single finite coupling constant, which sets the scale of masses in the theory, and which may be taken to be a constant energy density associated with the bubble surface. The equations of motion can be derived from an action principle, involving only the reduced set of variables, which is strikingly similar to that which generates the MIT bag model. Poincaré invariance of the resulting semi-classical theory of bubble dynamics is demonstrated.

The geometric formalism we develop is used to discuss the excited state spectrum of the BCDWY model. Though the levels cannot be computed exactly, a clear physical picture of bubble dynamics is developed. The most important qualitative feature of the bubble is its softness to deformations. Excited bubble states are highly deformed from sphericity. The equations of motion for a simple radial surface excitation of the spherical bubble are solved exactly. We estimate a mass ratio between the first radial excitation and the ground state that is very close to that of the Roper resonance to the nucleon.

The classical bubble theory is exactly and completely solvable in three space-time dimensions. The three-dimensional bubble is a closed string upon which quarks are bound. We discuss the physical properties of these solutions, paying particular attention to the way in which the softness of the bubble is reflected dynamically. The three-dimensional theory is quantized explicitly by introducing commutation relations among the normal mode amplitudes which define the classical solutions. We show that the operator algebra of this theory is Poincaré invariant and discuss the spectrum of states.

Preface

The work described in this thesis was carried out in the vibrant atmosphere of the Theory Group at S.L.A.C. I have benefitted greatly from numerous discussions with the members of this community, especially Marvin Weinstein and Henry Tye. I particularly want to express my appreciation to Sid Drell, who, as mentor of this community and as my advisor, has been a source of invaluable counsel and support to me.

To my wife, Linda Grisham, who has patiently suffered the burden of having a husband who thinks about quarks and bubbles at odd hours of the day or night, I am grateful.

I thank the Fannie and John Hertz Foundation for their generous support.

Finally, I want to express my appreciation of the efforts of those of my forbears who have walked these paths before, with great difficulty and little reward, in order that in our time and in the future, none shall be denied.

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Chapter 1

Introduction

Over the past decade and a half, much evidence supporting the picture of quarks as the fundamental building blocks of the observed hadrons has accumulated.

The strongest suggestions of a possible quark substructure to the hadrons arise from the successes of the SU(3) symmetry scheme first proposed by Gell-Mann[1] and developed and extended by many others. In this scheme, baryons or mesons of the same spin and parity are grouped into multiplets, each of which corresponds to an irreducible representation of SU(3). Hadrons within a given multiplet are distinguished only by "internal" quantum numbers which label the vectors in the corresponding representation of SU(3).

Of course, for such a classification to be physically sensible at all, it must be respected by the hadronic interactions. That is: the strongest hadronic interactions should be SU(3) symmetric, with only weaker hadronic, electromagnetic, and weak interactions distinguishing the members within a multiplet. As is now well known, this seems to be the case in nature. The relatively small mass splittings within multiplets and the success of the

Gell-Mann Okubo formula[2] in describing these splittings attests to the approximate SU(3) symmetry of strong interaction Hamiltonian. The SU(3) structure of electromagnetic and weak currents has been classified. The relations resulting from an application of the SU(3) Wigner-Eckhart theorem to the low energy weak and electromagnetic transitions of the hadrons have been verified experimentally[3]. Observed strong interaction scattering processes are consistent with the assumption of approximate SU(3) symmetry of the strong interactions.

In what sense does the success of SU(3) suggest the existence of quarks? All observed SU(3) hadron multiplets are octets (<u>8</u>) or decouplets (<u>10</u>). The fundamental representation of SU(3) is a triplet; hence all higher representations can be obtained as tensor products of a fundamental triplet. Perhaps the simplest way of understanding the SU(3) symmetry of hadrons theoretically is, then, to suppose that hadrons are bound states of SU(3) triplet "quarks" from which they derive their quantum numbers.

Indeed, there are indications from phenomenological work that quarks have an existence more substantial than that as mere group theoretic abstractions from SU(3). The parton picture of scaling in deep inelastic lepton scattering[4] suggests that, to a high energy electromagnetic or weak current probe, nucleons appear to be for the most part bound states of a few nearly free point-like objects of charges consistent with the required SU(3) quarks. The successful quark recombination rules (such as the so-called "Zweig's rule") in dual theories[5] picture hadrons as composed of quark-like constituents carrying SU(3) quantum numbers which are exchanged in inelastic strong scattering processes. Finally, the recently proposed theories of large transverse momentum hadron scattering[6] rely on a quark constituent interchange picture in order to predict asymptotic power-law behaviour of cross-sections from the dimensions of underlying fundamental quark field operators[7].

However attractive models based on quark constituents may be, there remain several outstanding problems that must be resolved before a real understanding of hadron dynamics in terms of quark constituents can be attained. First, at present energies, free quarks have never been observed or directly inferred to be present in any scattering process. No states with quark quantum numbers (fractional charge or baryon number) have been observed. Apparently, only states of three quarks (baryons) or quark-antiquark (mesons) can exist. Evidence from form factors and from the spin, parity, and SU(3) assignment of states indicates that quarks have spin 1/2 but that the wave functions of the ground state baryons and mesons are symmetric under quark

interchange. This is a violation of the spin-statistics theorem if there are but three distinct quarks.

Some of these difficulties have been resolved. Others remain puzzles. In particular the apparently anomolous statistics and the exclusion of all states with quark quantum numbers can be acheived by adopting the "color hypothesis"[8]: that there is a hidden SU(3) of "color" under which each quark transforms as a triplet; but that the dynamics are such that only color singlet states can be bound to form hadrons. The implications of this hypothesis and alternatives to it have been discussed at length in the literature. Further discussion will not be presented here. We merely note that, as it has been stated here, the color hypothesis does not itself provide any dynamical explanation of quark binding. It simply asserts that the dynamics of the theory should be such as to insure only the binding of color singlets.

Perhaps the problem whose resolution will afford us the greatest insight into the internal dynamics of hadrons is the problem of the non-observation of quarks. That is: how can it be that quarks form low mass bound states, while free quarks, if they exist at all, must have masses at least a factor ten higher than the observed hadrons?

In the following discussion and throughout this thesis, we adopt the point of view that a field theoretic description of quark dynamics is appropriate. We will assume that quark fields are among the fundamental fields of the theory. This is certainly not the only approach that could be taken , but it does have the advantage that Lorentz invariance and causal space-time structure are made manifest from the outset. Further, the sense in which quarks are "constituents" of the hadrons is also clear, even in the absence of asymptotic quark states : hadronic currents and interactions are written in terms of the elementary quark fields of the theory.

Within a conventional field-theoretic framework, quark binding and confinement are quite difficult to understand. Most techniques that have been developed to analyze field theories are based on perturbation expansions. As we shall see presently, quark confinement cannot be understood as a perturbative phenomenon. Perturbation theory begins with a structureless vacuum and the Fock space of free quark states created from it by the action of the interaction picture field operators. The true physical states are then expanded in this basis in a power series in the coupling constants. In order to describe the observed spectrum of hadronic states, we must find that the true spectrum of the theory consists of many low-lying quark bound states very far below the free quark threshold. This necessitates that either: (1) free quark states do not exist at all; or, (2) that free

quark states are much more massive, with this large "bare" mass being almost perfectly cancelled by binding energy in the hadronic bound states. In either case, perturbation theory gives no reasonable approximation.

The possibility that the fundamental fields do not create asymptotic states was first proposed by Schwinger based on his studies of quantum electrodynamics in two dimensions.[9] This idea is actively being investigated by many researchers using the techniques of the renormalization group. The idea upon which this approach is based is that, when perturbation theory is summed to all orders, the infrared singularities of Green's functions involving bare quark lines may become so severe as to prevent the existence of asymptotic quark states. Whether or not this "Schwinger Mechanism" occurs in any field theory in four dimensions is very difficult to determine. No calculable model of hadron structure based on it has ever been proposed.

In this thesis , we will be concerned with a field theory model of hadronic structure proposed by Bardeen, Chanowitz, Drell, Weinstein, and Yan[10] ("BCDWY") which takes the second approach. It considers a strongly coupled field theory in which a large, dynamically generated, quark mass is cancelled in hadronic bound states by a large binding energy. In two respects, a perturbative approach is an inadequate tool to develop an understanding of such a theory. First, because the coupling is strong, the perturbation expansion is not manifestly convergent order by order -- it is an expansion in powers of a large number. Further, since one must look at all terms in the perturbation expansion, the physics of strong binding is obscured. It is generally difficult to understand how a state which is a sum over many orders of perturbation theory, containing ever higher numbers of bare quarks, can be regarded effectively as a bound state of a few quarks; or how, in the Bjorken limit, the nucleon can appear to be composed of a few quasi-free partons. A non-perturbative approach to the problem of quark binding is essential.

The BCDWY model is one of several recently proposed "bag" models. Bag models generally provide a clear intuitive, if not mathematical, picture of how quark binding occurs. The fundamental idea is that the interactions are such that the vacuum is highly polarized in the presence of quarks. Quarks, which may have an extremely large bare mass, bind very strongly to extended, coherent, neutral, vacuum excitations ("bags"). A non-abelian colored gauge interaction is introduced, after Nambu[11], in such a way that this binding occurs only in color singlet quark states. These low mass, color singlet bound states are taken to be the hadrons.

Over a region of size on the order of that of the

vacuum excitation, quarks move as very nearly free quasi-particles. They cannot, however, be asymptotically separated as there is an energy associated with the size of the bag that contains them. This provides a clear picture of how quark binding and confinement can be made consistent with the idea of quasi-free parton constituents at short distances. Indeed, quarks <u>are</u> free particles over a region the size of a hadron compton wavelength.

Because quark interactions within the bag are taken to be small, hadrons are seen to be predominantly bound states of a few quasi-particles from which they derive their SU(3) quantum numbers. In bag theories, the masses, magnetic moments, charge radii, and static electromagnetic transition matrix elements of the hadrons are determined in terms of a single parameter in the exact SU(3) symmetry limit. Thus, these models are highly constrained in their ability to fit the data. That they give numbers for these quantities that are consistent with experiment is, perhaps, something of a triumph.

The picture of a bag described above is essentially classical. The outstanding problem in the present bag theories is to construct a systematic, calculable, and complete quantum theory of bags. The various bag theories differ in their approach to this question, as well as in many significant details of their models. The theory proposed by Chodos, Jaffe, Johnson, Thorn, and Weisskopf[1a] ("MIT Bag") begins with a classical field theory of quarks which are confined to four dimensional bounded domains in space-time by fiat. There is no dynamics of confinement postulated to antecede the formation of bags in this approach. The vacuum structure is taken to be such that quarks exist only in confined bag states which have a constant energy density associated with them. The problem here is to identify the independent classical degrees of freedom and then quantize the theory. This has been accomplished only in two dimensions. The numerical predictions of this theory derive from a semi-classical approximation which treats the quark fields as first quantized wave functions and the degrees of freedom corresponding to the bag surface as c-number functions. Creutz and Soh[12] have shown that there is a classical local field theory whose strong coupling limit has low energy states corresponding to MIT bags. Whether this is also true in the quantum theory is unknown. We shall return to a similar question in the context of the BCDWY theory.

The BCDWY theory starts from a quite different point of view. The Hamiltonian of a strongly coupled, renormalizable interacting quantum field theory is shown, via a variational calculation over a class of trial states, to possess low energy bound states of quarks and coherent

excitations of a field with vacuum quantum numbers. The variational equations for the functional parameters defining the trial states <u>are</u> the equations of motion of the corresponding semi-classical field theory. The bound state solutions of the BCDWY theory in n space-time dimensions are n-1 dimensional thin shells, which will be referred to here as "bubbles." BCDWY solve the semi-classical equations for the case of a static spherical bubble, and thereby obtain estimates of the masses and electromagnetic matrix elements of the ground state mesons and baryons.

A striking feature of this theory, at least semi-classically, is that, although the original field theory is characterized by three coupling constants, the bubble states are characterized by a single coupling, C, which, in analogy with the case of the MIT bag, can be taken to be an energy per unit area. Unlike the MIT theory, the BCDWY approach does admit bare quark states. However the bare quark mass is determined by a combination of the coupling constants that is independent of C. Thus, the threshold for quark production may be set arbitrarily high without affecting low energy hadron dynamics. In this sense, there are no free quarks in this model. A most intriguing question is whether this decoupling of the bare quark states from hadron dynamics in the strong coupling limit can be made as an operator statement in the Hilbert space of physical states.

The answer to this question is unknown.

In this thesis, we study the semi-classical BCDWY theory in detail. We discover that the only particle-like bound states whose mass remains finite in the strong coupling limit have field configurations corresponding to a thin shell, or "bubble." In the limit of infinite coupling, the bubble becomes a closed surface in space-time upon which quarks are trapped. We show that, in the strong coupling limit, the dynamics of such states may be expressed completely in terms of geometric variables which describe the bubble surface and quark fields defined on this surface. We derive the equations of motion of the bubble in this set of variables, and show that they are identical to those derived from an action principle that is strikingly similar to that which generates the MIT bag. The classical BCDWY field theory is antecedent to this "bubble theory" in precisely the same way that the Creutz-Soh theory is to the MIT bag.

The geometric formalism we develop is used to discuss the semi-classical excited state spectrum of the BCDWY model in four space-time dimensions. Though we cannot compute the levels exactly, we develop a clear physical picture of the dynamics of the theory. The most important qualitative feature of the bubble is its "softness"-- a bubble may suffer large deformations of shape at little cost in energy.

This has far-reaching physical consequences. The semi-classical excited states of the bubble are found to be highly non-spherical. Further, we expect that in a quantum theory of the bubble, the zero point motion of its surface will be large, smearing the sharp energy and charge distributions of the classical state over a finite volume of space. Surface excitations of the bubble lead to non-trivial excited states in the model. We solve the semi-classical equations for a radial surface excitation of the ground state. The ratio of the mass of this radial mode to the mass of the ground state is very close to the mass ratio of the Roper resonance and the nucleon.

The classical bubble theory is exactly and completely solvable in three space-time dimensions. The bubble surface is a spacially closed two dimensional hypersurface. The three dimensional bubble might be described as a closed string upon which a fermion is bound. We find that, with the proper choice of Lorentz frame and surface coordinates, all solutions to the classical three dimensional bubble theory can be expressed in terms of a countable number of normal mode amplitudes. The theory may be quantized explicitly by imposing commutation relations on these amplitudes. We exhibit such a quantization, and show that it leads to a Poincare invariant quantum theory of the bubble. The operator structure of this theory is very similar to that of the Neveu-Schwartz model in three dimensions. We discuss the spectrum of the quantum theory, and find it is that of a single particle with many possible internal excitations. The quantum theory in three dimensions will be seen to be consistent with our general picture of the bubble as a soft object.

Chapter II

In this chapter, we review the work of Bardeen, Chanowitz, Drell, Weinstein and Yan, with particular emphasis on those ideas which will be needed for subsequent developments. The discussion here is intended to be a summary; the reader is encouraged to refer to their paper (Ref 10) for a more complete and detailed analysis.

The BCDWY model for the binding of a single quark species is developed from the field theory defined by the Lagrangian:

$$L = \frac{1}{2}(\partial\sigma)^2 - \lambda(\sigma^2 - f^2)^2 + \bar{\psi}(i\not{\partial} - G\sigma)\psi$$

whose Hamiltonian is

$$H = \int dx \; \frac{1}{2}\left(\frac{\partial\sigma}{\partial t}\right)^2 + \frac{1}{2}(\nabla\sigma)^2 + \lambda(\sigma^2 - f^2)^2 + \psi^{+}(-i\vec{\alpha}\cdot\vec{\nabla} + G\sigma\gamma^0)\psi$$

This is a theory of a Dirac field ψ interacting, via the Yukawa coupling, with a neutral, quartically self-coupled scalar field, σ. The theory is characterized by three coupling constants: G, λ, and f. In four space-time dimensions, G and λ are dimensionless, while f has dimensions of energy.

This Lagrangian is symmetric under the discrete transformation:

$$\sigma \rightarrow -\sigma$$
$$\psi \rightarrow \gamma_5 \psi$$

In the quantum field theory defined by H and the canonical commutation relations of the fields, this γ_5 reflection symmetry is spontaneously broken. The classical potential of the σ field has minima at $\sigma = \pm f$, and we expect that the vacuum of the quantum theory will be such that the expectation value of σ is $\pm f$. As a convention, we choose that the expectation value of σ be $+f$.

By shifting to a new field , σ', whose vacuum expectation value is zero,

$$\sigma = \sigma' + f$$

we put the Lagrangian in a form that is suitable for perturbative analysis:

$$L = \tfrac{1}{2}(\partial\sigma')^2 - 4\lambda f^2 \sigma'^2 - 4\lambda f \sigma'^3 - \lambda \sigma'^4$$
$$+ \bar{\psi}(i\not{\partial} - Gf - G\sigma')\psi$$

In a perturbative approach, we would conclude that this Lagrangian describes the interaction of a σ meson of bare mass $M_\sigma = \sqrt{8\lambda}\, f$ with a Dirac particle of bare mass $M_Q = Gf$. We will consider a limit of couplings such that these masses are "large" (>>1 gev.).

As observed by BCDWY, there may be particle-like

excitations of the interacting theory with energies much lower than M_σ and M_Q . That this is possible may be seen from a simple heuristic arguement. A state of one quark at rest has energy Gf in perturbation theory because the σ field has value f in the vacuum. It is only in zero'th order perturbation theory, however, that the σ field is not free to respond to the presence of the quark in such a way as to lower the total energy of the state. A simple picture of such a lower energy state might be one where the σ field is depressed to zero in a region of finite size , R, where the quark is trapped (Fig 1). Inside this region, the quark is massless and will have only a kinetic energy of order 1/R. However, it will cost a potential energy of order $\lambda f^4 R^3$ to depress the σ field from f. In addition, there is a surface energy associated with the gradient of the σ field in the transition region, which we take to be negligible here. The total energy of the state can then be estimated to be

$$U \sim \frac{1}{R} + \lambda f^4 R^3$$

Minimizing over R, we find

$$R \sim \frac{1}{\lambda^{\frac{1}{4}} f}$$
$$U \sim \lambda^{\frac{1}{4}} f$$

If $\lambda^{\frac{1}{4}} \ll G$, this is a state of energy much lower than the bare quark mass.

In fact, such bag-like states are not the lowest quark states of the theory. It is energetically even more favourable for the σ field to go to the "wrong" vacuum expectation value, -f, inside a region of size R, making a transition back to f in a narrow region of thickness D (Fig 2). The quark can then be trapped in this thin transition region with an energy of order 1/R -- an effect that will be discussed in more detail below. The estimate for the total energy is:

$$U \sim \frac{1}{R} + R^2 D\left[\left(\frac{f}{D}\right)^2 + \lambda f^4\right]$$

Minimizing over R and D, we find

$$D \sim \frac{1}{\lambda^{\frac{1}{2}} f} \qquad R \sim \frac{1}{\lambda^{1/6} f}$$

$$U \sim \lambda^{\frac{1}{6}} f$$

In order for this estimate to be sensible, R>>D, which requires λ >>1. Then if $\lambda^{1/6} \ll G$, this shell state, or "bubble", is a lower energy 1 quark state than either the bare quark or the bag-like state. In the strong coupling limit -- G, λ large, f small, $\lambda^{1/6} \ll G$, $\lambda^{1/6} f \sim 1$ gev -- BCDWY use this mechanism of quark trapping in bubbles to build a model of hadron structure.

The equations which BCDWY use to describe the dynamics

of the theory are the static, semi-classical equations of motion. These equations are derived by BCDWY via an approximate variational calculation of the energy in a trial state of the quantum theory. This variational calculation will not be reproduced here. The reader is encouraged to consult Ref 10 for a detailed presentation and discussion.

The semi-classical equations of motion consist of the (one-particle) Dirac equation for Ψ in the presence of a classical σ field:

$$(i\partial\!\!\!/ - G\sigma)\Psi = 0$$

where the wave function must be normalized to unit charge

$$Q = \int dx\ \psi^+\psi = 1$$

and of the classical equation for the σ field in the presence of a fermion source:

$$-\partial^2\sigma + 4\lambda\sigma(f^2 - \sigma^2) = G\bar{\Psi}\Psi$$

In the "static" case, $\sigma = \sigma(\vec{x})$, $\Psi = \psi(\vec{x})e^{-iEt}$,

these reduce to to:

$$(-i\vec{\alpha}\cdot\vec{\nabla} + G\sigma\gamma^0)\Psi(\vec{x}) = E\,\Psi(\vec{x})$$

$$\nabla^2\sigma + 4\lambda\sigma(f^2 - \sigma^2) = G\bar{\Psi}\Psi$$

These differential equations are the classical

Euler-Lagrange equations of the theory. The system is "semi-classical" in the sense that ψ is interpreted as if it were a single-particle Dirac wave function: it is normalized to unit charge and negative energy fermion states are to be given the Dirac interpretation as positive energy anti-fermions. We note that the Dirac equation is one with a scalar potential, so that no Klein paradox arises -- the distinction between positive and negative energy states is always unambiguous.

In the BCDWY variational analysis of the static case, these semi-classical constraints arise naturally. The ψ and σ "fields" above are actually functional parameters that define a class of trial states over which the variation is carried out. The constraint Q=1 and the interpretation of the negative energy solutions as anti-quarks reflect the requirements that the trial state be normalized and have a definite charge.

The solution of the static semi-classical equations upon which the BCDWY model is based is an approximate one, valid in the strong coupling limit. An intuitive picture of this solution can be obtained from the examination of an exact solution to the system of coupled, non-linear equations in one space dimension[14].

Taking the representation of the gamma matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

this solution is :

$$\sigma(x) = f \tanh \sqrt{2\lambda}\, f\,(x - x_0)$$

$$\psi(x) = \frac{N}{\sqrt{2}} \left[\cosh \sqrt{2\lambda}\, f\,(x-x_0)\right]^{-\frac{G}{\sqrt{2\lambda}}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

where N is a normalization constant that insures Q=1, and x_0 =constant.

One finds:

$$E = 0$$

$$U = \frac{4}{3}\sqrt{2\lambda}\, f^3$$

$$\bar{\psi}\psi = 0$$

There are several aspects of this one dimensional solution which point toward more general features of the theory. First, because $\bar{\psi}\psi$ vanishes, the σ field equation is actually independent of ψ. The above solution for σ is the well-known "kink" solution of the spontaneously broken quartic scalar theory in one dimension.[15] The dynamics of the scalar field is determined primarily by its self-coupling, rather than by its coupling to fermion sources. This will remain true in higher dimensions. The width of the transition region of the σ field is on the order of the σ compton wavelength, which will always be small compared to $(1\,gev)^{-1}$.

Perhaps the most striking feature of the solution is that the Dirac energy is small even though the Dirac wave-function is very sharply peaked. Intuition based on the quantum mechanics of bosons would suggest that the energy should be comparable to the dominant Fourier components of the wave-function -- on the order of the bare quark mass.

Formally, this intuition need not be correct for fermions because the Hamiltonian is linear, rather than quadratic, in the momentum operator:

$$H = \alpha p + G\sigma\gamma^0 \qquad , \qquad p = -i\frac{\partial}{\partial x}$$

Mathematically, the zero energy of this bound state wave-function arises because the relative phase between its upper and lower components is such that their high momentum contributions to the energy exactly cancel. That such a cancellation is possible is evident from symmetry considerations: Except in the region near the kink, the Dirac equation is that of a free particle of mass +Gf on the right, and -Gf on the left. The solutions of the free Dirac equation are such that under $M \to -M$, positive energy solutions of momentum p are mapped into negative energy solutions of momentum -p. Then, as simple algebra shows, under $M \to -M$:

$$p \to -p$$

$$\alpha \to +\alpha$$

whence $$p\alpha \to -p\alpha$$

Thus one might expect that, for the symmetric ground state wave-function, the energy should be zero, since, in the sense described above, the Hamiltonian is "odd". This is indeed the case.

BCDWY solve the semi-classical equations approximately for a spherically symmetric state in three dimensions. Their arguement proceeds as follows. Assume that the solution will be a bubble of some radius R:

$$\sigma(r) \sim \begin{cases} -f & r \lesssim R - \frac{D}{2} \\ +f & r \gtrsim R + \frac{D}{2} \end{cases}$$

Then the Dirac equation is approximately that for a quark of "mass" -Gf inside the sphere and of "mass" +Gf outside. For $\frac{1}{R} \ll Gf$, there are bound state solutions with energy much less than Gf. For $r \sim R$, the wave function has the general form:

$$\psi = F(r)\begin{pmatrix} \varphi^{\ell}_{jm}(\theta,\varphi) \\ i\frac{\vec{\sigma}\cdot\vec{r}}{r}\varphi^{\ell}_{jm}(\theta,\varphi) \end{pmatrix} + O\left(\frac{D}{R}, \frac{1}{GfR}\right)$$

where φ^{ℓ}_{jm} is the Pauli two-component spinor of angular momentum (j,m) and parity $(-1)^{\ell}$. F(r) decreases exponentially near $r \sim R$: $F \sim e^{-Gf|r-R|}$. The detailed shape of F depends on the behaviour of σ in the transition region. However, the energy of such a state is independent of these details in the strong coupling limit.

$$E \approx \begin{cases} \frac{j+\frac{1}{2}}{R} & \text{if } j=\ell+\frac{1}{2} \\ -\frac{j+\frac{1}{2}}{R} & \text{if } j=\ell-\frac{1}{2} \end{cases}$$

$\bar{\Psi}\Psi$ is small near R:

$$\bar{\Psi}\Psi \sim |F(r)|^2 |\varphi^{\ell}_{jm}|^2 \frac{|r-R|}{R} \times O\left(\frac{D}{R}, \frac{1}{GfR}\right)$$

$\bar{\Psi}\Psi$ is spherically symmetric only if j=1/2. Thus, as in the one dimensional case, the behaviour of the σ field is determined primarily by its quartic self couplings rather than by the fermion source term.

$$\sigma(x) \approx f \tanh \sqrt{2\lambda} f (r-R)$$

The energy of this field configuration is

$$\begin{aligned} E_{\sigma} &= \int dx \; \tfrac{1}{2}(\nabla\sigma)^2 + \lambda(\sigma^2-f^2)^2 \\ &\approx \tfrac{4}{3}\sqrt{2\lambda}\, f^3 \cdot 4\pi R^2 \\ &= C \cdot 4\pi R^2 \qquad \text{where } C \equiv \tfrac{4}{3}\sqrt{2\lambda}\, f^3 \end{aligned}$$

So that for j=1/2, the total energy is

$$U \approx \frac{1}{R} + C \cdot 4\pi R^2$$

BCDWY show in the case of spherical symmetry that, if R is chosen to minimize the above expression for U, further corrections to the wave functions contribute to the energy terms of order $\frac{D}{R}, \frac{1}{GfR}$. The minimization gives:

$$R = (8\pi C)^{-\frac{1}{3}}$$

$$U = \frac{3}{2}(8\pi C)^{\frac{1}{3}}$$

In Chapter III, we give a general scheme for the calculation of bubble states in the strong coupling limit, which has the BCDWY solution as a special case. We therefore defer the proof of the accuracy of these estimates until then.

The picture of quark binding developed so far is exactly that whose general features were discussed in the introduction. The vacuum is highly polarizable. Even in the absence of quarks, bubble-like domains within which σ=-f, though unstable against collapse, can form with excitation energies that are small on the scale of the bare σ mass[16]. We have seen that quarks can bind strongly to the boundary surface of such a domain and stabilize the bubble.

So far, what we have is a model of quark binding only. In order to have a viable model of hadrons, we must account for the confinement of quark quantum numbers and the anomolous statistics of quark wave functions. This is accomplished by BCDWY through the introduction of color degrees of freedom. They assume that there are nine quarks which provide the fundamental representation of SU(3)xSU(3)' (color). These are all coupled to the field with the same Yukawa coupling discussed above for a single quark species. A set of strongly coupled color gauge fields are introduced. These gauge fields acquire a large mass in the vacuum via a Higgs mechanism. BCDWY then show that the set of coupling constants can be chosen so that this color gauge interaction unbinds all color non-singlets. These additional interactions do not affect the energy calculated for color singlets (at least in the semi-classical approximation). The reader is referred to Ref 10 for a full exposition of these ideas.

In practice, then, hadronic levels can be computed as if there is only the usual SU(3) of non-interacting quarks. The effect of the color interaction is that only states with zero triality and completely symmetric quark wave-functions do, in fact, correspond to physical hadrons.

The BCDWY spherically symmetric solution provides a model of the ground state baryons and mesons. For a multi-quark or quark-antiquark bubble, each particle's wave-function obeys the Dirac equation separately, while the source for the σ field is the sum of $G\bar{\Psi}\Psi$ over individual

particles. Thus, in the spherical state, the energy of an n particle system results from the minimization of:

$$U = \frac{n}{R} + C \cdot 4\pi R^2$$

so we have:

$$R = n^{\frac{1}{3}} (8\pi C)^{-\frac{1}{3}}$$

$$U = n^{\frac{2}{3}} \; \frac{3}{2} (8\pi C)^{\frac{1}{3}}$$

For mesons, n=2; for baryons, n=3. Each quark state in the bubble has j=1/2 and can be assigned m=1/2 or -1/2 independently. Therefore, the ground state mesons and baryons reproduce the SU(6) <u>35</u> and <u>56</u>, respectively, with the expected combinations of spin, parity and charge conjugation. We observe that SU(6) is an essentially non-relativistic symmetry, but is reproduced here in a model where quarks are ultra-relativistic. This will be true only in the ground state of the BCDWY model, where the independent quarks each have j=1/2.

C is the only parameter that need be specified to determine the ground state properties of baryons and mesons in the model. Of course, we are still working in the exact SU(3) limit, so that numerical comparisons can be subject to ambiguities. C may be determined from the central mass of the baryon <u>56</u>.

If we take: $M_{56} = 1150.8$ mev

then

$$C = 51.3 \quad \mathrm{Mev/fm^a}$$

$$R_{56} = .77 \quad \mathrm{fm}$$

The model gives $\frac{M_{35}}{M_{56}} = \left(\frac{2}{3}\right)^{2/3}$, or $M_{35} = 878$ mev. This is not inconsistent with the observed spectrum, but M_{35} is quite ambiguous due to SU(3) breaking.

The electromagnetic current, $\bar{\Psi} Q \gamma^\mu \psi$, is conserved in this model, so that magnetic moments and M1 transition matrix elements can be computed. Because SU(6) is exact, all the SU(6) relations between electromagnetic matrix elements of ground state hadrons in the same multiplet follow trivially. The non-trivial predictions of the model relate magnetic moments to masses and baryon moments to meson moments. The content of these non-trivial predictions can be written in terms of the more well measured matrix elements:

$$\mu_P = 3\left[\frac{e}{2M_{56}}\right] \quad \text{or} \quad \frac{\mu_P}{\mu_{\text{NUCLEAR}}} = 3\left(\frac{M_P}{M_{56}}\right)$$

$$\frac{\Gamma(\omega \rightarrow \pi\gamma)}{\Gamma(\Delta \rightarrow p\gamma)} = \left(\frac{2}{3}\right)^{2/3} = .76$$

Though, again, it is hard to take these numbers seriously before understanding the mass splittings, we note that they are in good agreement with experiment.

There is no partially conserved axial current in this model. This is a fundamental phenomenological element that

is absent in all bag models. There is no sense in which the pion is different (e.g. a goldstone boson) from the other members of the 35. One can, however, write down a pseudo-vector current composed of quark fields: $\bar{\Psi} Q_5 \gamma^\mu \gamma_5 \Psi$. The matrix elements of this current between ground state baryons at rest corresponds to $g_A = 5/9$. Experimentally, $g_A = 1.25$. The significance of this large discrepancy is unclear.

This is as far as the BCDWY model goes in its predictions of the hadron spectrum. In order to characterize higher states, one must understand non-spherical bubbles. To compute form factors and non-static properties, one must at least understand the motion and quantum dynamics of bubbles. There are two avenues of approach to the question of quantum corrections. One is to go back to the full field theory and to try to invent new techniques of analysis more powerful than the variational calculation of BCDWY. The hope would then be that the strong coupling limit of the full quantum theory will provide us with as simple and understandable a picture as the semi-classical theory does. Perhaps a more modest approach is to develop the semi-classical theory of bubbles to its fullest extent, with the hope that, by understanding the dynamics of bubbles, one may be led to the quantum strong coupling limit through the back door. This thesis is a first attempt in this direction.

Chapter III

In this chapter, we consider the problem of finding all bubble solutions to the BCDWY semi-classical theory. In the strong coupling limit,

$$\begin{array}{l} G \to \infty \\ \lambda \to \infty \\ f \to 0 \\ \lambda^{1/6} \ll G \end{array} \qquad C = \tfrac{4}{3}\sqrt{2\lambda}\, f^3 \quad \text{fixed}$$

we show that a bubble can be pictured as an infinitely thin shell-- a spacially closed hypertube in space-time-- upon which free quark fields are defined. The dynamics of bubbles, including their equations of motion, can be expressed completely in terms of these surface Dirac fields and the geometric variables describing the hypertube. What emerges is a simple and elegant picture of the semi-classical physics of bubbles.

Section A : The Static Bubble

We begin by considering the problem of extending the static solution of BCDWY from spherically symmetric bubbles to bubbles of a more general shape. We expect that, to the extent they can be pictured as static, higher baryon and meson resonances must be described by such non-spherical bubbles. In the following discussion, we consider only the binding of a single quark species. The extension to the multi-quark case is trivial.

The static semi-classical equations are:

$$(1) \quad \nabla^2\sigma + 4\lambda\sigma(f^2 - \sigma^2) = G\bar{\psi}\psi$$

$$(2) \quad (-i\vec{\alpha}\cdot\vec{\nabla} + G\sigma\gamma^0)\,\psi = E\psi$$

We proceed to solve these equations approximately, using a straightforward extension of the BCDWY technique for the sphere:

(i) We assume the solution will be a bubble of some as yet undetermined shape. We solve the σ field equation approximately for such a configuration.

(ii) We then find an approximate solution to the Dirac equation in the presence of this σ field. This gives the Dirac energy up to corrections which vanish in the strong coupling limit.

(iii) We show that if the shape of the bubble surface is chosen to minimize the total energy, all further corrections to the fields give vanishingly small corrections to the total energy in the strong coupling limit.

The approximations we use throughout this discussion give physical quantities to "lowest order" in a small parameter. This small parameter will be written schematically as "D/R", where

D $\sim$ bare quark or meson compton wavelength

R $\sim$ size of bubble

D/R will, in fact, turn out to be on the order of $\lambda^{-\frac{1}{3}}$ or $\lambda^{1/6}G^{-1}$.

We begin with the assumption that the solution of interest will turn out to be a bubble. There will be a region of space inside of which σ =-f, and outside of which σ =+f. The σ field will make a sharp transition between these values over a distance of order D at the boundary. We denote the boundary surface of the bubble by giving its points as functions of two "internal" coordinates, u^1, u^2 :

surface: $\vec{R}(u^\alpha)$ $\alpha = 1,2$

To be mathematically precise: we define the surface, R, as that closed surface in space at which the σ field goes through zero.

Because all fields will have a non-trivial spacial dependence only in a very thin shell about this surface, it is convenient to use a set of (non-cartesian) spacial coordinates centered about it:

$$\vec{X}(u^\alpha, \xi) = \vec{R}(u^\alpha) + \xi\, \hat{n}(u^\alpha)$$

$\hat{n}(u^\alpha)$= unit normal to surface at point $\{u^\alpha\}$.

The coordinates (u^1, u^2, ξ) are well defined only within a distance on the order of one radius of curvature away from the surface. We assume that the radii of curvature of the bubble surface are always large compared to D. This assumption has no effect whatsoever on the spectrum of low-lying excitations of the theory in the strong coupling limit. By increasing G and λ, D may be made arbitrarily small without affecting either the spectrum or the surface geometry. It may be, however, that with this assumption the dynamics of bubble-bubble scattering, which is determined by the detailed dynamics of the overlap of two bubble surfaces in the full field theory, becomes indeterminant in the limit where the surface becomes infinitely thin. We shall return to this and related questions in Chapter VI.

In the new coordinate system, we can write the gradient:

$$\vec{\nabla} = \vec{\nabla}_{\parallel} + \hat{n}\, \frac{\partial}{\partial \xi}$$

where $\vec{\nabla}_{\parallel}$ is the "tangential" gradient which, though it depends on ξ, involves only differentiations with respect to the u^α, and is tangent, as a vector, to the surface.

Consider the field equation for σ. We will choose, as our first approximation to σ, a function that satisfies the "largest" part of equation (1) near the surface. Because σ makes its transition from -f to +f in a distance

D , we expect:

$$\frac{\partial\sigma}{\partial\xi} \sim \frac{1}{D} f$$

while

$$\nabla_{\parallel}\sigma \sim \frac{1}{R} f$$

We also anticipate that, as in the case of the spherical solution, the fermion source term will be relatively unimportant in (1) -- an assertion which must be verified later to insure self-consistency. Thus our first approximation to (1) in the neighborhood of the surface is:

$$\frac{\partial^2\sigma}{\partial\xi^2} + 4\lambda\sigma(f^2-\sigma^2) = 0$$

This is the same as the equation for the kink of the one-dimensional theory. The solution of this equation which satisfies the boundary conditions and vanishes on the surface is unique:

$$\sigma(x) = \sigma(\xi) = f \tanh\sqrt{2\lambda}\, f\xi$$

Next, we solve the Dirac equation (2) in the presence of this σ field.

$$\left[-i\vec{\alpha}\cdot\vec{\nabla}_{\parallel} - i\hat{n}\cdot\vec{\alpha}\frac{\partial}{\partial\xi} + \gamma^0 G f \tanh\sqrt{2\lambda}\, f\xi\right]\Psi = E\Psi$$

We construct an approximate solution valid as $G\to\infty$, using a technique similar to one invented by A. Chodos for use in a different context[12]. We expect that the Dirac wave

function will fall of exponentially as $\sim e^{-Gf|\xi|}$ away from the surface. It is clear that such a Ψ is not an analytic function of 1/G as $1/G\to 0$. However, we can attempt to factor out the essential singularity in 1/G and then expand its coefficient in 1/G.

We write:

$$\Psi(u^\alpha,\xi) = N e^{+GF(\lambda,\xi)}\left[\Psi_0(u^\alpha,\xi) + \frac{1}{G}\Psi_1(u^\alpha,\xi)\right]$$

$$E = E_0 + \frac{1}{G}E_1$$

where: F, Ψ_0, E_0 are independent of G

N is a normalization constant

Ψ_0, Ψ_1 are finite near $\xi = 0$ as $G\to\infty$

E_0, E_1 are finite as $G\to\infty$

$\Psi_0 + \frac{1}{G}\Psi_1$ is the beginning of an expansion of the field in powers of 1/G. As will become evident, only the properties of the first term will be important, so we have not written out the corrections in full.

Substituting this form in the Dirac equation (1), we have:

$$\begin{aligned}
& G\left[-i\hat{n}\cdot\vec{\alpha}\frac{dF}{d\xi} + \gamma^0 f\tanh\sqrt{2\lambda}\, f\xi\right]\Psi_0(u^\alpha,\xi) \\
(3)\quad & + \left[-i\hat{n}\cdot\vec{\alpha}\frac{\partial}{\partial\xi} - i\vec{\alpha}\cdot\vec{\nabla}_{\parallel}\right]\Psi_0(u^\alpha,\xi) \\
& + \left[\gamma^0 f\tanh\sqrt{2\lambda}\, f\xi - i\hat{n}\cdot\vec{\alpha}\frac{dF}{d\xi}\right]\Psi_1(u^\alpha,\xi) \\
& = E_0\Psi_0(u^\alpha,\xi) + O\left(\frac{1}{G}\right)
\end{aligned}$$

This equation must be satisfied order by order in 1/G. The equation for the coefficient of G is :

$$[-i\hat{n}\cdot\vec{\alpha}\,\frac{dF}{d\xi} + \gamma^0 f \tanh\sqrt{2\lambda}\, f\xi]\,\psi_0(u^\alpha,\xi) = 0$$

In order for there to be any solution of this matrix equation such that $\psi_0 \neq 0$, we must have

$$\frac{dF}{d\xi} = \pm f \tanh\sqrt{2\lambda}\, f\xi$$

The requirement that F decrease with $|\xi|$ neccesitates that we take the "-" sign above. We have:

$$\text{(I)} \qquad (\gamma^0 + i\hat{n}\cdot\vec{\alpha})\,\psi_0 = 0$$

$$e^{+GF(\xi)} = [\cosh\sqrt{2\lambda}\, f\xi]^{-\frac{G}{\sqrt{2\lambda}}}$$

The equation between the terms of order unity in (3) becomes:

$$[-i\vec{\alpha}\cdot\vec{\nabla}_{\parallel} - i\hat{n}\cdot\vec{\alpha}\,\frac{\partial}{\partial\xi}]\,\psi_0(u^\alpha,\xi)$$

$$+ f\tanh\sqrt{2\lambda}\, f\xi\,(\gamma^0 + i\hat{n}\cdot\vec{\alpha})\,\psi_1(u^\alpha,\xi)$$

$$= E_0\,\psi_0(u^\alpha,\xi)$$

Multiplying by $(\gamma^0 + i\hat{n}\cdot\vec{\alpha})$, using (I) and the fact that $(\gamma^0 + i\hat{n}\cdot\vec{\alpha})^2 = 0$, and rearranging, we find

$$\frac{\partial\psi_0}{\partial\xi} = -k\,\psi_0$$

where: $$k \equiv \frac{1}{2}\,\vec{\nabla}_{\parallel}\cdot\hat{n}$$

The quantity k depends on the geometry alone, and has a simple geometric interpretation which will be discussed later. At $\xi = 0$, where the term involving ψ_1 vanishes, we have

$$\text{(II)} \qquad [-i\vec{\alpha}\cdot\vec{\nabla}_{\parallel} + ik\,\hat{n}\cdot\vec{\alpha}]\,\psi_0(u^\alpha,0) = E_0\,\psi_0(u^\alpha,0)$$

This is an eigenvalue equation for E_0 involving only the Dirac field on the surface. Thus, given only the geometry of the bubble surface, the Dirac energy can be computed, up to terms that vanish in the strong coupling limit, by solving (I) and (II).

Finally, we must see how the Dirac field feeds back through the equations of motion to determine the shape of the surface. We assert that if the bubble shape is such that the total energy is stationary under all local variations of surface geometry, then further corrections to the σ and ψ fields obtained above induce corrections to the energy which vanish in the strong coupling limit. Thus, if the total energy is minimized over bubble configurations, we have a full solution to the coupled field equations within our scheme of approximations.

The formal proof of this assertion relies on methods of differential geometry which have not yet been developed at this point in the exposition. A summary of these methods is contained in Appendix A; the full proof is presented in

Appendix B. Here, we simply sketch the main ideas of the proof.

We have obtained an approximate solution to the field equation for σ,

$$(4) \quad \sigma_0(x) = f \tanh \sqrt{2\lambda}\, f\xi$$

The effects of corrections to σ_0 may be investigated in perturbation theory. For a scalar field which differs slightly from σ_0, $\sigma = \sigma_0 + \delta\sigma$, the shift in the total energy is, to second order in $\delta\sigma$,

$$\Delta H = \int dx \; \tfrac{1}{2}(\nabla\delta\sigma(x))^2 + 2\lambda(3\sigma_0(x)^2 - f^2)\,(\delta\sigma(x))^2$$

$$+ \tfrac{1}{2}\int dx\,dy \;\; K(x,y)\; \delta\sigma(x)\,\delta\sigma(y) - \int dx\, J(x)\,\delta\sigma(x)$$

where

$$J(x) \equiv \;\; \nabla^2\sigma_0(x) + 4\lambda\,\sigma_0(x)\,(f^2 - \sigma_0(x)^2) - G\,\bar{\Psi}(x)\,\Psi(x)$$

$$K(x,y) \equiv \;\; \frac{\delta^2 E}{\delta\sigma(x)\,\delta\sigma(y)}$$

The expression for ΔH includes the first and second order shifts in the exact Dirac energy, E, due to the perturbation of the scalar field. The first order shift is contained in the term $-G\bar{\Psi}\Psi$ in J(x), while the second order shift is represented by the integral over the non-local kernal, K(x,y).

The leading corrections to σ_0 can be estimated by minimizing ΔH over all possible $\delta\sigma$. The "current," J(x), represents the deviation of the approximate solution σ_0 from an exact solution of the coupled equations and acts as a source term for $\delta\sigma$. The solutions to the minimization problem for the quadratic functional, ΔH, may be expressed in terms of the solutions to an associated linear eigenvalue problem. Let the quantities, Λ_β^2, $\Sigma_\beta(x)$, be defined by

$$[-\nabla^2 + 4\lambda(3\sigma_0^2 - f^2)]\,\Sigma_\beta(x) + \int dy\; K(x,y)\,\Sigma_\beta(y) = \Lambda_\beta^2\,\Sigma_\beta(x)$$

$$\int dx \;\; \Sigma_\beta^*(x)\,\Sigma_{\beta'}(x) = \;\; \delta_{\beta,\beta'}$$

where β runs over an index set which may have both continuous and discrete parts. Then we have:

$$(5) \qquad \delta\sigma(x) = \sum_\beta \frac{1}{\Lambda_\beta^2}\; J_\beta\,\Sigma_\beta(x)$$

$$\Delta H = -\frac{1}{2}\sum_\beta \frac{1}{\Lambda_\beta^2}\,|J_\beta|^2$$

where

$$J_\beta \equiv \int dx \;\; \Sigma_\beta^*(x)\,J(x)$$

The spectrum of Λ_β consists of a continuum, with threshold $\sqrt{8\lambda}\,f$, and of possible low Λ "bound states." In Appendix B, we show that the contribution of the continuum states to ΔH vanishes in the strong coupling limit. Thus, finite corrections to the total energy can arise only from the coupling of J(x) to very small Λ, bound state, eigenfunctions. These bound state eigenfunctions must be very sharply peaked in the neighborhood of the bubble surface. As with the Dirac field and the original scalar field, σ_0, they must satisfy the "large," ξ dependent, parts of the wave equation near the surface, independent of the eigenvalue Λ_β.

$$\left[\frac{\partial^2}{\partial\xi^2}+4\lambda(3\sigma_0(x)^2-f^2)\right]\Sigma_\beta(x)=0$$

This is precisely the equation for the translation mode of the one dimensional kink in ξ. The solution is:

$$\Sigma_\beta(x)=g_\beta(u^\alpha)\frac{\partial}{\partial\xi}\sigma_0(\xi)$$

where $g_\beta(u^\alpha)$ is determined by the remaining terms in the eigenvalue equation, which are not so sharply dependent on ξ. Any $\delta\sigma$ which is a superposition of such eigenfunctions corresponds to an infinitesimal variation of the position of the bubble surface in space.

$$\delta\sigma=\delta g(u^\alpha)\frac{\partial}{\partial\xi}\sigma_0(\xi)$$

corresponds to

$$\delta\vec{R}(u^\alpha)=\delta g(u^\alpha)\,\hat{n}(u^\alpha)$$

Thus we have the result that the only corrections to the form of the scalar field, σ_0, that lead to corrections to the energy which are finite in the strong coupling limit correspond to motions of the surface itself, rather than to changes in the shape of the σ field near the surface. Our calculation of the total energy is accurate in the strong coupling limit, then, if and only if the total energy we compute is stationary under all local variations of the

bubble surface.

The total field energy is the sum of the Dirac energy, E, and the energy associated with the σ field configuration. To lowest order in D/R, the σ energy is given by

$$\begin{aligned}E_\sigma&=\int d^3x\ \tfrac{1}{2}(\nabla\sigma)^2+\lambda(\sigma^2-f^2)^2\\&\approx\int da\left[\int d\xi\ \tfrac{1}{2}\left(\frac{\partial\sigma_0}{\partial\xi}\right)^2+\lambda(\sigma_0^2-f^2)^2\right]\\&=C\,a\end{aligned}$$

where $C\equiv\frac{4}{3}\sqrt{2\lambda}\,f^3$

and a =area of bubble

Thus, the σ field energy is simply proportional to the area of the bubble surface, with the combination of the couplings

$$C=\frac{4}{3}\sqrt{2\lambda}\,f^3$$

playing the role of a constant energy density per unit area.

Let us summarize what we have found. We have shown that, in the strong coupling limit, the low-lying static bubble solutions of the BCDWY theory can be described in terms of a highly reduced set of dynamic variables. This reduction can be easily understood intuitively. In the limit of very large coupling constants, only a very special class of solutions exist which retain low energies. The requirement that the energy be small forces these solutions to mimic, locally, the one dimensional kink. The only degrees of freedom that remain are those that describe how these local one dimensional kinks are patched together continuously in three dimensions. These degrees of freedom can be taken to be the surface geometry, which locates the kinks in space, and the surface Dirac field, which defines how the quark is apportioned among kinks.

We have seen that the static field equations can be written in terms of this reduced set of variables as follows:

$$(\mathrm{I}) \quad (\gamma^0 + i\hat{n}\cdot\vec{\alpha})\,\Psi(u^\alpha) = 0$$

$$(\mathrm{II}) \quad (-i\,\vec{\alpha}\cdot\vec{\nabla}_{\parallel} + ik\hat{n}\cdot\vec{\alpha})\,\Psi(u^\alpha) = E\,\Psi(u^\alpha)$$

$$(\mathrm{III}) \quad \delta_{geometry}\,(E + Ca) = 0$$

Although it is still a complicated system of coupled non-linear partial differential equations, this system is a considerable improvement over the original field equations. In Chapter IV, we will discuss some exact and approximate solutions of these equations and their implications for the physics of excited states in the BCDWY model.

Section B: Bubble Dynamics

We now turn to the general question of non-static bubbles. Given our geometric picture of the low-lying static solutions of the BCDWY field equations as closed surfaces in space, we certainly expect that there should be similar bubble solutions of the general time dependent theory. Indeed, if the theory is to be Lorentz invariant, it must admit moving bubbles which correspond to boosted static bubbles. In addition, it is reasonable to suspect that there may exist bubble states of the theory which can be pictured as rotating and vibrating.

The static bubble is a closed surface in space. A bubble, in general, may be thought of as a hypertube in space-time. Such a hypertube would be infinitely extended in time and a closed surface in space. (Fig 3). The hypertube corresponding to a static bubble would be a time-like cylinder generated by a fixed spacial surface. The generalization of equations (I), (II), and (III) should be a system of equations that relate the local geometric structure of such a surface to the quark fields defined on it.

The general equations of motion of bubbles may be derived from the time dependent BCDWY field equations in much the same way as in the static case. Starting with the assumption that the low-lying states will correspond to bubbles of some shape, one can solve the field equations approximately and derive the restrictions imposed by the requirement that further corrections be small. It is simpler, however, to generalize (I), (II), and (III) directly from the static case, keeping clearly in mind our geometric picture of the bubble as a hypertube imbedded in space-time. We can then readily verify, a posteriori, that the bubbles so obtained generate self-consistent approximate solutions to the original field equations in the strong coupling limit.

We expect the general equations to be Lorentz invariant and local on the hypertube. They should give the static equations as a special case. Equations (I) and (II) have immediate and natural generalizations to the non-static case which satisfy these requirements. At each point, the bubble surface has a space-like unit outward normal vector, n^{ν}. In the static case, $n^{\nu}=(0,\hat{n})$. Equation (I) is :

$$(\mathrm{I}) \qquad i\not{n}\,\psi=\psi$$

We take this equation over, in the same form, to the general case. Equation (II) for the static bubble can be re-written:

$$(i\not{\partial}_{\parallel}+ik\not{n})\left[\psi(u^{a})e^{-iE\tau}\right]=0$$

where:

$$(\partial_{\parallel})_{\nu}=\left(\frac{\partial}{\partial t},\vec{\nabla}_{\parallel}\right)$$ = tangential gradient to static hypertube

and

$$k\equiv\tfrac{1}{2}(\partial_{\parallel})_{\nu}(n^{\nu})$$

In the general case, we write:

$$(\mathrm{II}) \qquad (i\not{\partial}_{\parallel}+ki\not{n})\psi=0$$

where

$(\partial_{\parallel})_{\nu}$ =tangential gradient to hypertube

and

$$k\equiv\tfrac{1}{2}\,(\partial_{\parallel})_{\nu}n^{\nu}$$

On the moving bubble, of course, the Dirac field no longer need have only a simple exponential phase dependence on time.

We must generalize (III). Equation (III) is presently stated as a variational principle: that the total energy be stationary under arbitrary variations of the static surface geometry. The natural extension of such a variational principle for the static energy is to an action principle in the more general case. The static bubble has a constant energy density, C, associated with its surface. We take the generalization of this to be that the general bubble have a constant action density, -C, associated with its hypersurface. The contribution of the Dirac field to the action should be such that the Dirac equation (II) follows from the variation of the action with respect to Ψ. Thus, we are led to the following system:

$$(6)\qquad \begin{array}{ll} (\mathrm{I}) & i\not{\kappa}\psi=\psi \\ (\mathrm{II}),(\mathrm{III}) & \delta\int da\left[\bar{\Psi}(i\gamma_{\shortparallel}+ik\not{\kappa})\psi-c\right]=0 \end{array}$$

where

da =invariant element of "area" on the hypertube

The variation in (6) is to be carried out over both Ψ and the geometric variables which define the bubble surface.

In order to proceed further with the analysis of these equations and the physics they represent, we must develop a more economical and mathematically precise language with which to describe the geometry of hypersurfaces. The required language is that of differential geometry; in particular, that of the differential geometry of a time-like surface imbedded in a Minkowski space of one higher dimension. In the following brief discussion, we give the basic mathematical definitions and geometric concepts that will be needed. Proofs of some of the results and a more detailed mathematical discussion have been relegated to

Appendix A.

The surfaces whose geometry is of interest are n-1 dimensional hypertubes imbedded in n dimensional Minkowski space. Our initial physical model has assumed n=4, but as we shall see, the theory is sensible for other values of n. The internal geometry of such a surface is induced by its imbedding in Minkowski space. We can represent the surface by giving the coordinates of its points as functions of n-1 "internal" coordinates, $\{u^\alpha\}$:

Surface:

$$X^\mu = R^\mu(u^\alpha)$$

Our notation will be such that the greek letters $\alpha,\beta,\gamma,\delta$ run from 0,...,n-2 while μ,ν,λ,σ run from 0,...,n-1. The choice of internal coordinates is arbitrary. The geometric quantities we will be most concerned with will therefore be tensors whose indices correspond to these internal coordinates. The equations we will write will be tensor equations that are manifestly "covariant" under general coordinate transformations.

The fundamental tensors which describe the surface are as follows:

Tangent Vectors:

$$\tau_\alpha{}^\mu = \frac{\partial R^\mu}{\partial u^\alpha}$$

Induced Metric:

$$g_{\alpha\beta} = \tau_\alpha\cdot\tau_\beta = \tau_\alpha{}^\mu\,\tau_{\beta\mu}$$

<u>Outward</u> <u>Unit</u> <u>Normal</u>: $n^\mu(u^\alpha)$: $n\cdot\tau_\alpha = 0, \quad n^2 = -1$

<u>Coefficients</u> <u>of</u> <u>Curvature</u>:

$$h_{\alpha\beta} = -n\cdot\tau_{\alpha|\beta} = n_{|\alpha}\cdot\tau_\beta = h_{\beta\alpha}$$

where we use the notation:

$$A_{|\alpha} = \frac{\partial A}{\partial u^\alpha}$$

for any quantity, A.

The induced metric tensor $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$ will be used, in the usual way, to transform between the covariant and contravariant forms of tensors. This metric is "induced" in the following sense: if V^μ is a tangent vector,

$$V^\mu = V^\alpha \tau_\alpha{}^\mu$$

the length of V^μ in Minkowski space can be written in terms of its components

$$V^\mu V_\mu = (V^\alpha \tau_\alpha{}^\mu)(V^\beta \tau_{\beta\mu}) = g_{\alpha\beta} V^\alpha V^\beta = V^\alpha V_\alpha$$

The invariant element of "area" on the surface is

$$da = d^{n-1}u\,\sqrt{|g|} \quad , \quad g \equiv \det(g_{\alpha\beta})$$

The n vectors $\{\tau_\alpha{}^\mu, n^\mu\}$ form a local "n-bien" in terms of which any Minkowski vector can be expanded:

$$(\tau^\alpha)^\mu(\tau_\alpha)^\nu - n^\mu n^\nu = \eta^{\mu\nu} = \text{Minkowski Metric}$$

The tensor $h_{\alpha\beta}$, sometimes called the "second fundamental form," describes the local curvature of the surface. At any point, the principal values of $h^\alpha{}_\beta$ are the reciprocal radii of curvature of the surface. For a time-like direction, this reciprocal radius of curvature is proportional to the normal acceleration of the corresponding spacial surface at the point. The quantity k, which we have introduced already, is:

$$k = \tfrac{1}{2}(\partial_{||})_\mu n^\mu = \tfrac{1}{2}(\tau^\alpha{}_\mu)\,\partial_\alpha n^\mu = \tfrac{1}{2} h^\alpha{}_\alpha$$

Thus, k is proportional to the mean curvature of the surface at each point.

The flat Minkowski space induces natural laws of parallel transport along the surface for both vectors and spinors. For a coordinate shift δu^γ, these are:

<u>Vectors</u>: $$\delta V^\alpha = -\left\{{\alpha \atop \beta\gamma}\right\} V^\beta \delta u^\gamma$$

where the "Christoffel symbol" is:

$$\left\{{\alpha \atop \beta\gamma}\right\} = \tfrac{1}{2} g^{\alpha\delta}\left[g_{\beta\delta|\gamma} + g_{\delta\gamma|\beta} - g_{\beta\gamma|\delta}\right]$$

<u>Spinors</u>:

$$\delta\psi = -\tfrac{i}{2}\sigma^{\mu\nu} n_\mu n_{\nu|\gamma}\,\psi\,\delta u^\gamma$$

The parallel transport law for spinors is just such that the quantity, $\bar\psi\gamma^\alpha\psi$, parallel transports as a vector. These give corresponding "covariant derivatives" for vectors and spinors:

$$V^\alpha{}_{\|\beta} = V^\alpha{}_{|\beta} + \left\{ {\alpha \atop \beta\gamma} \right\} V^\gamma$$

$$D_\alpha \psi = \left[\partial_\alpha + \frac{i}{2} \sigma^{\mu\nu} n_\mu n_{\nu|\alpha} \right] \psi$$

A little algebra gives the following relations, which will be of some use to us later:

$$\not{D} = \gamma^\mu (\tau^\alpha{}_\mu) D_\alpha = \not{\partial}_{\|} + k \not{n}$$

$$\not{D} \not{n} = - \not{n} \not{D}$$

$$\tau_{\alpha|\beta} = \left\{ {\gamma \atop \alpha\beta} \right\} \tau_\gamma + h_{\alpha\beta} n$$

$$\tau_{\alpha\|\beta} = h_{\alpha\beta} n$$

$$V^\alpha{}_{\|\alpha} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\, V^\alpha \right)_{|\alpha} \quad , \text{ for any } V^\alpha$$

We are now equipped to continue our analysis of the equations of motion (6).

We re-write the equations of motion:

$$(\mathrm{I}) \quad i \not{n} \psi = \psi$$

$$(\mathrm{II}) \quad i \not{D} \psi = 0$$

$$(\mathrm{III}) \quad \delta_{\text{geometry}} \int du \sqrt{|g|}\; L = 0$$

where:

$$L \equiv \bar{\Psi} i \not{D} \psi - C$$

The Dirac equation (II) has a clear interpretation as that of a free massless fermion confined to a curved surface. The equation of constraint (I) on the Dirac field is consistent with the equation of motion (II) by virtue of relation (7).

The equation of motion that arises from the variation in (III) is now straightforward to derive. The calculation is somewhat long, however, and is presented in Appendix C. The result is that, under the variation

$$R^\mu(u^\alpha) \longrightarrow R^\mu(u^\alpha) + \delta R^\mu(u^\alpha)$$

after using (I) and (II),

$$\frac{1}{\sqrt{|g|}} \delta\left(\sqrt{|g|}\, L \right) = - T^{\alpha\beta} \left(\tau_\beta \cdot \delta R_{|\alpha} \right)$$

where

$$T^{\alpha\beta} \equiv C g^{\alpha\beta} - \mathrm{Im}\, \bar{\Psi} \gamma^\alpha \partial^\beta \psi$$

As we shall see presently, $T^{\alpha\beta}$ is the canonical energy-momentum tensor of the bubble.

The corresponding equation of motion is

$$0 = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\; T^{\alpha\beta} \tau_\beta{}^\mu \right)_{|\alpha} = T^{\alpha\beta}{}_{\|\alpha}\, \tau_\beta{}^\mu + h_{\alpha\beta} T^{\alpha\beta} n^\mu$$

The tangential component of this equation, $T^{\alpha\beta}{}_{\|\alpha}=0$, is trivial and follows from (II). This simply reflects the fact that an infinitesimal tangential variation of $R^{\mu}(u^{\alpha})$ is equivalent to an infinitesimal coordinate transformation -- the surface itself is unchanged. The non-trivial normal component of this equation provides us with the third equation of motion in local form:

$$\text{(III)} \qquad h_{\alpha\beta}T^{\alpha\beta}=0$$

Accepting, for the moment, that $T^{\alpha\beta}$ is the energy-momentum tensor of the theory, this equation has a simple physical interpretation. Schematically, (III) is

$$0 = \sum_{i} \frac{T(i)}{R(i)}$$

where (i) runs over the principal directions of curvature, R(i) is a radius of curvature, and T(i) is the corresponding diagonal element of the stress tensor. For each (i) that corresponds to a spacial component, T(i)/R(i) is just the contribution of that component of surface stress to the normal force density of the spacial surface. For the time-like component, T(i)/R(i) is proportional to the product of the energy density and the normal acceleration of the surface. Thus, (III) is nothing more than Newton's Second Law for the case of a relativistic hypersurface.

The proof that a solution to (I), (II), and (III)

generates an approximate solution to the field equations proceeds in the same fashion as in the static case. We introduce a space-time coordinate system in a neighborhood of the hypertube:

$$X^{\mu}(u^{\alpha},\xi) = R^{\mu}(u^{\alpha}) + \xi\, n^{\mu}(u^{\alpha})$$

In terms of these coordinates, the fields can be written

$$\sigma(x) = f \tanh \sqrt{2\lambda} f \xi$$

$$\psi(x) = N\left[\cosh \sqrt{2\lambda} f\xi\right]^{-\frac{G}{\sqrt{2\lambda}}} \quad \psi(u^{\alpha},\xi)$$

where: $\frac{\partial}{\partial\xi}\psi(u^{\alpha},\xi) = -k\,\psi(u^{\alpha},\xi)$

$$\psi(u^{\alpha},\xi=0) = \psi(u^{\alpha}) = \text{surface field}$$

We recall from the static theory that there are essentially two conditions that must be satisfied in order for a bubble to give a good approximate solution of the field equations. The first is that the normal dependence of the fields be exactly that in equation (8). This insures that the state is a low-lying one relative to the bare particle masses. The second is that the geometry of the bubble surface be so chosen that the correction terms generated from the fields of (8) decouple locally from the normal translation modes of the fields. That this second condition is satisfied for the solutions of the general theory follows, almost trivially, from the following observation: The second condition is equivalent to the requirement that the action be stationary up to terms of order D/R for the fields (8). The original action,

$$S=\int dx \left[\bar{\Psi}(i \not{\partial}-G\sigma)\Psi + \tfrac{1}{2}(\partial\sigma)^2 - \lambda(\sigma^2-f^2)^2 \right]$$

can be computed approximately in terms of the surface fields. The Lagrangian is very strongly peaked near the bubble surface. The integral over ξ in the action can be carried out to lowest order in the D/R (see Appendix D). One finds

$$S \approx \int_{\text{hypertube}} du\sqrt{|g|}\ \left[\bar{\Psi} i \not{D} \Psi - C \right]$$

This is precisely the surface action functional used to generate the equations of motion (I), (II), and (III). Thus, the condition is automatically the satisfied for solutions of the bubble equations of motion derived from this action principle.

In the static theory, we showed that the energies of bubbles could be computed in terms of surface field variables alone. It should come as no suprise that, in the general case, all conserved currents and their charges can be expressed completely in terms of these variables in the strong coupling limit. There are two ways this can be demonstrated. One is by going back to the expressions for

the momentum, angular momentum, and charge of the system as integrals over the fields. We note that the densities of these quantities are strongly peaked in the neighborhood of the bubble surface. The integrals over ξ can be done to lowest order in D/R. The remaining expressions give the conserved charges in terms of integrals over the hypertube. These calculations are presented in Appendix D.

Another approach is to derive the conserved charges directly from the action via Noether's theorem. We take this approach in the following discussion. If the Lagrangian density, $\sqrt{|g|}\,L$, is invariant under some transformation

$$R^{\mu} \longrightarrow R^{\mu} + \delta R^{\mu}$$

$$\Psi \longrightarrow \Psi + \delta\Psi$$

then the current,

$$\delta K^{\alpha} \equiv -\frac{1}{\sqrt{|g|}}\left[\frac{\partial(\sqrt{|g|}L)}{\partial \Psi_{|\alpha}}\delta\Psi + \frac{\partial(\sqrt{|g|}L)}{\partial \bar{\Psi}_{|\alpha}}\delta\bar{\Psi} + \frac{\partial(\sqrt{|g|}L)}{\partial \tau_{\alpha}{}^{\mu}}\delta R_{\mu} \right]$$

is conserved:

$$0 = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\,\delta K^{\alpha}\right)_{|\alpha} = \delta K^{\alpha}{}_{\|\alpha}$$

After some algebra, we find that this current can be written

$$\delta K^{\alpha} = T^{\alpha\beta}\tau_{\beta}{}^{\mu}\delta R_{\mu} - \frac{i}{2}\left[\bar{\Psi}\Gamma^{\alpha}\delta\Psi - \delta\bar{\Psi}\Gamma^{\alpha}\Psi \right]$$

The following are the symmetries, currents, and charges of the theory:

Fermion Number: $\delta R^\mu = 0$, $\delta\Psi = -i\,\delta\theta\,\Psi$

$$\delta K^\alpha \equiv J^\alpha \delta\theta = \bar\Psi \mathcal{Z}^\alpha \delta\theta$$

$$J^\alpha = \bar\Psi \mathcal{Z}^\alpha \Psi$$

$$Q = \int d\Sigma_\alpha \sqrt{|g|}\; J^\alpha$$

Energy-momentum: $\delta R^\mu = \delta a^\mu = \text{constant}$, $\delta\Psi = 0$

$$\delta K^\alpha \equiv \mathcal{T}^{\alpha\mu}\delta a_\mu = T^{\alpha\beta}(\tau_\beta)^\mu \delta a_\mu$$

$$\mathcal{T}^{\alpha\mu} = T^{\alpha\beta}\tau_\beta{}^\mu$$

$$P^\mu = \int d\Sigma_\alpha \sqrt{|g|}\; T^{\alpha\beta}\tau_\beta{}^\mu$$

Lorentz Rotations: $\delta R^\mu = \delta\omega^\mu{}_\nu R^\nu$, $\delta\Psi = -\frac{i}{4}\delta\omega^{\mu\nu}\sigma_{\mu\nu}\Psi$

$$\delta K^\alpha \equiv \tfrac{1}{2}\delta\omega_{\mu\nu} M^{\alpha\mu\nu}$$
$$= \tfrac{1}{2}\delta\omega_{\mu\nu}\left[R^\mu \mathcal{T}^{\alpha\nu} - R^\nu \mathcal{T}^{\alpha\mu} + \tfrac{1}{4}\bar\Psi\{\mathcal{Z}^\alpha, \sigma^{\mu\nu}\}\Psi\right]$$

$$M^{\alpha\mu\nu} = R^\mu \mathcal{T}^{\alpha\nu} - R^\nu \mathcal{T}^{\alpha\mu} + \tfrac{1}{4}\bar\Psi\{\mathcal{Z}^\alpha, \sigma^{\mu\nu}\}\Psi$$

$$M^{\mu\nu} = \int d\Sigma_\alpha \sqrt{|g|}\; M^{\alpha\mu\nu}$$

The integrals above are to be taken over any closed space-like submanifold ("space-like cut") of the hypertube

(Fig 4). The differential, $d\Sigma_\alpha$ is the oriented element of area defined by:

$$d\Sigma_{(\alpha)} \wedge du^{(\alpha)} = d^{n-1}u \quad (\text{no sum on } \alpha)$$

The theory we have developed is manifestly Lorentz invariant and generally covariant. Mathematically, this is a trivial consequence of the fact that all quantities are represented as tensors under Lorentz transformations and under internal coordinate transformations. We note that the spinor Ψ is a spinor only in Minkowski space; it is a scalar with respect to surface coordinate transformations. One immediate consequence of Lorentz invariance is that static solutions, which have zero spacial momentum, correspond to particles of mass equal to their energy.

We observe that the conserved currents are tangential to the surface at each point. This is a physically and mathematically sensible result. If a current had a normal component, one would hardly expect that its charge could be conserved on the surface. Mathematically, only a tangential current can be integrated over a space-like cut to produce a coordinate invariant result. The condition which insures that the conserved currents are tangential is equation (1). This equation of constraint on the Dirac field severely restricts the possible fermionic currents that can be constructed. Essentially, we have a two-component fermion.

From (I) and the relation $\{i\not n, \not e^{\alpha}\}=0$, we have

$$\bar\Psi \not e^{\alpha_1} \cdots \not e^{\alpha_n} \Psi = 0 \qquad \text{if } n \text{ is even}$$

$$\bar\Psi \not e^{\alpha_1} \cdots \not e^{\alpha_n} \gamma_5 \Psi = 0 \qquad \text{if } n \text{ is odd}$$

Thus (I) guarantees that the usual fermion current agrees with the Noether current derived above:

$$\bar\Psi \gamma^{\mu} \Psi = \bar\Psi [\not e^{\alpha} \tau_{\alpha}{}^{\mu} - \not n\, n^{\mu}] \Psi = J^{\alpha} \tau_{\alpha}{}^{\mu}$$

In contrast, the "axial current" $\bar\Psi \gamma^{\mu} \gamma_5 \Psi$ is purely normal:

$$\bar\Psi \gamma^{\mu} \gamma_5 \Psi = \bar\Psi [\not e^{\alpha} \tau_{\alpha}{}^{\mu} - \not n\, n^{\mu}] \gamma_5 \Psi = [\bar\Psi(-i\gamma_5)\Psi] n^{\mu}$$

This axial current cannot be "conserved" in any sense in this theory, nor can a Lorentz and coordinate invariant integral over it even be defined. Every current constructed from the Dirac field can be expressed in terms of the vector and pseudo-scalar currents. These expressions are given for the standard currents in Table I.

The generalization of the bubble equations to the case of several quark species is completely straightforward. Each quark field appears in the action separately.

$$S = \int du \sqrt{|g|} \left(\sum_a \bar\Psi_a i \not D \Psi_a - C \right)$$

Therefore, each quark field obeys the equations of motion (I) and (II), while the fermion contribution to the stress

tensor in (III) is the sum over all species. As in the static case, there are no direct quark-quark interactions in the low-lying color singlet states.

We have now developed a complete semi-classical theory of bubble dynamics. The strong coupling limit of the BCDWY field theory has been taken, leaving a theory of extended geometric objects upon which quarks are permanently confined. The theory has been shown to be Lorentz invariant, and all conserved currents and charges have been constructed. The Lagrangian of this theory looks exactly like that of the MIT bag model. The crucial difference is that our action arises as an integral over a hypertube imbedded in a higher dimensional space. Such an imbedding is non-trivial, so that the geometric degrees of freedom of the surface are dynamic variables. In the following chapters, two principal questions will concern us. First, what can we learn from this theory about the spectrum of states in the four dimensional BCDWY model? This question will not be answered in a general way, but we will develop a clearer physical picture of the properties of the states. The second question is how can we go beyond the semi-classical theory to a true quantum theory of bubbles? We discuss general features we might expect in the quantized theory. Further, it will be shown that the theory can be explicitly quantized in three space-time dimensions. This quantum theory will prove instructive as to the general quantum mechanical problem.

Chapter IV

In the preceding chapters, we have developed a simple and elegant geometric formulation of the BCDWY theory in the strong coupling limit. This formulation is, however, of little practical value unless it facilitates our understanding the physical properties of the model. In this chapter, we use the formalism we have developed to discuss the spectrum of hadronic states predicted by the BCDWY model. The problem of computing the exact spectrum of the theory in four space-time dimensions is a formidable one and remains unsolved. In the following analysis, we make use of various "approximations" in order to render the problem tractable. What emerges is not a numerical tabulation of hadronic masses, but rather, we hope, a clearer picture of the physical characteristics of hadronic states.

We shall see that the most striking property of bubbles is their softness: a bubble can suffer extreme deformations of shape at very little cost in energy. This fact has far reaching implications in our model. Not least among these is that hadronic states, though pictured classically as thin shells, must necessarily be very smeared out by the quantum fluctuations of their surfaces. Thus, for example, the thin shell picture does not require that hadronic form factors be oscillatory as are the fourier transforms of rigid thin shells. Further, the softness of bubbles affords us some insight into how scaling might occur in this model. From the equations of motion, it is clear that quarks move freely within the surface. Because the bubble surface may easily be deformed, a quark trapped on it is nearly free to move short distances in the normal direction by dragging the surface along with it. The "softness" of the bubble is simply the statement that the energy required to deform the bubble surface is on the order of a few tenths of the total bubble energy. Thus, it need not be suprising that quarks can appear to be nearly free particles at momentum transfers on the order of a few hadron masses. One might expect that this would be reflected in deep inelastic lepton scattering as "precocious" scaling.

In Section A, we discuss the relationship of the semi-classical and quantum spectra of the theory. In Section B, we discuss the semi-classical spectrum of static single quark states in two and three dimensions. We discover that, though excited bubble states are highly non-spherical, their energies are not very different than the energies estimated for corresponding states on a rigid sphere. In Section C, we consider multi-quark bubbles and discuss qualitatively the excited hadronic states expected in the model. Section D is devoted to the analysis of a

three dimensional spherically sysmetric surface excitation of the ground state bubble. This radial mode is found to have a mass enticingly close to that of the Roper resonance.

Section A: The Semi-Classical Approximation

The calculations presented in this chapter are based on the semi-classical theory of bubbles derived in Chapter III. Before proceeding, it is essential to understand, as best as we can, the relationship of the semi-classical and quantum theories.

What we should be computing is the spectrum of states in a quantum theory of bubbles which corresponds to the classical theory defined by the action III.(6). At present, we have given no prescription for constructing such a theory canonically, though in the next chapter we shall see that such a quantum theory can be constructed explicitly in three space-time dimensions. We assume in this discussion that a full quantum theory does, in fact, exist. The states of this quantum theory will, presumably, transform as particle representations of the Poincaré group. The mass spectrum of particles will be discrete. The particle eigenstates will be states of definite momentum which are completely delocalized in space.

In the full quantum theory, both the Dirac field and the surface variables, $R^{\mu}(u^{\alpha})$, will be "quantized." But, in the semi-classical theory, only the quantum nature of the Dirac field is taken into account; the bubble surface is treated purely classically. An immediate consequence is that semi-classical states of definite energy and momentum are completely localized surfaces in space. A potentially more troubling aspect of the semi-classical "approximation" is that, because the surface variables are classical, the spectrum of surface excitations is continuous. A third quantum effect that is neglected semi-classically is the effect on the spectrum of quantum fluctuations (zero-point motion) of the surface and of the filled negative energy Dirac sea. In order to use the semi-classical approximation to discuss the spectrum, all these effects must be understood, or at least appreciated.

The zero-point energy and sum over the negative energy Dirac sea will each have a divergent term proportional to the area of the bubble. Because they are proportional to the area, these terms can be considered as a renormalization of the bubble constant C. There may be finite corrections which remain after this divergent term is removed. Such corrections will not be taken into account numerically in

the analysis of this chapter.

The states of the semi-classical theory are localized in space. We expect quantum fluctuations to smear them out. We might guess that this smearing would be significant only over states of nearby classical energies. That such is the case is suggested, for example, by the path integral formulation of quantum mechanics. The paths that will be most important are those along which the action does not differ greatly from the action along a closed classical orbit. Thus, in the case of a very soft object like the bubble, we can expect that the true quantum states will bear little or no resemblance to any single classical surface we might compute. Can we, then, reasonably expect that a semi-classical estimate of the energy be realistic? The answer is both yes and no. We cannot suppose that a semi-classical approximation includes all the terms which contribute to the energy. But, because the bubble is soft, we may hope that the effects of quantum fluctuations, though large, are nearly the same for semi-classical states of similar shapes. In this case, the semi-classical approximation may qualitatively reflect the relative differences between energy levels.

In order to obtain a realistic estimate of the particle masses from the semi-classical theory, we must decide what to do with the continuous spectrum of surface excitations. In the spirit of the preceding discussion, we ignore them in the first part of our analysis, fixing our attention on static bubbles. In the last section of this chapter, we find a spherically symmetric, time-dependent solution to the equations of motion corresponding to a "breathing" mode of the ground state. We quantize this simple surface excitation in the WKB approximation. We see that, indeed, the magnitude of the surface fluctuation is large. The ratio of the mass of this state to that of the ground state bubble is within 3% of the ratio of the mass of Roper resonance to that of the nucleon.

Section B: Static Single Quark Bubbles

We begin, then, by looking for solutions of the static bubble equations. Consider first the case where the bubble contains only a single quark. The static equations are:

$$(\mathrm{I})\quad i\beta\psi=\psi$$

$$(\mathrm{II})\quad (-i\vec{\alpha}\cdot\vec{\nabla}_{\parallel}+k i\,\hat{n}\cdot\vec{\alpha})\psi=E\psi$$

$$(\mathrm{III})\quad \delta[E+Ca]=0 \quad \text{or} \quad h_{\alpha\beta}T^{\alpha\beta}=0$$

For a static surface, the geometric formalism introduced in Chapter III simplifies considerably. Taking internal coordinates $u^0=\tau, u^1, u^2$, we have

$$R^{\mu}(\tau,u^{\alpha})=(\tau,\vec{R}(u^{a}))\,,\quad \tau_0{}^{\mu}=(1,\vec{0})\,,\quad \tau_a{}^{\mu}=(0,\vec{\tau}_a)$$

$$g_{\alpha\beta}=\begin{pmatrix}1&0&0\\0&\multicolumn{2}{c}{-g_{ab}}\\0&&\end{pmatrix}\qquad g_{ab}\equiv\vec{\tau}_a\cdot\vec{\tau}_b$$

$$n^{\mu}=(0,\hat{n}(u^{a}))$$

$$h_{\alpha\beta}=\begin{pmatrix}0&0&0\\0&\multicolumn{2}{c}{-h_{ab}}\\0&&\end{pmatrix}\,,\qquad k=\tfrac{1}{2}h^{a}{}_{a}$$

We adopt the notation that a,b,c,d,... refer to space-like indices (1,2), while i,j,k,l... refer to the indices in Euclidean space (1,2,3). The geometric objects which we defined in Chapter III for a time-like surface are defined in the same way, up to a sign, for the spacial surface, $\vec{R}(u^a)$. In particular, g_{ab} is the spacial metric tensor and will be used to raise and lower indices on tensors.

By virtue of equation (I) we can write the Dirac field in terms of a two component spinor, χ :

$$\psi=\frac{1}{\sqrt{2}}\begin{pmatrix}\chi\\ i\hat{n}\cdot\vec{\sigma}\chi\end{pmatrix}$$

where we have used the Dirac representation of the gamma matrices. In terms of χ, the Dirac equation is

$$H\chi = E\chi$$

where the two component Hamiltonian is

$$H = k - i\vec{\sigma}\cdot(\hat{n}\wedge\vec{\nabla}_{\parallel})$$

The conserved currents of the theory can be written in terms of χ :

$$J^0 = \chi^+\chi \quad , \qquad J^a = \chi^+\vec{\sigma}\cdot(\hat{n}\wedge\vec{\tau}^a)\chi$$

$$T^{00} = C + E\chi^+\chi$$

$$T^{0a} = \mathrm{Im}\,(\chi^+\partial^a\chi) + \tfrac{1}{2}h^{ab}J_b$$

$$T^{a0} = -EJ^a$$

$$T^{ab} = -Cg^{ab} + \tfrac{1}{2}h^{ab}\chi^+\chi + \mathrm{Im}\,\chi^+\vec{\sigma}\cdot(\hat{n}\wedge\vec{\tau}^a)\partial^b\chi$$

The normalization of χ is

$$Q = \int du\sqrt{|g|}\;\chi^+\chi = 1$$

so the total energy is

$$U = \int du\sqrt{|g|}\;T^{00} = E + Ca$$

As we expect, the total spacial momentum can be shown to be zero:

$$P^i = \int du\sqrt{|g|}\;T^{0i}$$

but:

$$\begin{aligned} 0 &= (M^{a0i})_{\parallel a} \\ &= \frac{\partial}{\partial\tau}\left[\tau T^{0i} - R^i T^{00} + \tfrac{1}{4}\bar{\Psi}\{\Sigma^0, \sigma^{0i}\}\Psi\right] + M^{a0i}{}_{\parallel a} \\ &= T^{0i} + M^{a0i}{}_{\parallel a} = T^{0i} + \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\,M^{a0i}\right)_{|a} \end{aligned}$$

Thus: $\quad P^i = \int du\,\left(-\sqrt{|g|}\,M^{a0i}\right)_{|a} = 0$

Finally, we can write the local form of the energy variation condition

$$2ck = \tfrac{1}{2}(h^{ab}h_{ab})\chi^+\chi + h_{ab}\,\mathrm{Im}\,\chi^+\vec{\sigma}\cdot(\hat{n}\wedge\vec{\tau}^a)\partial^b\chi$$

The system of coupled equations (II), (III) is very difficult to solve exactly or approximately in three dimensions. Before attacking the three dimensional problem, it is instructive to consider the two dimensional case, where we find an exact general solution is available.

In two space dimensions, the bubble is a closed curve in the x-y plane (Fig 5). We can choose the single parameter describing this curve as its length

$$\vec{R} = \vec{R}(\ell)$$

$$\hat{e} = \frac{d\vec{R}}{d\ell} = \text{unit vector} \qquad \hat{n} \wedge \hat{e} = \hat{z}$$

The curvature is then

$$k = \frac{1}{2}\hat{e} \cdot \frac{d\hat{n}}{d\ell} = \frac{1}{2}\frac{d\Phi}{d\ell}$$

where Φ is the angle of the normal with respect to some fixed direction in the plane (Fig 6).

The Dirac equation is

$$\left[\frac{1}{2}\frac{d\Phi}{d\ell} - i\sigma_3 \frac{d}{d\ell}\right]\chi = E\chi$$

which may be integrated immediately to yield

$$\chi(\ell) = e^{i\sigma_3\left[E\ell - \frac{1}{2}(\Phi(\ell) - \Phi(0))\right]}\,\chi(0)$$

χ must be single valued, so we have

$$\chi(L) = \chi(0) \qquad \text{where } L = \text{total length}$$

or

$$2n\pi = EL - \frac{1}{2}\left[\Phi(L) - \Phi(0)\right] = EL - \pi$$

where n is an integer. The Dirac energy is

$$E = \frac{2\pi m}{L} \quad , \quad m \equiv n + \frac{1}{2}$$

and the normalized Dirac wave-function can be written

$$\chi = \frac{1}{\sqrt{L}}\, e^{i\sigma_3\left(E\ell - \frac{1}{2}\Phi(\ell)\right)}\, u$$

where u is a fixed unit spinor.

The Dirac energy depends only on the perimeter of the bubble, L, not on its shape. There are paired positive and negative energy levels of the same magnitude. There is no zero energy mode. These results can be readily understood geometrically. The static Dirac equation on the two dimensional bubble is just the equation for a spinor which is parallel transported around a closed curve, up to a phase, $e^{i\sigma_3 E\ell}$. On a one dimensional manifold, there can be no intrinsic curvature. From the point of view of a quark trapped on a one dimensional curve, the geometry in the neighborhood of any one point is equivalent to the geometry in the neighborhood of any other point. This leads to a "translation" invariance along the curve. For spinors, this translation is realized by parallel transport, under which the spinor changes only in phase. Because the quark has spin 1/2, transport around a closed path induces a phase change of π , which must be compensated by the factor EL. Hence, the energy cannot vanish.

We interpret negative energy quark states as positive energy anti-quarks. The total bubble energy is, then,

$$U = \frac{2\pi|m|}{L} + CL$$

Minimizing over L, we have

$$L = \left[\frac{2\pi |m|}{C}\right]^{\frac{1}{2}}$$

$$U = (8\pi C)^{1/2} |m|^{1/2}$$

It is straightforward to check that, if L is chosen to minimize U as above, equation (III) is satisfied at each point on the bubble surface.

The two dimensional bubble is, then, extremely soft. Static bubble states occur only with perimeters fixed by the Dirac quantum number m; but bubbles of all shapes with this perimeter are degenerate classically. In Chapter V, we will see that the spectrum of the full quantum theory does not have such an infinite degeneracy. The reflection of the bubble's softness there lies in the large quantum fluctuations of the surface. We shall see that the three dimensional bubble is also soft, but not so soft that all shapes are degenerate.

We note that there is one quantity which does depend on the bubble shape. This is the angular momentum, J_3,

$$J_3 \equiv M^{12} = \int d\ell \; [R^1 T^{0(2)} - R^2 T^{0(1)}]$$

where (1) and (2) refer to a spacial index, i.

$$T^{0i} = \mathrm{Im}\left[\frac{u^+}{\sqrt{L}} \; i\sigma_3 (E - k) \frac{u}{\sqrt{L}}\right] + \frac{1}{2}(2k) u^+ \sigma_3 u$$

$$= \frac{E}{L} \langle \sigma_3 \rangle \quad , \text{where} \quad \langle \sigma_3 \rangle \equiv u^+ \sigma_3 u$$

Then

$$J_3 = \frac{E}{L} \langle \sigma_3 \rangle \int d\ell \; [\vec{R} \wedge \hat{e}]_3 = \frac{E}{L} \langle \sigma_3 \rangle A$$

where A is the total area of the bubble, and, of course, depends on its shape. Using the expression for E, we can re-write this result:

$$J_3 = |m| \langle \sigma_3 \rangle \left[\frac{A}{\pi (\frac{L}{2\pi})^2}\right]$$

or

$$J_3 = (8\pi C)^{-1} U^2 \langle \sigma_3 \rangle \quad \left[\frac{A}{\pi (\frac{L}{2\pi})^2}\right]$$

The ratio $A/[\pi (\frac{L}{2\pi})^2]$ is the ratio of the area of the bubble to the maximum area it could have, given perimeter L. The state of maximum area is a circle, which is unique. Thus, the maximum possible angular momentum of a state of energy U is

$$J_{3\,max}(U^2) = (8\pi C)^{-1} U^2$$

In a Regge picture, this is the statement that the leading Regge trajectory is non-degenerate, and linear in (mass)^2 with slope $(8\pi C)^{-1}$.

Unfortunately, the static bubble equations in three dimensions are not so easily solved. The only known exact solution is a spherically symmetric one corresponding to the approximate solution of the field equations found by BCDWY. It is simply a very difficult technical problem to simultaneously solve the Dirac equation and satisfy the condition that the total energy be minimal under local variations of the surface. In principle, however, we can find all solutions to the static equations as follows: (1) Solve the Dirac equation exactly on a general closed spacial surface, $\vec{R}(u^\alpha)$. Because the surface is compact, the Dirac spectrum is discrete and the energy levels can be labelled by two discrete parameters, M_1, M_2. These Dirac energies will be continuous functionals of the surface variables: $E_{M_1 M_2}[\vec{R}(u^\alpha)]$. (2) Choose which levels are to be occupied by quarks or anti-quarks. (3) Minimize the total energy functional,

$$U[\vec{R}(u^\alpha)] = C\,a[\vec{R}(u^\alpha)] + \sum_{\substack{\text{occupied}\\ \text{levels}}} E_{M_1 M_2}[\vec{R}(u^\alpha)]$$

in the space of functions $\vec{R}(u^\alpha)$.

Such a procedure is much too difficult to be carried out in practice. It suggests, however, a practical scheme for finding the energy levels approximately. Namely, we attempt to carry out the above procedure, not on a general surface, but over a class of surfaces sufficiently limited that the Dirac equation is tractable. We will choose a form for the bubble surface that depends on several real

parameters, solve for the Dirac energy as a function of these parameters, then minimize the total energy over the parameters that define the surface. Because the total energy functional is positive definite, this variational estimate of the energy is always an upper bound on the energy of the lowest bubble state with the assumed Dirac quantum numbers M_1, M_2. The accuracy of such a variational estimate depends entirely on whether the trial surfaces we consider are sufficiently "near" the true solution. This in turn depends, as a practical matter, on how well we understand the character of the distortions of the excited states of the theory.

We begin by considering the simplest possible trial surface-- a sphere. We re-derive the results of the BCDWY approximate solution of the static field equations, now expressed in the geometric language of bubble theory. Let the sphere have radius R and be coordinatized by the usual polar angles θ, φ. Then we have

$$g_{ab} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \qquad \hat{n} = \hat{r}(\theta, \varphi)$$

$$h_{ab} = \frac{1}{R} g_{ab} \qquad \mathcal{K} = \frac{1}{R}$$

$$\vec{\nabla}_{\parallel} = \frac{1}{R}\left(\hat{\theta} \frac{\partial}{\partial\theta} + \frac{\hat{\varphi}}{\sin\theta} \frac{\partial}{\partial\varphi} \right)$$

The two component Dirac Hamiltonian is

$$H = \frac{1}{R} - i\vec{\sigma}\cdot(\hat{r}\wedge\vec{\nabla}_{11}) = \frac{1}{R}(1+\vec{L}\cdot\vec{\sigma})$$

its normalized eigenfunctions and eigenvalues are

$$\chi = \frac{1}{R}\varphi^{\ell}_{jm}$$

$$E = \begin{cases} \frac{j+\frac{1}{2}}{R} & \text{if } j=\ell+\frac{1}{2} \\ -\frac{j+\frac{1}{2}}{R} & \text{if } j=\ell-\frac{1}{2} \end{cases}$$

We interpret states with $j=\ell+1/2$ as quarks, those with $j=\ell-1/2$ as anti-quarks. The total energy is :

$$U = \frac{j+\frac{1}{2}}{R} + C\,4\pi R^2$$

Minimizing over the parameter R, we have

$$R = (8\pi C)^{-\frac{1}{3}}(j+\tfrac{1}{2})^{1/3} \equiv R_0\,(j+\tfrac{1}{2})^{1/3}$$

$$U = \frac{3}{2R_0}(j+\tfrac{1}{2})^{2/3}$$

This gives the best approximation to the energy of single quark states with the quantum numbers (j,m) over spherical surfaces.

The local equation for the minimization of the total energy is

$$0 = -h_{\alpha\beta}T^{\alpha\beta} = h_{ab}\,T^{ab} \equiv F$$

F = outward normal force density

but

$$F = -2CK + \tfrac{1}{2}(h^{ab}h_{ab})\chi^+\chi + h_{ab}\,\mathrm{Im}\,\chi^+\vec{\sigma}\cdot(\hat{n}\wedge\vec{\tau}^{a})\partial^b\chi$$
$$= -\frac{2C}{R} + \frac{1}{R^2}\left|\frac{1}{R}\varphi^{\ell}_{jm}\right|^2 + \frac{1}{R}\frac{1}{R^2}\,\mathrm{Im}\,\varphi^{\ell+}_{jm}\,i\vec{\sigma}\cdot\vec{L}\,\varphi^{\ell}_{jm}$$

$$(1)\qquad F = -\frac{2C}{R} + \frac{E}{R^3}|\varphi^{\ell}_{jm}|^2$$

This vanishes locally only if j=1/2 so that $|\varphi^{\ell}_{jm}|^2 = \frac{1}{4\pi}$ is independent of θ,φ . For j=1/2, the solution obtained by varying over spherical trial surfaces is exact. In the bubble with j=1/2, the net surface tension vanishes locally. Physically, this reflects the exact balance of the uniform surface tension C and the fermi pressure due to the quark field.

For j>1/2, the surface tension and fermi pressure balance only on the average; there is a tension induced normal force that will tend to distort the surface from sphericity. From (1), we see that this force tends to push the surface out where the quark density is high, and allows the surface to collapse where the quark density is low (Fig 7). A particularly simple example is the case of a quark of maximal z-component angular momentum, m=j=l+1/2. The normal force density is:

$$F = 2\left[\frac{8\pi C}{\ell+1}\right]^{\frac{1}{3}} C\left[\frac{(\ell+1)(2\ell+1)!!}{(2\ell)!!}(\sin\theta)^{2\ell} - 1\right]$$

This is a force which is axially symmetric and has a single peak in the equatorial plane. It will tend to stretch the sphere at the equator and depress it at the poles. The force densities associated with quark states with $|m|<j$ have one or more azimuthal nodes, and tend to distort the sphere to rather more complicated shapes. (Fig 8)

The angular dependence of these force densities on the sphere suggests the shapes we should use for trial surfaces in a variational estimate of excited state energies. We note that, because the force densities differ for spherical quark states of the same j but different m, the surfaces which actually minimize the total energy will presumably be of different shapes. Thus, it appears, the semi-classical spectrum will not necessarily consist of $(2j+1)$-fold degenerate levels corresponding to particle states of the same j but different m. This result, though disturbing, is not terribly surprising. It is again a consequence of the semi-classical treatment of the surface degrees of freedom. In the full quantum theory, the surfaces corresponding to states of the same j but different m will, because their shapes differ, have slightly different energies associated with their quantum fluctuations. This relative shift will

precisely cancel the semi-classical splitting, and restore rotational invariance to the spectrum.

We will sidestep this problem by considering only quark states corresponding to $|m|=j$, and interpreting the resulting energies as estimates of the energy of the quantum multiplet of spin j. We can adduce several arguements for this interpretation. The surfaces corresponding to $|m|=j$ states are simple and smooth. Those corresponding to other values of m will be complicated and "bumpy." As a practical matter, it is extremely difficult to do the required variational calculations for surfaces of very complicated shapes. It takes many variational parameters and correspondingly many hours of computer time. Further, because these surfaces are "bumpy," we expect the effects of their quantum fluctuations to be relatively more important than they are for smooth surfaces like the sphere or the $|m|=j$ surfaces. Thus, the most relatively consistent way of neglecting quantum fluctuations is to estimate the energies using states which have smooth surfaces. Finally, as we shall see, the effects of distortions of the surface are numerically small for the low-lying excited states. In no case will our variational estimate of the energy be more than 10% lower than the value estimated from the sphere. Thus, whatever approximation we make, we commit no gross numerical error.

As a simple trial surface that is smooth and flattened at the poles, we use the oblate spheroid:

$$\vec{R}(\theta,\varphi) = R\left[\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \sqrt{1-d^2}\cos\theta\right]$$

$$\text{where} \qquad 0 \le d \le 1$$

This surface depends on two parameters: R which determines its overall size, and d which determines its shape. For d=0, the surface is a sphere. As d increases from zero, the spheroid becomes flatter and flatter, until, at d=1, it is an infinitely thin "pancake." The area of the spheroid is

$$a = \frac{1}{2}\left[1 + \frac{1-d^2}{2d}\ln\frac{1+d}{1-d}\right]4\pi R^2$$

The Hamiltonian of the surface is

$$H = \frac{1}{R\sqrt{1-d^2\sin^2\theta}}\left[\frac{1}{2}\sqrt{1-d^2}\left(1+\frac{1}{1-d^2\sin^2\theta}\right)\right.$$

$$\left.-i\sigma\cdot\hat{\varphi}\frac{\partial}{\partial\theta} + i\left[\cot\theta(\cos\varphi\sigma_1+\sin\varphi\sigma_2)-\sigma_3\sqrt{1-d^2}\right]\frac{\partial}{\partial\varphi}\right]$$

Because the surface is axially symmetric, the z-component of angular momentum is conserved.

$$[H, J_3] = 0 \qquad \text{where} \qquad J_3 = -i\frac{\partial}{\partial\varphi} + \frac{1}{2}\sigma_3$$

Thus, we can choose our Dirac eigenstates to be states of definite J_3.

The remaining diagonalization of the Hamiltonian must be done numerically. The level on the spheroid which corresponds to j=m on the sphere is simply the lowest positive energy state in the sector $J_3 = m$. We compute the total energy, U, of a spheroidal bubble occupied by a single quark of spin m, and minimize it over R at fixed d. The ratio of this energy to the corresponding energy estimate on the sphere,

$$\rho_m(d) = \frac{U(d)}{\frac{3}{2R_0}\left(m+\frac{1}{2}\right)^{2/3}}$$

is plotted as a function of d for m=3/2 and m=5/2 in Fig 9.

We see immediately that, in both cases, the total energy decreases monotonically as a function of d. Indeed, these calculations show that the energy of the spheroid is lowest in the limit where it becomes a completely flattened disk. Despite the fact that such a disk has very large curvature at its edge, the Dirac energy remains small. This result is actually quite general-- the static Dirac equation can be solved on surfaces with sharp edges. In the limit that an edge becomes infinitely sharp, the Dirac equation gives a boundary condition at the edge. This point, and its possible application to the numerical study of more general surfaces, is discussed more fully in Appendix E.

The oblate spheroid is not an adequate trial surface. It takes into account the tendency of the surface to spread at the equator, but does not allow for sufficient depression at the poles. We note, however, that although the energy decreases uniformly as the spheroid flattens, the numerical size of the decrease is rather small. Even the completely flattened disk has energy down by less than 10% from that estimated on the sphere.

We want to find a trial surface which is both spread at the equator and dips inward at the poles. We could begin to consider surfaces that are defined by three or more parameters, but it is computationally more straightforward to continue to work with two parameter surfaces as long as possible. A simple two parameter surface in which the region near the poles is completely depressed is the torus (Fig 10). This surface may be regarded as one where the poles have dipped in so far as to create a hole through the center.

We coordinatize the torus as follows:

$$\vec{R}(\theta,\varphi) = b\left[(\gamma+\sin\theta)\cos\varphi,\ (\gamma+\sin\theta)\sin\varphi,\ \cos\theta\right]$$

where: $0 \le \theta < 2\pi \qquad \gamma \ge 1$

$0 \le \varphi < 2\pi$

b is the radius of the circular vertical cross-sections of the torus; γb is the radius of the torus in the x-y plane. The area of the torus is

$$a = 4\pi^2\gamma b^2$$

The surface Hamiltonian is

$$H = \frac{1}{b}\left[\frac{\frac{1}{2}\gamma+\sin\theta}{\gamma+\sin\theta}\,(-i\,\sigma\cdot\hat{\varphi}\,\frac{\partial}{\partial\theta}) + \frac{i}{\gamma+\sin\theta}\,\sigma\cdot\hat{\theta}\,\frac{\partial}{\partial\varphi}\right]$$

where:

$$\hat{\varphi} = (-\sin\varphi,\ \cos\varphi,\ 0)$$

$$\hat{\theta} = (\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta)$$

As before, $J_3 = -i\frac{\partial}{\partial\varphi} + \frac{1}{2}\sigma_3$ commutes with H, and we can work in a sector of definite J_3, $J_3 = m$. The state corresponding to j=m on the sphere is again the lowest positive energy state in this sector.

This Hamiltonian must also be diagonalized numerically. We compute the total energy, U, of single quark states with m=3/2 and m=5/2, and minimize over b at fixed γ. The ratio

$$\rho_m(\gamma) = \frac{U_m(\gamma)}{\frac{3}{2R_0}(m+\frac{1}{2})^{2/3}}$$

is plotted versus γ in Fig 11. The minima of the total energy in γ are:

	γ	ρ_m
m=3/2	2.09	.973
m=5/2	4.04	.910

The energy estimate of the m=5/2 state here is lower than the corresponding estimate on a flattened disk and suggests that single quark bubbles of spin 5/2 and larger will have a toroidal shape. The energy estimate for the spin 3/2 bubble on the torus is larger than the estimate on the flattened disk. Presumably, the m=3/2 state is extremely depressed at the poles, but still connected.

Despite the radical differences in their shape and topology, we see that the energies of low-lying single quark states on spheres and on toruses are not very different. We interpret this as a reflection of the "softness" of the three dimensional bubble. This three dimensional result is analogous to the complete shape degeneracy of the two dimensional bubble. In order to estimate static energies more accurately, we should consider trial surfaces defined by more than two adjustable parameters. As a practical matter, as long as we are interested in only the energies of low-lying states, the computational difficulties involved in such calculations are not justified by the results we would hope to obtain. For single quark states of spin less than 5/2, we have seen that the correction to the energy due to distortions is less than 10%. Three quarks of spin 5/2 could combine to form baryonic states of maximum spin 15/2. There are not yet observed hadrons of such high spin, nor are the experimental masses of the higher resonances known to within 10%. Also, we have neglected the effects of SU(3)

breaking, which must be sizeable in the higher multiplets. Further, as we shall see in the case of the radial mode, quantum fluctuations may be expected to give corrections to the energy levels at least as large as those due to static distortions of the bubble shape.

Section C: The Hadronic Spectrum

So far, we have considered only bubbles containing single quarks. The baryons and mesons in the BCDWY model contain three quarks and quark-antiquark, respectively, in color singlet states. We discuss briefly the physics of these multi-quark bubbles in the static picture, noting the differences between the multiplet structure predicted by this model and that given from SU(6)xO(3) symmetry.

We have seen in Chapter II that the ground state baryons and mesons occur in this model in the same SU(3), spin, and parity combinations as predicted from SU(6). This is a trivial result. The ground state baryons, for example, are formed by all possible symmetric combinations of three, independent, SU(3) triplet, $j=1/2$ quarks on the sphere. Schematically, ground state baryons=(3,1/2)x(3,1/2)x(3,1/2) (symmetric)

=(10,3/2)+(8,1/2)

=$\underline{56}$ L=0 of SU(6)xO(3)

where the notation (A,B) refers to

A= SU(3) representation

B= spin

Each quark is in an l=0 state on the sphere, and therefore has positive parity. Thus the ground state baryons are all of positive parity.

Similarly, the ground state mesons are formed from all possible combinations of a j=1/2 quark and antiquark.

ground state mesons=(3,1/2)x($\bar{3}$,1/2)

=(8,1)+(1,1)+(8,0)+(1,0)

=$\underline{35}$ L=0 + $\underline{1}$ L=0 of SU(6)xO(3)

The quark and anti-quark wavefunctions have opposite parity on the bubble, so all these are negative parity states. The neutral members of these multiplets have charge conjugation determined by their spin, $C=(-1)^S$.

In the static picture, the lowest excited baryon states are formed by promoting one of the quarks to a higher orbital excitation, for example, j=3/2. This, of course, will tend to distort the bubble from sphericity. But, because the other occupied quark states tend to make the

surface remain spherically symmetric, this distortion is not so large as in the case of the single quark bubble. Indeed, the energy shifts due to distortions can be estimated variationally as was done in the single quark case. It is found that the corrections to the energy are typically on the order of 3% for mesons and 1% for baryons in the first excited state. These shifts are negligible relative to other corrections that we are neglecting. We will, therefore, treat the excited states as if they were spherical for the purpose of counting the possible SU(3) and spin multiplets.

For the baryons, then, the first excited states in this model will consist of two quarks with j=1/2, and a third with j=3/2 with an overall wavefunction that is symmetric under quark interchange. The mass of these states is predicted on the sphere to be

$$U=\frac{3}{2R_0}(4)^{2/3}=\left[\frac{3}{2R_0}3^{2/3}\right]\left(4/3\right)^{2/3}$$

It is a straightforward exercise in group theory to show that there are 252 possible states:

first excited states=(3,1/2)x(3,1/2)x(3,3/2) (symmetric)

=(10,5/2)+(8,5/2)

+(10,3/2)+2(8,3/2)+(1,3/2)

+(10,1/2)+(8,1/2)

$= 252$ states.

Because the quark with $j=3/2$ has $l=1$, these are all negative parity states.

The corresponding odd parity first orbital excitation in the SU(6)xO(3) scheme is the $\underline{70}$ L=1:

$$\underline{70}\ L=1 = (8,5/2)+(10,3/2)+2(8,3/2)+(1,3/2)$$
$$+(10,1/2)+2(8,1/2)+(1,1/2)$$
$$= 210 \text{ states.}$$

Relative to this structure, our spectrum contains an additional (10,5/2) and is missing the (1,1/2) and one (8,1/2). Table II summarizes these results, along with the conventional particle assignments to the $\underline{70}$ L=1. We see that the degeneracy of these levels in the SU(6)xO(3) scheme or in our model must be badly broken. Without understanding this breaking, it is difficult to decide which, if either, of the two pictures fits the data more closely.

Like the baryon states, the first excited meson states will consist of a quark antiquark pair with one particle in the $j=3/2$ state. The possible mesons that can be formed are

$$(3,1/2)\times(\bar{3},3/2)=(3,3/2)\times(\bar{3},1/2)$$
$$=(8,2)+(1,2)+(8,1)+(1,1)$$

$= 72$ states

These are all states of positive parity. The corresponding SU(6)xO(3) level is $\underline{35}$ L=1 which contains

$$\underline{35}\ L=1 \ =(8,2)+(1,2)+2(8,1)+(1,1)+(8,0)+(1,0)$$

Relative to the prediction of SU(6)xO(3), we are missing (8,1), (8,0) and (1,0). This is not terribly significant, as this meson multiplet is even more badly broken than is the first excited baryon multiplet. A much more disturbing feature of the predicted meson spectrum is that, because the quark and anti-quark are treated as independent particles, states exist in which either the quark or the antiquark is promoted to $j=3/2$. Thus the states

$$|q\, j=\tfrac{1}{2};\ \bar{q}\ j=\tfrac{3}{2}\rangle \pm |q\ j=\tfrac{3}{2};\ \bar{q}\ j=\tfrac{1}{2}\rangle$$

are independent states of the same SU(3) and spin quantum numbers, but whose neutral members have opposite charge conjugation. A completely satisfactory resolution of this problem is presently unavailable.

We make several observations on this point. First, as compared with SU(6)xO(3), the BCDWY theory tends to predict larger degenerate hadron super-multiplets at each level. The reason can be traced to the extra degrees of freedom associated with the bubble surface. In non-relativistic

SU(6)xO(3), the center of mass of a hadron state is determined as the center of mass of the quarks which constitute it. The quark wave functions are the wave functions of the quarks relative to this center of mass. Thus, for example, mesons are characterized by a single quark-antiquark relative wave function, and there is no charge conjugation doubling. In contrast, the center of mass in the static BCDWY theory is determined as the rest frame of the bubble surface, relative to which each quark is described by a independent wave function. This is the source of the additional states in the baryon spectrum, and of the doubling of the meson spectrum. One might expect that, when the quantum nature of the surface and the effects of the negative energy fermi sea are properly taken into account, the complete independence of the quark and the antiquark on the bubble surface will disappear, and along with it, the charge conjugation degeneracy of the meson spectrum. No calculational method which takes into account such quantum effects presently exists.

A more practically useful observation is that, as long as all physical operators have definite transformation properties under charge conjugation, it is consistent to simply delete one set of states from the theory. The only operators of physical interest which do not have this property are the weak currents. As we have seen in Chapter

III, the axial current and PCAC are not adequately represented in the semi-classical BCDWY theory in any case.

We note that, though we have removed most exotics through the introduction of the color interaction, there still remains one class of stable exotics. These are di-baryons (6 quarks in a color singlet state), di-mesons (two quark-antiquark pairs), and tri-mesons (three quark-antiquark pairs). For example, the estimated mass of the dibaryon is

$$M_{DIBARYON} = \frac{3}{2R_0}\, 6^{2/3} = 2^{-\frac{1}{3}}\,[2\, M_{56}]$$

Thus, the di-baryon appears to be bound relative to two free baryons. Similarly, di- and tri-mesons seem bound. Our confidence in these simple energy estimates decreases as the estimated energy increases. We note two as yet unaccounted for mechanisms which may serve to unbind these states. One, suggested by K. Johnson in the context of the MIT model, is that the finite effects of the surface zero-point motion and the negative energy Dirac sea may provide an additional relative energy to the exotics and cause them to be unstable. A second possibility is that, because di-baryons and di-mesons have different SU(3) quantum numbers than do the usual ground state hadrons, SU(3) breaking forces that account for mass splittings may also unbind these states.

The order of magnitude of the necessary mass shift of the exotics (20%) is not inconsistent with the observed size of SU(3) breaking. However, no theory that exploits this possibility has yet been devised.

The ratio of the energy of the first excited baryon multiplet to the ground state energy is estimated from the sphere to be $\left(\frac{4}{3}\right)^{2/3} = 1.21$. From Table II, we see that the experimental ratio, though rather uncertain, seems to be a bit larger. We have seen that the distortions of the bubble have only a small effect on its energy. Qualitatively, however, we expect that the quantum fluctuations of the more highly curved excited state surfaces will increase the energy of this multiplet relative to the ground state. Indeed, we might hope that it would be possible to compute the splittings induced between states of different j within the degenerate super-multiplets of the model. Unfortunately, no techniques presently exist which can account for these effects quantitatively.

Section D: The Radial Mode

We now consider another kind of excitation of the bubble: one in which the surface as a whole is excited, rather than just the quark. The general problem of estimating the spectrum of surface excitations is technically beyond us, in three space dimensions. In this section, we discuss only the simplest possible such excitation-- a spherically symmetric 'breathing' mode.

We begin with the semi-classical time dependent equations of motion. Let us assume that there is a solution of these equations whose surface is a sphere of time dependent radius R(t):

$$R^{\mu}(\tau,\theta,\varphi) = (\tau,\ R(\tau)\,\hat{r}(\theta,\varphi))$$

Defining

$$\dot{R} = \frac{dR}{d\tau} \equiv \tanh \omega(\tau)$$

we have

$$g_{\alpha\beta} = \begin{pmatrix} \frac{1}{\cosh^2\omega} & 0 & 0 \\ 0 & -R^2 & 0 \\ 0 & 0 & -R^2\sin^2\theta \end{pmatrix}, \quad \sqrt{|g|} = \frac{R^2\sin\theta}{\cosh\omega}$$

$$n^{\mu} = (\sinh\omega,\ \cosh\omega\ \hat{r})$$

$$h^{\alpha}{}_{\beta}=\begin{pmatrix} \cosh w\,\dot{w} & 0 & 0 \\ 0 & \frac{\cosh w}{R} & 0 \\ 0 & 0 & \frac{\cosh w}{R} \end{pmatrix}$$

We take a form for the Dirac field that has "l=0" and automatically satifies (I).

$$\psi=\frac{1}{\sqrt{2}}\left(1+i\alpha\right)\begin{pmatrix} F(\tau) \\ 0 \end{pmatrix}$$

where F(t) is some two component spinor. The Dirac equation becomes:

$$\dot{F}(\tau)=-\frac{k}{\cosh^2 w}\left(i+\sinh w\right)F(\tau)$$

whose integral is

$$F(\tau)=e^{-\int_0^\tau d\tau\,\frac{k(i+\sinh w)}{\cosh^2 w}}\,F(0)$$

$$=\frac{R(0)\sqrt{\cosh w(0)}}{R(\tau)\sqrt{\cosh w(\tau)}}\,e^{-i\int_0^\tau d\tau\,\frac{R}{\cosh^2 w}}\,F(0)$$

The normalization condition,

$$1=\int d\theta\, d\varphi\,\frac{R^2\sin\theta}{\cosh w}\,\bar{\psi}\gamma^0\psi$$

allows the wave-function to be written

$$F(\tau)=\frac{1}{\sqrt{4\pi R(\tau)^2\cosh w(\tau)}}\,e^{-i\int_0^\tau\frac{d\tau}{\cosh w}\left[\frac{1}{R}+\frac{1}{2}\dot{w}\right]}\,u$$

where u is a fixed two component unit spinor.

We see that the Dirac equation is solvable exactly for an arbitrary R(t). Equation (III) will determine which of these surfaces are actually allowed dynamical states. Putting the solution for ψ into equation (III), we have

$$0=1-R\dot{w}-8\pi C R^3\left(1+\tfrac{1}{2}R\dot{w}\right)$$

This equation can be more simply expressed in rescaled variables.

$$R_0=(8\pi C)^{-\frac{1}{3}}=\text{ground state radius}$$

$$\tau=\Upsilon R_0$$

$$R(\tau)=\rho(\Upsilon)R_0$$

we have:

$$\frac{dw}{d\Upsilon}=\frac{1-\rho^3}{\rho(1+\frac{1}{2}\rho^3)}\qquad \frac{d\rho}{d\Upsilon}\equiv\dot{\rho}=\tanh w$$

this can be integrated once to give

$$(2)\qquad \epsilon=\frac{1}{\sqrt{1-\dot{\rho}^2}}\left(\frac{2}{3\rho}+\frac{1}{3}\rho^2\right)$$

where is a constant. A straightforward integration of the energy density shows that

$$\text{total energy}=U=\epsilon\cdot\frac{3}{2R_0}$$

Thus ϵ is the total energy of the radial mode measured in units of the static ground state energy.

If $\epsilon=1$, we recover the static solution, $\rho=1,\ \dot{\rho}=0$.

For $\epsilon < 1$, there are no solutions. There exists a unique solution for each $\epsilon > 1$ in which ρ is periodic, with turning points determined by

$$\epsilon = \frac{2}{3\rho} + \frac{1}{3}\rho^2$$

The equation for ρ is similar to that for a relativistic particle in a scalar potential

$$V(\rho) = \frac{2}{3\rho} + \frac{1}{3}\rho^2 \qquad \text{shown in Fig 12.}$$

We note that the total energy is continuous. As emphasized in Section A, this is an effect due to the classical treatment of the surface degrees of freedom. In order to get some idea of the level structure of the radial modes, we will "quantize" this excitation in the WKB approximation.

We treat the equation for ρ as if it were, indeed, the equation of motion of a relativistic particle in a potential. We take the expression for the total energy (2) to be the Hamiltonian

$$H = \frac{1}{\sqrt{1-\dot\rho^2}}\left(\frac{2}{3\rho} + \frac{1}{3}\rho^2\right)$$

The most general Lagrangian from which this H could have been derived is

$$L(\rho,\dot\rho) = -\sqrt{1-\dot\rho^2}\left[\frac{2}{3\rho}+\frac{1}{3}\rho^2\right] + f(\rho)\,\dot\rho$$

where $f(\rho)$ is some undetermined function.

The canonical momentum conjugate to ρ is

$$P = \frac{\partial L}{\partial \dot\rho} = \frac{\dot\rho}{\sqrt{1-\dot\rho^2}}\left[\frac{2}{3\rho} + \frac{1}{3}\rho^2\right] + f(\rho)$$

The WKB approximation gives the discrete energy levels from the quantization condition

$$2\pi(n+\tfrac{1}{2}) = \oint_{\text{orbit}} P\,d\rho = 2\int_{\rho_{min}}^{\rho_{MAX}} d\rho\,\sqrt{\epsilon_n^2 - \left[\frac{2}{3\rho}+\frac{1}{3}\rho^2\right]^2}$$

$$n = \text{integer}$$

This equation can be easily solved numerically. The first few values of ϵ_n and the corresponding turning points are given in Table III.

In the lowest state, n=0, we see that the effects of its zero-point motion are very large. The radius fluctuates by a factor of two about its static value. The energy in the surface excitation is 60% of the static ground state energy. This is a quite dramatic illustration of the softness of the bubble dynamically and suggests that if fluctuations are properly accounted for, the bubble will be quite smeared out in space.

The n=1 state is the lowest radial excitation of the bubble. Its energy is a factor ϵ_1/ϵ_0 =1.60 higher than that of the ground state. It is easy to convince oneself that, in the case of several quarks in the bubble, all the energies of the radial mode simply rescale. Thus, the model predicts radial excitations of baryons and mesons with energy 1.6 times higher than the ground state energies.

No radially excited meson candidates have been confirmed experimentally. There is, however, a presumed radial excitation of the nucleon-- the Roper resonance-- of mass 1470 mev. We note that 1470/940=1.56 . In the face of our inability to derive many solid numerical predictions of excited state masses, this is a pleasing numerical coincidence.

In this chapter, we have tried to learn as much as possible about the spectrum of the BCDWY model in the semi-classical approximation. We have developed a clear intuitive picture of the dynamics of bubbles. The most important feature of this picture is the softness of bubbles to deformation. We have seen that, although the excited states are expected to correspond to highly distorted surfaces, their spectrum can be estimated and is in qualitative agreement with SU(6)xO(3) and with the data for the baryons. In the example of the radial mode, we have seen that quantum fluctuations of the surface are large, and that dynamical surface excitations of the bubble are

important in determining the full spectrum of excited states.

We believe that when quantum corrections are taken into account, the theory may provide a natural framework within which scaling, the behaviour of form factors, and perhaps even hadronic interactions can be understood. At present, however, no calculational methods adequate to handle the quantum theory of the surface in four space-time dimensions exists. We are therefore unable to systematically analyze either the corrections to the spectrum, form factors, or the question of scaling.

Chapter V

In Chapter IV, we discovered that the equations for static bubbles can be solved exactly in two space dimensions. The shape degeneracy of these solutions provided the first indications of the softness of the bubble. In this chapter, we examine the bubble theory in two space and one time dimensions. We will see that all classical solutions to the general, time dependent theory can be constructed explicitly. In the general case, the bubble executes a complicated, but periodic, oscillation in time. The quarks trapped on it are massless and move along light-like lines imbedded in the surface. As in the static case, there is a degeneracy over an infinite class of "shapes" of the bubble. By choosing a special coordinate system and a particular Lorentz frame, we can represent all possible solutions to the theory in terms of a countable number of independent "normal mode" amplitudes. We exhibit a set of commutation relations among these amplitudes which provide a Poincaré invariant quantum theory of the single bubble. The operator algebra of this quantum theory is similar to that of the Neveu-Schwartz model[17]. Indeed, the bubble in three space-time dimensions is a two dimensional object that closely resembles a closed string upon which a

fermion is trapped.

The spectrum of the quantum theory is not uniquely determined by the operator algebra alone. As is the case with other dual theories in three dimensions, the spectrum depends critically on how the infinite normal ordering terms which arise in the theory are handled. We discuss the possible spectra of the theory. We find that the bubble has a discrete mass spectrum, with states corresponding to "mesons" (even fermion number) or to "baryons" (odd fermion number), but not both. We discover that the classical infinite degeneracy of all levels disappears in the quantum theory. The softness of the bubble is reflected in the size of the quantum fluctuations of its surface and in its exponentially growing density of states.

All of the work described in this chapter was done in collaboration with Dr. Henry Tye of SLAC, without whose insight into the string-like nature of the three dimensional bubble no solution of the theory would have been possible.

We begin by finding all solutions to the classical bubble equations in three space-time dimensions. We consider only the case of a single quark species. The extension to many species is completely straightforward. These equations are:

$$\text{(1)}\quad \begin{aligned} &(\mathrm{I})\ \ i\not{n}\psi = \psi \\ &(\mathrm{II})\ \ (i\partial_{\parallel} + k_L n)\psi = 0 \\ &(\mathrm{III})\ \ h_{\alpha\beta}T^{\alpha\beta} = 0 \end{aligned}$$

In three space-time dimensions, both the geometry of the bubble and the gamma matrix algebra can be simplified tremendously. The bubble surface is two dimensional. Our notation will be:

$$R^{\mu}(u^0, u^1) = R^{\mu}(\tau, \sigma) \qquad u^0 \equiv \tau\ ,\ u^1 \equiv \sigma$$

with $\quad \dot{A} \equiv \frac{\partial A}{\partial \tau} \quad , \quad A' \equiv \frac{\partial A}{\partial \sigma}$

for any quantity A

We choose the orientation of the internal coordinates such that

$$\text{(2)}\qquad n^{\mu} = -\frac{1}{\sqrt{-g}}\,\varepsilon^{\mu\nu\lambda}\,(\tau_0)_{\nu}\,(\tau_1)_{\lambda}$$

In three dimensions, we need only three matrices satisfying the anti-commutation relations

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\,\eta^{\mu\nu}\ ,\quad \mu = 0,1,2$$

We choose these to be 2x2 matrices rather than the usual 4x4 gamma matrices.

$$\text{(3)}\qquad \gamma^0 = \sigma_1 \qquad \gamma^1 = i\sigma_3 \qquad \gamma^2 = -i\sigma_2$$

whose algebra is

$$\gamma^{\mu}\gamma^{\nu} = \eta^{\mu\nu} - i\,\varepsilon^{\mu\nu\lambda}\gamma_{\lambda}$$

That such a choice is possible is obvious mathematically. Its significance in the theory becomes clear if we begin with a 4x4 representation of the usual gamma matrices.

$$\text{(4)}\qquad \begin{aligned} &\gamma^0 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix},\quad \gamma^1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix},\quad \gamma^2 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \\ &\gamma^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\qquad\qquad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \end{aligned}$$

In this representation, the Dirac equation involves only $\gamma^0, \gamma^1, \gamma^2$. Thus, the two component spinors ψ_+ and ψ_- decouple from each other. Because the fermion moves in a single plane, there is an extra conserved "charge" whose matrix is $\gamma^3\gamma_5$. To choose a two component representation of the gamma matrices is to impose the condition that the Dirac field be an eigenstate of $\gamma^3\gamma_5$ with eigenvalue +1.

The theory we obtain by making this choice is a consistent and complete theory of a fermion trapped on the bubble surface. That this is so is not completely obvious. In a three dimensional theory where the fermion is free to move throughout space-time, charge conjugation and time-reversal invariance cannot be realized in the two component representation. For example, charge conjugation must be represented by a matrix C with the property

$$(5) \qquad C\gamma^{\mu *}C^{-1} = \gamma^{\mu}$$

In the representation (4), we must take $C = \gamma_5\gamma^1$. Thus, C does not commute with $\gamma^3\gamma_5$. Because the fermion in the bubble theory is confined to the two dimensional bubble surface, however, the requirement (5) need be satisfied only by the tangential components of the gamma matrices:

$$C\gamma^{\alpha *}C^{-1} = \gamma^{\alpha}$$

This condition can be realized in the two component representation by $C = i\not{n}\gamma^1$. The two component representation is "complete" for the bubble theory in the sense that P, C, and T can all be realized for the fermi field.

We must also emphasize that a bubble theory based on a four component Dirac spinor is not an altogether trivial extension of the two component theory. It may be viewed as a theory of two independent two component quark fields, ψ_+ and ψ_-, trapped on the bubble surface. However, the two component version of equation (I) is different for these two spinors:

$$(6) \qquad \begin{aligned} i\not{n}\psi_+ &= \psi_+ \\ i\not{n}\psi_- &= -\psi_- \end{aligned}$$

Though the Dirac equations for these two spinors separate

completely, both interact with the surface through equation (III). Because of the difference in sign in equation (6), the effects on the surface of the two spinor fields do not add in a simple way. At this time, the general solution to the four component theory has not been obtained.

The equations of motion (1) are a rather complicated system of coupled non-linear partial differential equations. They are quite difficult to solve directly. The procedure by which the general solution will be obtained is rather intricate. Our goal in the following discussion is to reduce the equations of motion by partial integrations as far as possible to algebraic equations relating functions which describe the surface and Dirac degrees of freedom. We proceed as follows: First, exploiting some special geometric properties of two dimensional surfaces and rather general properties of the equations of motion, we show that coordinates may be chosen in which the bubble surface has a particularly simple form. Using this result, we find that the Dirac equation can be integrated to give the Dirac field everywhere on the surface explicitly in terms of the variables describing the surface geometry and independent "initial" values of the Dirac field along a curve in the surface. Finally, we show that the surface equation (III) gives an algebraic relation between the initial data for the Dirac field and the surface variables.

Because the system is invariant under arbitrary coordinate transformations

$$u^\alpha \to f^\alpha(u^\beta)$$

we are at liberty to choose a system of coordinates which simplify the equations. Further, as we shall see below, the requirement that a solution to the equations of motion exists at all places very strong constraints on the geometric structure of the surface. These constraints arise, essentially, from the causal structure of the free, massless Dirac field on the surface.

We will show that a coordinate system can be chosen such that

$$(7) \qquad \dot{R}'(\tau,\sigma) = 0$$

and

$$(8) \qquad R'(\tau,\sigma)^2 = 0$$

Some parts of the proof involve rather tedious and unilluminating algebra. These are relegated to Appendix F. Below, we sketch the main ideas of the derivation.

A special property of two dimensional manifolds which we rely on to choose coordinates is that any symmetric tensor of signature (1,-1) can be brought into off-diagonal form by a coordinate transformation. This would, for example, allow us to choose the metric to be off-diagonal. It is more useful, however, to work in coordinates where the

stress tensor is of a simple form.

It can easily be shown (see Appendix F) that the stress tensor,

$$(9) \qquad T^{\alpha\beta} = C g^{\alpha\beta} - \mathrm{Im}\, \bar{\Psi} \Sigma^\alpha \partial^\beta \Psi$$
$$\equiv C g^{\alpha\beta} + \gamma^{\alpha\beta}$$

is symmetric and has signature (1,-1). The symmetry of $T^{\alpha\beta}$ reflects the absence of spin in two dimensions. In general, a spin-dependent divergence must be added to the canonical stress tensor to form the symmetric "improved" stress tensor. In two dimensions, however, the canonical fermion stress tensor is already "improved."

We choose coordinates such that

$$(10) \qquad T^{\alpha\beta} = \frac{C}{T}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $T(\tau,\sigma)$ depends on the details of the solution. The coordinate system is not uniquely determined by the condition (10). We still have "conformal" invariance: (10) is invariant under coordinate transformations of the form

$$(11) \qquad \begin{aligned} \tau &\to f(\tau) \\ \sigma &\to g(\sigma) \end{aligned}$$

So far, we have used nothing but the coordinate invariance of our desription of the bubble surface. We now show that a necessary condition for the field equations to be solvable is that $R^\mu(\tau,\sigma)$ satisfy (7) and (8).

We begin by considering the algebraic relations between the fermion current J^{α} and the stress tensor $T^{\alpha\beta}$. A trivial result which follows from the two component representation of the gamma matrices is:

if ψ is any spinor satisfying $\bar{\psi}\psi = 0$,

then $(\bar{\psi}\gamma^{\nu}\psi)\gamma_{\nu}\psi = 0$

This has the immediate consequences:

$$(12) \qquad J^{\alpha} J_{\alpha} = 0$$

$$(13) \qquad J_{\alpha} T^{\alpha\beta} = C J^{\beta}$$

$$(14) \qquad J_{\alpha} J_{\beta} T^{\alpha\beta} = 0$$

$$\text{where} \qquad J^{\alpha} = \bar{\psi}\gamma^{\alpha}\psi$$

Assuming that J^{α} is not zero, these relations allow us to determine some components of the metric tensor in terms of T.

With the stress tensor of the form (10), equation (14) implies $J_0 J_1 = 0$. We shall see below that the choice of orientation (2) and the condition $i\not{\alpha}\psi = \psi$ require that $J_1 = 0$. Putting this result in (13) and comparing both sides, we find:

$$g^{00} = 0 \qquad\qquad g^{01} = \frac{1}{T}$$

So the metric has the form

$$g^{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{T} \\ \frac{1}{T} & -A \end{pmatrix} \qquad\qquad g_{\alpha\beta} = \begin{pmatrix} AT^{2} & T \\ T & 0 \end{pmatrix}$$

$$\sqrt{-g} = T$$

where A is not determined by this analysis. We note that

$$0 = g_{11} = R'(\sigma,\tau)^{2}$$

This is condition (8).

The stress tensor is divergenceless.

$$0 = T^{\alpha\beta}{}_{\|\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,T^{\alpha\beta}\right)_{|\alpha} + \left\{ {\beta \atop \gamma\delta} \right\} T^{\gamma\delta}$$

$$= \frac{1}{T}\left(T T^{\alpha\beta}\right)_{|\alpha} + \frac{2C}{T}\left\{ {\beta \atop 01} \right\}$$

But $T T^{\alpha\beta}$ is a constant, so we have

$$\left\{ {\beta \atop 01} \right\} = 0 \qquad , \qquad \beta = 0,1$$

The equation of motion (III) of the surface is

$$h_{\alpha\beta} T^{\alpha\beta} = 0 \quad , \quad \text{or} \quad h_{01} = 0$$

The condition (7) now follows immediately:

$$\dot{R}' = \tau_{0\|1} = h_{01} n + \left\{ {\beta \atop 01} \right\} \tau_{\beta} = 0$$

We now turn the problem around. Starting with a coordinate system satisfying (7) and (8), we derive the solutions to the bubble equations. Equation (7) implies the

the surface is of the form

$$(15) \qquad R^\mu(\tau,\sigma) = Q^\mu(\tau) + S^\mu(\sigma)$$

defining: $\quad q^\mu(\tau) = \tau_0{}^\mu = \dot{Q}^\mu \; , \quad s^\nu(\sigma) = \tau_1{}^\nu = S'^\nu$
we have:

$$(8) \qquad s(\sigma)^2 = 0$$

$$(16) \qquad g_{\alpha\beta} = \begin{pmatrix} q^2 & q\cdot s \\ q\cdot s & 0 \end{pmatrix}, \quad g^{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{q\cdot s} \\ \frac{1}{q\cdot s} & -\frac{q^2}{(q\cdot s)^2} \end{pmatrix}$$

$$\tau^0 = \frac{1}{q\cdot s}\, s \quad , \quad \tau^1 = \frac{1}{q\cdot s}\, q - \frac{q^2}{(q\cdot s)^2}\, s$$

$$(17) \qquad n^\mu = -\frac{1}{q\cdot s}\, \varepsilon^{\mu\nu\lambda}\, q_\nu s_\lambda$$

In the following discussion, we assume that $q^2 > 0$. In fact, we shall see that the equations of motion imply that q^2 is proportional to the fermion energy density, which, as a classical function, is not positive definite. We will proceed as if the energy density is positive, and we will find that our solutions are self-consistent in the quantum theory after normal ordering is taken into account.

So far, our analysis of the surface geometry has been local. The global condition that the surface $R^\mu(\tau,\sigma)$ be a closed hypertube in space-time places further constraints on Q^μ and S^μ. Geometrically, equation (15) asserts

that the hypertube is a surface that is swept out by moving a rigid light-like curve, $S^\mu(\sigma)$, along some time-like curve, $Q^\mu(\tau)$. At each point on the two dimensional surface, there are but two light-like directions. Because the hypertube is closed, the light-like curve $S^\mu(\sigma)$ contained in it must spiral up the tube, intersecting $Q^\mu(\tau)$ infinitely many times (Fig 13). It is clear that, if the surface is to be swept out by the motion of $S^\mu(\sigma)$ along $Q^\mu(\tau)$, each of these points of intersection must be equivalent geometrically, except for an overall time-like translation, Λ^μ. After choosing appropriate coordinates τ and σ from one interval to the next, we clearly have the result that $Q^\mu(\tau)$ and $S^\mu(\sigma)$ are "semi-periodic" functions:

$$Q^\mu(\tau+\tau_0) = Q^\mu(\tau) + \Lambda^\mu$$

$$S^\mu(\sigma+\sigma_0) = S^\mu(\sigma) + \Lambda^\mu$$

where

τ_0, σ_0 = fixed periods

Λ^μ = a constant time-like translation

From this analysis, it is clear that the coordinates (τ,σ) and $(\tau+\tau_0, \sigma-\sigma_0)$ correspond to the same point on the hypertube. Later, we will choose ranges for τ and σ so as to bring the coordinates into one-to-one correspondence with the points of the surface.

We proceed to solve the surface Dirac equation in terms of the coordinates (15). The two component Dirac field has only one complex degree of freedom by virtue of equation (1). Using (2) and (17) we find

$$(18) \quad \not{s}\not{q} = \not{\Sigma}_1 \not{\Sigma}_0 = q\cdot s\,(1+i\alpha)$$

We can rewrite (1) in the equivalent form

$$(19) \quad \not{s}(\sigma)\,\psi(\tau,\sigma)=0$$

The Dirac equation becomes

$$0 = \not{\Sigma}_1\,(i\not{\Sigma}^0\partial_0 + i\not{\Sigma}^1\partial_1 + \not{R})\,\psi$$

or $$0 = (1+i\alpha)\,\psi'$$

or $$(20) \quad \psi' = \frac{i}{2}\not{A}'\psi = \frac{i}{2}\,\frac{\varepsilon_{\mu\nu\lambda}\,q^{\mu}(\tau)\,s^{\nu}(\sigma)\,s'^{\lambda}(\sigma)}{[q(\tau)\cdot s(\sigma)]^2}\,\not{q}(\tau)\,\psi$$

Because $\not{q}$ is independent of σ, this may be integrated directly:

$$(21) \quad \psi(\tau,\sigma) = \exp\frac{i}{2}\left[\int_0^{\sigma} d\sigma_1\,\frac{\varepsilon_{\mu\nu\lambda}\,q^{\mu}(\tau)\,s^{\nu}(\sigma_1)\,s'^{\lambda}(\sigma_1)}{[q(\tau)\cdot s(\sigma_1)]^2}\,\not{q}(\tau)\right]\psi(\tau,0)$$

Given "initial" data, $\psi(\tau,0)$, equation (21) propagates ψ away from the curve $\sigma=0$, along a family of parallel light-like lines. The initial data is not entirely free of constraints. First, equation (19) must be satisfied:

$$\not{s}(0)\;\psi(\tau,0)=0$$

Also, because the points (τ,σ_0) and $(\tau+\tau_0,0)$ are the same, $\psi(\tau,0)$ must satisfy the "periodicity" condition:

$$(22) \quad \psi(\tau+\tau_0,0) = \exp\frac{i}{2}\left[\int_0^{\sigma_0} d\sigma_1\,\frac{\varepsilon_{\mu\nu\lambda}\,q^{\mu}(\tau)\,s^{\nu}(\sigma_1)\,s'^{\lambda}(\sigma_1)}{[q(\tau)\cdot s(\sigma_1)]^2}\,\not{q}(\tau)\right]\psi(\tau,0)$$

The phase integral in (22) is Lorentz invariant, and may most easily be evaluated, for a given τ, in a Lorentz frame where $q^{\mu}=(\sqrt{q^2},\vec{0})$. We find

$$\frac{i}{2}\int_0^{\sigma_1} d\sigma_1\,\frac{\varepsilon_{\mu\nu\lambda}\,q^{\mu}(\tau)\,s^{\nu}(\sigma_1)\,s'^{\lambda}(\sigma_1)}{[q(\tau)\cdot s(\sigma_1)]^2}\,\not{q}(\tau) = \frac{i}{2}\,\Delta\Phi_{\vec{s}}(\sigma_1)\,\gamma^0$$

where $\Delta\Phi_{\vec{s}}(\sigma_1)$ is the angle through which the spacial part of s^{μ} has rotated as σ varies from 0 to σ_1. Over a full period σ_0, this angle is 2π, so (22) becomes

$$\psi(\tau+\tau_0,0) = -\psi(\tau,0)$$

Thus $\psi(\tau,0)$ must be "anti-periodic" with period τ_0.

The physical and geometric interpretation of these solutions to the Dirac equation is clear. The Dirac field is parallel transported up the surface along the light-like curves $S^{\mu}(\sigma)$. This is simply the motion of a free massless fermion trapped on a curved surface. On the hypertube, there are two disconnected families of light-like lines, which spiral up the surface in either the "left-handed" or the "right-handed" sense. The condition (I), in the two component representation, insures that the orbits of all quarks in the bubble surface have the same "handedness." We note that, by equation (6), a bubble theory based on four component spinors contains both left- and right-handed quarks. This is the reason that the structure of the four component theory is rather more complicated. As in the static case, parallel transport once around the tube gives a phase factor -1.

We can now understand qualitatively how the "causal structure" of the Dirac equation induces the periodicity of the surface motion. The Dirac field energy propagates along light-like curves. These curves must wrap around the surface over and over again. Thus the initial distribution of Dirac field energy must be reconstructed after the light-like curves have come once around the bubble. As we have seen generally above and shall see explicitly below, to the extent that the surface is determined by the quark energy distribution, the surface motion is then forced to be periodic.

We now consider the explicit form of equation (III) in terms of $q^\nu(\tau), s^\nu(\sigma)$ and $\Psi(\tau,0)$. From equations (19) and (20) we find that the only non-zero component of the fermion stress tensor is :

$$\Upsilon^{11} = -\frac{1}{[q(\tau)\cdot s(\sigma)]^2}\,\mathrm{Im}\,\bar{\Psi}(\tau,\sigma)\not{q}(\tau)\,\dot{\Psi}(\tau,\sigma)$$

Using (21), we can show after some algebra that this is the same as

$$\Upsilon^{11}(\tau,\sigma) = -\frac{1}{[q(\tau)\cdot s(\sigma)]^2}\,\mathrm{Im}\,\bar{\Psi}(\tau,0)\not{q}(\tau)\,\dot{\Psi}(\tau,0)$$

Then equation (III) is

$$0 = h_{11}T^{11} = h_{11}\left[\Upsilon^{11} - \frac{Cq^2}{[q\cdot s]^2}\right]$$

Thus equation of motion for the surface is

$$C\,q^2 = -\mathrm{Im}\,\bar{\Psi}(\tau,0)\not{q}(\tau)\,\dot{\Psi}(\tau,0)$$

Let us summarize what we have obtained. The surface is described by two periodic vector fields, q^ν, s^ν . The Dirac field is specified by the anti-periodic function, $\Psi(\tau,0)$. The conditions these functions must satisfy in order that they give a solution to the theory are:

$$q^\nu(\tau+\tau_0) = q^\nu(\tau)$$

$$(21)\qquad s^\nu(\sigma+\sigma_0) = s^\nu(\sigma)$$

$$\Psi(\tau+\tau_0,0) = -\Psi(\tau,0)$$

$$(22)\qquad C\,q^2 = -\mathrm{Im}\,\bar{\Psi}(\tau,0)\not{q}(\tau)\,\dot{\Psi}(\tau,0)$$

$$(23)\qquad s(\sigma)^2 = 0$$

$$(24)\qquad \not{s}(0)\,\Psi(\tau,0) = 0$$

$$(25)\qquad \int_0^{\tau_0} d\tau\, q^\nu(\tau) = \int_0^{\sigma_0} d\sigma\, s^\nu(\sigma) \equiv \Lambda^\nu$$

With the exception of equation (25), this is a system of algebraic relations among q^μ, Δ^ν and $\Psi(\tau,0)$. Before constructing all solutions to this system explicitly, we discuss some of its general properties. First, we count the number of independent functional degrees of freedom of the system. Each of the vectors $q^\mu(\tau)$ and $\Delta^\nu(\sigma)$ has three real components of which two are independent by (22) and (23). $\Psi(\tau,0)$ has one complex degree of freedom by (24). Apparently the system is described by four real and one complex degrees of freedom. However, because the equations are invariant under conformal transformations, there are two real degrees of freedom which correspond merely to changes of internal coordinates rather than to physically different states. Thus, all physically distinguishable solutions to the bubble equations are described by two real and one complex functions. These may be taken to be: one real function to specify each of q^μ and Δ^ν, and one complex function that determines $\Psi(\tau,0)$.

The charge and momentum can be simply expressed in terms of q^μ, Δ^ν, and $\Psi(\tau,0)$. These quantities are computed as integrals over any closed space-like curve in the surface. Along such a curve, as τ varies from τ to $\tau+\tau_0$, σ goes from σ to $\sigma-\sigma_0$. We have

$$\sqrt{-g}\; d\Sigma_\alpha J^\alpha = d\tau\, \bar{\Psi}(\tau,0)\, q(\tau)\, \Psi(\tau,0)$$

$$Q = \int_0^{\tau_0} d\tau \; \bar{\Psi}(\tau,0)\, q(\tau)\, \Psi(\tau,0) \tag{26}$$

$$\sqrt{-g}\; d\Sigma_\alpha T^{\alpha\beta} \tau_\beta^{\ \nu} = C[\, q^\nu d\tau - \Delta^\nu d\sigma]$$

$$P^\nu = C \int_{\substack{\text{space-like}\\ \text{cut}}} q^\nu d\tau - \Delta^\nu d\sigma = 2C\Lambda^\nu \tag{27}$$

The result which is analogous to the shape degeneracy of the static bubble in three dimensions is now apparent. The energy and charge are independent of Δ^ν. Δ^ν is functionally independent of q^ν and $\Psi(\tau,0)$, being constrained only by the "initial" condition (24) and through its integral (25). Thus, the moving bubble states are degenerate over all "shapes" of Δ^ν. As in the static case, the angular momentum will depend on Δ^ν through its first moment.

We proceed to construct the independent solutions of the algebraic equations (21-24). In order to eliminate the conformal degrees of freedom, we must specify a "conformal gauge" by choosing one component of q^μ and Δ^ν to have a definite functional dependence on τ and σ. Unfortunately, any such choice also destroys the manifest Lorentz invariance of the theory. We use the notation

$$X^+ = X^0 + X^2 \qquad\qquad X^- = X^0 - X^2$$

for any vector, x, in Minkowski space. We specify the conformal gauge by the choice

$$(28) \qquad R^+(\tau,\sigma) = \frac{p^+}{2c}(\tau+\sigma)$$

so $\quad q^+ = s^+ = \frac{p^+}{2c}$

where $\quad p^+ =$ constant

and $\quad \tau_0 = \sigma_0 = 1$

A Lorentz frame and conformal gauge can always be found such that (28) holds.

Next, we make use of their periodicity to expand q' and s' in Fourier series:

$$(29) \qquad q'(\tau) = \sqrt{\frac{\pi}{c}} \sum_{n=-\infty}^{\infty} a_n e^{-2\pi i n \tau}$$

$$(30) \qquad s'(\sigma) = \sqrt{\frac{\pi}{c}} \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i n \sigma}$$

The coefficients a_n and c_n must satisfy

$$(31) \qquad a_n^* = a_{-n} \;, \quad c_n^* = c_{-n}$$

Condition (24) implies that $\psi(\tau,0)$ must be of the form

$$(32) \qquad \psi(\tau,0) = \left[\frac{4c}{p^+}\, q\cdot s\right]^{-\frac{1}{2}} \left(\frac{2ic}{p^+} s^1(0)\right) F(\tau)$$

where the overall factor which multiplies the spinor has been chosen for convenience. We expand F in a Fourier

series:

$$(33) \qquad F(\tau) = \sum_m{}' \; b_m e^{-2\pi i m \tau}$$

The sum in (33) is over half-odd integral m, so that F is anti-periodic.

We can now use (22) and (23) to compute q^- and s^- in terms of a_n, c_n, b_m. We find

$$(34) \qquad q^- = \frac{4\pi}{p^+} \sum_n \mathcal{L}_n e^{-2\pi i n \tau}$$

$$(35) \qquad s^- = \frac{4\pi}{p^+} \sum_n \bar{\mathcal{L}}_n e^{-2\pi i n \sigma}$$

where:

$$(36) \qquad \mathcal{L}_n \equiv \sum_m{}' \left(m + \frac{n}{2}\right) b_m^+ b_{m+n} + \frac{1}{2}\sum_k a_{-k} a_{k+n}$$

$$(37) \qquad \bar{\mathcal{L}}_n \equiv \frac{1}{2}\sum_k c_{-k} c_{k+n}$$

The representation given in (28-37) satisfies all of the algebraic constraints (21-24). There remains the integral constraint (25). The + component of (25) is satisfied trivially. The 1 component of (25) requires $c_0 = a_0$, an identification which we assume henceforth. The - component of the integrals in (25) gives:

$$(38) \qquad \mathcal{L}_0 = \bar{\mathcal{L}}_0$$

This is a constraint which involves all of the normal mode amplitudes and reduces the total number of degrees of freedom by 1. We will not use this condition to eliminate any one of the normal mode amplitudes. In the quantum theory, (38) cannot be imposed as an operator condition, but rather, must be imposed as a "weak" constraint on the physical states.

We can express the coordinate functions and the conserved charges of the bubble in terms of the normal mode amplitudes a_n, c_n, b_m. Before doing so, it is useful to first specify the range over which Υ and σ can vary. We make this choice as follows:

let
$$\tau \equiv \Upsilon + \sigma$$
$$\hat{\sigma} \equiv \frac{1}{2}(\Upsilon - \sigma)$$

we choose
$$-\infty < \tau < \infty$$
$$-\frac{1}{2} \le \hat{\sigma} \le \frac{1}{2}$$

This choice is useful because t acts as a "time," or evolution parameter, along the bubble. Unlike curves of constant Υ, the curves t=constant are closed space-like curves in the bubble surface.

We may write the momentum as follows:

$$P^+ = p^+$$

$$(39) \qquad P^1 = 2\sqrt{\pi C}\, a_0$$

$$P^- = \frac{4\pi C}{p^+}\left(\mathcal{L}_0 + \bar{\mathcal{L}}_0\right)$$

so the mass of the bubble is

$$m^2 = 4\pi C\left[\mathcal{L}_0 + \bar{\mathcal{L}}_0 - a_0^2\right]$$

The coordinates of the surface are:

$$R^+(\Upsilon, \sigma) = x^+(\tau)$$

$$(40) \qquad R^1(\Upsilon, \sigma) = x^1(\tau) + \sqrt{\frac{\pi}{C}} \sum_{n\neq 0} \frac{i}{2\pi n}\left[a_n e^{-2\pi i n \Upsilon} + c_n e^{-2\pi i n \sigma}\right]$$

$$R^-(\Upsilon, \sigma) = x^-(\tau) + \frac{4\pi}{p^+} \sum_{n\neq 0} \frac{i}{2\pi n}\left[\mathcal{L}_n e^{-2\pi i n \Upsilon} + \bar{\mathcal{L}}_n e^{-2\pi i n \sigma}\right]$$

where;

$$x^+(\tau) = \frac{p^+}{2C}\tau$$

$$(41) \qquad x^1(\tau) = \frac{p^1}{2C}\tau + \bar{x}^1$$

$$x^-(\tau) = \frac{p^-}{2C}\tau + \bar{x}^-$$

and $\bar{x}^1, \bar{x}^-$ are constants of integration

The angular momenta are:

$$M^{+1} = x^+ p^1 - x^1 p^+$$

$$(42) \qquad M^{+-} = x^+ p^- - x^- p^+$$

$$M^{1-} = x^1 p^- - x^- p^1 + \frac{4\pi}{p^+}\sqrt{\pi C} \sum_{n\neq 0} \frac{i}{2\pi n}\left[a_n \mathcal{L}_{-n} + \mathcal{L}_{-n} a_n + c_n \bar{\mathcal{L}}_{-n} + \bar{\mathcal{L}}_{-n} c_n\right]$$

And the fermion number is:

$$(43) \qquad Q = \sum_m b_m^+ b_m$$

We now have an explicit representation of all solutions to the classical bubble theory in three space-time dimensions. In this representation, a bubble state is completely defined by giving the classical normal mode amplitudes, a_n, c_n, b_m, and the quantities p^+, x^1, x^-. The amplitudes which appear in $\mathcal{M}^2$ -- $a_n, c_n, n \neq 0$; and b_m -- describe the internal excitations of the bubble. p^+, p^1, and the initial values of x^1, x^- give the momentum and position of the bubble.

The static states described in Chapter IV can now easily be recovered. For these states, the τ coordinate can be taken to be the time in the rest frame of the bubble. Then, in order for the bubble to be static, we must take $a_n = 0$ for all n. A Q=1 positive energy state of the quark field corresponds to $b_m = 1$ for some value of m>0. c_n can be chosen arbitrarily, subject only to the constraint (38). p^+ must be chosen to be m so that the bubble will be at rest. The mass of such a state is

$$\mathcal{M}^2 = 4\pi C[\mathcal{L}_0 + \bar{L}_0] = 8\pi C m$$

in agreement with the calculations of Chapter IV.

Because all classical solutions of the theory are available to us, we may construct a quantum theory of the three dimensional bubble explicitly. The quantization of

the bubble which we present is neither canonical or gauge invariant. Whether a general scheme for quantizing the bubble theory is possible in three and especially in higher dimensions is currently under investigation. The spectrum of states of the quantum theory is plagued with normal ordering ambiguities similar to those which arise in dual theories in three space-time dimensions. Despite these, we find that the quantization is revealing as to how the physical properties of the bubble, which we have discussed at length classically, are reflected in a true quantum theory.

We would expect any quantum theory of the bubble to induce simple commutation relations between the normal mode amplitudes of the classical theory. In the following discussion, we will "quantize" the bubble by introducing a set of fundamental commutation relations among the normal mode amplitudes. Our guide in choosing these commutation relations will be the requirement that the canonical Poincaré and charge operators have the correct algebra.

We observe that the structure of the Poincaré generators of the bubble in terms of the normal mode amplitudes is quite similar to the corresponding structure in the Neveu-Schwartz model. The quantum theory which we develop, despite the differences in its interpretation, looks quite similar to the Neveu-Schwartz theory in three space-time dimensions. We will discuss these similarities and differences in more detail later.

We require, specifically, that the commutation relations guarantee:

(1) That the quark have fermion number 1,

$$[Q, b_m^+] = b_m^+$$

(2) That the canonical momentum and angular momentum operators, (39) and (42), satisfy the correct Poincare algebra.

(3) That the constraint (38) imposed weakly on states is consistent with Poincaré invariance. That is, that $\mathcal{L}_0 - \bar{\mathcal{L}}_0$ commute with all the Poincaré generators.

Rather than outline the derivation of the correct commutation relations from the requirements (1), (2), and (3), we will begin with the fundamental commutation relations and sketch the verification of the operator algebra.

We take the commutation relations of the normal mode amplitudes to be:

$$(44) \quad \begin{aligned} [a_n, a_k] &= n\,\delta_{n,-k} \\ [c_n, c_k] &= n\,\delta_{n,-k} \end{aligned}$$

$$(45) \quad \{b_m, b_n^+\} = \delta_{m,n}$$

with all other commutators vanishing. Equation (45) gives the correct charge commutator immediately.

$$[Q, b_m^+] = b_m^+$$

From (44) and (45), we have, formally,

$$(46) \quad [\mathcal{L}_n, \mathcal{L}_m] = (n-m)\,\mathcal{L}_{n+m}$$

$$(47) \quad [\bar{\mathcal{L}}_n, \bar{\mathcal{L}}_m] = (n-m)\,\bar{\mathcal{L}}_{n+m}$$

The relations (46) and (47) do not, however, survive normal ordering. If we choose to normal order all operators with respect to the "vacuum," $|0\rangle$, defined by

$$(48) \quad \left.\begin{aligned} a_n|0\rangle &= 0 \\ c_n|0\rangle &= 0 \end{aligned}\right\} n>0 \qquad \left.\begin{aligned} b_m|0\rangle &= 0 \\ d_m|0\rangle &= 0 \end{aligned}\right\} m>0$$

where $d_m \equiv b_{-m}^+$

Then we find

$$(49) \quad \begin{aligned} :\mathcal{L}_n: &= \mathcal{L}_n - \Lambda\,\delta_{n,0} \\ :\bar{\mathcal{L}}_n: &= \bar{\mathcal{L}}_n - \bar{\Lambda}\,\delta_{n,0} \end{aligned}$$

where $\Lambda, \bar{\Lambda}$ are the (infinite) constants:

$$\Lambda = \sum_{n>0} \tfrac{1}{2} n - \sum_{m>0} m$$

$$\bar{\Lambda} = \sum_{n>0} \tfrac{1}{2} n$$

The commutation relations of the normal ordered operators are

$$(46') \quad [:\mathcal{L}_n:, :\mathcal{L}_m:] = (n-m) :\mathcal{L}_{n+m}: + \frac{1}{6}\delta_{n,-m}(n^3-n)$$

$$(47') \quad [:\bar{\mathcal{L}}_n:, :\bar{\mathcal{L}}_m:] = (n-m) :\bar{\mathcal{L}}_{n+m}: + \frac{1}{12}\delta_{n,-m}(n^3-n)$$

We also have

$$(50) \quad \begin{aligned} [\mathcal{L}_n, a_K] &= -K a_{K+n} \\ [\bar{\mathcal{L}}_n, c_K] &= -K c_{K+n} \end{aligned}$$

$$(51) \quad \begin{aligned} [\mathcal{L}_n, b_m] &= -(m+\tfrac{n}{2}) b_{m+n} \\ [\mathcal{L}_n, b_m^+] &= (m-\tfrac{n}{2}) b_{m-n}^+ \end{aligned}$$

We must also define commutators involving the momenta P^ν and coordinates $X^\nu(\tau)$. These are determined by the requirement that P^ν generate translations of the bubble. The commutation relations must be such that

$$(52) \quad [\delta G, R^\nu] = i\,\delta R^\nu$$

where $\delta G = P^\mu \delta a_\mu$ is the generator of the infinitesimal translation δa^ν, and δR^ν is the infinitesimal shift in R^ν. The representation of the bubble surface we have chosen is not Poincaré invariant. In order to maintain the gauge condition

$$R^+(\tau,\sigma) = x^+(\tau) = \frac{p^+}{2c}\tau$$

we must perform a conformal transformation along with the translation:

$$\tau \rightarrow \tau - \frac{2c}{p^+}\,\delta a^+$$

Thus, the total shift δR^ν is

$$\delta R^\nu = \delta a^\nu - \frac{2c}{p^+}\frac{\partial R^\nu}{\partial \tau}\,\delta a^+$$

Through (52) this gives the commutation relations of the momenta p^ν and the coordinates variables $x^\nu(\tau)$. The non-vanishing commutators are:

$$(53) \quad \begin{aligned} [p^1, x^1] &= -i \\ [p^-, x^1] &= -\frac{2ip^1}{p^+} \\ [p^+, x^-] &= 2i \\ [p^-, x^-] &= -\frac{2ip^-}{p^+} \end{aligned}$$

In deriving (53), we note that the relations (50), (51) have been used. For example,

$$\frac{\partial}{\partial \tau}\Big(\sum_n a_n e^{-2\pi i n\tau}\Big) = 2\pi i\,[\mathcal{L}_0, \sum_n a_n e^{-2\pi i n\tau}]$$

We take the classical expressions (42) as the definition of the Lorentz generators, with the additional assumption that products of non-commuting operators in (42) are to be hermitean symmetrized. For example, we take

$$M^{+-} = x^+p^- - \frac{1}{2}[x^-p^+ + p^+x^-]$$

It is straightforward to show that the Poincaré algebra is satisfied, both formally and for the normal ordered operators. We note that in the case of dual theories, whose quantization is virtually indentical to the bubble quantization we have described, it is well known that normal ordering can lead to anomolies in the Lorentz algebra. These anomolies occur in the commutators between the various "transverse" Lorentz generators of the string, M^{i-}. For a string-like object imbedded in only three space-time dimensions, however, there is but one transverse mode (i=1), and no such anomolies can arise.

From the commutation relations (46) and (47), it can easily be verified that $\mathcal{L}_0 - \bar{\mathcal{L}}_0$ commutes with the charge and with all of the Poincaré generators. Thus, the constraint that physical states obey

$$(54) \qquad (\mathcal{L}_0 - \bar{\mathcal{L}}_0)\,|\psi\rangle = 0$$

is consistent with Poincaré invariance of the theory.

We may also introduce an operator which corresponds to the spin of the bubble:

$$W = \frac{1}{2}\varepsilon_{\mu\nu\lambda}\, p^{\mu} M^{\nu\lambda}$$
$$= -4\pi\sqrt{\pi c}\sum_{n\neq 0}\frac{i}{2\pi n}\left[a_n\mathcal{L}_{-n} + \mathcal{L}_{-n}a_n + c_n\bar{\mathcal{L}}_{-n} + \bar{\mathcal{L}}_{-n}c_n\right]$$

Classically, in the rest frame of the bubble,

$$W = p^0 M^{12} = \mathcal{M} J_3$$

W commutes with both Q and $\mathcal{M}^2$, and is unchanged by normal ordering.

We have now exhibited a self-consistent quantized operator algebra corresponding to the bubble theory in three space-time dimensions. There is a conserved fermion number, Q, and the theory has been shown to be Poincaré invariant.

It remains for us to analyze the physical states of the theory. At this point, we run into rather unpleasant, and unavoidable, ambiguities associated with the normal ordering of operators. In terms of the infinite constants Λ and $\bar{\Lambda}$, the normal ordered operators are:

$$(56) \qquad \mathcal{M}^2 = 4\pi c\,[\,:\mathcal{L}_0 + \bar{\mathcal{L}}_0: - a_0^2 + \Lambda + \bar{\Lambda}\,]$$

$$(57) \qquad \mathcal{L}_0 - \bar{\mathcal{L}}_0 = \;:\mathcal{L}_0 - \bar{\mathcal{L}}_0: + \Lambda - \bar{\Lambda}$$

$$(58) \qquad \begin{aligned} :\mathcal{L}_0: &= \sum_{n>0} a_n^+ a_n + \tfrac{1}{2}a_0^2 + \sum_{m>0} m\,(b_m^+ b_m + d_m^+ d_m) \\ :\bar{\mathcal{L}}_0: &= \sum_{n>0} c_n^+ c_n + \tfrac{1}{2}a_0^2 \end{aligned}$$

$$(59) \qquad :Q: = \sum_{m>0}(b_m^+ b_m - d_m^+ d_m)$$

$$(60) \qquad :W: = W$$

States of definite particle number are clearly eigenstates of $:Q:$, $:\mathcal{L}_0:$, and $:\bar{\mathcal{L}}_0:$.

The Hilbert space of the theory becomes well defined only after we have assigned finite values to the constants

$$(61) \qquad \begin{aligned} \lambda_1 &= \Lambda + \bar{\Lambda} \\ \lambda_2 &= \Lambda - \bar{\Lambda} \end{aligned}$$

The actual values we choose for λ_1 and λ_2 are completely arbitrary-- they are unconstrained by the operator algebra. Further, no matter what the value of λ_2, the condition (54) will place severe restrictions on the spectrum. We see from (58) that $:\bar{\mathcal{L}}_0 - \frac{1}{2}a_0^2:$ has only integral eigenvalues, while $:\mathcal{L}_0 - \frac{1}{2}a_0^2:$ has integral eigenvalues if Q is even and half-odd integral eigenvalues if Q is odd. Therefore, if λ_2 is an integer, we can form only states of even fermion number ("mesons"). If λ_2 is a half-odd integer, all states must have odd fermion number ("baryons"). Whether or not such ambiguities remain in other possible variants of the theory, such as the four-component theory, is presently unknown.

We will discuss only two of the infinite number of possible choices of λ_1 and λ_2. Our guide in the selection of λ_1 and λ_2 will be the classical theory. The operators $\mathcal{L}_0$ and $\bar{\mathcal{L}}_0$ appear on an equal footing in the mass operator. $\mathcal{L}_0$ is the contribution of the fermion and the "a" surface excitations to the bubble energy, $\bar{\mathcal{L}}_0$ is the contribution of the "c" surface excitations. In the

classical theory, $\mathcal{L}_0$ and $\bar{\mathcal{L}}_0$ contribute equally to the mass, and we can write:

$$\mathcal{M}^2 = 8\pi C \mathcal{L}_0$$

In the case of the static classical bubble, there are no "a" excitations, and the "c" excitations are forced to be non-vanishing in the presence of any fermions to satisfy (38). We maintain these features in any quantum theory defined by choosing $\lambda_1 = -\lambda_2 < 0$. We will consider the spectrum of the simplest such cases,

$$(63) \qquad \lambda_1 = -\lambda_2 = -\tfrac{1}{2} \qquad \text{"baryons"}$$

$$(64) \qquad \lambda_1 = -\lambda_2 = 0 \qquad \text{"mesons"}$$

In the case of "baryons", $\lambda_2 = -\lambda_1 = 1/2$, the mass levels of the Q=1 states are exactly those of the classical static theory:

$$\mathcal{M}^2 = 8\pi C \ell_0$$

where ℓ_0 = eigenvalue of $:\mathcal{L}_0 - \frac{1}{2}a_0^2:$, a positive half-odd integer. We see that the degeneracy of each of these levels is finite. Table IV lists the degenerate states comprising the first few levels.

The breaking of the semi-classical degeneracy of levels over all bubble shapes is an easily understood quantum effect. Classically, the only constraint on $\{c_n\}$ is (38). Because each classical normal mode coefficient can take on a continuum of values, this constraint can be satisfied by an uncountably infinite number of combinations of $\{C_n\}$. In the quantum theory, however, the energy associated with each mode becomes discrete, so there are only a finite number of combinations of occupation numbers which sum to any given finite energy, ℓ_0 +1/2. In the quantum theory, the softness of the bubble becomes apparent in two ways. First, the size of the fluctuations in the surface coordinates is always comparable to the size of the bubble itself; a result which follows from the absence of any dimensionless parameters which might serve to set an independent scale for the size of fluctuations. Also, as simple combinatorics indicates, as the excitation energy increases, the degeneracy of the levels increases exponentially as $e^{m^2/(8\pi C)}$.

The spin operator, W, can be diagonalized simultaneously with Q and m^2. Table IV also indicates the the eigenstates and eigenvalues of W among the three lowest levels of the Q=1 "baryon" spectrum. Because of the normal ordering ambiguities, we have not been able to relate the eigenvalues of W directly to a "physical" spin of the bubble. The source of the difficulty is precisely the same as that which leaves λ_1 and λ_2 undetermined. There are no non-trivial commutation relations between W and other operators of the theory which might serve to fix the scale

of W when the theory is made finite by normal ordering. More concretely, we observe that classically the spin can be written

$$J_3 = \frac{W}{\sqrt{m^2}}$$

In the normal ordered quantum theory, we have no analogous result.

Next, we consider the spectrum of "mesons," taking $\lambda_1 = -\lambda_2 = 0$. The mass levels are:

$$m^2 = 8\pi C \ell_0$$

$$\ell_0 = 0, 1, 2, \ldots$$

The states corresponding to the first three meson levels are given in Table V, along with the corresponding eigenvalues and eigenstates of W. The "meson" spectrum has many of the qualitative features of the "baryon" spectrum. We remark upon only two special aspects of it. First, the lowest state is the state that we have called the "vacuum." This "vacuum" is not, then, the usual vacuum state of a multi-particle theory. It is, rather, the lowest lying state in the spectrum of a single particle with many possible internal excitations. With the choice of normal ordering parameters we have made, the "vacuum" is a massless bubble state, and has no classical analog. Second, we note that the meson spectrum contains states which correspond to bubbles containing no quarks at all. These are purely surface excitations, and are analogous to the excitations of a closed dual string in three dimensions.

Because the normal ordering ambiguities are so severe, we learn little more of use from an examination of further details of the spectrum. We turn instead to a discussion of several qualitative features of the quantum theory which are suggestive, perhaps, of features a quantum theory of the four dimensional bubble might possess.

The quantum theory we have constructed is the theory of a single "particle," which has many possible internal excitations. In a theory which is to reflect more accurately the properties of the real world, we must have mechanisms by which these particles scatter and are created and destroyed. One might hope that, in analogy with the string theory, such mechanisms are already implicit in the formulation of the classical bubble theory.

An attractive classical picture of bubble-bubble interactions is that bubbles interact with each other by fusing or fissioning when their surfaces touch. (Fig 14) Such a picture is the analogy in bubble theory of the fission and fusion of MIT bags or of dual strings. Generally, string, bubble, and bag theories have classical solutions which correspond to such processes. For the bubble, such a solution would be characterized by the existence of surface singularities at which the evolution of

the classical bubble becomes indeterminate. Of the possible solutions for the evolution of the system is one in which a single bubble state emerges and others which correspond to the formation of new bubbles.

Mandelstam has shown, in the string theory, that string-string scattering amplitudes can be computed via a path integral which includes paths corresponding to the classical fission and fusion of strings[18]. It seems evident that such a procedure might be formulated for the bubble in three space-time dimensions.

In principle at least, we are in a position to compute the form factor of the bubble in three dimensions. The operator whose matrix elements give the form factor is the fourier transform of the current density:

$$J^{\mu}(q) = :\int d^2u\,\sqrt{-g}\; e^{-q_{\mu}R^{\mu}(u)}\,\bar{\psi}(u)\gamma^{\mu}\psi(u):$$

The normal ordering difficulties which arise in any attempt to evaluate a finite matrix element of this operator are severe, and have not been resolved[19].

As we have observed, the bubble in three space-time dimensions is a two dimensional object closely resembling a closed string. The only conventional dual model which includes fermion-like degrees of freedom is the Neveu-Schwartz model[17]. The Neveu-Schwartz model, when interpreted geometrically[20], is the theory of an open string with which a Majorana spinor field is associated. The quanta of such a Majorana field are not fermions. The states of the Neveu-Schwartz theory are all "mesons." The bubble, as we have seen, is analogous to a closed string upon which a fermion is trapped. There is a conserved charge, and bubble theories exist whose states are "baryons."

For both the string and the bubble, the three dimensional case lacks sufficient resemblance to the real world. Because there is but one component of angular momentum, and hence no algebraic constraints on the normal ordering terms, the spectrum of states remains ambiguous. String models are extended to higher dimensions in the most obvious way. That is, the string in higher dimensions is regarded as a two dimensional hypersurface imbedded in some higher dimensional Minkowski space. One then finds that the quantum theory is anomoly free only in peculiar space-time dimensions. (10 for the Neveu-Schwartz model, 26 for the conventional dual string).

The physical picture of the bubble, which becomes a closed string in three dimensions, suggests that perhaps the proper generalization of the string from three to four dimensions is the BCDWY theory. Whether a quantum theory of the four dimensional BCDWY model can be constructed which might make this observation useful is presently an open question.

Chapter VI

This chapter briefly summarizes the principal ideas of this thesis and discusses some possible extensions of the results and techniques developed here.

We have shown that the classical BCDWY field theory reduces to the theory of bubbles for low mass states in the strong coupling limit. This bubble theory is very different in appearance from conventional field theories. As a theory of hadrons as extended objects, it clearly belongs to the same "family" as do the theories of the string and the bag. The string, bubble, and bag theories picture hadrons, respectively, as two, three, and four dimensional objects imbedded in four dimensional Minkowski space. Our geometric formulation of the bubble theory shares with general relativity the qualitative feature of being a theory of non-interacting matter (quarks) moving in a curved space-time whose geometry is determined by the distribution of energy-momentum.

We have used the bubble theory to characterize the hadronic spectrum predicted in the BCDWY model. The "approximations" used for these computations largely ignored the quantum mechanical nature of the surface degrees of freedom of the bubble. Interactions responsible for SU(3) breaking, electromagnetic interactions, and possible residual color interactions (e.g., color charge fluctuation energies) were also neglected. Nevertheless, the resulting level spacing is largely consistent with the data. Our sole attempt to take into account the quantum effects of the surface in four dimensions resulted in an estimate of the mass of the first radial excitation of the spherical bubble, which, for the baryons, agrees with the mass of the Roper resonance to a few percent. Though the spectrum we have estimated is not inconsistent with the data, the calculations we have presented here hardly constitute compelling evidence that the bubble theory provides the "true" picture of low-lying single hadron states.

We demonstrated that, classically, the bubble is a very "soft" dynamical system. We discussed qualitatively how we might expect this softness to be manifested in a fully quantized bubble theory in four dimensions. We argued that form factors of hadrons need not show a dip structure that reflects the thin shell nature of their quark densities in the static picture, and that structure functions might well exhibit scaling in deep inelastic lepton scattering. Because the four dimensional quantum problem is unsolved, we have been unable to back up these guesses with a formal mathematical analysis. We mention these ideas only to show that the thin shell picture of the BCDWY theory is not, a

priori, inconsistent with well known experimental results.

We discovered that the field equations of the bubble theory can be solved exactly and completely for the single bubble in three space-time dimensions. The integrability of the equations follows from the special simplicity of the geometry of two dimensional surfaces. That the bubble theory is solvable in three dimensions is analogous to the result that the MIT bag is exactly solvable in two dimensions, where it too becomes a two dimensional object. From the set of all classical solutions to the theory, a Poincaré invariant quantum theory of the three dimensional bubble was explicitly constructed. Though the spectrum of this theory is not uniquely determined by its operator algebra alone, the spectrum becomes well-defined after two arbitrary normal ordering parameters are specified. The normal ordered theory is free of ghosts and divergences. However, the theory we have constructed is the quantum theory of only a single bubble. It apparently does not allow for bubble-bubble interactions. Whether a ghost-free quantum theory of interacting bubbles can be constructed is not yet known.

The states of the three dimensional quantum theory have the character anticipated for the quantum states of a "soft" object: the quantum fluctuations of the bubble surface are of the same order of magnitude as the size of the bubble, and the density of states grows exponentially. In principle, form factors can be expressed in terms of the matrix elements of the operators of this quantum theory. However, it is not clear whether the calculation is free of divergences.

The work described in this thesis leaves many interesting and important questions about the BCDWY model unanswered. There are also, evidently, many dangling loose ends which it might be profitable to pursue. In the following brief discussion, we will identify a few of these, with the hope that we can at least understand what might be needed to resolve these questions and what we might learn from the answers.

First, we still have not determined whether the BCDWY model provides a numerically accurate phenomenological description of low-lying hadronic states. What is required is a better approximation to the bubble theory than used in this thesis. One must take into account quantum effects due to surface motion and the filled Fermi sea and also allow for SU(3) breaking quark-quark interactions within the bubble. A first step toward understanding the effects of surface fluctuations on bubble energies might be to treat the surface motion in perturbation theory about static solutions to the theory. Any such treatment would be, at best, a crude approximation, as there is actually no small

parameter available in which to expand. However, such a procedure has the advantages that it can be done, with sufficient patience, on a computer, and it should take into account, at least qualitatively, the effects of surface fluctuations.

The calculation of form factors and structure functions in the four dimensional theory requires a more complete and detailed understanding of the quantum mechanics of bubbles than is presently available. It has been argued in the case of the MIT bag theory that the "cavity approximation," in which the bag is treated as a rigid object, is adequate for the approximate calculation of structure functions of the hadron bag states. It is quite clear, in the case of the bubble, that surface motion cannot be neglected in such a fashion -- the bubble is not at all rigid.

As discussed in this thesis, the geometric formulation of the bubble theory arises from the consideration of the strong coupling limit of the BCDWY field theory. However, the bubble theory can obviously be formulated, at least classically, without reference to the BCDWY model or the interpretation of the bubble surface as a domain boundary. Once freed of this interpretation, it is clear that we can write down many possible theories of fields defined on surfaces which are analogous to the bubble theory. The simplest cases which might be of interest are those of bubbles with scalar or vector fields defined on them. The motivation for discussing such possibilities is twofold. First, we may be able to construct more sophisticated models of quark binding in which there are non-trivial quark-quark interactions mediated by surface vector fields. An interaction mediated by a surface field has a technical advantage over one mediated by fields which extend away from the bubble surface in that it can still be discussed in the intrinsic surface coordinates of the original bubble theory. A second reason to consider such theories is that it is possible that they may be more easily analyzed, in four dimensions, than is the bubble containing quarks. We may be able to get some insight into the effects of surface motion and its relation to the surface field energy distribution in such a simpler version of the bubble theory.

Perhaps the most intriguing question suggested by this work arises from the original observation that the classical field theory which defines the BCDWY model becomes equivalent to the bubble theory for low-lying states in the strong coupling limit. Superficially, these seem to be theories of quite a different sort. The BCDWY field theory is a conventional theory of interacting "quarks" and "mesons." A quantized version of the theory exists and is perturbatively renormalizable. The quantum theory

presumably manifests the features usually associated with a renormalizable field theory: Lorentz invariance, local causality, analyticity and crossing symmetry of scattering amplitudes. As discussed in the Introduction, though renormalizable quantum field theories have such desirable features, it has remained a very difficult problem to construct a calculable field theoretic model of hadron structure. The strong coupling bound state problem appears very complex when analyzed using conventional field theoretic techniques. Indeed, this observation would appear to hold, at first glance, in the case of the BCDWY model. The low energy bound states which are to be interpreted as hadrons are very complicated superpositions of the bare quark and meson states. However, as we have seen in this thesis, the bubble theory provides an equivalent classical description, in the strong coupling limit, in which these bound states are seen to be "simple."

In and of itself, the classical bubble theory describes the dynamics of an extended geometric object upon which quark fields are defined. Whether a canonical quantum theory of the bubble can be defined, free of ghosts and divergences, is not clear, a priori. Because the bubble theory is a "gauge" theory and involves non-polynomial interactions among the fundamental fields defining the geometry, naive canonical quantization procedures fail to insure a sensible quantum theory. Further, it is not clear how bubble-bubble interactions are to be treated in the quantized theory. In the three dimensional example, where we have been able to sidestep the problems associated with canonical quantization, the quantum theory we derived is that of a single bubble. That there are no interactions in this theory can be traced, in part at least, to our pervasive assumption that all bubble states have the topology of a single hypertube. Classical scattering solutions must correspond to more complicated topologies. One expects that a quantum bubble theory whose currents are "local" in Minkowski space must include non-trivial bubble-bubble interactions. We do not yet have such a quantum theory in hand.

An obvious question is whether the classical equivalence of the bubble theory and the BCDWY field theory carries over to the quantum case. That is, is there a strong coupling limit of the BCDWY field theory which corresponds to a fully quantized, interacting theory of bubbles? We do not know the answer to this question, though the tools we need to investigate it may be in hand. If such an equivalence could be demonstrated, it would represent, we believe, a definite step forward in our understanding of strong coupling field theory, the quantum dynamics of extended objects, and their relation to the problems of hadronic structure.

Appendix A

In this appendix, the differential geometry of hypersurfaces will be discussed in more detail. The ideas presented here are quite standard. The reader is urged to consult any of the references cited at the end of this thesis for more exhaustive analyses.

The surfaces of interest to us are n-1 dimensional time-like hypertubes imbedded in an n dimensional Minkowski space. We represent such a hypersurface by giving its points as functions of n-1 independent "internal coordinates," u^α :

$$\underline{\text{surface}}: \quad x^\mu = R^\mu(u^\alpha) \qquad \begin{matrix} \mu = 0,\dots,n-1 \\ \alpha = 0,\dots,n-2 \end{matrix}$$

The internal coordinates are quite arbitrary, except that we require that the functions $R^\mu(u^\alpha)$ be differentiable. Implicitly, we assume that the surface is sufficiently "smooth" that such coordinate systems can always be found. The geometric description of the surface must be "covariant" under general coordinate transformations:

$$u^\alpha = u^\alpha(v^\beta) \;;\quad R^\mu(u^\alpha) \longrightarrow \tilde{R}^\mu(v^\beta) \equiv R^\mu(u^\alpha(v^\beta))$$

where the functions $u^\alpha(v^\beta)$ are differentiable.

Such a hypersurface is a Riemannian manifold whose intrinsic geometry may be characterized by a metric tensor and the associated affine connection and curvature tensor. We assume that the reader has some familiarity with the basic concepts of Riemannian geometry. The discussion of this appendix will focus on the relation of the Riemannian description of the internal geometry of the surface and the nature of its imbedding in Minkowski space.

A basis for the local description of the imbedding of the hypersurface is the set of "tangent vectors":

$$(1) \qquad \tau_\alpha^{\ \mu} \equiv \frac{\partial R^\mu}{\partial u^\alpha} = R^\mu{}_{|\alpha}$$

where we adopt the notation $A_{|\alpha} \equiv \frac{\partial A}{\partial u^\alpha}$, for any quantity, A. The n-1 vectors, $\tau_0^{\ \mu} \ldots \tau_{n-2}^{\ \mu}$, form a linearly independent set of vectors in Minkowski space which span the tangent space to the surface at the point $R^\mu(u)$. The quantities $\tau_\alpha^{\ \mu}$ also constitute a mixed tensor ("n-1 bien") which transforms as a vector under Lorentz transformations, and independently as a vector under general coordinate transformations.

The metric in Minkowski space, $\eta_{\mu\nu}$, induces a metric on the hypersurface,

$$(2) \qquad g_{\alpha\beta} = \eta_{\mu\nu}\, \tau_\alpha^{\ \mu}\, \tau_\beta^{\ \nu} = \tau_\alpha \cdot \tau_\beta$$

$g_{\alpha\beta}$ is so defined that for any two tangent vectors, $U^\mu = U^\alpha \tau_\alpha^{\ \mu}$ and $V^\mu = V^\beta \tau_\beta^{\ \mu}$, the dot product $U \cdot V$ can be

written in terms of the components in the tangent basis as

$$U \cdot V = U^\alpha V^\beta\, \tau_\alpha \cdot \tau_\beta = U^\alpha V^\beta g_{\alpha\beta}$$

The Riemannian geometry of the hypersurface derives from the induced metric, $g_{\alpha\beta}$. $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$ will be used, in the usual way, to transform between covariant and contravariant tensors (in the indices $\alpha, \beta, \gamma, \delta \ldots$). We note that the invariant element of surface area is

$$(3) \qquad da = \sqrt{|g|}\; d^{n-1}u,$$

where g=det($g_{\alpha\beta}$).

The imbedding of the surface in Minkowski space is also characterized by the unit normal vector, $n^\mu(u^\alpha)$.

$$(4) \qquad \begin{array}{ll} (a) & n \cdot \tau_\alpha = 0 \quad \text{for all } \alpha \\ (b) & n^2 = -1 \end{array}$$

That n^μ is space-like may be taken as the definition of a "time-like" hypertube. Equations (4) determine n^μ only up to an overall sign. Because the surfaces we are concerned with are closed spacially, we can specify n^μ uniquely by choosing it to be the outward unit normal to the surface at each point. The n vectors, $\{\tau_\alpha^{\ \mu}, n^\mu\}$, are linearly independent and therefore constitute a basis for all Minkowski vectors at the point u^α.

$$(5) \qquad (\tau^\alpha)^\mu (\tau_\alpha)^\nu - n^\mu n^\nu = \eta^{\mu\nu}$$

Because n^ν is a unit vector, its derivatives with respect to u^α must be tangent vectors.

$$n^\nu{}_{|\alpha} = h_{\alpha\beta}\, \tau^{\beta\nu}$$

where

$$(6) \qquad h_{\alpha\beta} = n_{|\alpha}\cdot\tau_\beta = -n\cdot\tau_{\beta|\alpha} = -n\cdot\tau_{\alpha|\beta} = h_{\beta\alpha}$$

The eigenvalues of $h^\alpha{}_\beta$ are the inverse principal radii of curvature of the surface. In the time-like principal direction of curvature, the inverse radius is the normal spacial acceleration of a given point on the surface in its local rest frame. The quantity k, which appears in the surface Dirac equation, is proportional to the mean curvature of the surface:

$$k = \tfrac{1}{2}(\partial_{11})_\nu n^\nu = \tfrac{1}{2}\tau^\alpha{}_\nu\, n^\nu{}_{|\alpha} = \tfrac{1}{2} h^\alpha{}_\alpha$$

The tensor, $h_{\alpha\beta}$, gives a representation of the local imbedding of the surface in flat space in terms of the internal coordinates. $g_{\alpha\beta}$ and $h_{\alpha\beta}$ are sometimes referred to, respectively, as the first and second fundamental forms of the surface.

On a Riemannian manifold with metric $g_{\alpha\beta}$, there is a natural law of parallel transport which leaves inner products invariant.

$$\delta V^\alpha = -\left\{ {\alpha \atop \beta\gamma} \right\} V^\beta \delta u^\gamma$$

for an infinitesimal translation, δu^γ, of a vector V^α. The "Christoffel Symbol" is:

$$(7) \qquad \left\{ {\alpha \atop \beta\gamma} \right\} = \tfrac{1}{2} g^{\alpha\delta}\left[g_{\beta\delta|\gamma} + g_{\delta\gamma|\beta} - g_{\beta\gamma|\delta} \right]$$

For a hypersurface imbedded in flat space, this law has a simple geometric interpretation. Let V^ν be a tangent vector at the point u^α. If we slide V^ν along the surface to the point $u^\alpha + \delta u^\alpha$, V^ν is shifted infinitesimally in the normal direction so as to remain a tangent vector at the new point.

$$V^\nu \longrightarrow V^\nu + \delta V^\nu$$

so that

$$(n_\nu + \delta n_\nu)\cdot(V^\nu + \delta V^\nu) = 0$$

$$\text{whence:} \quad \delta V^\nu = n^\nu(\delta n\cdot V)$$

In terms of the components of V^ν in the tangent basis, we have,

$$\begin{aligned} \delta V^\alpha &= \delta V^\nu \tau^\alpha{}_\nu + V^\nu \delta\tau^\alpha{}_\nu \\ &= \tau^\alpha{}_{|\gamma}\cdot\tau_\beta\, V^\beta \delta u^\gamma \end{aligned}$$

So we identify,

$$(8) \qquad \left\{ {\alpha \atop \beta\gamma} \right\} = -\tau^\alpha{}_{|\gamma}\cdot\tau_\beta = \tau^\alpha\cdot\tau_{\beta|\gamma}$$

It is straightforward to check that (18) follows from (2) and (7). Using (6) and (8), we have the useful identity,

$$(9) \qquad \tau_{\alpha|\beta} = \left\{ {\gamma \atop \alpha\beta} \right\} \tau_\gamma + h_{\alpha\beta}\, n$$

We also note from (7) that

$$(10) = \quad \left\{ {\alpha \atop \alpha\beta} \right\} = (\ln\sqrt{|g|})_{|\beta}$$

Given the law of parallel transport, we can define covariant derivatives of tensor fields in the standard fashion:

$$V^\alpha(u+\delta u) = V^\alpha(u) + [V^\alpha{}_{\|\beta} - \left\{ {\alpha \atop \beta\gamma} \right\} V^\gamma]\,\delta u^\beta$$

or

$$(11) \qquad V^\alpha{}_{\|\beta} = V^\alpha{}_{|\beta} + \left\{ {\alpha \atop \beta\gamma} \right\} V^\gamma$$

We note, from (9),

$$\tau_{\alpha\|\beta} = \tau_{\alpha|\beta} - \left\{ {\gamma \atop \alpha\beta} \right\} \tau_\gamma = h_{\alpha\beta}\, n$$

The Riemann curvature tensor of the surface describes its "intrinsic" curvature. It arises from the commutator of covariant derivatives. For example:

$$V^\alpha{}_{\|\beta\|\gamma} - V^\alpha{}_{\|\gamma\|\beta} = R^\alpha{}_{\delta\beta\gamma}\, V^\delta$$

where

$$(12) \quad R^\alpha{}_{\beta\gamma\delta} \equiv \left\{ {\alpha \atop \beta\gamma} \right\}_{|\delta} - \left\{ {\alpha \atop \beta\delta} \right\}_{|\gamma} + \left\{ {\alpha \atop \delta\tau} \right\}\left\{ {\tau \atop \beta\gamma} \right\} - \left\{ {\alpha \atop \gamma\tau} \right\}\left\{ {\tau \atop \beta\delta} \right\}$$

A straightforward calculation using (8) and (9) gives

$$(13) \qquad R_{\alpha\beta\gamma\delta} = h_{\alpha\gamma} h_{\beta\delta} - h_{\alpha\delta} h_{\beta\gamma}$$

Because $n_{|\alpha|\beta} = n_{|\beta|\alpha}$, we also have the relation

$$(14) \qquad h_{\alpha\beta\|\gamma} = h_{\alpha\gamma\|\beta}$$

It can be shown that, if $h_{\alpha\beta}$ is any tensor satisfying (13) and (14) on a Riemannian manifold with metric $g_{\alpha\beta}$, then there exists a hypersurface (unique up to Lorentz transformations) whose fundamental forms are $g_{\alpha\beta}$ and $h_{\alpha\beta}$. This result is the Gauss-Codazzi theorem. It allows us to discuss the geometry of the imbedded hypersurface purely in terms of tensor fields in the internal coordinate space.

We also consider the properties of spinor fields defined on the hypersurface. The spinor fields of interest are taken to be spinors in Minkowski space only. They are scalars under general coordinate transformations. We represent the gamma matrices on the surface by the quantities:

$$\not{\tau}_\alpha = \gamma_\mu \tau_\alpha{}^\mu$$
$$\not{n} = \gamma_\mu n^\mu$$

we have

$$(15) \qquad \begin{aligned} &\{\not{n}, \not{\tau}_\alpha\} = 0 \\ &\{\not{\tau}_\alpha, \not{\tau}_\beta\} = 2 g_{\alpha\beta} \end{aligned}$$

There exists a natural parallel transport of spinors along the surface, defined in such a way that the quantity $\bar\psi \gamma^\nu \psi$ transports as a vector when the spinor ψ is parallel transported. This is:

$$(16) \qquad \delta\Psi = -\frac{i}{2}\sigma^{\mu\nu} n_\mu \delta n_\nu \Psi$$

The matrix $1-\frac{i}{2}\sigma^{\mu\nu}n_\mu\delta n_\nu$ is just the infinitesimal Lorentz rotation matrix corresponding to the rotation of the normal vector under the translation, δu^γ, along the surface.

We can form a "covariant derivative" for spinors using this law of parallel transport:

$$(17) \qquad D_\alpha\Psi = [\partial_\alpha + \frac{i}{2}\sigma^{\mu\nu} n_\mu n_{\nu|\alpha}]\Psi$$

The surface operator corresponding to the free Dirac operator, $\partial\!\!\!/$, in Minkowski space, is:

$$(18) \qquad D\!\!\!/ \equiv \gamma^\alpha D_\alpha = \gamma^\alpha\partial_\alpha + \frac{i}{2}\gamma^\alpha\sigma^{\mu\nu}n_\mu n_{\nu|\alpha}$$

With some algebraic manipulation, this can be re-written,

$$(19) \qquad D\!\!\!/ = \partial\!\!\!/_{\parallel} + k\, n\!\!\!/$$

where: $\partial\!\!\!/_{\parallel} \equiv \gamma^\alpha\partial_\alpha$

We note that, from (6),(15), and (19),

$$D\!\!\!/\, n\!\!\!/ = -n\!\!\!/\, D\!\!\!/$$

Thus, the free surface Dirac equation, $iD\!\!\!/\Psi = 0$, is consistent with the constraint $i n\!\!\!/\Psi = \Psi$.

Finally, we note the structure of conservation laws for currents defined on the surface. If K^α is any divergenceless current,

$$0 = K^\alpha{}_{\|\alpha} = K^\alpha{}_{|\alpha} + \left\{{\alpha \atop \alpha\beta}\right\}K^\beta = \frac{1}{\sqrt{|g|}}(\sqrt{|g|}\,K^\alpha)_{|\alpha}$$

then we have

$$0 = \int_\Omega d^{n-1}u\sqrt{|g|}\;K^\alpha{}_{\|\alpha} = \int_{\partial\Omega} d\Sigma_\alpha\sqrt{|g|}\;K^\alpha$$

where Ω is any region of the surface, $\partial\Omega$ is its oriented boundary, and $d\Sigma_\alpha$ is, for each α, the n-2 form defined by

$$d\Sigma_{(\alpha)}\wedge du^{(\alpha)} = d^{n-1}u \qquad \text{(no sum on } \alpha\text{)}$$

On a time-like hypertube, this result implies that the "charge"

$$Q = \int_C d\Sigma_\alpha\sqrt{|g|}\;K^\alpha$$

where C = any closed space-like n-2 dimensional sub-surface, is "conserved" -- that is, independent of C.

Appendix B

This appendix presents the details of the proof, sketched in Chapter III, that further corrections to the surface fields of the static bubble theory correct the energy by terms which vanish in the strong coupling limit ("SCL").

It was shown in Chapter III that, in the presence of the scalar field,

$$(1) \qquad \sigma_0(x) = f \tanh(\sqrt{2\lambda}\, f \xi)$$

the leading finite term in the Dirac energy, E, can be computed by solving the surface Dirac equation. The equation for the σ field follows from the requirement that the total energy,

$$(2) \qquad H = \int d^3x \; \tfrac{1}{2}(\nabla\sigma)^2 + \lambda(\sigma^2 - f^2)^2 + E_{DIRAC}[\sigma]$$

be stationary under all variations of σ. In this appendix, we show that, in the neighborhood of an arbitrary σ field of the form (1), the only local variations, $\delta\sigma$, which result in finite corrections to H in the SCL correspond to changes in the geometry of the bubble. It follows that, if the bubble shape is chosen to minimize H, any further corrections to the σ field induce corrections to H which vanish in the SCL. For convenience, we work in four space-time dimensions. In other dimensions, the power counting in some of our estimates will change, but the form of the arguement is exactly the same.

We adopt the notation introduced in Chapter III and Appendix A. The region of space near the bubble surface will be coordinatized by $\{u^1, u^2, u^3 \equiv \xi\}$:

$$\vec{x}(u) = \vec{R}(u^1,u^2) + \xi\, \hat{n}(u^1,u^2)$$

This coordinatization is characterized by the triplet of vectors,

$$\vec{T}_a \equiv \frac{\partial \vec{x}}{\partial u^a} \qquad , \qquad a = 1,2,3$$

These vectors are analogous to the tangent vectors of an imbedded surface. We have

$$(3) \qquad \begin{aligned} \vec{T}_\alpha &= \vec{z}_\alpha + \xi\, h_{\alpha\beta}\, \vec{z}^{\beta} \qquad \alpha,\beta = 1,2 \\ \vec{T}_3 &= \hat{n} \end{aligned}$$

The Euclidean metric in the new coordinates is:

$$(4) \qquad \begin{aligned} G_{ab} &\equiv \vec{T}_a \cdot \vec{T}_b \\ G_{33} &= 1 \quad , \quad G_{3\alpha} = 0 \\ G_{\alpha\beta} &= g_{\alpha\beta} + 2\xi\, h_{\alpha\beta} + \xi^2 h_{\alpha\gamma} h^{\gamma}{}_{\beta} \end{aligned}$$

For the calculations of this appendix, we need only consider the lowest order expansion of physical and geometric quantities in ξ. Because the densities of physical quantities are exponentially cut-off for $\xi \gtrsim D$, an expansion in ξ becomes, effectively, an expansion in D/R. To first order in ξ/R, we have:

$$G^{33}=1 \quad , G^{3\alpha}=0$$
$$G^{\alpha\beta}= g^{\alpha\beta}-2\xi h^{\alpha\beta}$$
$$\vec{T}^3=\hat{n}, \quad \vec{T}^{\alpha}=\vec{\tau}^{\alpha}-\xi h^{\alpha\beta}\vec{\tau}_\beta$$

Let Ψ_0 be the exact solution to the Dirac equation in the presence of the kink field (1). Let $\mathcal{E}_0$ be the exact Dirac energy. From Chapter III we have:

$$\Psi_0= N[\cosh\sqrt{2\lambda}\, f\xi]^{-\frac{G}{\sqrt{2\lambda}}}\,[\psi_0+\tfrac{1}{G}\psi_1]$$
$$\mathcal{E}_0= E_0+\tfrac{1}{G}E_1+\dots$$
$$(6) \qquad (\gamma^0+i\,\hat{n}\cdot\alpha)\,\psi_0=0$$
$$\frac{\partial}{\partial\xi}\psi_0(u^\alpha,\xi)=-k\,\psi_0(u^\alpha,\xi)$$
$$N^{-2}=\int d\xi\,[\cosh\sqrt{2\lambda}\, f\xi]^{-2\frac{G}{\sqrt{2\lambda}}}$$

$$(-i\,\alpha\cdot\nabla_{\parallel}+ki\,\hat{n}\cdot\alpha)\psi_0+\sigma(\gamma^0+i\hat{n}\cdot\alpha)\psi_1=E_0\psi_0$$

For a small perturbation of the σ field, $\sigma=\sigma_0+\delta\sigma$, the change in the total energy is, to second order in $\delta\sigma$,

$$(7) \qquad \Delta H=\int d^3x\ \tfrac{1}{2}(\nabla\delta\sigma)^2-\nabla^2\sigma_0\,\delta\sigma$$
$$+4\lambda\sigma_0(\sigma_0^2-f^2)\,\delta\sigma+2\lambda(3\sigma_0^2-f^2)(\delta\sigma)^2$$
$$+\Delta\mathcal{E}_0$$

The shift in the Dirac energy may be computed in perturbation theory:

$$(8) \qquad \Delta\mathcal{E}_0=\int d^3x\, G\bar{\Psi}_0\Psi_0+\tfrac{1}{2}\iint d\vec{x}\,d\vec{y}\ K(x,y)\,\delta\sigma(x)\,\delta\sigma(y)$$

where:

$$K(x,y)\equiv 2\sum_{n\neq 0}\frac{G\bar{\Psi}_0(x)\Psi_n(x)\,G\bar{\Psi}_n(y)\Psi_0(y)}{\mathcal{E}_0-\mathcal{E}_n}$$

where $\{\Psi_n,\mathcal{E}_n\}$ is the complete set of Dirac eigenfunctions and energy levels in the presence of the σ_0 field.

The condition that ΔH be minimal over possible $\delta\sigma$ provides an estimate of the correction to the σ field. $\delta\sigma$ must satisfy:

$$(9) \qquad [-\nabla^2+4\lambda\sigma_0(3\sigma_0^2-f^2)]\,\delta\sigma(x)+\int d\vec{y}\ K(x,y)\,\delta\sigma(y)=J(x)$$

where

$$J(x)\equiv\nabla^2\sigma_0+4\lambda\sigma_0(f^2-\sigma_0^2)-G\bar{\Psi}_0\Psi_0$$

The "current" J(x) is exactly the quantity which appears on the left side of the σ field equation and should be zero for an exact solution. The deviation of σ_0 from a true solution induces corrections to the σ field through (9).

In principle, we may solve (9) in terms of the solutions to the associated eigenvalue problem:

$$[-\nabla^2+4\lambda\sigma_0(3\sigma_0^2-f^2)]\,\Sigma_\beta(x)+\int dy\,K(x,y)\,\Sigma_\beta(y)=\Lambda_\beta^2\,\Sigma_\beta(x)$$
$$(10) \qquad \int dx\ \Sigma_\beta^*(x)\,\Sigma_{\beta'}(x)=\delta_{\beta,\beta'}$$

where β is an index labelling both the discrete and continuous parts of the spectrum of (10). In terms of $\{\Sigma_\beta(x),\Lambda_\beta^2\}$, the solution of (9) can be written

(11)

(a) $J_\beta \equiv \int dx\, \Sigma_\beta^*(x) J(x)$

(b) $\delta\sigma(x) = \sum_\beta \frac{1}{\Lambda_\beta^2} J_\beta \Sigma_\beta(x)$

(c) $\Delta H = -\frac{1}{2}\sum_\beta \frac{1}{\Lambda_\beta^2} |J_\beta|^2$

At distances from the bubble surface large compared to D, the eigenvalue equation becomes

$$[-\nabla^2 + 8\lambda f^2]\, \Sigma_\beta(x) = \Lambda_\beta^2\, \Sigma_\beta(x)$$

We may identify two parts to the spectrum: (1) a continuous spectrum with threshold $\Lambda^2 = 8\lambda f^2$, and (2) a possible discrete spectrum of "bound state" eigenfunctions which decrease asymptotically like $e^{-\xi(\sqrt{8\lambda}f)}$. We can estimate the order of magnitude of the contribution of the continous part of the spectrum to ΔH from (11c). We will see that this contribution vanishes in the SCL.

First, however, we must compute J(x) near the bubble surface. Using (1) and (2), we find

$$\nabla^2\sigma_0 + 4\lambda\sigma_0(f^2 - \sigma_0^2) = 2k\sqrt{2\lambda}\, f^2\, \mathrm{sech}^2(\sqrt{2\lambda} f\xi) \equiv J_\sigma(x)$$

From (6)

$$J_D \equiv G\bar\Psi_0\Psi_0 \simeq N^2[\cosh\sqrt{2\lambda}f\xi]^{-2\frac{G}{\sqrt{2\lambda}}}(\bar\psi_0\psi_1 + \bar\psi_1\psi_0)$$

$$= N^2[\cosh\sqrt{2\lambda}f\xi]^{-2\frac{G}{\sqrt{2\lambda}}}\ \mathrm{Re}\ \psi_0^+(\gamma^0 + i\hat{n}\cdot\alpha)\psi_0$$

$$= N^2[\cosh\sqrt{2\lambda}f\xi]^{-2\frac{G}{\sqrt{2\lambda}}}\ \mathrm{Re}\ \frac{1}{\sigma}\psi_0^+(i\alpha\cdot\nabla_\parallel - k i\hat{n}\cdot\alpha)\psi_0$$

$$= N^2[\cosh\sqrt{2\lambda}f\xi]^{-2\frac{G}{\sqrt{2\lambda}}}\ \frac{\xi}{\tanh\sqrt{2\lambda}f\xi}\ \mathrm{Im}\ h_{\alpha\beta}\bar\psi_0 \tau^\alpha\partial^\beta\psi_0$$

where we have used the first order expansion of the tangential gradient near the surface:

$$\vec\nabla_\parallel = \vec{T}^\alpha\partial_\alpha \approx (\vec\tau^\alpha - \xi h^{\alpha\beta}\vec\tau_\beta)\partial_\alpha$$

Both J_σ and J_D are peaked sharply in a region of width $D \sim 1/(\lambda^{1/2} f)$ near the bubble surface. If we let R denote a typical radius of curvature of the bubble, we have the estimates in this thin region:

$$J_\sigma \sim \frac{1}{R}\lambda^{\frac{1}{2}} f$$

$$J_D \sim (\lambda^{\frac{1}{2}} f)\frac{1}{(\lambda^{\frac{1}{2}} f^2)}\ \frac{1}{R^2}\frac{1}{R^2} \sim \frac{1}{fR^4}$$

If, as will be seen to be the case presently, $R \sim \frac{1}{\lambda^{1/6} f}$, both contributions to the current will be of the same order of magnitude: $\lambda^{2/3} f^3$.

The order of magnitude of the contribution of the continuum eigenfunctions to ΔH can then be estimated

$$|\Delta H_{\mathrm{continuum}}| = \frac{1}{2}\left(\sum_\beta \frac{1}{\Lambda_\beta^2}|J_\beta|^2\right)\Big|_{\Lambda_\beta^2 \ge 8\lambda f^2}$$

$$\le \frac{1}{16\lambda f^2}\sum_\beta |J_\beta|^2 = \frac{1}{16\lambda f^2}\int dx\, J(x)^2$$

$$\sim \frac{1}{\lambda f^2}\left(R^2\frac{1}{\lambda^{\frac{1}{2}} f}\right)[\lambda^{2/3} f^3]^2 \sim \lambda^{-\frac{1}{2}} f \ll \lambda^{1/6} f$$

Thus, the continuum eigenfunctions, and indeed any eigenfunctions whose eigenvalue is of order $\Lambda \sim \lambda^{\frac{1}{2}} f$, contribute terms to the energy shift which vanish in the SCL.

Next, we must consider the effect of very low Λ "bound state" eigenfunctions. As we noted earlier, such bound state eigenfunctions must drop off exponentially in a distance D from the bubble surface. As discussed in the cases of σ_0 itself and of the Dirac field, the requirement that the eigenvalue be small places strong constraints on the ξ dependence of the eigenfunction near the surface. Specifically, the terms of order $\lambda \rho^2$ in (10) must cancel. It is easy to see that the kernal K(x,y) contributes no such terms. Thus, the leading part of the bound state eigenfunction must obey:

$$(13) \qquad \left[-\frac{\partial^2}{\partial \xi^2} + 4\lambda\sigma_0(3\sigma_0^2-\rho^2)\right]\Sigma_\beta = 0$$

Equation (13) is just the equation for the infinitesimal translation mode of a one dimensional kink in ξ. The bound state eigenfunctions have the form:

$$(14) \qquad \Sigma_\beta(x) = g_\beta(u^\alpha)\,\frac{d\sigma_0(\xi)}{d\xi}$$

where the functions $g_\beta(u^\alpha)$ and the eigenvalues Λ_β^2 are determined by the remaining finite terms in (10).

We have shown that the only corrections to the scalar field which induce finite energy corrections in the SCL have the general form:

$$\delta\sigma(x) = \delta G(u^\alpha)\,\frac{d\sigma_0}{d\xi}$$

where: $$\delta G(u^\alpha) \equiv \sum_\beta \frac{J_\beta}{\Lambda_\beta^2}\, g_\beta(u^\alpha)$$

Such an infinitesimal shift does not change the local kink-like configuration of the field σ_0. Rather, it corresponds to a shift in the position of the kinks in space-- in other words, a variation of the surface geometry.

$$\delta\sigma(x) = \delta G(u^\alpha)\,\frac{d\sigma_0}{d\xi}$$

is equivalent to

$$(15) \qquad \delta\vec{R}(u^\alpha) = \delta G(u^\alpha)\,\hat{n}(u^\alpha)$$

Thus, if the bubble surface is such that the total energy is stationary under all local variations of its geometry (15), further corrections to the fields change the energy by terms which vanish in the SCL.

Appendix C

This appendix presents the details of the derivation of equation (III) from the variation of the action over possible bubble surfaces.

The surface Lagrangian is

$$L = \bar{\Psi} i \not{\partial} \psi - C = \bar{\Psi}(i \not{\partial}_{\parallel} + i k \not{\alpha}) \psi - C$$

and the action is

$$S = \int du \sqrt{|g|} \, [\bar{\Psi}(i \not{\partial}_{\parallel} + i k \not{\alpha}) \psi - C]$$

We take ψ to be an arbitrary spinor satisfying the constraint (I), $i \not{\alpha} \psi = \psi$. It is convenient, for the derivation of the equations of motion, to modify the Lagrangian by the addition of a divergence:

$$L' = L - \tfrac{i}{2}[\bar{\Psi} \not{\tau}^{\alpha} \psi]_{\parallel \alpha}$$

$$= \tfrac{i}{2}[\bar{\Psi} \not{\tau}^{\alpha} \partial_{\alpha} \psi - (\partial_{\alpha} \bar{\Psi}) \not{\tau}^{\alpha} \psi] - C$$

$$= -\mathrm{Im}(\bar{\Psi} \not{\partial}_{\parallel} \psi) - C$$

Adding this divergence to L will not, of course, change the equations of motion. Indeed, in this case, the divergence we have added is identically zero for a ψ which satisfies the surface Dirac equation.

Under a general infinitesimal variation

$$(2) \qquad R^{\nu}(u^{\alpha}) \longrightarrow \tilde{R}^{\nu}(u^{\alpha}) = R^{\nu}(u^{\alpha}) + \delta R^{\nu}(u^{\alpha})$$

we have

$$(3) \qquad \begin{aligned} \delta \tau_{\alpha} &= \delta R_{|\alpha} \\ \delta g_{\alpha\beta} &= \tau_{\alpha} \cdot \delta R_{|\beta} + \tau_{\beta} \cdot \delta R_{|\alpha} \\ \delta g^{\alpha\beta} &= -\tau^{\alpha} \cdot \delta R^{|\beta} - \tau^{\beta} \cdot \delta R^{|\alpha} \\ \delta \sqrt{|g|} &= \sqrt{|g|} \, \tau^{\alpha} \cdot \delta R_{|\alpha} \\ \delta \tau^{\alpha} &= (n \cdot \delta R^{|\alpha}) n - (\tau^{\alpha} \cdot \delta R^{|\beta}) \tau_{\beta} \end{aligned}$$

Using (3), we find

$$(4) \qquad \begin{aligned} \delta S &= \int du \sqrt{|g|} \, [L' \tau^{\alpha} \cdot \delta R_{|\alpha} - \mathrm{Im} \bar{\Psi} \delta \not{\tau}^{\alpha} \partial_{\alpha} \psi] \\ &= \int du \sqrt{|g|} \, [-T^{\alpha\beta} \tau_{\beta} \cdot \delta R_{|\alpha} - \mathrm{Im} \bar{\Psi} \not{\alpha} \partial^{\alpha} \psi \, (n \cdot \delta R_{|\alpha})] \end{aligned}$$

where

$$(5) \qquad T^{\alpha\beta} \equiv -L' g^{\alpha\beta} - \mathrm{Im} \, \bar{\Psi} \not{\tau}^{\alpha} \partial^{\beta} \psi$$

The equation of motion follows from an integration by parts:

$$0 = \delta S = \int du \sqrt{|g|} \, \delta R_{\nu} [(T^{\alpha\beta} \tau_{\beta}{}^{\nu})_{\parallel \alpha} + (n^{\nu} \mathrm{Im} \bar{\Psi} \not{\alpha} \partial^{\alpha} \psi)_{\parallel \alpha}]$$

or

$$(6) \qquad \begin{aligned} 0 &= (T^{\alpha\beta} \tau_{\beta}{}^{\nu})_{\parallel \alpha} + (n^{\nu} \mathrm{Im} \bar{\Psi} \not{\alpha} \partial^{\alpha} \psi)_{\parallel \alpha} \\ &= [T^{\alpha\beta}{}_{\parallel \alpha} + h^{\alpha\beta} \mathrm{Im} \bar{\Psi} \not{\alpha} \partial_{\alpha} \psi] \tau_{\beta}^{\nu} + [h_{\alpha\beta} T^{\alpha\beta} + \mathrm{Im}(\bar{\Psi} \not{\alpha} \partial^{\alpha} \psi)_{\parallel \alpha}] n^{\nu} \end{aligned}$$

The tangential component of equation (6) is identically zero, using the equations of motion (I) and (II) of the Dirac field.

$$T^{\alpha\beta}{}_{\|\alpha} + h^{\alpha\beta}\,\mathrm{Im}\,\bar{\Psi}\gamma\partial_\alpha\psi = 0$$

As mentioned in Chapter III, this result reflects the fact that a tangential variation, $\delta R^\nu = \delta V^\alpha E_\alpha^\nu$, is equivalent to a coordinate transformation: $u^\alpha \to u^\alpha - \delta V^\alpha$. That the action is invariant under such a transformation is a trivial result of its coordinate invariance.

The normal component of (6) provides the third equation of motion:

$$(7) \qquad h_{\alpha\beta}T^{\alpha\beta} + \mathrm{Im}\,(\bar{\Psi}\gamma\,\partial^\alpha\psi)_{\|\alpha} = 0$$

Using the constraint (I) and the Dirac equation (II), we have

$$T^{\alpha\beta} = C g^{\alpha\beta} - \mathrm{Im}\,\bar{\Psi}\gamma^\alpha\partial^\beta\psi$$

$$\mathrm{Im}\left(\bar{\Psi}\gamma\partial^\alpha\psi\right) = 0$$

Therefore, the surface equation of motion is

$$(\mathrm{III}) \qquad h_{\alpha\beta}T^{\alpha\beta} = 0$$

as indicated in Chapter III.

Appendix D

In this appendix, we compute some of the conserved "charges" of the bubble theory, directly from the original field theoretic expressions, in the strong coupling limit ("SCL").

As in Appendix B, we use a coordinate system for the region near the bubble surface defined by

$$(1) \qquad x^\nu(u^\alpha, \xi) = R^\nu(u^\alpha) + \xi n^\nu(u^\alpha)$$

In these coordinates, the approximate fields are:

$$(2) \qquad \sigma(x) = f\tanh(\sqrt{2\lambda}\,f\xi)$$

$$(3) \qquad \Psi(x) = N[\cosh\sqrt{2\lambda}\,f\xi]^{-\frac{G}{\sqrt{2\lambda}}}\,\psi(u^\alpha, \xi)$$

with $\psi(u^\alpha, \xi)$ satisfying the surface Dirac equation, and

$$i\gamma\psi = \psi$$

$$\frac{\partial}{\partial\xi}\psi = -\frac{1}{2}(\partial_\|)_\nu n^\nu\psi$$

The action and all conserved charges of the theory are expressed as spacial integrals of densities which are very sharply peaked near the surface. The integrals over ξ can be performed explicitly, to lowest order in D/R, in the SCL. Such calculations are completely straightforward. We illustrate them here for the cases of the action, S, and the energy momentum, p^μ, only.

The coordinate system (1) is characterized by a set of basis vectors, $T_a{}^\mu$, and a metric, G_{ab}, as discussed in Appendix B. As before, we need only work to first order in ξ, as higher order terms in ξ represent, effectively, an expansion in D/R. We have,

$$T_\alpha{}^\mu = \tau_\alpha{}^\mu + \xi n^\mu{}_{|\alpha} \quad , \quad T_3{}^\mu = n^\mu$$

$$G_{\alpha\beta} = g_{\alpha\beta} + 2\xi h_{\alpha\beta} \quad , \quad G^{\alpha\beta} = g^{\alpha\beta} - 2\xi h^{\alpha\beta}$$

$$G_{33} = G^{33} = -1 \quad , \quad G_{3\alpha} = G^{3\alpha} = 0$$

$$\sqrt{|G|} = \sqrt{|g|}\,(1 + 2k\xi)$$

$$T^{\alpha\mu} = (\tau^{\alpha\mu} - \xi n^{\mu|\alpha}), \quad T^{3\mu} = -n^\mu$$

$$\partial^\mu = (T^\alpha)^\mu \partial_\alpha = (\tau^{\alpha\mu} - \xi n^{\mu|\alpha})\partial_\alpha - n^\mu \frac{\partial}{\partial\xi}$$

Consider "surface fields" defined by equations (2) and (3), where the surface is taken to be of arbitrary shape and the Dirac field, ψ, satisfies $\bar{\psi}\psi = 0$, but not necessarily the surface Dirac equation. The action integral is:

$$S = \int dx \left[\bar{\Psi}(i\partial\!\!\!/ - G\sigma)\Psi + \tfrac{1}{2}(\partial\sigma)^2 - \lambda(\sigma^2 - f^2)^2 \right]$$

Using (3), we have, to leading order in D/R,

$$\int d\xi \sqrt{|G|}\, \bar{\Psi}(i\partial\!\!\!/ - G\sigma)\Psi$$

$$\approx \int d\xi \sqrt{|G|}\, N^2 [\cosh\sqrt{2\lambda} f\xi]^{-\frac{2G}{\sqrt{2\lambda}}} \bar{\psi}(i\gamma)\psi$$

Now,

$$\partial\!\!\!/\psi \approx (\partial\!\!\!/_{\parallel} + k n\!\!\!/)\psi_0$$

where $\quad \partial\!\!\!/_{\parallel} = \gamma^\alpha \partial_\alpha$

The support for the integral over ξ lies in a region of width D near the surface. For quantities whose ξ dependence is much slower, this sharp peak merely picks out the value at $\xi = 0$. Thus,

$$\int d\xi \sqrt{|G|}\, \bar{\Psi}(i\partial\!\!\!/ - G\sigma)\Psi$$

$$\approx \int d\xi \sqrt{|G|}\, N^2 [\cosh\sqrt{2\lambda} f\xi]^{-\frac{2G}{\sqrt{2\lambda}}} \bar{\psi}(i\partial\!\!\!/_{\parallel} + k i n\!\!\!/)\psi$$

$$\approx \sqrt{|g|}\, \bar{\psi}\, i D\!\!\!\!/\,\psi$$

after using the normalization condition,

$$\frac{1}{N^2} = \int d\xi \, [\cosh\sqrt{2\lambda} f\xi]^{-\frac{2G}{\sqrt{2\lambda}}}$$

The scalar contribution to the action can be treated similarly,

$$\tfrac{1}{2}(\partial\sigma)^2 - \lambda(\sigma^2 - f^2) \approx \tfrac{1}{2}\left(-n\frac{\partial\sigma}{\partial\xi}\right)^2 - \lambda(\sigma^2 - f^2)^2 = -2\lambda f^4 \operatorname{sech}^4\sqrt{2\lambda} f\xi$$

so

$$\int d\xi \sqrt{|G|}\left[\tfrac{1}{2}(\partial\sigma)^2 - \lambda(\sigma^2 - f^2)^2\right] \approx -\tfrac{4}{3}\sqrt{2\lambda} f^3 \sqrt{|g|} = -C\sqrt{|g|}$$

Thus, we have the surface form of the action:

$$S \approx \int du \sqrt{|g|}\left[\bar{\psi}\, i D\!\!\!\!/\,\psi - C\right]$$

This is exactly the expression for the action in terms of surface variables that was derived heuristically in Chapter III. We clearly have the result that the variation of this surface action over possible surface Dirac fields and over possible surface geometries is equivalent to the variation of the field theory action over field configurations of the form (2), (3).

The calculation of the energy-momentum, P^μ, follows similar lines. The energy momentum is given in the field theory by

$$P^\mu = \int_V dS_\nu T^{\nu\mu}$$

where

$$T^{\mu\nu} \equiv \partial^\mu\sigma\partial^\nu\sigma - Lg^{\mu\nu} - \mathrm{Im}\,\bar{\Psi}\gamma^\mu\partial^\nu\psi$$

and V is any space-like surface in Minkowski space. If we choose such a spacelike surface to be one evolved from a space-like cut, W, of the bubble by translation in ξ, we have:

$$dS_\mu = d\xi\, d\Sigma_\alpha (T^\alpha)_\mu$$

and

$$P^\nu = \int_W d\Sigma_\alpha \int d\xi \sqrt{|G|}\, (T^\alpha)_\mu T^{\mu\nu}$$

Now,

$$\begin{aligned}(T^\alpha)_\mu T^{\mu\nu} &\approx (\tau^\alpha)_\mu T^{\mu\nu} \\ &= -L(\tau^\alpha)^\nu - \mathrm{Im}\,\bar{\Psi}\not{\tau}^\alpha\partial^\nu\psi \\ &\approx -L(\tau^\alpha)^\nu - N^2[\cosh\sqrt{2\lambda}f\xi]^{-\frac{2G}{\sqrt{2\lambda}}}\,\mathrm{Im}\,\bar{\Psi}\not{\tau}^\alpha\tau_\beta{}^\nu\partial^\beta\psi\end{aligned}$$

so we have

$$P^\nu \approx \int_W d\Sigma_\alpha \int d\xi\sqrt{|G|}\left\{-Lg^{\alpha\beta} - N^2(\cosh\sqrt{2\lambda}f\xi)^{-\frac{2G}{\sqrt{2\lambda}}}\,\mathrm{Im}\,\bar{\Psi}\not{\tau}^\alpha\partial^\beta\psi\right\}\tau_\beta{}^\nu$$

Performing the integrals over ξ, to lowest order, we have

$$\begin{aligned}P^\nu &= \int_W d\Sigma_\alpha\sqrt{|g|}\,[Cg^{\alpha\beta} - \mathrm{Im}\,\bar{\Psi}\not{\tau}^\alpha\partial^\beta\psi]\,\tau_\beta{}^\nu \\ &= \int_W d\Sigma_\alpha\sqrt{|g|}\; T^{\alpha\beta}\tau_\beta{}^\nu\end{aligned}$$

This is the result derived canonically in Chapter III.

The fermion charge and angular-momentum integrals may be computed in the same way, and are found to agree with the results derived from Noether's theorem applied to the surface action.

Appendix E

In this appendix, we briefly discuss a mathematical observation that may facilitate numerical estimation of energy levels for static bubbles of more complicated shapes. We will see that the three dimensional surface Dirac equation may be solved on surfaces which have a one dimensional sharp edge. Though one of the principal radii of curvature is singular along such an edge, the Dirac energy remains finite and the Dirac equation goes over into a boundary condition relating the fields on either side of it.

The static, two component, Dirac equation of Chapter IV is

$$(1) \qquad [k - i\vec{\sigma}\cdot(\hat{n}\wedge\vec{\nabla}_{\shortparallel})]\chi = E\chi$$

We consider this equation in the neighborhood of a very sharp, but smooth, edge on the surface (Fig 15). At any given point on the edge, we can establish locally geodesic internal coordinates which also diagonalize $h^{\alpha}{}_{\beta}$:

$$(2) \qquad \begin{aligned} g_{\alpha\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ g_{\alpha\beta|\gamma} &= 0 \qquad \text{at a fixed point, } u^{\alpha} \\ h^{\alpha}{}_{\beta} &= \begin{pmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_2} \end{pmatrix} \end{aligned}$$

then $\vec{\tau}_1, \vec{\tau}_2$ are unit vectors, whose orientation we define by $\vec{\tau}_1\wedge\vec{\tau}_2 = \hat{n}$. We will suppose that the edge is such that $\vec{\tau}_1, \hat{n}$ vary rapidly in u^1 as the edge is crossed.

We consider the Dirac equation in the limit as this edge becomes infinitely sharp: $R_1 \to 0$. If the Dirac energy is to remain finite, the terms of order $1/R_1$ on the left side of (1) must cancel. These terms come from the $1/R_1$ contribution to k and from the derivative of ψ with respect to u^1.

$$(3) \qquad 0 = \left[\tfrac{1}{2}\vec{\tau}_1\cdot\frac{\partial\hat{n}}{\partial u^1} - i\sigma(\hat{n}\wedge\vec{\tau}^1)\frac{\partial}{\partial u^1}\right]\chi = \left[\tfrac{1}{2}\frac{\partial\Phi}{\partial u^1} - i\sigma\vec{\tau}_2\frac{\partial}{\partial u^1}\right]\chi$$

where Φ is the angle of the normal about $\vec{\tau}_2$ (Fig 15). This is the same as the Dirac equation on the surface of the two dimensional static bubble. The integral of (3) in the neighborhood of the edge is

$$(4) \qquad \chi(u_1', u_2) = e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\tau}_2(\Phi(u_1',u_2)-\Phi(u_1,u_2))}\chi(u_1, u_2)$$

In the limit of an infinitely sharp edge, we integrate (4) across the edge and find the boundary condition:

$$(5) \qquad \chi(2) = e^{-\frac{i}{2}\Delta\vec{\Phi}_{21}\cdot\vec{\sigma}}\chi(1)$$

where (2) and (1) label the two sides of the edge, and $\Delta\vec{\Phi}_{21}$ is the vector rotation angle of the normal as the edge is crossed from side (1) to side (2).

This result may be used to estimate numerically the energies of bubbles of more complicated shapes than were discussed in Chapter IV. One may consider surfaces which consist of simple surface elements patched together with sharp edges. The Dirac equation can be solved on each subsurface independently. The remaining problem is to satisfy the boundary condition (5), which can be expressed as a finite dimensional algebraic constraint relating the parameters which define the solutions on each sub-surface. Generally, such an algebraic problem is more easily handled numerically than is the problem of diagonalizing the Dirac Hamiltonian on a complex surface.

It is clear that surfaces with sharp edges will not solve the bubble equations exactly. However, because the Dirac equation is sensible on such surfaces, one might hope that they will make reasonable trial surfaces for numerical approximations.

Appendix E

We prove miscellaneous results refered to in the analysis of the bubble in three space-time dimensions contained in Chapter V.

First, we show that if $T^{\alpha\beta}$ is any symmetric tensor with signature (1,-1) on a two dimensional Riemann manifold, then a local coordinate system can be found in which $T^{\alpha\beta}$ is off-diagonal: $T^{00}=T^{11}=0$. Let $\{v^0, v^1\}$ be the desired coordinates, and $\tilde{T}^{\alpha'\beta'}$ the tensor $T^{\alpha\beta}$ in these coordinates.

$$\tilde{T}^{\alpha'\beta'} = T^{\alpha\beta} \frac{\partial v^{\alpha'}}{\partial u^\alpha} \frac{\partial v^{\beta'}}{\partial u^\beta}$$

We require:

$$(1) \qquad 0 = \tilde{T}^{00} = T^{\alpha\beta} \frac{\partial v^0}{\partial u^\alpha} \frac{\partial v^0}{\partial u^\beta}$$

$$0 = \tilde{T}^{11} = T^{\alpha\beta} \frac{\partial v^1}{\partial u^\alpha} \frac{\partial v^1}{\partial u^\beta}$$

Both V^0 and V^1 must have gradients which satisfy the homogeneous quadratic constraint (1). Because $T^{\alpha\beta}$ has signature (1,-1) the solutions of (1) are such that the gradient must lie on a degenerate hyperbola (analogous to the light cone) in the tangent space to the surface at each point. There are two independent real solutions to the quadratic equation at each point. These generate two functionally independent solutions to the differential equation, which can be taken to be v^0 and v^1.

Next, we must show that the energy-momentum tensor, $T^{\alpha\beta}$, satisfies the conditions for this theorem -- that $T^{\alpha\beta}$ is symmetric and has signature (1,-1).

$$T^{\alpha\beta} = C g^{\alpha\beta} + \gamma^{\alpha\beta} \tag{2}$$

where, $\gamma^{\alpha\beta} \equiv -\mathrm{Im}\,\bar{\Psi}\,\gamma^\alpha \partial^\beta \Psi$

The metric tensor is symmetric, so the symmetry of $T^{\alpha\beta}$ will follow if we show that $\gamma^{\alpha\beta}$ is symmetric. It is sufficient to show that $\gamma^{\alpha\beta}$ is symmetric at any given point in some coordinate system. A tensor which is symmetric at a point in one coordinate system is symmetric at that point in all coordinate systems. At a given point, u^α, we choose locally geodesic coordinates:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad g_{\alpha\beta|\gamma} = 0 \tag{3}$$

We want to show that $\gamma^{01} - \gamma^{10} = 0$ at the point u^α. We have:

$$\gamma^{01} - \gamma^{10} = -\mathrm{Im}\,\bar{\Psi}\,(\gamma^0\partial^1 - \gamma^1\partial^0)\Psi \tag{4}$$

Using the Dirac equation,

$$[i\,\gamma^0\partial_0 + i\,\gamma^1\partial_1 + k]\,\Psi = 0$$

and the relation, valid in the two component representation of the gamma matrices, following from (3):

$$\gamma^1\gamma^0 = -i\alpha$$

we have

$$\partial_1\Psi = [i\alpha\partial_0 - ik\,\gamma^1]\Psi$$

then (4) becomes

$$\gamma^{01} - \gamma^{10} = -\mathrm{Im}\,\bar{\Psi}[-(\gamma^1 + \gamma^0 i\alpha)\partial_0 + ik\,\gamma^0\gamma^1]\Psi$$

But

$$\gamma^0\gamma^1 = i\alpha \quad \text{and} \quad \bar{\Psi}(-i\alpha)\Psi = 0$$

$$\gamma^0\, i\alpha = \gamma_1 = -\gamma^1$$

So $\gamma^{01} - \gamma^{10} = 0$ at the point u^α. This is sufficient to show that $\gamma^{\alpha\beta}$ is symmetric at all points in all coordinate systems.

We must show that $T^{\alpha\beta}$ has signature (1,-1). It is sufficient to show that det($T^{\alpha\beta}$)<0, for then the eigenvalues of $T^{\alpha\beta}$ have opposite sign. Because $T^{\alpha\beta}$ is a 2x2 matrix, we have

$$\det(T^{\alpha\beta}) = \det[\,C g^{\alpha\beta} + \gamma^{\alpha\beta}]$$

$$= \frac{C^2}{g^2} + \frac{C}{g}\, g_{\alpha\beta}\gamma^{\alpha\beta} + \det(\gamma^{\alpha\beta})$$

From the Dirac equation, $g_{\alpha\beta}\gamma^{\alpha\beta} = 0$. We can also show that det($\gamma^{\alpha\beta}$)=0. As indicated in Chapter V, and as will proven below, the Dirac current, $J_\alpha = \bar{\Psi}\gamma_\alpha\Psi$, is light-like and satisfies

$$J_\alpha \Upsilon^{\alpha\beta} = 0$$

It follows immediately that $\Upsilon^{\alpha\beta}$ is of the form

$$\Upsilon^{\alpha\beta} = \Lambda\, J^\alpha J^\beta$$

where Λ is some scalar field. Then

$$\det(\Upsilon^{\alpha\beta}) = \Lambda^2[(J^0)^2(J^1)^2 - (J^0J^1)^2] = 0$$

so $$\det(T^{\alpha\beta}) = \frac{c^2}{g} < 0 \quad , \text{since } g<0.$$

Finally, we must verify the assertion in Chapter V that

If $\bar{\Psi}\Psi = 0$,

$$(\bar{\Psi}\gamma^\nu\Psi)\gamma_\nu\Psi = 0$$

for any two component spinor, Ψ. This result follows from the observation that, for any two component spinor, Ψ, there exists a unique real unit vector, $\hat{m}$, such that:

$$\hat{m}\cdot\vec{\sigma}\,\Psi = \Psi$$

Using the 2x2 representation of the gamma matrices introduced in Chapter V,

$$\gamma^0 = \sigma_1 \qquad \gamma^1 = i\sigma_3 \qquad \gamma^2 = -i\sigma_2$$

we have

$$\bar{\Psi}\gamma^\nu\Psi = \Psi^+\Psi(1, \hat{m}_2, \hat{m}_3)$$
$$\bar{\Psi}\Psi = (\Psi^+\Psi)\,\hat{m}_1 = 0$$

then

$$(\bar{\Psi}\gamma^\nu\Psi)\gamma_\nu\Psi = \Psi^+\Psi[\sigma_1 - i\hat{m}_2\sigma_3 + i\hat{m}_3\sigma_2]\Psi$$
$$= (\Psi^+\Psi)\sigma_1[1 - \hat{m}_2\sigma_2 - \hat{m}_3\sigma_3]\Psi$$
$$= (\Psi^+\Psi)\sigma_1(1 - \hat{m}\cdot\sigma)\Psi$$
$$= 0$$

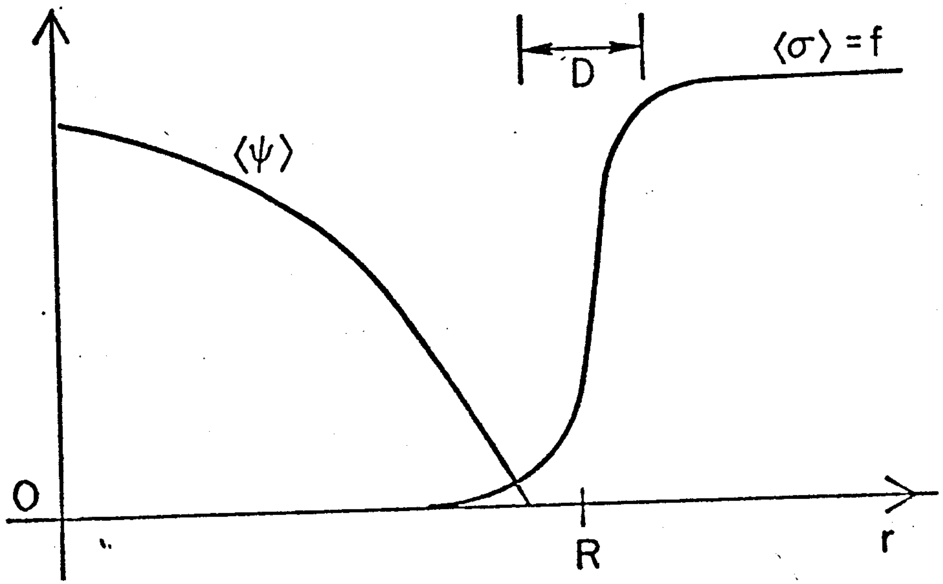


FIGURE 1

A schematic representation
of a "bag-like" state

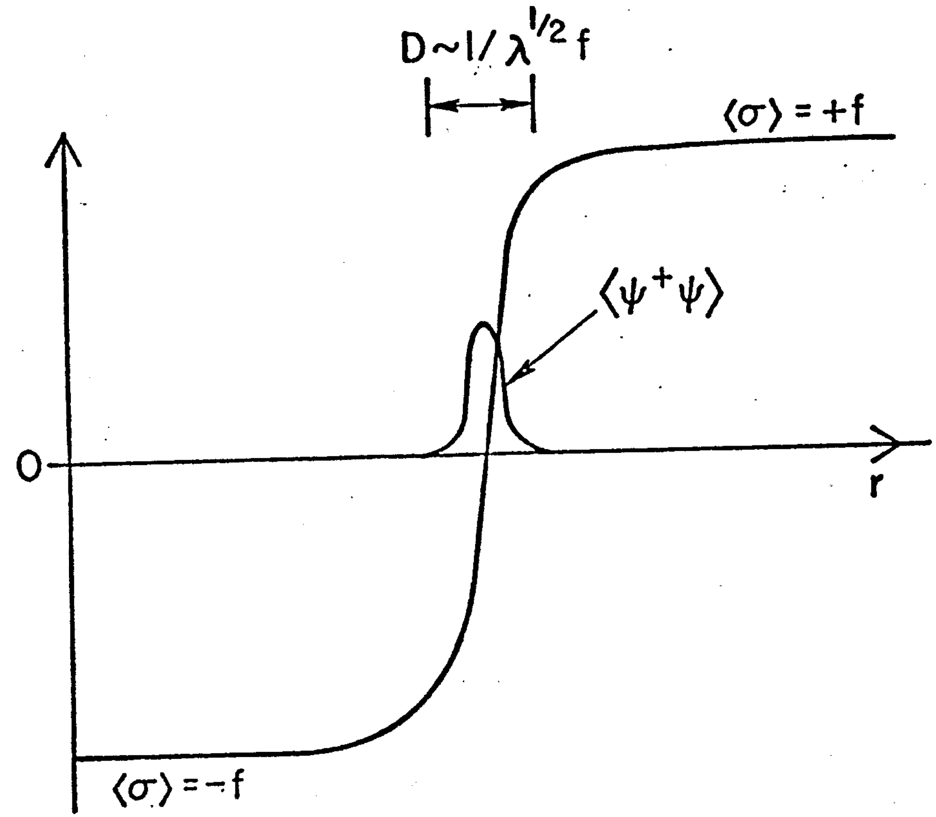


FIGURE 2

A schematic representation
of a bubble state

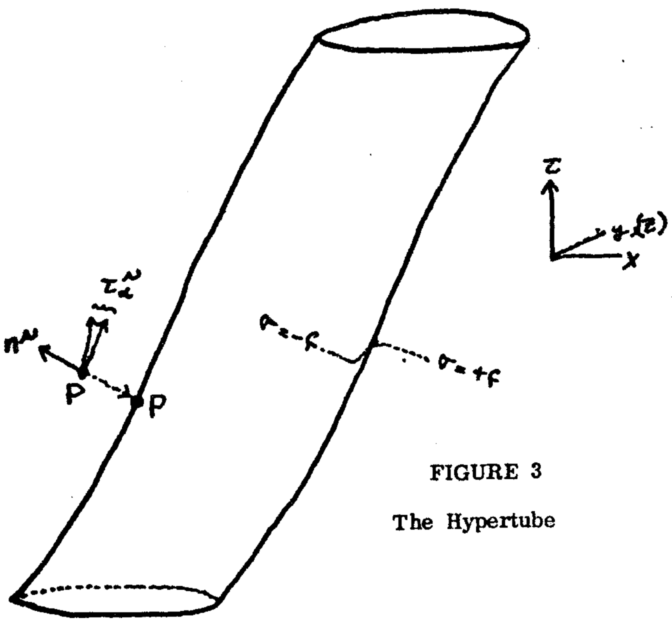
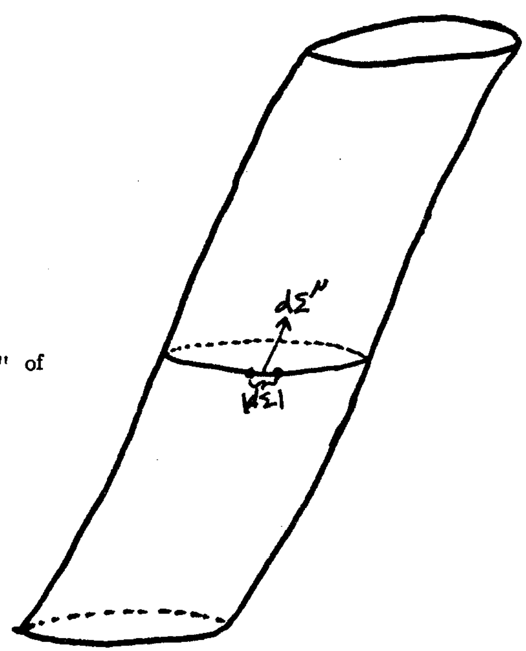


FIGURE 3

The Hypertube

FIGURE 4

A "space-like cut" of the hypertube



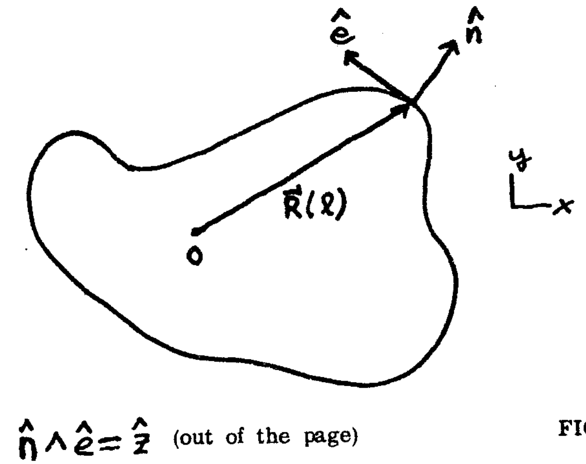


FIGURE 5

A two dimensional static bubble

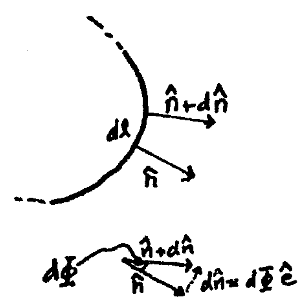


FIGURE 6

The motion of the normal along a two dimensional static bubble

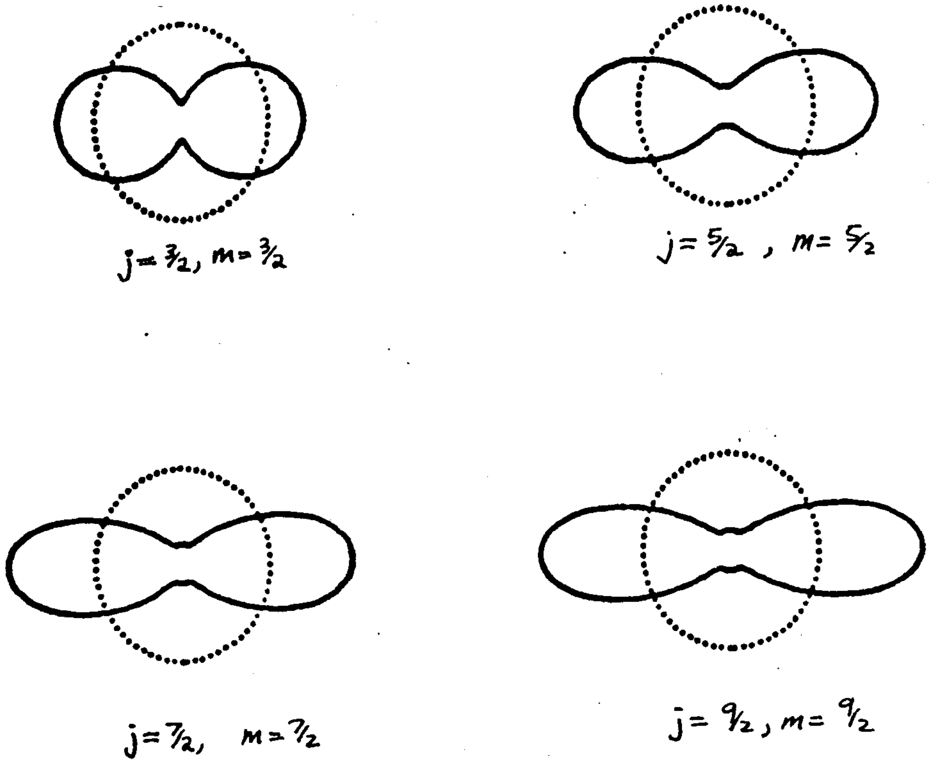


FIGURE 7

The normal force density on a spherical bubble for quark states with j=m. The dotted line is the zero of force.

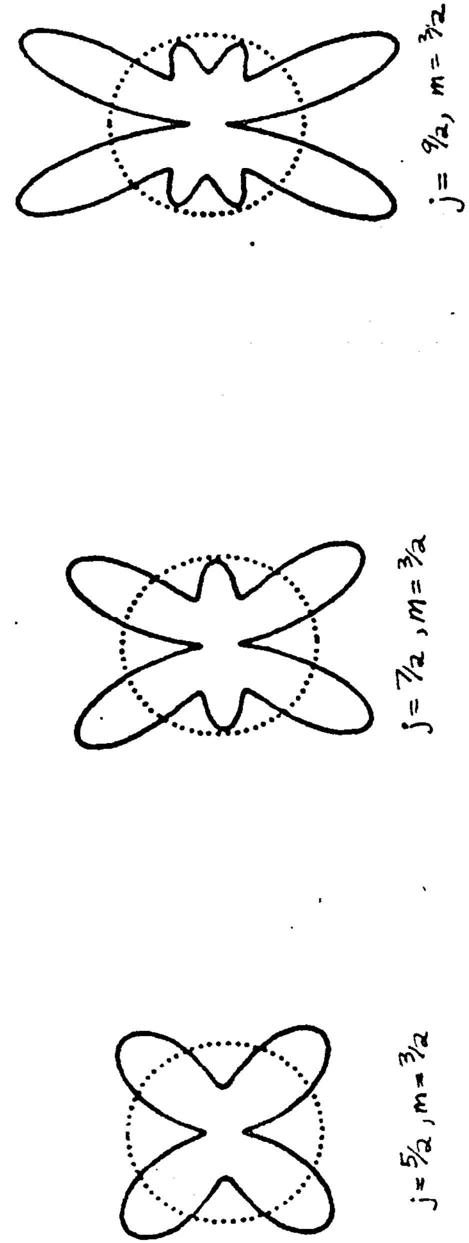


FIGURE 8

The normal force density on a spherical bubble for various quark states.

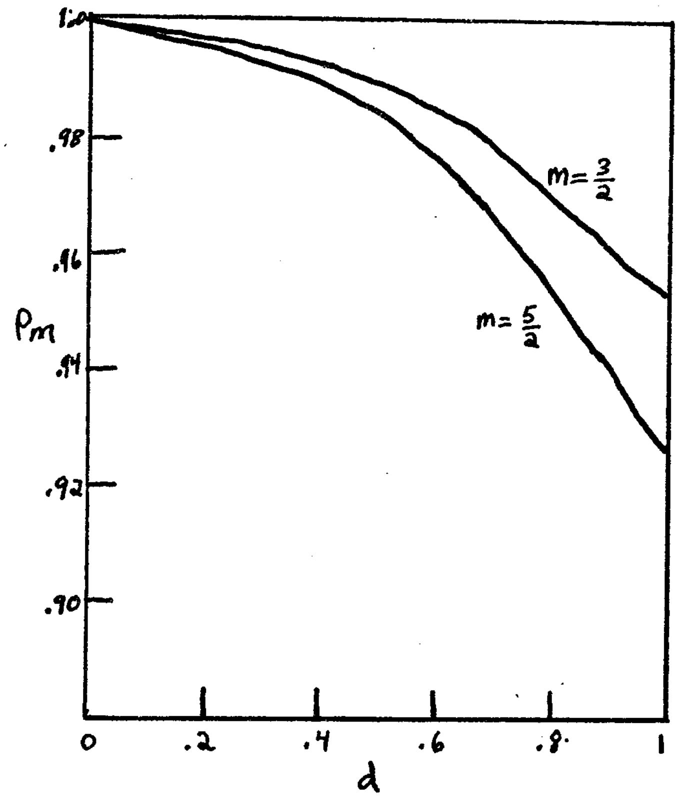


FIGURE 9

ρ_m vs d for the oblate spheroid

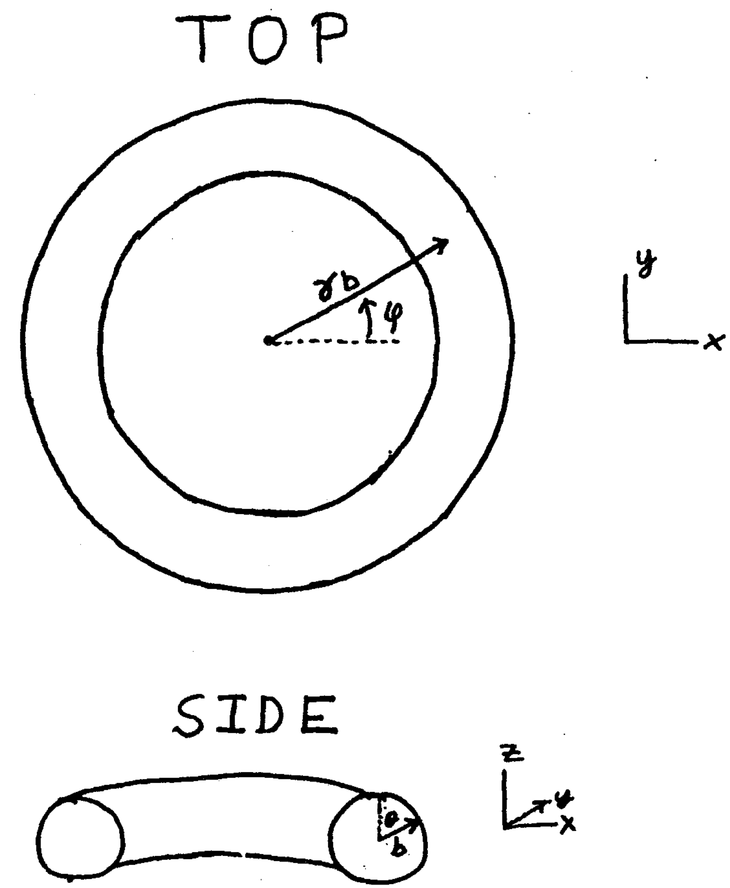


FIGURE 10

Coordinates on the torus

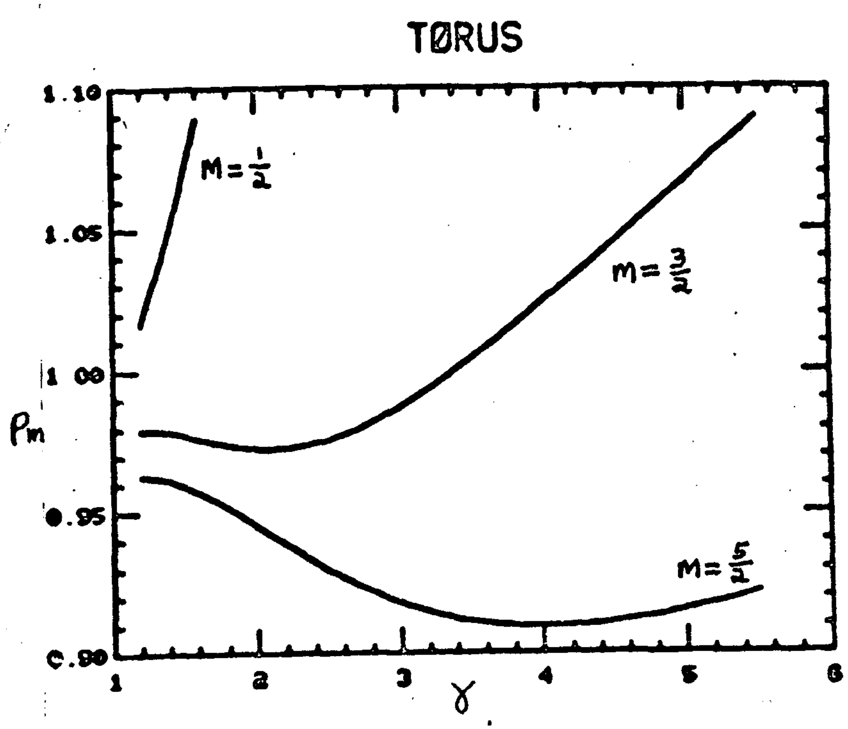


FIGURE 11

ρ_m vs γ for the torus

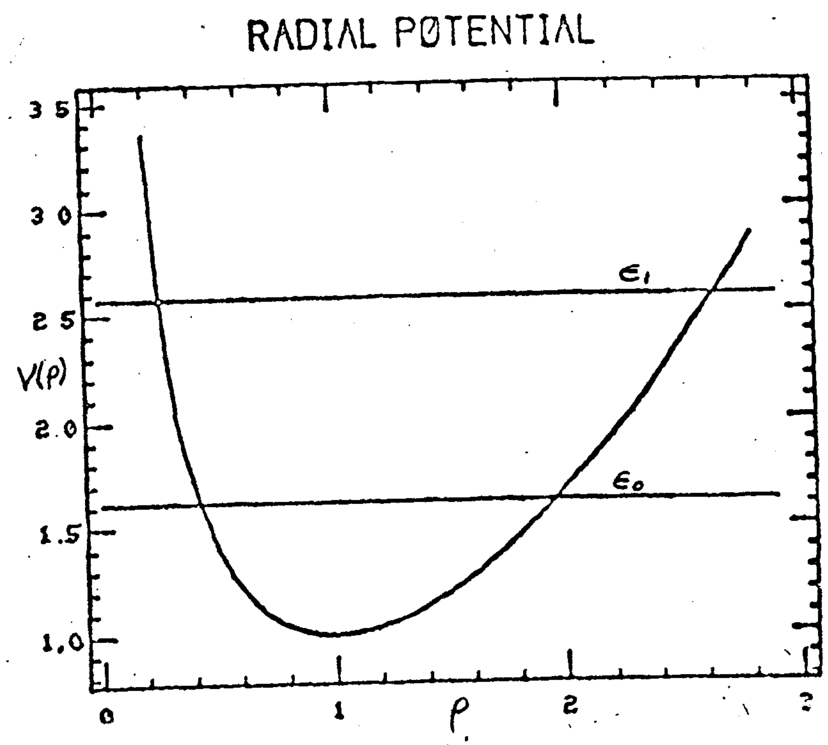


FIGURE 12

The scaled radial potential, $V(\rho)$

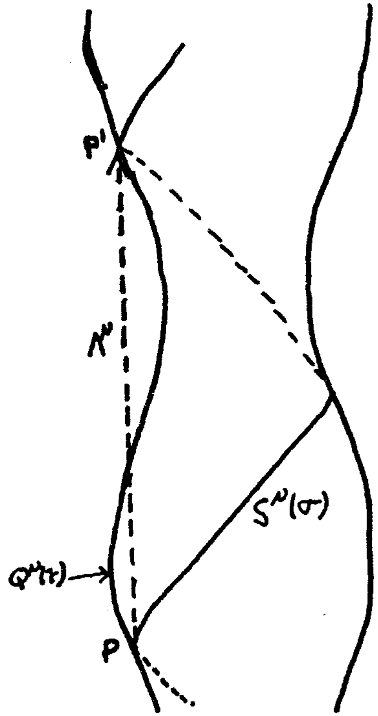


FIGURE 13

A three dimensional bubble generated by curves $Q^\mu(\tau)$ and $S^\mu(\sigma)$. Points P and P′ are equivalent.

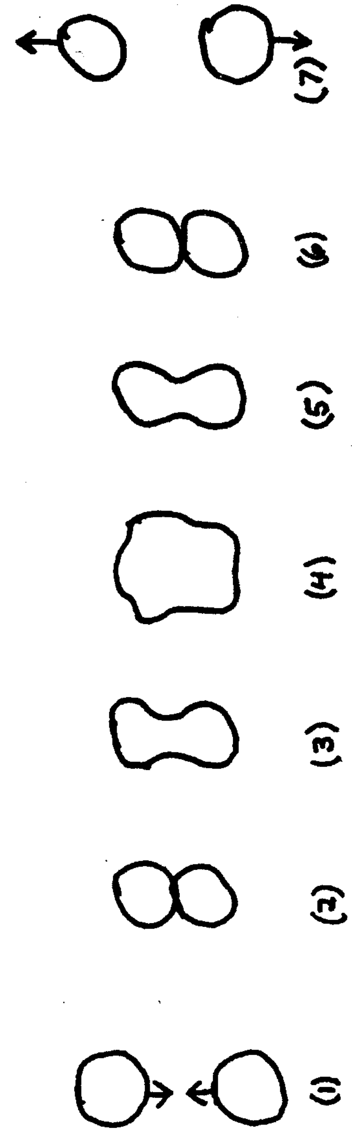


FIGURE 14

Bubble-bubble scattering by fusion and fission.

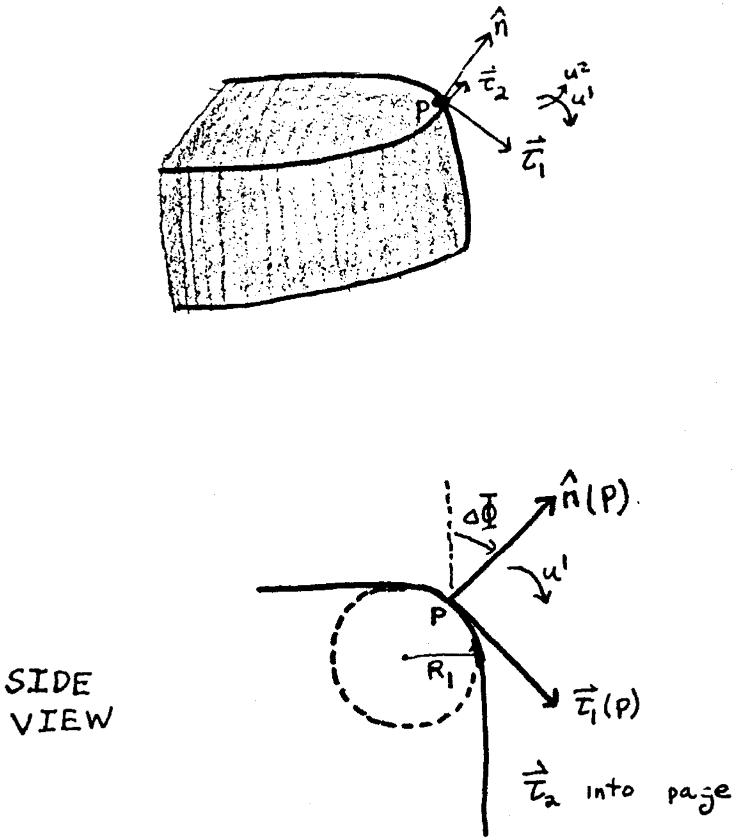


FIGURE 15

A three dimensional static bubble with a sharp edge.

TABLE I

The Dirac Currents on the Bubble

The condition (I) allows all Dirac currents to be expressed in terms of the tangent vector and pseudo-scalar currents:

$$\bar{\psi}\psi = 0 \qquad \bar{\psi}\gamma^\mu\psi \equiv J^\alpha \tau_\alpha^\mu$$

$$\bar{\psi} i\gamma_5\psi \equiv P \qquad \bar{\psi}\gamma^\mu\gamma_5\psi = -P n^\mu$$

$$\bar{\psi}\sigma^{\mu\nu}\psi = J^\alpha (n^\mu \tau_\alpha^\nu - n^\nu \tau_\alpha^\mu)$$

TABLE II

The States of the $\underline{70}$, L=1 of SU(6)†

SU(3)	J^P	Observed States			Bubble Model
10	$5/2^-$				Extra state
10	$3/2^-$	$\Delta(1700)$	$\Sigma(1580)$		
10	$1/2^-$	$\Delta(1610)$	$\Sigma(1740)$		
8	$5/2^-$	$N^*(1670)$	$\Sigma(1765)$	$\Lambda(1830)$	
8	$3/2^-$	$N^*(1710)$	$\Sigma(1940)$	$\Lambda(—)$	
8	$1/2^-$	$N^*(1660)$	$\Sigma(—)$	$\Lambda(1670)$	one missing
8	$1/2^-$	$N^*(1510)$	$\Sigma(—)$	$\Lambda(—)$	
8	$3/2^-$	$N^*(1520)$	$\Sigma(1660)$	$\Lambda(1690)$	
1	$3/2^-$			$\Lambda(1520)$	
1	$1/2$			$\Lambda(1405)$	missing

†From: R. J. Cashmore, "Resonances: Experimental Review," Proceedings of the Summer Institute on Particle Physics, Vol. I, SLAC Report No. 179 (1974).

TABLE III

Excitation Energies and Turning Points for the Radial Mode

n	ϵ_n	ρ_{min}	ρ_{max}
0	1.615	.429	1.956
1	2.577	.261	2.641
2	3.381	.198	3.081

TABLE IV

The Low-Lying "Baryon" States
of the Three Dimensional Bubble

ℓ_0 = eigenvalue of $:\mathcal{L}_0 - \frac{1}{2}a_0^2:$
$\bar{\ell}_0$ = eigenvalue of $:\bar{\mathcal{L}}_0 - \frac{1}{2}a_0^2:$
ω = eigenvalue of $W/\sqrt{8\pi c}$

$m^2 = 8\pi c\, \ell_0$

ℓ_0	$\bar{\ell}_0$	ω	State Vector
$1/2$	1	0	$b^+_{1/2}\, c_1^+ \lvert 0\rangle$
$3/2$	2	$-5/\sqrt{2}$	$\frac{1}{2\sqrt{2}}[b^+_{3/2} - i b^+_{1/2} a_1^+][c_2^+ - i(c_1^+)^2]\lvert 0\rangle$
		$-1/\sqrt{2}$	$\frac{1}{2\sqrt{2}}[b^+_{3/2} + i b^+_{1/2} a_1^+][c_2^+ - i(c_1^+)^2]\lvert 0\rangle$
		$+1/\sqrt{2}$	$\frac{1}{2\sqrt{2}}[b^+_{3/2} - i b^+_{1/2} a_1^+][c_2^+ + i(c_1^+)^2]\lvert 0\rangle$
		$+5/\sqrt{2}$	$\frac{1}{2\sqrt{2}}[b^+_{3/2} + i b^+_{1/2} a_1^+][c_2^+ + i(c_1^+)^2]\lvert 0\rangle$

TABLE V

The Low-Lying "Meson" States
of the Three Dimensional Bubble

ℓ_0	$\bar{\ell}_0$	ω	State Vector
0	0	0	$\lvert 0\rangle$
1	1	0	$a_1^+ c_1^+ \lvert 0\rangle$
		0	$b^+_{1/2} d^+_{1/2} c_1^+ \lvert 0\rangle$
2	2	$\pm\frac{3}{\sqrt{2}} - \sqrt{5}$	$C_\pm[\frac{i}{2} a_2^+ + \frac{3}{\sqrt{40}}(a_1^+)^2 + \frac{1}{\sqrt{40}} b^+_{1/2} d^+_{3/2} + \frac{1}{\sqrt{40}} d^+_{1/2} b^+_{3/2}]\lvert 0\rangle$
		$\pm\frac{3}{\sqrt{2}} - 2$	$C_\pm[-\frac{i}{\sqrt{2}} b^+_{1/2} d^+_{1/2} a_1^+ + \frac{1}{2} b^+_{1/2} d^+_{3/2} - \frac{1}{2} d^+_{1/2} b^+_{3/2}]\lvert 0\rangle$
		$\pm\frac{3}{\sqrt{2}}$	$C_\pm[-\frac{1}{\sqrt{20}}(a_1^+)^2 + \frac{3}{\sqrt{20}} b^+_{1/2} d^+_{3/2} + \frac{3}{\sqrt{20}} d^+_{1/2} b^+_{3/2}]\lvert 0\rangle$
		$\pm\frac{3}{\sqrt{2}} + 2$	$C_\pm[\frac{i}{\sqrt{2}} b^+_{1/2} d^+_{1/2} a_1^+ + \frac{1}{2} b^+_{1/2} d^+_{3/2} - \frac{1}{2} d^+_{1/2} b^+_{3/2}]\lvert 0\rangle$
		$\pm\frac{3}{\sqrt{2}} + \sqrt{5}$	$C_\pm[\frac{i}{2} a_2^+ - \frac{3}{\sqrt{40}}(a_1^+)^2 - \frac{1}{\sqrt{40}} b^+_{1/2} d^+_{3/2} - \frac{1}{\sqrt{40}} d^+_{1/2} b^+_{3/2}]\lvert 0\rangle$

where:

$$C_\pm \equiv \frac{1}{2}[c_2^+ \pm i(c_1^+)^2]$$

References

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References for Appendix A

There are many good mathematical texts on differential geometry, at various levels of sophisticaton. We mention but two of them, which we have found useful,

L.P. Eisenhart, _Riemannian Geometry_, Princeton University Press, Princeton (1949).

J.D. Struik, _Lectures on Classical Differential Geometry_, Addison-Wesley, Reading, Mass. (1961).

For applications of the techniques of differential geometry to physical problems somewhat similar to those here, the reader can consult the various texts on the general theory of relativity. We find the discussion of Adler, et. al., particularly lucid.

R.J. Adler, M. Bazin, and M. Schiffer, _Introduction to General Relativity_, McGraw-Hill, New York (1965).