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QUANTUM-MECHANICAL FEW-BODY SCATTERING EQUATIONS WITH HALF-ON-SHELL ENERGY-INDEPENDENT SUBSYSTEM INPUT*

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*Ph.D. Dissertation

ABSTRACT

We present new equations for three- and four-body scattering, within the context of non-relativistic quantum mechanics and a Hamiltonian scattering theory.

For the three-body case we present Faddeev-type equations which, although obtained from the rigorous Faddeev theory, only require two-body bound state wavefunctions and half-off-shell transition amplitudes as input. In addition, their "effective potentials" are independent of the three-body energy, and can easily be made real after an angular momentum decomposition. The equations are formulated in terms of physical transition amplitudes for three-body processes, except that in the breakup case the partial-wave amplitudes differ from the corresponding full amplitudes by a Watson final-state-interaction factor.

We also present new equations for four-body scattering, obtained by generalizing our three-body formalism to the four-body case. These equations, although equivalent to those of Faddeev-Yakubovskii, are expressed in terms of singularity-free transition amplitudes, and their energy-independent effective potentials require only half-on-shell subsystem transition amplitudes (and bound state wavefunctions) as input. However, due to the detailed index structure of the Faddeev-Yakubovskii formalism, the result of our generalization is considerably more complicated than in the three-body case.

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Chapter One

INTRODUCTION

Our understanding of physical processes in the microscopic realm is largely based on quantum theory, developed in the 1920's and 1930's by Bohr, Schrödinger, Heisenberg and others.

In the decades that followed, as mastery of the new theory and the amount of relevant experimental data increased, attention was turned to the problem of understanding the nature of the interactions among subatomic (or nuclear) particles. The physical process that was then studied more and more extensively was that of scattering, in which a beam of particles of one kind is made to hit a target (usually stationary in the laboratory) composed of one or more kinds of particles.

The theory that was developed to describe such a process was quantum scattering theory.[1] During the 1940's and 1950's this theory was developed in great mathematical detail for the case of two-body scattering, that is a scattering in which the beam and the target are each composed of a single kind of particle. For such physical processes, the theory was developed to a very satisfactory state, especially after the introduction of the Lippmann-Schwinger (LS) equation,[2] which marked the transition from a theory based on differential equations to one based on integral equations.

In the late 1950's, attempts were made at generalizing scattering theory to the case of three-body scattering, first with great confidence in the power of the already-developed two-body theory, and then more haltingly as serious difficulties were encountered

repeatedly. These difficulties, related to a basic difference in the asymptotic nature of the two- and three-body scattering descriptions, were eventually pinpointed as arising from the non-uniqueness of the solutions of the three-body LS equations.[3]

In the late 1950's and early 1960's many successful special models for the three-body problem were presented, meeting these difficulties within the restrictions of each model: A zero-range model by Skornyakov and Ter-Martirosyan,[4] separable models by Mitra,[5] Sitenko and Kharchenko,[6] and a non-relativistic field-theory model by Amado.[7]

A rigorous mathematical solution to the general problem of three-body scattering within a Hamiltonian theory was finally presented in the early 1960's by Faddeev,[8] who was able to solve the above-mentioned difficulties by introducing different three-body entities and the new set of coupled integral equations they satisfy. Faddeev's work was recast into a form more suitable to meet the practical requirements of scattering data by Lovelace,[9] who first used the concept of pole dominance, and later by Alt, Grassberger and Sandhas,[10] with their quasi-particle approach. Extensive calculations applying these equations to specific physical systems were carried out in the last two decades.[11]

One of the most characteristic features of the Faddeev equations is that they are expressed in terms of amplitude <u>components</u>, i.e. in terms of splittings of the three-body entities considered in the LS equations. This is a consequence of the fact that to properly handle the asymptotic structure of the three-body problem, Faddeev intro-

duces the definition of channel, in which only two of the particles are assumed to interact, while the third particle is free.

As a result, the Faddeev equations have as input the channel two-body transition matrices of each two-body subsystem. Because of the kinematics involved, these two-body t-matrices appear in the kernels off the energy shell; that is, the energy parameter corresponds to neither of the momenta arguments in the two-body t-matrix. In this way, the three-body energy parameter appears in the kernel not only through the physical singularities (Green's functions), but also through the two-body subsystem t-matrices. Thus the kind of analytic structure that appears in the LS equations is significantly changed when going into the Faddeev theory. Furthermore, since the experimental phase shifts of the two-body subsystems are connected only to the fully-on-shell two-body t-matrices, the input to the Faddeev equations is far removed from the experimental data, and thus highly model-dependent.

Using a general representation of the off-shell two-body t-matrices that separates the on-shell from the off-shell pieces,[12] Noyes deals with this problem by introducing a further decomposition of the three-body amplitudes into "interior" and "exterior" parts, so that only the "interior" amplitudes involve off-shell pieces of the two-body t-matrices.[13] However, since the interior and exterior amplitudes remain coupled to each other in this formulation, the off-shell character of the kernel is not completely eliminated.

In this work, we deal with this difficulty in Faddeev's work by focusing on a different aspect of the three-body problem that is

seemingly disconnected from the one mentioned above. Most of the approaches we referred to define the amplitudes of the theory by using an appropriate plane-wave basis in which to expand the relevant state vectors of the theory. This is understandable, as historically this choice is the most direct generalization of the two-body case: In two-body scattering, the only natural basis that exists is of course that of the eigenstates of the free Hamiltonians, i.e. the plane-wave basis corresponding to a single free particle. Obviously, a plane-wave representation is also natural in three-body scattering; however, other natural bases are also available in this case, namely the complete sets of eigenstates of the channel Hamiltonians $H=H_o+V_\beta$, $\beta=1,2,3$. (An analogous freedom of choice of course holds in the four-body case.) Such projections of the three-body wavefunction components onto channel eigenfunctions were first considered by Noyes.[14]

It is by exploiting this freedom of choice for the three- and four-body case that we arrive at the main results of this work: We present here three- and four-body scattering equations that result from expanding the corresponding wavefunction components onto eigenfunctions of the appropriate channel Hamiltonians, rather than considering the plane-wave projections of such components.

For the three-body case, we expand the three-body Faddeev wavefunction components onto the two-body channel eigenstates,[15] and show that this representation is actually more natural than the plane-wave representation, and leads to a considerably simplified formulation of the three-body theory.

This three-body approach leads to a new pair of amplitudes $\mathscr{H}_{\beta\alpha}$

and $\mathscr{E}_{\beta\alpha}$, that represent the nonsingular parts of the three-body wavefunction in a simpler way than the amplitudes introduced by Faddeev do. The main advantage of this formulation, however, lies in the fact that the integral equations for the new set of amplitudes $\mathscr{H}$ and $\mathscr{E}$ are significantly simpler in structure: their effective potentials are independent of the three-body energy, and they only require two-body half-on-shell transition amplitudes and bound state wavefunctions as input. Additional convenient features become apparent after an angular momentum decomposition: by a simple redefinition of the partial wave components of the amplitude $\mathscr{E}$, the effective potentials can be made real, and the breakup scattering amplitude is seen to exhibit explicitly a Watson final-state-interaction factor in each channel.

The reasons for these simplifications can be physically understood as follows: much of the complicated structure of the plane wave projections of the three-body wavefunction is not due to true three-body dynamics, but is simply a reflection of the "spectator" two-body channel dynamics. By considering these plane wave projections, the channel dynamics are mixed with the true three-body dynamics in a complicated way. If however we expand each Faddeev component of the full wavefunction into the complete set of eigenfunctions of the spectator Hamiltonian <u>in its own channel</u>, the two-body channel dynamics are automatically treated in a natural way by these spectator complete sets; as a consequence, the three-body entities one is left to consider when solving the three-body problem get appreciably simplified.

For the case of four-body scattering, the situation is significantly more complicated; nevertheless, we obtain our equations by following a similar approach.[16] The resulting four-body equations exhibit essentially the same features as our three-body equations.

To carry out our four-body generalization, we choose the formulation due to Yakubovskii,[17] obtained by generalizing Faddeev's three-body theory. Yakubovskii's approach is the most well-established four-body theory, in particular because its equivalence with the Schrödinger equation has been demonstrated, so the possibility of spurious solutions can be ruled out.

The most characteristic feature of this formalism, and also its main weakness, is its very detailed classification of the clustering properties of the four-body system.

In some alternative approaches (such as that due to Sloan[18]), a less detailed index structure is considered, for instance using only a two-cluster classification of the four particles. As compared to Yakubovskii's, the resulting equations exhibit in general a more complicated structure, and their connection with the Schrödinger equation remains unclear.

A common feature of all these formalisms is that they have been developed almost exclusively at the formal operator level: the actual complexity involved (such as the singularity structure of the considered entities) is therefore not explicitly shown.

In our generalization we establish a four-body formalism based on the Faddeev-Yakubovskii (FY) theory in a way that makes the actual structure of the formalism more evident. With our three-body results

in mind, we carry out a similar singularity analysis of the four-body kernel. As in the three-body case, this task is considerably simplified by using the complete sets of eigenstates of the channel Hamiltonians. The analysis turns out to be particularly straightforward for FY entities labeled by two-cluster indices only - such as the wavefunction components $\Psi^{\sigma} = \sum_{\beta} \Psi^{\sigma}_{\beta}$, Ψ^{σ}_{β} being the conventional four-body FY component - and leads very naturally to new singularity-free amplitudes for four-body scattering.

In order to obtain equations for such amplitudes, however, the FY formalism requires that we also analyze the wavefunction component Ψ^{σ}_{β} itself; i.e., it requires that the singularity analysis be made taking into account the full index structure of the formalism. Unfortunately, this more detailed analysis turns out to be less straightforward than the first; in addition to the physical transition amplitudes, we are forced to introduce a nonphysical amplitude which, although not present in the full four-body wavefunctions, still appears in the dynamical equations.

Nevertheless, the set of equations we are led to exhibit essentially the same features as our corresponding three-body equations: namely, a multi-channel Lippmann-Schwinger structure with energy-independent effective potentials that require a simplified subsystem input (i.e., only half-on-shell subsystem scattering amplitudes and bound state wavefunctions).

In Chapter 2 we present a brief summary of the relevant two-body operators and eigenstates, as well as the relations and equations that will be useful in later chapters.

In Chap. 3 we review the difficulties faced by the direct generalization of two-body theory to the case of three-body scattering, summarize some elements of integral equations theory that have relevance to the subject, and outline Faddeev's solution to the quantum-mechanical three-body problem.

In Chap. 4 we present one of the main results of this work, namely our half-on-shell three-body equations. We obtain the half-off-shell amplitudes $\mathscr{H}$ and $\mathscr{E}$ from the projections of the Faddeev components into channel eigenstates, generalize to the corresponding fully-off-shell amplitudes, and derive the equations they satisfy. We consider the angular momentum decomposition of these equations for the S-wave case, and show how the $\mathscr{E}$-amplitude can be redefined so as to produce equations with real effective potentials. The amplitudes for processes starting from three free particles are briefly considered. In the appendix we prove that the amplitudes we have defined are free from primary singularities.[19]

In Chap. 5, we explore the question of whether the amplitudes of our three-body theory are unitary. We outline the symmetry properties of our amplitudes under time reversal, and define the amplitudes that arise from expanding the outgoing-wave three-body wavefunction components. We define the three-body S-matrix components in terms of the new amplitudes, and explore the conditions imposed on the latter by the requirement of S-matrix unitarity. We prove a general unitarity relation for the operators of our theory, and find some of the unitarity relations they imply for the scattering amplitudes we have defined.

Finally in Chapter 6, we present the generalization of our three-body approach to the four-body case. We briefly present a straightforward route to derive the FY equations, and introduce the four-body channel eigenstates. We analyze the singularities of the partially-summed wavefunction components, and identify the physical scattering amplitudes. The fully-split FY components are then analyzed, and we find the equations they satisfy. Finally, we generalize our amplitudes to the fully-off-shell case, connect our amplitudes to the operator formulation, and discuss the complications encountered. In the appendix we prove that our amplitudes yield the physical scattering amplitudes.

Chapter Two

TWO-BODY SCATTERING

I. The two-body scattering problem

In this chapter we outline the main results of two-body scattering theory that are relevant to the material presented in later chapters, and write down some results that will become useful later.

For the case of two-particle scattering, the mathematical difficulties referred to in Chap. 1 can be easily handled by appropriate specialized techniques, introduced during the early development of scattering theory. The starting point for these techniques is stationary scattering theory, in which it is required that we find solutions of the two-body Schrödinger equation,

$$h\,|\psi_e\rangle = (h_o + v)\,|\psi_e\rangle = e\,|\psi_e\rangle \tag{1.1}$$

where $h=h_0+v$ is the two-body Hamiltonian operator for the system. In momentum space, Eq. (1.1) has a representation

$$\tilde{p}^2\psi_{\vec{k}}(\vec{p}) + \int v(\vec{p}-\vec{p}\,')\psi_{\vec{k}}(\vec{p})\,d^3p' = \tilde{k}^2\psi_{\vec{k}}(\vec{p}) \tag{1.2}$$

while, if we assume local potentials, it has a coordinate-space representation

$$-\frac{\nabla^2}{2\mu}\,\psi_{\vec{k}}\,(\vec{x}) + v(\vec{x})\;\psi_{\vec{k}}\,(\vec{x}) = \tilde{k}^2\psi_{\vec{k}}(\vec{x}). \tag{1.3}$$

In the above expressions, μ is the reduced mass of the two-body system, $\vec{k}$ is the relative momentum corresponding to an energy eigenvalue $e=k^2/2\mu$, $\vec{x}$ is the vector distance between the two particles in configuration space, $\vec{p}$ is the relative momentum, and $\tilde{p}^2=p^2/2\mu, \tilde{k}^2=k^2/2\mu$. The interaction between the particles is usually represented by a potential operator v, with $v(\vec{x})=\langle\vec{x}|v|\vec{x}\rangle$ and $v(\vec{p}-\vec{p}\,')=\langle\vec{p}\,'|v|\vec{p}\rangle$. Finally,

we have denoted $\langle\vec{x}|\psi_{\vec{k}}\rangle$ and $\langle\vec{p}|\psi_{\vec{k}}\rangle$ by $\psi_{\vec{k}}(\vec{x})$ and $\psi_{\vec{k}}(\vec{p})$ respectively.

Within this framework, the appropriate boundary conditions arising from a general time-dependent scattering theory are incorporated into the stationary formulation by requiring that the solution $\psi_{\vec{k}}(\vec{x})$ be of the form

$$\psi_{\vec{k}}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} + w_{\vec{k}}(\vec{x}), \tag{1.4}$$

where $w_{\vec{k}}(\vec{x})$, the <u>scattered wave</u>, must satisfy specific boundary conditions for $|\vec{x}|\to\infty$, namely the <u>radiation condition</u>

$$w_{\vec{k}}(\vec{x}) \sim f_{\vec{k}}(\vec{x})\,\frac{e^{ikx}}{x}\,, \quad |\vec{x}|\to\infty\,. \tag{1.5}$$

For the problem outlined above, a rigorous formulation has been developed by several authors; in particular, a complete mathematical description of the solution to the two-body problem has been given by Povzner and Kato,[20] who prove the existence of solutions $\psi_{\vec{k}}(\vec{x})$ under certain restrictions imposed on the potential functions $v(\vec{x})$. By showing that the functions $\psi_{\vec{k}}(\vec{x})$ (together with the eigenfunctions corresponding to the discrete spectrum of the operator h), constitute a <u>complete set</u> of eigenfunctions, they make it possible to expand an arbitrary function into a generalized Fourier integral in terms of these functions. That is, they show how to construct a general solution of the two-body scattering problem with arbitrary boundary conditions. A fully detailed exposition of both the time-dependent and the time-independent formulations can be found for example in Ref. 1.

II. The Lippmann-Schwinger Equation and The Free-Particle Green's Function

The fact that the boundary conditions (1.4) and (1.5) appear as an additional requirement on the solutions to Eqs. (1.2) or (1.3) is a

natural consequence of having formulated the scattering problem in terms of differential equations. It was therefore desirable to reformulate the theory in such a way so as to have those boundary conditions "built-in" into the new formulation. This was accomplished in the 1950's by several authors, notably with the introduction of the Lippmann-Schwinger (LS) integral equation,[2] which greatly facilitated further progress of scattering theory.

The change from a differential equation to an integral equation with built-in boundary conditions can be accomplished in the following way: Consider again Eq. (1.1), rearranged as follows:

$$(h_o - e)|\psi\rangle = -v|\psi\rangle \ . \tag{2.1}$$

Multiplying both sides by the inverse operator $(h_o - e)^{-1}$, we immediately obtain the LS equation for the two-body wavefunction,

$$|\psi\rangle = |\phi_o\rangle - (h_o - e)^{-1} v|\psi\rangle \ , \tag{2.2}$$

where $|\phi_o\rangle$ is a solution of the equation

$$(h_o - e)|\phi_o\rangle = 0, \tag{2.3}$$

i.e. it is an eigenstate of the free-particle Hamiltonian with the same energy eigenvalue $e = k^2/2\mu$. (Sometimes we will write $|\vec{k}\rangle$ instead of $|\phi_o\rangle$.) Since the operator $(h_o - e)^{-1}$ becomes singular at points corresponding to the eigenvalues of h_o, it is necessary to generalize its definition by introducing a complex energy parameter z, i.e. by defining

$$g_o(z) = (h_o - z)^{-1}, \tag{2.4}$$

known as the resolvent of the free particle Hamiltonian, or as the free-particle Green's function for the scattering problem. As long as $\mathrm{Im}\, z \neq 0$, the operator $g_o(z)$ remains well behaved, and the physical solutions can be recovered by an appropriate limit procedure. It can be

shown that once $|\phi_o\rangle$ is chosen and a specific prescription for the limit Im $z \to 0$ is given, the eigenfunctions are completely determined by Eq. (2.2); i.e., the boundary conditions have been successfully incorporated into the dynamical equations to be solved.

The nature of the limit prescription can be seen as follows: In a scattering situation, we have an incoming plane wave (representing the incident beam) impinging upon a scattering target (usually stationary in the lab frame). The scattering of projectile and taget produce a spherical wave centered at the target's position, so that we end up with a scattering solution of the type given in Eq. (1.4), i.e.

$$\psi_{\vec{k}}(\vec{k}) = e^{i\vec{k}\cdot\vec{x}} + f_{\vec{k}}(\vec{x})\,\frac{e^{\pm ikx}}{x} \tag{2.5}$$

Choosing the positive sign to the right in (2.5) fixes an <u>outgoing-wave</u> boundary condition; i.e., an incoming plane wave and an outgoing scattered wave. A negative sign, on the other hand, yields the time-reversed solution, i.e. an incoming "scattered" wave which ends up as an outgoing plane wave.

It can be shown that this choice can be made at the level of the Green's function (2.4) in Eq. (2.2); i.e. that choosing $z \to e+io$ yields the outgoing-wave solution, and conversely for $z \to e-io$. We can thus label Eq. (2.2) more explicitly,

$$|\psi^{\pm}\rangle = |\phi_o\rangle - g_o(e\pm io)v|\psi^{\pm}\rangle \ . \tag{2.6}$$

The factor $g_o(z)v$ in (2.6) is known as the <u>LS kernel</u>, since it appears in all LS-type integral equations.

Eq. (2.6) determines the eigenfunctions of the two-body Hamiltonian. With appropriate restrictions on the kernel, this LS-type integral equation obeys the <u>Fredholm alternative</u>[21]: That is, we can either find

scattering wavefunctions $\psi_{\vec{k}}(\vec{p})$ as solutions of (2.6) with a nonzero inhomogeneous term $|\phi_o>$, and e>0, or, bound state wavefunctions $\phi_\kappa(\vec{p})$ (of energy $-\kappa^2$), as solutions to the homogeneous version of (2.6) with e<0. But we can never find the two kinds of solutions coexisting at a given energy.

More explicitly, that the Fredholm alternative exists for Eq. (2.6) implies that we can solve (2.6) in two different cases: Either for e>0 as the inhomogeneous case (here we write $|\phi_o>$ as $|\vec{k}>$)

$$|\psi^{\pm}_{\vec{k}}> = |\vec{k}> - g_o(\tilde{k}^2 \pm i\sigma)v|\psi^{\pm}_{\vec{k}}> \tag{2.7a}$$

or as the homogeneous case for e<0,

$$|\phi_\kappa> = -g_o(-\kappa^2)v|\phi_\kappa> \tag{2.7b}$$

but never both kinds simulatenously. Physically, the existence of the Fredholm alternative for the LS equation reflects the fact that if we scatter two particles that were originally free, we can never end up with a bound state of the pair. That is, the bound-state and the scattering state solutions are kinematically separated from each other.

III. The Eigenfunctions and the Resolvent of the Two-Body Hamiltonian

As is clear from the above considerations, the complete set of eigenfunctions of h is composed of the scattering wavefunctions with a continuum set of positive quantum numbers $\vec{k}$ corresponding to an energy eigenvalue $e=k^2/2\mu$, and the bound-state wavefunctions with a set of discrete negative eigenvalues $-\kappa_i^2$. Assuming only one bound state for simplicity, the completeness relation for the eigenfunctions of h takes the form

$$|\phi_\kappa><\phi_k| + \int d^3k|\psi^{\pm}_{\vec{k}}><\psi^{\pm}_{\vec{k}}| = 1 , \tag{3.1}$$

where it will be recalled that for plane-wave states the corresponding completeness relations is given simply by

$$\int d^3k \; |\vec{k}\rangle\langle\vec{k}| = 1 \tag{3.2}$$

The actual solutions to Eqs. (2.7), however, are usually obtained in terms of other operators that are more convenient and also more central to scattering theory. One of them is the <u>resolvent of the full two-body Hamiltonian</u>, i.e.

$$g(z) = (h-z)^{-1} = (h_o+v-z)^{-1} \; . \tag{3.3}$$

From the definition (3.3), it is clear that, for real potentials,

$$g^{\dagger}(z) = g(z^*) \tag{3.4}$$

(where † implies the hermitian adjoint operation, and the asterisk * implies complex conjugation), and also that

$$g(z_1)-g(z_2) = (z_1-z_2)g(z_1)g(z_2). \tag{3.5}$$

Eq. (3.5) is known as <u>Hilbert's identity</u>, or as the <u>first resolvent equation</u>.

The importance of the operator g(z) is that, once it is known, the eigenfunctions of h can be obtained by a direct limit procedure (instead of finding it by solving (2.6)). To see this we manipulate Eq. (2.6) again, "solving" for $|\psi\rangle$:

$$|\psi\rangle = \left[1+g_o(z)v\right]^{-1}|\phi_o\rangle. \tag{3.6}$$

But

$$g_o^{-1}(1+g_o v) = g_o^{-1}+v = h_o+v-z = g^{-1}, \tag{3.7}$$

so that

$$|\psi\rangle = (1+g_o v)^{-1}|\phi_o\rangle = g\, g_o^{-1}|\phi_o\rangle \; . \tag{3.8}$$

Furthermore, $g_o^{-1}|\phi_o\rangle = (h_o-e\pm i\varepsilon)|\phi_o\rangle = \pm i\varepsilon\phi_o\rangle$, so we finally find that

$$|\psi^{\pm}\rangle = \lim_{\varepsilon\to 0^{+}} \mp i\varepsilon g(e\pm i\varepsilon)|\phi_o\rangle . \tag{3.9}$$

Eq. (3.9) indicates that given the initial state $|\phi_o\rangle$ and the sign of the limit prescription, the solution $|\psi^{\pm}\rangle$ is completely determined by the Green's function of the problem.

Replacing (3.8) into (2.6) immediately yields the LS integral equation for g(z),

$$g = g_o - g_o v g. \tag{3.10a}$$

Had we started operating with bras rather than kets in the foregoing equations, we would have obtained the equivalent equation

$$g = g_o - g v g_o. \tag{3.10b}$$

Applying the operator g(z) to Eq. (3.1), we obtain the <u>spectral decomposition</u> of the resolvent operator, i.e. (assuming a single bound state of energy $-\kappa^2$)

$$g(z) = \frac{|\phi_\kappa\rangle\langle\phi_\kappa|}{z+\kappa^2} + \int d^3k \, \frac{|\psi^{\pm}_{\vec{k}}\rangle\langle\psi^{\pm}_{\vec{k}}|}{\tilde{k}^2 - z} \tag{3.11}$$

in terms of the complete set of eigenfunctions of h. Note that in (3.11) the complex-plane structure arising from the complex variable z is explicitly exhibited.

IV. The two-body t matrix

The remaining important operator in two-body stationary scattering theory is encountered when analyzing the singularity structure of the momentum-space representation of the two-body wavefunction (2.7a):

$$\psi^{\pm}_{\vec{k}}(\vec{p}) = \langle\vec{p}|\psi^{\pm}_{\vec{k}}\rangle = \langle\vec{p}|\vec{k}\rangle - \langle\vec{p}|g_o(e\pm io)v|\vec{k}\rangle. \tag{4.1}$$

Using (3.2), we can write

$$\langle\vec{p}|g_o(e\pm io)v|\psi^{\pm}_{\vec{k}}\rangle = \int\langle\vec{p}|g_o(e\pm io)|\vec{p}\,'\rangle\langle\vec{p}\,'|v|\psi^{\pm}_{\vec{k}}\rangle d^3p\,'$$

$$= \frac{\delta(\vec{p}-\vec{p}\,')}{\tilde{p}'^2-\tilde{k}^2\mp io} \langle\vec{p}\,'|v|\psi^{\pm}_{\vec{k}}\rangle d^3p' =$$

$$= \frac{\langle\vec{p}|v|\psi^{\pm}_{\vec{k}}\rangle}{\tilde{p}^2-\tilde{k}^2\mp io} \tag{4.2}$$

(since the free-particle Green's function is diagonal in a plane-wave representation). Thus

$$\psi^{\pm}_{\vec{k}}(\vec{p}) = \delta(\vec{p}-\vec{k}) - \frac{\langle\vec{p}|v|\psi^{\pm}_{\vec{k}}\rangle}{\tilde{p}^2-\tilde{k}^2\mp io} . \tag{4.3}$$

The matrix element $\langle\vec{p}|v|\psi^{\pm}_{\vec{k}}\rangle$ is directly connected with the scattering amplitude $f_{\vec{k}}$ of Eq. (2.5), as can be seen by obtaining the Fourier transform of (4.3) to reproduce (2.5). Because of this, plane-wave matrix elements of a <u>transition operator t</u>[1] are defined through

$$\langle\vec{p}|v|\psi^{\pm}_{\vec{k}}\rangle = \langle\vec{p}|t(\tilde{k}^2\pm io)|\vec{k}\rangle = t(\vec{p},\vec{k};\tilde{k}^2\pm io), \tag{4.4}$$

such that

$$\langle\vec{p}|t(\tilde{k}^2\pm io)|\vec{k}\rangle = f_{\vec{k}}(\vec{p}), \tag{4.5}$$

thus related to the differential cross-section by

$$\frac{d\sigma}{d\Omega} \sim |\langle\vec{k}|t(\tilde{k}^2+io)|\vec{k}\rangle|^2 . \tag{4.6}$$

Now from the definition (4.4) we see that

$$t(\tilde{k}^2\pm io)|\vec{k}\rangle = v|\psi^{\pm}_{\vec{k}}\rangle \quad ; \tag{4.7}$$

furthermore, since replacing (3.10b) into (3.8) yields

$$|\psi_{\vec{k}}\rangle = (1-gv)|\vec{k}\rangle \tag{4.8}$$

we obtain, replacing (4.7) into (4.8),

$$t = v-vgv \quad , \tag{4.9}$$

which defines the transition operator t in terms of the Green's function g. Replacing (4.9) into (3.10a) right-multiplied by v, and into (3.10b) left-multiplied by v, we obtain the identities

$$gv = g_o t \tag{4.10a}$$

$$vg = tg_o \ , \tag{4.10b}$$

which again replaced into Eqs. (3.10) yield

$$g = g_o - g_o t g_o \ , \tag{4.11}$$

which is the inverse relation to (4.9). Finally, with the aid of (4.10) we can rewrite (4.8) as

$$|\psi_{\vec{k}}^{\pm}\rangle = (1-g_o t)|\vec{k}\rangle \tag{4.12}$$

which (equivalently to (4.3) with (4.4)) yields the expression for the momentum-space representation of the wavefunction in terms of the matrix elements of the transition operator t:

$$\psi_{\vec{k}}^{\pm}(\vec{p}) = \delta(\vec{p}-\vec{k}) - \frac{t(\vec{p}, \vec{k}, \tilde{k}^2 \pm io)}{\tilde{p}^2 - \tilde{k}^2 \mp io} \tag{4.13}$$

Eq. (4.13) shows the advantage of solving for matrix elements of t rather than for the functions $\psi_{\vec{k}}$: the wavefunction $\psi_{\vec{k}}$ has a δ-function singularity in the first term, <u>and</u> a pole (the physical singularity) in the second term. Eq. (4.13) clearly shows that the function $t(\vec{p}, \vec{k}, \tilde{k}^2 \pm io)$ is less singular than $\psi_{\vec{k}}$.

Integral equations for t are obtained by replacing (4.10) into (4.9), giving

$$t = v - vg_o t \tag{4.14a}$$

$$t = v - tg_o v \ . \tag{4.14b}$$

So far, the matrix elements $\langle\vec{p}|t(\tilde{k}^2 \pm io)|\vec{k}\rangle$ have been defined only with the energy argument being equal to the energy of the initial-state ket $|\vec{k}\rangle$, i.e. $\tilde{k}^2$. In view of (4.9), however, a simple generalization allows the definition of a t-matrix operator for arbitrary argument z, i.e.

$$t(z) = v - vg(z)v \ . \tag{4.15}$$

The functions

$$t(\vec{p}, \vec{k}; z) = \langle\vec{p}|t(z)|\vec{k}\rangle \tag{4.16}$$

are called the <u>fully-off-shell</u> matrix elements of t, in the sense that $p^2 \neq k^2 \neq \mathrm{Re}z$. When $z=\tilde{k}^2 \pm io$, as in Eq. (4.4), the matrix elements (4.16) are said to be <u>half on shell</u>. The physical transition amplitudes are obtained from the <u>fully-on-shell</u> matrix elements $t(\vec{k}, \vec{k}, \tilde{k}^2+io)$ $=\langle\vec{k}|t(\tilde{k}^2+io)|\vec{k}\rangle$, as in Eq. (4.6).

Using (3.4) and (4.9), we see that

$$t^{\dagger}(z) = t(z^*) \ , \tag{4.17}$$

which implies that

$$t^*(\vec{p}, \vec{k}, z) = t(\vec{k}, \vec{p}, z^*). \tag{4.18}$$

In the situations that are usually encountered, in which time-reversal invariance and detailed balance are conserved, we also have evidently,

$$t(\vec{p}, \vec{k}; z) = t(\vec{k}, \vec{p}; z) \tag{4.19}$$

in which case, by taking the complex conjugate of (4.13) (and using (4.18)) we obtain

$$[\psi^{\pm}_{\vec{k}}(\vec{p})]^* = \psi^{\mp}_{\vec{k}}(\vec{p}). \tag{4.20}$$

V. The Analytic Structure of the t-matrix

The singularity structure of the matrix element $t(\vec{k}, \vec{k}; \tilde{k}^2+io)$ can be seen to include a so-called "right-hand cut" on the positive real axis ($0<k<\infty$), and a pole in the negative real axis for each bound state. Additional structure results from the specific nature of the potential functions $v(\vec{x})$.

This analytic structure is best studied considering the off-shell matrix elements (4.16); using Hilbert's identity for g(z) (Eq. (3.5)) multiplied to the right and to the left by v, and recalling (4.15) and (4.10), we can write

$$t(z_1)-t(z_2) = (z_2-z_1)t(z_1)g_o(z_1)g_o(z_2)t(z_2) \tag{5.1}$$

which is known as the <u>operator unitarity relations</u> for t. In terms of the matrix elements (4.16), Eq. (5.1) yields the fully-off-shell unitarity relations for matrix elements of t,

$$t(\vec{q},\vec{q}';z_1)-t(\vec{q},\vec{q}';z_2)=(z_2-z_1)\int d^3k\ \frac{t(\vec{q},\vec{k};z_1)}{(\tilde{k}^2-z_1)}\frac{t(\vec{k},\vec{q}';z_2)}{(\tilde{k}^2-z_2)}\ . \tag{5.2}$$

By taking appropriate limits for z_1 and z_2, the singularity structure mentioned above can be made evident. In particular, if we take the special case $\vec{q}=\vec{q}'$, $z_1=\tilde{q}^2+io$, $z_2=\tilde{q}^2-io$, and recall that $\lim_{\varepsilon\to 0} \frac{2i\varepsilon}{x^2+\varepsilon^2} = 2\pi i\delta(x)$, we obtain from (5.2),

$$t(\vec{q},\vec{q};\tilde{q}^2+io)-t(\vec{q},\vec{q};\tilde{q}^2-io)= -2\pi i\int d^3k\ t(\vec{q},\vec{k};\tilde{q}^2+io)\ \times$$
$$\times\ \delta(\tilde{k}^2-\tilde{q}^2)t(\vec{k},\vec{q};\tilde{q}^2-io), \tag{5.3}$$

which gives the discontinuity of the forward scattering amplitude across the positive real axis, i.e. in the physical region. Eq. (5.3) is known as the <u>on-shell unitarity relation</u> for the t-matrix.

It was also mentioned before that t has a simple pole in z at each bound state energy. We conclude this chapter by seeing this explicitly. If we assume for simplicity a single bound state of energy $-\kappa^2$, we will have[8]

$$t(z) = \frac{c(z)}{z+\kappa^2} + t^R(z)\ , \tag{5.4}$$

where $t^R(z)$ is nonsingular at $z=-\kappa^2$. To find c(z) we make use of expressions (4.15) and (3.11) to obtain

$$t(z)= v-v\left\{\frac{|\phi_\kappa\rangle\langle\phi_\kappa|}{z-x^2} + \int d^3k'\ \frac{|\psi^{\pm}_{\vec{k}'}\rangle\langle\psi^{\pm}_{\vec{k}'}|}{\tilde{k}'^2-z}\right\} v, \tag{5.5}$$

so that

$$c(z) = v|\phi_\kappa\rangle\langle\phi_\kappa|v\ , \tag{5.6a}$$

$$t^R(z) = v - \int d^3k' \frac{v|\psi^{\pm}_{\vec{k}'}\rangle\langle\psi^{\pm}_{\vec{k}'}|}{\tilde{k}'^2 - z} \quad . \tag{5.6b}$$

Taking matrix elements of (5.6), we get

$$c(\vec{p},\vec{k},z) = \langle\vec{p}|v|\phi_\kappa\rangle\langle\phi_\kappa|v|\vec{k}\rangle \tag{5.7a}$$

$$t^R(\vec{p},\vec{k};z) = \langle\vec{p}|v|\vec{k}\rangle - \int d^3k' \frac{\langle\vec{p}|v|\psi^{\pm}_{\vec{k}'}\rangle\langle\psi^{\pm}_{\vec{k}'}|v|\vec{k}\rangle}{\tilde{k}'^2 - z} \quad . \tag{5.7b}$$

Furthermore, recalling that $|\phi_\kappa\rangle$ satisfies the equation $(h_o + v + \kappa^2)|\phi_\kappa\rangle = 0$ and that $v|\psi^{+}_{\vec{k}}\rangle = v(1 - g_o t)|\vec{k}\rangle = t|\vec{k}\rangle$, we find

$$t(\vec{p},\vec{k};z) = \frac{\Phi_\kappa(\vec{p})\Phi^*_\kappa(\vec{k})}{z + \kappa^2} + t^R(\vec{p},\vec{k};z) \tag{5.8}$$

where

$$\Phi_\kappa(\vec{p}) = -(\tilde{p}^2 + \kappa^2)\phi_\kappa(\vec{p}) \tag{5.9}$$

is known as the <u>vertex function</u> for the bound state, and the remainder is given by

$$t^R(\vec{p},\vec{k};z) = v(\vec{p}-\vec{k}) - \int d^3k' \frac{t(\vec{p},\vec{k}';\tilde{k}'^2 \pm io)\, t(\vec{k}',\vec{k};\tilde{k}'^2 \mp io)}{\tilde{k}'^2 - z} \quad . \tag{5.10}$$

Chapter Three

THE THREE-BODY PROBLEM, AND FADDEEV'S SOLUTION

The extensive work done on the two-body problem (briefly and partially summarized in Chap. 2) gave credence to the idea that the three-body problem could be approached in an analogous manner. The only modifications that were expected to arise were those due to trivial kinematical differences related to the larger number of particles involved.

This confidence was however not justified, as was found when persistent difficulties were encountered during the 1950's in solving the three-body scattering problem. It was only in the early 1960's that a satisfactory mathematical solution to the general problem was found by Faddeev,[8] who was able to prove rigorously that a theory for three-body scattering could be formulated so as to satisfy the requirements of general scattering theory.

To examine this in some detail, we start with the Schrödinger equation for the three-body case,

$$(H_o + V - E)|\Psi\rangle = 0 \ , \tag{1.1}$$

where we now use capital letters to differentiate operators in the appropriate three-body Hilbert space from those defined in a two-body space, which are represented by lower case letters as in Chap. 2. In (1.1),

$$V = V_1 + V_2 + V_3 \tag{1.2}$$

is the sum of the three interactions between the pairs of particles, i.e.

$$V_1 = V(2,3) \quad V_2 = V(3,1) \quad V_3 = V(1,2) \ . \tag{1.3}$$

Following the ideas of the previous chapter, we define the three-

body free-particle Green's function as

$$G_o(z) = (H_o - z)^{-1}, \tag{1.4}$$

and the corresponding full Green's function as

$$G(z) = (H_o + V - z)^{-1} \tag{1.5}$$

The three-body transition operator T(z) is defined to be, also in analogy to the two-body case,

$$T(z) = V - VG(z)V \tag{1.6}$$

and obeys the equations

$$T(z) = V - VG_o(z)T(z) \tag{1.7a}$$

$$T(z) = V - T(z)G_o(z)V , \tag{1.7b}$$

known as the <u>three-body LS equations for T</u>.

Inspired by the confidence with which the well-developed two-body scattering formalism was generally held, it was assumed that, by means of similar techniques, Eqs. (1.7) would yield all scattering amplitudes for three-body scattering, with only the minor changes arising from the change from a two- to a three-body Hilbert space.

These assumptions turned out to be incorrect, however: most of the work in the three-body problem carried out during the 1950's was beset by unexpected difficulties and ambiguities.

In 1957, Foldy and Tobocman[3] pointed out that the difficulties might arise from the fact that for the three-body case, the LS equations (1.7) did not yield unique solutions. It was not however until the work of Faddeev that this was studied in detail, the reasons for this non-uniqueness well understood, and a satisfactory new set of equations that yielded correct scattering amplitudes was found.

There are several ways of understanding the problems that beset the attempts at solving Eqs. (1.1) or (1.7); we will give here only some of

them, as well as a sketch of the solution given by Faddeev.

II. Three-Body Kinematics

First we must outline the kinematical definitions that are used in the three-body problem. In the two-body problem, we describe the situation by a single vector variable ($\vec{p}$ or $\vec{x}$) in center-of-mass (CM) coordinates (usually referred to as the coordinates of the reduced two-body problem). In the three-body case we have instead the lab momenta of the three-body particles, which (when transforming to the CM system) result in two independent momenta plus the overall CM momentum.

This is done for instance by defining the following quantities in terms of the three CM momenta of the particles, i.e. $\vec{k}_1$, $\vec{k}_2$, $\vec{k}_3$:

$$q_3 = \frac{m_2\vec{k}_1 - m_1\vec{k}_2}{m_1 + m_2}$$

$$\vec{p}_3 = \frac{(m_1+m_2)\vec{k}_3 - m_3(\vec{k}_1+\vec{k}_2)}{M}$$

$$\vec{K} = \vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0 \tag{2.1}$$

In (2.1), $M=m_1+m_2+m_3$, $\vec{q}_3$ is the relative momentum of the pair of particles labeled by 1 and 2, $\vec{p}_3$ is the momentum of the unpaired particle (here particle 3) relative to the center of mass of the pair, and $\vec{K}$ is the overall CM momentum.

This change of variables can obviously be done in three different ways, according to which particle is left unpaired; thus there are three equivalent coordinate systems in terms of which we can describe the kinematics of the (CM) three-body problem. These equivalent systems are of course not independent, so in fact any one is sufficient to specify the kinematic configuration of the system. In practice, the choice is made by selecting the system which is "natural" to the three-

body entity being described.

As an illustration, we give here the relationship between $\vec{p}_\gamma$, $\vec{q}_\gamma$ and $\vec{p}_\beta$, $\vec{q}_\beta$, (α, β, γ cyclic):

$$\vec{q}_\gamma = -\frac{m_\beta}{m_\alpha+m_\beta}\vec{q}_\beta - \frac{m_\alpha M}{(m_\alpha+m_\beta)(m_\alpha+m_\gamma)}\vec{p}_\beta$$

$$\vec{p}_\gamma = \vec{q}_\beta - \frac{m_\gamma}{m_\alpha+m_\gamma}\vec{p}_\beta \ . \tag{2.2}$$

In terms of the variables (2.1), a momentum-space representation of the three-body Hamiltonian H is given by

$$H = \frac{K^2}{2M} + \tilde{p}_\beta^2 + \tilde{q}_\beta^2 + V_1+V_2+V_3, \quad \beta=1,2,3 \tag{2.3}$$

The index β refers to the unpaired particle's label, and the notation clearly indicates that the term $\tilde{p}^2 + \tilde{q}^2$ is independent of the choice of β. In (2.3),

$$\tilde{p}_\beta^2 = \frac{p^2}{2n_\beta}, \quad \tilde{q}_\beta^2 = \frac{q_\beta^2}{2\mu_\beta} \tag{2.4}$$

with

$$n_\beta = \frac{m_\beta(m_\alpha+m_\gamma)}{M}$$

$$\mu_\beta = \frac{m_\alpha m_\gamma}{m_\alpha+m_\gamma}, \tag{2.5}$$

where α,β,γ is any cyclic choice of the pairs 1,2,3. The first term in (2.3) can of course always be decoupled from the internal motion of the system.

The kind of kinematic variables we have just described leads very naturally to a description of the three-body system in terms of <u>channels</u>, which as we will see are a central feature of Faddeev's solution. A channel is defined as a configuration of the three particles in which

only two particles (labeled, say, as the pair β) are interacting, while the third particle is assumed to be free.

III. Relevant Integral Equations Theory

Before we can outline the problems encountered when trying to solve the three-body LS equations (1.7),

$$T(z) = V - VG_0(z)T(z) \ , \tag{3.1}$$

we must briefly review some relevant aspects of the theory of integral equations.

In Chapter 2 it was mentioned that a feature of the two-body LS equations (Eqs. (2.7) of Chap. 2) was that they obeyed the Fredholm alternative. In fact, this is a necessary and sufficient condition for the LS equations to have unique solutions.

For this to occur, however, the kernel of the equations must obey certain restrictive conditions, namely the conditions that are necessary for the Fredholm theory of integral equations to be applicable. To see this condition in more detail, let us generically describe an integral equation as

$$f(x) = f_0(x) + \int K(x,x')f(x') \ , \tag{3.2}$$

where $\{x\}$ symbolizes the complete set of variables that describe the physical system, $f(x)$ is the object we wish to solve for, $K(x,x')$ is the kernel of the integral equation, and $f_0(x)$ is the inhomogeneous term, also called the "driving" term.

It can be shown that in order for (3.2) to be solvable by the Fredholm method[22] (i.e., that the Fredholm alternative applies), it is sufficient to require that the kernel $K(x,x')$ be an "$\mathscr{L}^2$ kernel", more precisely known as a <u>Hilbert-Schmidt kernel</u>.[22] This means that K must satisfy

$$\|K\|_s \equiv \int |K(x,x')|^2 \, dx \, dx' < \infty \, , \tag{3.3}$$

where we denote by $\|K\|_s$ the <u>Hilbert-Schmidt norm</u> of the kernel K. This should be compared to the usual operator norm $\|K\|$,

$$\|K\| = \max_{\{f\}} \frac{(Kf,Kf)}{(f,f)} \tag{3.4}$$

where (f,f) is the inner product defined in the appropriate space the operator K acts upon, and f is any arbitrary function in that space. To require that the norm (3.4) be finite is a less restrictive condition than (3.3), since it can be shown that

$$\|K\|_s \geq \|K\| \, . \tag{3.5}$$

That the kernel K be Hilbert-Schmidt is, however, only a sufficient condition for the Fredholm method to be applicable. A more general type of kernel that still allows a Fredholm solution is called a <u>compact</u> (or completely continuous) kernel.

To define a compact kernel, we assume that normalized wave-packet states $\psi_i(x)$, i=1,2,... can be constructed from some complete set of wavefunctions in the Hilbert space of the state vectors of the system. The kernel K is said to be compact if for <u>any</u> infinite set of the ψ_i,

$$K\psi_i \equiv \int K(x,x')\psi_i(x')dx' \tag{3.6}$$

contains a subset converging to a limit.[22] In general it is not easy to prove that a kernel is compact, but all $\mathscr{L}^2$ kernels are compact; since the latter condition is easier to prove, $\mathscr{L}^2$ theory is ordinarily used when applicable.

As a simple relevant example, let us mention that a δ-function kernel, i.e. $K(x,x') = \delta(x-x\ )$, can be easily proven to have a finite operator norm (3.4) but an infinite Hilbert-Schmidt norm (3.3).

IV. The Difficulties

We can now state very simply the difficulties that beset the attempts to solve the LS equations (3.1).

In the two-body case, the LS equation for t has a kernel that - for a wide class of potentials - is compact, so that for such a class the LS equation yields unique solutions for the physical transition amplitudes.

Eq. (3.1), in contrast, does not have a compact (or $\mathscr{L}^2$) kernel, so the Fredholm alternative does not hold. As a result, (3.1) has no unique solutions. The source for this difficulty can be easily found by looking for example at a piece of the operator $VG_o(z)$, namely the piece $V_1G_o(z)$. The kernel for this piece is

$$\frac{v_1(\vec{q}_1-\vec{q}_1') \; \delta(\vec{p}_1-\vec{p}_1')}{(\tilde{q}^2+\tilde{p}^2-z)} \tag{4.1}$$

which contains a δ-function that <u>is not removed</u>, no matter how many times we iterate it. In fact, the kernel of $V_1G_oV_1G_o$ contains the same δ-function, and it is therefore impossible to reduce Eq. (3.1) to an equation with a compact kernel.

Physically, this arises from the fact that there are pieces in the kernel of the equation in which one particle "just rides through", i.e. acts only as a spectator, giving rise to the δ-function factor. This factor just expresses the conservation of momentum of the spectator particle.

It must be emphasized that the problem does not <u>only</u> arise in Eq. (3.1), i.e. that the difficulty can't be simply eliminated by considering instead of (3.1) some other three-body scattering equation derived by analogy to the two-body case. The difficulty arises because the asymptotic structure of the three-body problem, as we will see, is

intrinsically different from that of the two-body problem, and this different structure must be dealt with appropriately.

To illustrate this, we show how a different view of the same difficulty becomes apparent when we consider the three-body Schrödinger equation (Eq. (1.1)),

$$(H_o + \sum_{\alpha=1}^{3} V_\alpha - E)|\Psi^+> = 0,$$

which we rearrange (for an arbitrary index α) to read

$$(H_o + V_\alpha - E)|\Psi^+) = -(V_\beta + V_\gamma)|\Psi^+>, \quad \alpha=1,2,3 \tag{4.2}$$

Defining the channel <u>two-body Green's function</u> as

$$G_\alpha(z) = (H_o + V_\alpha - z)^{-1} \tag{4.3}$$

we can invert (4.2) to obtain

$$|\Psi^+> = C_\alpha|\chi_\alpha> - G_\alpha(E+io)(V_\beta+V_\gamma)|\Psi^+> \quad \alpha=1,2,3 \tag{4.4}$$

where $|\chi_\alpha>$ is a solution of

$$(H_o + V_\alpha - E)|\chi_\alpha> = 0 \ . \tag{4.5}$$

$|\chi_\alpha>$ is the <u>channel two-body eigenfunction</u> of the pair α, i.e. the eigenfunctions of the α subsystem, in which particle α is a spectator.

Obviously, we can carry out this procedure in three different ways (one for each channel), so there is no way to determine how much of each of the $|\chi_\alpha>$ goes into the full solution $|\Psi^+>$ (i.e. the values of the C_α) until one has solved the problem in a different way.[23] A priori, any of the equations (4.4) has a built-in ambiguity in its boundary conditions.

In yet another way, we can express the difficulty by saying that, again due to the nature of its kernel, the homogeneous version of (4.4) will <u>also</u> have solutions in the scattering region. This helps to

clarify the physical reasons for the difficulty: In the two-body case, conservation of energy separates the scattering region from the bound-state region. In the three-body case, on the other hand, this is no longer true, since the energy released upon formation of a bound state in a given channel can be given to the third particle. In this way, in the Schrödinger or the LS equation, the different subregions become kinematically accessible, and the Fredholm alternative is inapplicable.

V. The Faddeev Equations

The foregoing difficulties were eliminated by Faddeev by replacing Eqs. (3.1) or (4.4) by suitable sets of coupled integral equations for three-body entities different from those considered above.

For the LS equations (3.1), Faddeev defines the new operator $M_{\beta\alpha}$ as

$$M_{\beta\alpha}(z) = \delta_{\beta\alpha}V_\beta - V_\beta G(z)V_\alpha , \tag{5.1}$$

such that upon summation over the channel indices,

$$\sum_{\beta\alpha} M_{\beta\alpha}(z) = V - VG(z)V = T(z), \tag{5.2}$$

i.e. we recover the usual three-body T-matrix. Comparison of (5.1) and (3.1) yields

$$M_{\beta\alpha}(z) = \delta_{\beta\alpha}V_\alpha - V_\beta G_o(z) \sum_\gamma M_{\gamma\alpha}(z). \tag{5.3}$$

Eqs. (5.3) are no better than the original LS-equations we started from, since their kernel is essentially the same. If however we shift the diagonal piece of the equation, $V_\alpha G_o(z)M_{\beta\alpha}(z)$, from the right to the left,

$$\left[1+V_\beta G_o(z)\right]M_{\beta\alpha}(z) = \delta_{\beta\alpha}V_\alpha - V_\beta G_o(z) \sum_{\gamma\neq\alpha} M_{\gamma\alpha}(z), \tag{5.4}$$

and then find $M_{\beta\alpha}$ by <u>inverting</u> the operator $\left[1+V_\beta G_o(z)\right]$, which it will be recalled from Eq. (3.7) of Chap. 2 has an inverse

$$\left[1 + V_\beta G_o(z)\right]^{-1} = G_o^{-1}(z) G_\beta(z) \ , \tag{5.5}$$

we obtain the equations

$$M_{\beta\alpha} = \delta_{\beta\alpha} G_o^{-1} G_\alpha V_\alpha - G_o^{-1} G_\beta V_\beta G_o \sum_{\gamma\neq\alpha} M_{\gamma\alpha} \ , \tag{5.6}$$

where we have omitted writing the z-dependence of the operators in (5.6). Recalling now Eq. (4.10) from Chap. 2, i.e. that

$$G_\beta V_\beta = G_o t_\beta \ , \tag{5.7}$$

where t_β is the two-body t-matrix in channel β suitably defined in the three-body Hilbert space, we can write (5.6) as

$$M_{\beta\alpha} = \delta_{\beta\alpha} t_\beta - t_\beta G_o \sum_\gamma \bar{\delta}_{\beta\gamma} M_{\gamma\alpha} \tag{5.8a}$$

or equivalently,

$$M_{\beta\alpha} = \delta_{\beta\alpha} t_\beta - \sum_\gamma M_{\beta\gamma} \bar{\delta}_{\gamma\alpha} G_o t_\alpha \tag{5.8b}$$

with $\bar{\delta}_{\beta\alpha}=1-\delta_{\beta\alpha}$. Eqs. (5.8) are the <u>Faddeev equations</u> for the Faddeev operator $M_{\beta\alpha}$.

The kernel of Eqs. (5.8) is different from the LS kernel, due to the inversion (5.5) that was used to obtain them. The δ-function that survives at all orders in the LS kernel disappears after one iteration from the Faddeev kernel $t_\beta G_o \bar{\delta}_{\beta\alpha}$: The factor $\bar{\delta}_{\beta\alpha}$ forbids the appearance of repeated channel indices upon iteration, and the δ-function is eliminated by the integrations involved in the first iteration.

The process leading from (5.3) to (5.8), through which we insure that the new kernel has the desired characteristics, is called removing disconnected pieces from the kernel. The name arises from the fact that, as opposed to the three-body LS equations, no disconnected processes (i.e. those in which one of the particles "rides through" as a spectator) are allowed in (5.8). The kernel of (5.8) is therefore said to be three-

body connected (after one iteration).

The splitting of $T(z)$ into the nine pieces $M_{\beta\alpha}(z)$ has its counterpart at the wavefunction level, where Faddeev defines[24]

$$|\Psi^{\pm}\rangle = \sum_{\beta=1}^{3} |\Psi^{\pm}_{\beta}\rangle \tag{5.9}$$

where the $|\Psi^{\pm}_{\beta}\rangle$ are the Faddeev components of the full three-body wavefunction $|\Psi^{\pm}\rangle$. The splitting (5.9) corresponds to a classification of the full wavefunction into three pieces, according to which particle pair (β) interacts last.

Instead of the system of equations (4.2) or (4.4), Faddeev postulates the system of coupled equations

$$(H_o+V_\beta-E)|\Psi^{+}_{\beta}\rangle = -V_\beta \sum_{\gamma} \bar{\delta}_{\gamma\beta}|\Psi^{+}_{\gamma}\rangle \quad \beta=1,2,3 \tag{5.10}$$

so that inversion now yields

$$|\Psi^{\pm}_{\beta}\rangle = -G_\beta(E\pm io)V_\beta \sum_{\gamma} \bar{\delta}_{\gamma\beta}|\Psi^{\pm}_{\beta}\rangle + G_\beta|\chi_\beta\rangle \tag{5.11}$$

or, again using (5.7),

$$|\Psi^{\pm}_{\beta}\rangle = C_\beta|\chi_\beta\rangle - G_o(E\pm io)t_\beta(E\pm io) \sum_{\gamma} \bar{\delta}_{\gamma\beta}|\Psi^{\pm}_{\beta}\rangle, \tag{5.12}$$

where $|\chi_\beta\rangle$ is again the channel eigenfunction defined in (4.5).

As opposed to any of of Eqs. (4.4), here the boundary conditions can be uniquely specified by simply requiring that $C_\beta=\delta_{\beta\alpha}$, where α is an arbitrary (fixed) channel index. This choice of boundary conditions determines the full three-body wavefunction $|\Psi^{+}\rangle$. More explicitly, from

$$|\Psi^{+}_{\beta(\alpha)}\rangle = \delta_{\beta\alpha}|\chi_\alpha\rangle - G_o(E+io)t_\beta(E+io) \sum_{\gamma} \bar{\delta}_{\beta\gamma}|\Psi^{+}_{\gamma(\alpha)}\rangle \quad \beta=1,2,3 \tag{5.13}$$

we obtain a <u>specific</u> full wavefunction $|\Psi^{+}_{(\alpha)}\rangle$, i.e.

$$|\Psi^{+}_{(\alpha)}\rangle = \sum_{\beta} |\Psi^{+}_{\beta(\alpha)}\rangle , \tag{5.14}$$

as that three-body scattering solution that arises from an initial state of an interacting pair α and a third free particle. Eqs. (5.13) are the Faddeev equations for the Faddeev components of such a full three-body wavefunction.

Through this procedure, Faddeev defines <u>four different scattering solutions</u>, according to the initial state from which each arise (a bound state in each channel with the corresponding third free particle, <u>and</u> the state of three free particles)[25]:

$$\Psi^{\pm}_{(\alpha)}(\vec{p}\vec{q};\vec{p}^{(o)}_{\alpha}\kappa_{\alpha};\ E\pm io) = \sum_{\gamma} \Psi^{\pm}_{\gamma(\alpha)}(\vec{p}\vec{q};\vec{p}^{(o)}_{\alpha}\kappa_{\alpha}\ ;\ E\pm io) \quad \alpha=1,2,3$$

$$\Psi^{\pm}_{(o)}(\vec{p}\vec{q};\vec{p}^{(o)}\vec{q}^{(o)};\ E\pm io) = \sum_{\gamma} \Psi^{\pm}_{\gamma(o)}(\vec{p}\vec{q},\vec{p}^{(o)}\vec{q}^{(o)}\ ;\ E\pm io) \tag{5.15}$$

In the second line of (5.15), the initial state of three free particles has an energy $E=\tilde{p}^{(o)2}+\tilde{q}^{(o)2}$, while the channel initial state in the first line has energy $E=\tilde{p}^{(o)2}_{\alpha}-\kappa^{2}_{\alpha}$, corresponding to a bound state in channel α with energy $-\kappa^{2}_{\alpha}$ and a third free particle of momentum $\vec{p}^{(o)}_{\alpha}$.

The Faddeev approach in fact solves the three-body problem by successfully constructing the Hilbert space corresponding to the continuous spectrum of H from four different pieces,[25]

$$\hat{\mathcal{H}} \equiv \hat{\mathcal{H}}_{o} + \hat{\mathcal{H}}_{1} + \hat{\mathcal{H}}_{2} + \hat{\mathcal{H}}_{3}\ , \tag{5.16}$$

so that each of the wavefunctions (5.15) are the eigenfunctions of H in the proper subspace. In addition, we of course also have the (kinematically inaccessible) discrete spectrum of H, corresponding to three-body bound states, with a Hilbert space $\mathcal{H}_{d}$.[25]

Returning to the Faddeev equations (5.8) for $M_{\beta\alpha}$, we obtain equations for the corresponding amplitudes by taking plane-wave matrix elements of (5.8). In the two-body case, such plane-wave bra and kets

were simply given by $|\vec{p}\rangle$; analogously, in the three-body space a plane-wave state of the three particles in the CM system described by $|\vec{p}_\beta \vec{q}_\beta\rangle$, where as was noted before the kinematic variables in one channel are connected to those in any other channel by the linear relationships (2.2).

We will now need the matrix elements of $t_\beta(z)$, which it will be remembered was the "spectator" two-body t-matrix in channel p embedded in the three-body space. This matrix element is readily obtained by realizing that for a given three-body energy z, the (interaction) energy available to the β pair is $z-\tilde{p}_\beta'^2$, $\vec{p}_p'$ being the momentum of the third (spectator) particle. Thus

$$\langle\vec{p}_\beta\ \vec{q}_\beta|t_\beta(z)|\ \vec{p}_\beta'\ \vec{q}_\beta'\rangle = \delta(\vec{p}_\beta-\vec{p}_\beta')t_\beta(\vec{q}_\beta\vec{q}_\beta';\ z-\tilde{p}_\beta'^2)\ , \tag{5.17}$$

and the Faddeev equations (5.8) for the amplitudes $\mathcal{M}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha'\vec{q}_\alpha';z)$ read

$$\mathcal{M}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\ p_\alpha'q_\alpha';\ z) = \delta_{\beta\alpha}\delta(\vec{p}_\alpha-\vec{p}_\alpha')t_\alpha(\vec{q}_\alpha\vec{q}_\alpha';\ z-\tilde{p}_\alpha'^2) -$$

$$-\sum_\gamma \bar{\delta}_{\gamma\alpha}\int d^3p_\gamma'' d^3q_\gamma''\ \delta(\vec{p}_\gamma-\vec{p}_\gamma'')t_\gamma(\vec{q}_\gamma\vec{q}_\gamma'',\ z-\tilde{p}_\gamma''^2)\ \times$$

$$\times\ \frac{1}{\tilde{p}_\gamma''^2+\tilde{q}_\gamma''^2-z}\times\mathcal{M}_{\gamma\alpha}(p_\gamma''q_\gamma'';\ p_\alpha'q_\alpha';\ z)\ . \tag{5.18}$$

VI. The Scattering Amplitudes

With the Faddeev operators defined in (5.1) as the splitting of the full three-body T-matrix, it is unclear how the amplitudes $\mathcal{M}_{\beta\alpha}$ themselves are related to the physical transition amplitudes for three-body scattering.

To investigate this, Faddeev first defines an operator $W_{\beta\alpha}(z)$ as

$$W_{\beta\alpha}(z) = M_{\beta\alpha}(z) - \delta_{\beta\alpha}t_\beta(z). \tag{6.1}$$

$W_{\beta\alpha}$ is the connected piece of the operator $M_{\beta\alpha}$, since the (discon-

nected) single-scattering terms have been substracted.

It is in fact for the operator $W_{\beta\alpha}$ that Faddeev carries out his rigorous mathematical proofs,[26] also proving that the residues at the physical singularities in $W_{\beta\alpha}$ yield the physical transition amplitudes.[25,27] Explicitly, it is proven that we can write for the amplitudes $\mathscr{W}_{\beta\alpha}$

$$\mathscr{W}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha'\vec{q}_\alpha';z) = \mathscr{F}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha'\vec{q}_\alpha';z) +$$

$$+ \mathscr{G}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha';z)\,\frac{\Phi_\kappa^{\alpha*}(\vec{q}_\alpha')}{z+\kappa_\alpha^2-\tilde{p}_\alpha'^2} + \frac{\Phi_\kappa^{\beta}(\vec{q}_\beta)}{z+\kappa_\beta^2-\tilde{p}_\beta^2}\,\tilde{\mathscr{G}}_{\beta\alpha}(\vec{p}_\beta;\vec{p}_\alpha'\vec{q}_\alpha';z)$$

$$+ \frac{\Phi_\kappa^{\beta}(\vec{q}_\beta)}{z+\kappa_\beta^2-\tilde{p}_\beta^2}\,\mathscr{H}_{\beta\alpha}(\vec{p}_\beta,\vec{p}_\alpha';z)\,\frac{\Phi_K^{\alpha*}(\vec{q}_\alpha')}{z+\kappa_\alpha^2-\tilde{p}_\alpha'^2} \tag{6.2}$$

where the vertex functions Φ have been defined in Eq. (5.9) of Chap. 2, and the functions $\mathscr{F}$, $\mathscr{G}$, $\tilde{\mathscr{G}}$ and $\mathscr{H}$ have no primary singularities.[19,27]

Thus, in Faddeev's approach, the scattering amplitudes are obtained as the residues of the operator $W_{\beta\alpha}$. $\mathscr{F}$ is shown to be related to the 3→3 amplitude, while both $\mathscr{G}$ and $\mathscr{H}$ are related to the 2→3 and 2→2 amplitudes.

For the case of most physical interest, i.e. an initial state consisting of a bound state plus a third free particle, $\mathscr{G}$ and $\mathscr{H}$ are the relevant quantities. Instead of writing equations for $W_{\beta\alpha}$, Osborn and Kowalski considered the Faddeev equations for $\mathscr{G}$ and $\mathscr{H}$ directly.[27] Both Eqs. (6.2) and the Osborn-Kowalski (OK) equations follow from making use of Eqs. (5.8) of Chap. 2, i.e. of a splitting of the two-body t-matrix into a term t^P containing the bound-state pole, and a remainder t^R.

A different approach, presented by Alt, Grassberger and Sandhas (AGS),[10] introduces a different Faddeev operator $U_{\beta\alpha}(z)$ that yields the

scattering amplitudes of the three-body problem directly, when taking appropriate matrix elements between initial and final states: The 2→2 amplitude is given directly by $\langle \vec{p}_\beta \phi^\beta_\kappa | U_{\beta\alpha}(z) | \vec{p}_\alpha \phi^\alpha_\kappa \rangle$, while the breakup amplitude 2→3 is given by $\frac{1}{2}\sum_\gamma \langle \vec{p}\vec{q} | U_{\gamma\alpha}(z) | \vec{p}_\alpha \phi^\alpha_\kappa \rangle$ (assuming an initial bound state in channel α).

The equations the operators $U_{\beta\alpha}(z)$ satisfy are

$$U_{\beta\alpha}(z) = -\bar{\delta}_{\beta\alpha} G_o^{-1}(z) - \sum_\gamma \bar{\delta}_{\beta\gamma} t_\gamma(z) G_o(z) U_{\gamma\alpha}(z). \tag{6.3}$$

The Faddeev equations in any of the forms described above (as well as other forms developed by other authors) solve in principle the three-body scattering problem. After Faddeev's work in the early 1960's, considerable work has been done to apply these (or other similarly derived equations) to systems of practical interest, and extensive calculations have been carried out to this effect.[11]

Nevertheless, all these equations suffer from the same difficulty: the two-body t-matrix in the Faddeev equations (or, equivalently, the remainder t^R in the OK equations) are <u>fully-off-shell</u>. In particular, when $p_\beta \to \infty$, the two-body energy in the two-body subsystem $z - \tilde{p}^2_\beta \to -\infty$. That is, to solve any of the various forms of the Faddeev equations, it is necessary to know the fully-off-shell two-body t-matrices for arbitrarily large negative energies, i.e. in a nonphysical region.

As we shall see explicitly in Chap. 4, the equations we present in this work completely eliminate this difficulty, and in addition we obtain other convenient simplifications over the above approaches.

Chapter Four

THREE-BODY EQUATIONS WITH HALF-ON-SHELL INPUT

I. Introduction

We present in this chapter one of the two main results of this work, namely a new set of equations for three-body scattering. The remaining main result is the corresponding generalization to four-body scattering, presented in Chapter 6.

In two-body scattering theory, as mentioned before, the only natural basis is the eigenstates of the free Hamiltonian h_0, i.e., the plane wave basis $\{|\vec{p}\rangle\}$. When expressing the outgoing wave scattering state vector $|\psi^+_{\vec{k}}\rangle$ in such a basis, $\psi^+_{\vec{k}}(\vec{p})=\langle\vec{p}|\psi^+_{\vec{k}}\rangle$, one is naturally led to a representation in terms of a less singular amplitude, such as is found in Eq. (4.13) of Chap. 2:

$$\psi^+_{\vec{k}}(\vec{p}) = \delta(\vec{p}-\vec{k}) - \frac{t(\vec{p},\vec{k},\tilde{k}^2+i0)}{\tilde{p}^2-\tilde{k}^2-i0} , \tag{1.1}$$

where $\tilde{k}^2=\frac{k^2}{2\mu}$, and where $t(\vec{p},\vec{k},\tilde{k}^2+i0)$ is just the plane wave matrix element of the transition operator $t(z)$ (its on-shell value, i.e., the residue at the scattering pole $\tilde{p}^2=\tilde{k}^2$ in (1.1) yields the physical transition amplitude).

In three-body scattering theory the situation is more complicated. Also in this case a plane wave representation (corresponding to the eigenstate $|\vec{p}\vec{q}\rangle$ of H_0) is natural. A detailed analysis of the singularity structure of such a representation for the three-body wavefunction, $\langle\vec{p}\vec{q}|\Psi^+\rangle$, has been carried out by Faddeev, as referred to in the previous chapter, and leads to an expression similar to (1.1), but now in terms of a pair of amplitudes $\mathscr{H}_{\beta\alpha}$ and $\mathscr{G}_{\beta\alpha}$ (described in the Appendix). Just as in the two-body case, these amplitudes are closely related to the

physical transition amplitudes. One can then of course consider the Faddeev equations these amplitudes satisfy, i.e., a three-body counterpart of the Lippmann-Schwinger equation for the two-body transition amplitude; these equations have recently been advocated by Osborn and Kowalski.[27]

However, in the three-body case other natural bases are also available, namely the complete sets of channel eigenstates $\left\{ |\vec{p}_\beta \phi^\beta_\kappa\rangle;\ |\vec{p}_\beta \psi^-_{\vec{q}_\beta}\rangle \right\}$ of the channel Hamiltonians $H_\beta = H_0 + V_\beta$; $\beta = 1,2,3$.[14]

In this chapter we consider the expansion of the three-body Faddeev wavefunction components in such a basis. We show that this representation is actually more natural than the plane wave representation mentioned before, and leads to a considerably simplified formulation of the three-body theory.

II. The Amplitudes $\mathscr{H}_{\beta\alpha}$ and $\mathscr{E}_{\beta\alpha}$

In this section we restrict ourselves to scattering processes starting from an initial state of one free particle and a two-body bound state. For this case we consider the Faddeev equation for the β-component of the three-body wavefunction:

$$|\Psi^+_{\beta(\alpha)}\rangle = \delta_{\beta\alpha} |\vec{p}^{(0)}_\alpha \phi^\alpha_\kappa\rangle - G_0(E+i0)\, t_\beta(E+i0) \sum_{\gamma\neq\beta} |\Psi^+_{\gamma(\alpha)}\rangle \tag{2.1}$$

where $|\vec{p}^{(0)}_\alpha \phi^\alpha_\kappa\rangle$ describes the initial state, i.e., a bound state in channel α and a third free particle, and E is the total energy in the initial state, $E = \frac{p^{(0)2}_\alpha}{2\mu_\alpha} - \kappa^2_\alpha$. Here and below we consider only one bound state per channel. Defining the complete set of channel eigenstates in channel β by $\left\{ |\vec{p}_\beta \phi^\beta_\kappa\rangle,\ |\vec{p}_\beta \psi^-_{\vec{q}_\beta}\rangle \right\}$, , where $|\psi^-_{\vec{q}_\beta}\rangle$ is the incoming two-body scattering state with momentum $\vec{q}_\beta$, we obtain for the projection of (2.1) onto these states (recall that $G_0 t_\beta = G_\beta V_\beta$)

$$<\vec{p}_\beta \phi^\beta_\kappa | \Psi^+_{\beta(\alpha)}> = \delta_{\beta\alpha}\delta^3(\vec{p}_\alpha - \vec{p}^{\,(0)}_\alpha) - \frac{\mathscr{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}^{\,(0)}_\alpha;E+i0)}{\tilde{p}^2_\beta - \kappa^2_\beta - E - i0}$$

$$<\vec{p}_\beta \psi^-_{\vec{q}_\beta} | \Psi^+_{\beta(\alpha)}> = -\frac{\mathscr{E}_{\beta\alpha}(\vec{p}_\beta,\vec{q}_\beta;\vec{p}^{\,(0)}_\alpha;E+i0)}{\tilde{p}^2_\beta + \tilde{q}^2_\beta - E - i0} \tag{2.2}$$

where

$$\mathscr{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}^{\,(0)}_\alpha;E+i0) = <\vec{p}_\beta \phi^\beta_\kappa | V_\beta | \sum_{\gamma\neq\beta} \Psi^+_{\gamma(\alpha)}>$$

$$\mathscr{E}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}^{\,(0)}_\alpha;E+i0) = <\vec{p}_\beta \psi^-_{\vec{q}_\beta} | V_\beta | \sum_{\gamma\neq\beta} \Psi^+_{\gamma(\alpha)}> \quad . \tag{2.3}$$

In (2.3), $\tilde{p}^2_\beta = \frac{p^2_\beta}{2n_\beta}$, and $\tilde{q}^2_\beta = \frac{q^2_\beta}{2\mu_\beta}$ with $n_\beta = \frac{m_\beta(m_\gamma + m_\alpha)}{(m_\alpha + m_\beta + m_\gamma)}$ and $\mu_\beta = \frac{m_\alpha m_\gamma}{m_\alpha + m_\gamma}$.

Using (2.2), the expansion of the plane wave projections of the Faddeev components of the three-body wavefunction is obtained as

$$<\vec{p}_\beta\vec{q}_\beta|\Psi^+_{\beta(\alpha)}> = \delta_{\beta\alpha}\delta^3(\vec{p}_\alpha - \vec{p}^{\,(0)}_\alpha)\,\phi^\alpha_\kappa(\vec{q}_\alpha) - \frac{\phi^\beta_\kappa(\vec{q}_\beta)}{\tilde{p}^2_\beta - \kappa^2_\beta - E - i0}\,\mathscr{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}^{\,(0)}_\alpha;E+i0)$$

$$- \int d^3\vec{q}'_\beta\, \psi^-_{\vec{q}'_\beta}(\vec{q}_\beta)\frac{1}{\tilde{p}^2_\beta + \tilde{q}'^2_\beta - E - i0}\,\mathscr{E}_{\beta\alpha}(\vec{p}_\beta\vec{q}'_\beta;\vec{p}^{\,(0)}_\alpha;E+i0) \tag{2.4}$$

Equation (2.4) constitutes a three-body-generalization of Eq. (1.1) since the amplitudes $\mathscr{H}$ and $\mathscr{E}$ of (2.3) are shown in the Appendix to be free from elastic, rearrangement and breakup poles; i.e., they are the amplitudes in terms of which we will now formulate the three-body theory.

We first note how these amplitudes are related to the physical transition amplitudes: recalling that the residues of the wavefunctions at the elastic or rearrangement and breakup poles are essentially the corresponding transition amplitudes, we directly see from (2.4) that the

elastic or rearrangement amplitude is simply given by the on-shell value of $\mathscr{H}_{\beta\alpha}$. In addition, it can be shown that the residue at the breakup pole $\tilde{p}_\beta^2+\tilde{q}_\beta^2 = E$ in (2.4) is the on-shell value of $\mathscr{E}_{\beta\alpha}$, so that the breakup amplitude is given by $\sum_\beta \mathscr{E}_{\beta\alpha}$.

Having established the on-shell connection between the amplitudes $\mathscr{H}$ and $\mathscr{E}$ and the physical transition amplitudes, we look into the relationship between our amplitudes and the matrix elements of the more familiar three-body transition operators. For this purpose we recall that in the wavefunction formalism, the three-body operators $K_{\beta\alpha}$ generate the Faddeev components out of the initial state wavefunction,[28] i.e.,

$$|\Psi^+_{\beta(\alpha)}\rangle = \left[\delta_{\beta\alpha} - G_0(E+i0)\ K_{\beta\alpha}(E+i0)\right] |\vec{p}_\alpha^{\,(0)}\, \phi_\kappa^\alpha\rangle \ , \tag{2.5}$$

where $E = \tilde{p}_\alpha^{(0)2} - \kappa_\alpha^2$. If we take projections of (2.5) onto channel eigenstates and use the relation $G_0 K_{\beta\alpha} = -G_\beta V_\beta G_0 U_{\beta\alpha}$, where $U_{\beta\alpha}$ is the AGS transition operator,[10] we find upon comparison with (2.4) that

$$\begin{aligned} \mathscr{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}_\alpha^{\,(0)};\, E+i0) &= -\langle \vec{p}_\beta \phi_\kappa^\beta | V_\beta G_0(E+i0)\, U_{\beta\alpha}(E+i0)\, | \vec{p}_\alpha^{\,(0)} \phi_\kappa^\alpha\rangle \\ \mathscr{E}_{\beta\alpha}(\vec{p}_\beta\, \vec{q}_\beta;\vec{p}_\alpha^{\,(0)};\, E+i0) &= -\langle \vec{p}_\beta \psi^-_{\vec{q}_\beta} | V_\beta G_0(E+i0)\, U_{\beta\alpha}(E+i0)\, | \vec{p}_\alpha^{\,(0)} \phi_\kappa^\alpha\rangle \ . \end{aligned} \tag{2.6}$$

In the on-shell limit $\langle \vec{p}_\beta \phi_\kappa^\beta | V_\beta G_0(E+i0) = -\langle \vec{p}_\beta \phi_\kappa^\beta |$, so we see that the expression for $\mathscr{H}_{\beta\alpha}$ in (2.6) reduces to the familiar expression for the elastic and rearrangement transition amplitude in terms of $U_{\beta\alpha}$.

In addition, the half-on-shell singularity-free amplitude $\mathscr{K}_{\beta\alpha}$ that in Faddeev's treatment[8,27] yields the breakup amplitude component can be written in operator form as

$$\mathscr{K}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha^{\,(0)};E+i0) = \langle \vec{p}_\beta \vec{q}_\beta | t_\beta(E+i0)\, G_0(E+i0)\, U_{\beta\alpha}(E+i0)\, | \vec{p}_\alpha^{\,(0)} \phi_\kappa^\alpha\rangle \tag{2.7}$$

The breakup amplitude component is obtained by taking the function $\mathscr{K}_{\beta\alpha}$ fully-on-shell, i.e., for $\tilde{p}_\beta^2 + \tilde{q}_\beta^2 = \tilde{p}_\alpha^{(0)2} - \kappa_\alpha^2 = E$. Since in that case

$< \vec{p}_\beta \vec{q}_\beta | t_\beta(E+i0) = <\vec{p}_\beta \psi^-_{q_\beta} | V_\beta$, we again obtain here that, on-shell, $\mathscr{E}_{\beta\alpha}$ yields the β-component of the breakup amplitude.

The factors $V_\beta G_0$ on the left in the amplitudes (2.6) are present to insure that the half-off-shell amplitudes $\mathscr{H}_{\beta\alpha}$ and $\mathscr{E}_{\beta\alpha}$ do not contain singularities (poles) in the off-shell variable $\vec{q}_\beta$.

The off-shell extensions of the amplitudes (2.6) are defined as

$$\begin{aligned} \mathscr{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}^{(0)}_\alpha;z) &= <\vec{p}_\beta \phi^\beta_\kappa | V_\beta G_0(z)\, U_{\beta\alpha}(z)\, G_0(z)\, V_\alpha | \vec{p}^{(0)}_\alpha \phi^\alpha_\kappa> \\ \mathscr{E}_{\beta\alpha}(\vec{p}_\beta\, \vec{q}_\beta;\vec{p}^{(0)}_\alpha;z) &= <\vec{p}_\beta \psi^-_{\vec{q}_\beta} | V_\beta\, G_0(z)\, U_{\beta\alpha}(z)\, G_0(z)\, V_\alpha | \vec{p}^{(0)}_\alpha \phi^\alpha_\kappa> \ . \end{aligned} \tag{2.8}$$

In the appendix it is shown that the amplitudes (2.8) are free from elastic, rearrangement and breakup poles. In fact, $\mathscr{H}_{\beta\alpha}$ in (2.8) coincides with Faddeev's fully-off-shell amplitude $\mathscr{H}_{\beta\alpha}$. On the other hand, the amplitude $\mathscr{E}_{\beta\alpha}$ in (2.8) and the Faddeev fully-off-shell amplitude $\mathscr{K}_{\beta\alpha}$ are different; it is this different choice of off-shell extensions that enables us to write remarkably simple Faddeev equations for $\mathscr{H}$ and $\mathscr{E}$, as we show in the next section.

III. Equations for $\mathscr{H}_{\beta\alpha}$ and $\mathscr{E}_{\beta\alpha}$

Inserting the expansion (2.4) into Faddeev's equations (2.1) a system of coupled integral equations for the half-on-shell amplitudes $\mathscr{H}$ and $\mathscr{E}$ can be immediately obtained. However, as it will be more convenient for the discussion of their properties, we present here the corresponding equations for the fully-off-shell amplitudes.

Such equations can be obtained from the Faddeev equations for the operators $U_{\beta\alpha}$,[10]

$$U_{\beta\alpha}(z) = -\bar{\delta}_{\beta\alpha}\, G_0^{-1}(z) - \sum_\gamma \bar{\delta}_{\beta\gamma}\, t_\gamma(z)\, G_0(z)\, U_{\gamma\alpha}(z) \tag{3.1}$$

where $\bar{\delta}_{\beta\gamma}=1-\delta_{\beta\gamma}$. Multiplying (3.1) with the appropriate operators and taking the matrix elements indicated by the definitions (2.8), (again,

recall that $G_0 t_\beta = G_\beta V_\beta$ and that the channel eigenstates form a complete set) we get

$$\mathcal{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}_\alpha^{(0)},z) = \mathcal{H}^{(0)}_{\beta\alpha}(\vec{p}_\beta;\vec{p}_\alpha^{(0)};z) -$$

$$-\sum_{\gamma\neq\beta}\int d^3\vec{p}'_\gamma \mathcal{V}^{\mathcal{H}\mathcal{H}}_{\beta\gamma}(\vec{p}_\beta;\vec{p}'_\gamma)\frac{1}{\tilde{p}'^2_\gamma-\kappa^2_\gamma-z}\mathcal{H}_{\gamma\alpha}(\vec{p}'_\gamma;\vec{p}_\alpha^{(0)};z) -$$

$$-\sum_{\gamma\neq\beta}\iint d^3\vec{p}'_\gamma d^3\vec{q}'_\gamma \mathcal{V}^{\mathcal{H}\mathcal{E}}_{\beta\gamma}(\vec{p}_\beta;\vec{p}'_\gamma\vec{q}'_\gamma)\frac{1}{\tilde{p}'^2_\gamma+\tilde{q}'^2_\gamma-z}\mathcal{E}_{\gamma\alpha}(\vec{p}'_\gamma\vec{q}'_\gamma;\vec{p}_\alpha^{(0)};z) \tag{3.2}$$

$$\mathcal{E}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha^{(0)};z) = \mathcal{E}^{(0)}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha^{(0)};z) -$$

$$-\sum_{\gamma\neq\beta}\int d^3\vec{p}'_\gamma \mathcal{V}^{\mathcal{E}\mathcal{H}}_{\beta\gamma}(\vec{p}_\beta\vec{q}_\beta;\vec{p}'_\gamma)\frac{1}{\tilde{p}'^2_\gamma-\kappa^2_\gamma-z}\mathcal{H}_{\gamma\alpha}(\vec{p}'_\gamma;\vec{p}_\alpha^{(0)};z) -$$

$$-\sum_{\gamma\neq\beta}\iint d^3\vec{p}'_\gamma d^3\vec{q}'_\gamma \mathcal{V}^{\mathcal{E}\mathcal{E}}_{\beta\gamma}(\vec{p}_\beta\vec{q}_\beta;\vec{p}'_\gamma\vec{q}'_\gamma)\frac{1}{\tilde{p}'^2_\gamma+\tilde{q}'^2_\gamma-z}\mathcal{E}_{\gamma\alpha}(\vec{p}'_\gamma\vec{q}'_\gamma;\vec{p}_\alpha^{(0)};z)\ ,$$

where the "effective potentials" $\mathcal{V}$ are given by

$$\mathcal{V}^{\mathcal{H}\mathcal{H}}_{\beta\gamma}(\vec{p}_\beta;\vec{p}'_\gamma) = -\frac{\Phi^\beta_\kappa(\vec{q}^{(1)}_\beta)\,\Phi^\gamma_\kappa(\vec{q}^{(2)}_\gamma)}{(\tilde{q}^{(2)}_\gamma)^2+\kappa^2_\gamma}$$

$$\mathcal{V}^{\mathcal{H}\mathcal{E}}_{\beta\gamma}(\vec{p}_\beta;\vec{p}'_\gamma\vec{q}'_\gamma) = \Phi^\beta_\kappa(\vec{q}^{(1)}_\beta)\ \psi^-_{\vec{q}'_\gamma}(\vec{q}^{(2)}_\gamma) \tag{3.3}$$

$$\mathcal{V}^{\mathcal{E}\mathcal{H}}_{\beta\gamma}(\vec{p}_\beta\vec{q}_\beta;\vec{p}'_\gamma) = -t_\beta(\vec{q}_\beta;\vec{q}^{(1)}_\beta;\tilde{q}^2_\beta+i0)\ \frac{\Phi^\gamma_\kappa(\vec{q}^{(2)}_\gamma)}{(\tilde{q}^{(2)}_\gamma)^2+\kappa^2_\gamma}$$

$$\mathcal{V}^{\mathcal{E}\mathcal{E}}_{\beta\gamma}(\vec{p}_\beta\vec{q}_\beta;\vec{p}'_\gamma\vec{q}'_\gamma) = t_\beta(\vec{q}_\beta;\vec{q}^{(1)}_\beta;\tilde{q}^2_\beta+i0)\ \psi^-_{\vec{q}'_\gamma}(\vec{q}^{(2)}_\gamma)\ .$$

The driving term $\mathcal{H}^{(0)}_{\beta\alpha}$ ($\mathcal{E}^{(0)}_{\beta\alpha}$) vanishes if $\alpha=\beta$, and is otherwise obtained from the expression for $\mathcal{V}^{\mathcal{H}\mathcal{H}}_{\beta\gamma}$ ($\mathcal{V}^{\mathcal{E}\mathcal{H}}_{\beta\gamma}$) in (3.3), taking $\gamma=\alpha$ and replacing κ^2_γ by $\tilde{p}^{(0)2}_\alpha-z$. In (3.3), $\vec{q}^{(1)}_\beta=(m_\gamma/m_\alpha+m_\gamma)\vec{p}_\beta+\vec{p}'_\gamma$, $\vec{q}^{(2)}_\gamma=-\vec{p}_\beta-(m_\beta/m_\alpha+m_\beta)\vec{p}'_\gamma$,

and Φ_κ^β is the two-body bound state vertex function, defined as $\Phi_\kappa^\beta(\vec{q}_\beta) = -(\tilde{q}_\beta^2 + \kappa_\beta^2)\,\phi_\kappa^\beta(\vec{q}_\beta)$.

As was mentioned before, we can see in (3.2) and (3.3) how the formulation of the three-body theory gets simplified when it is expressed in terms of the new pair of amplitudes $\mathscr{H}$ and $\mathscr{E}$. In fact, Eqs. (3.2) have the following features:

(i) The effective potentials are all independent of the energy parameter z. This fact simplifies the structure of the equations and has obvious computational advantages.

(ii) The input consists solely of two-body bound state wavefunctions and half-off-shell transition amplitudes. The completely off-shell amplitudes — in particular for arbitrarily large negative energies — occuring in the usual treatments of the Faddeev equations, are therefore completely eliminated.

Additional convenient features become evident after an angular momentum decomposition of Eqs. (3.2) is carried out. This is discussed in detail in the next section.

IV. Angular Momentum Decomposition

In this section we consider the angular momentum decomposition of Eqs. (3.2). Since the properties we want to discuss are present in all terms of such a decomposition, we only consider the simplest situation, i.e., the S-wave case: We assume that the total angular momentum J is zero, and that only s-wave two-body interactions are present.

It will be remembered from Eqs. (2.8) that the breakup amplitude component $\mathscr{E}_{\beta\alpha}$ is obtained by projecting to the left onto scattering channel eigenstates $\langle\psi_\beta^-|$. As is well-known from two-body scattering theory,[29] the coordinate space representation of this solution can be

expressed in the s-wave case as

$$\psi^{-}_{q_\beta}(r) = \frac{q_\beta \phi_{q_\beta}(r)}{\mathscr{L}_{-}(q_\beta)} \quad , \tag{4.1}$$

where $\phi_{q_\beta}(r)$ is the s-wave regular solution to the partial wave Schrödinger equation (satisfying boundary conditions at the origin $\phi_{q_\beta}(0)=0$, $\phi'_{q_\beta}(0)=1$) and $\mathscr{L}_{-}(q_\beta)$ is the two-body Jost function. A similar relation holds of course in every partial wave.

In this way, we see from (4.1) that a Jost function factor $1/\mathscr{L}_{\beta+}$ can naturally be extracted from each partial wave component of $\mathscr{E}_{\beta\alpha}$. Redefining these amplitudes accordingly,

$$\mathscr{E}_{\beta\alpha} = \frac{1}{\mathscr{L}_{\beta+}} \hat{\mathscr{E}}_{\beta\alpha} \quad , \tag{4.2}$$

the new amplitudes $\hat{\mathscr{E}}_{\beta\alpha}$ are obtained in each partial wave by projecting onto the regular solutions rather than onto the scattering solutions. The resulting equations for the amplitudes $\mathscr{H}_{\beta\alpha}$ and $\hat{\mathscr{E}}_{\beta\alpha}$ in the S-wave case are:

$$\begin{aligned}
\mathscr{H}_{\beta\alpha}(p_\beta;p_\alpha^{(0)};z) &= \mathscr{H}^{(0)}_{\beta\alpha}(p_\beta;p_\alpha^{(0)};z) \\
&\quad - \sum_{\gamma\neq\beta} \int_0^\infty p'^2_\gamma \, dp'_\gamma \; \mathscr{V}^{\mathscr{H}\mathscr{H}}_{\beta\gamma}(p_\beta;p'_\gamma) \frac{1}{\tilde{p}'^2_\gamma - \kappa^2_\gamma - z} \mathscr{H}_{\gamma\alpha}(p'_\gamma;p_\alpha^{(0)};z) \\
&\quad - \sum_{\gamma\neq\beta} \int_0^\infty\!\!\int_0^\infty p'^2_\gamma \, dp'_\gamma \, q'^2_\gamma \, dq'_\gamma \; \hat{\mathscr{V}}^{\mathscr{H}\mathscr{E}}_{\beta\gamma}(p_\beta;p'_\gamma,q'_\gamma) \\
&\qquad \frac{\dfrac{1}{|\mathscr{L}_{+}(q'_\gamma)|^2}}{\tilde{p}'^2_\gamma + \tilde{q}'^2_\gamma - z} \hat{\mathscr{E}}_{\gamma\alpha}(p'_\gamma,q'_\gamma;p_\alpha^{(0)};z)
\end{aligned}$$

$$\begin{aligned}
\hat{\mathscr{E}}_{\beta\alpha}(p_\beta,q_\beta;p_\alpha^{(0)};z) &= \hat{\mathscr{E}}^{(0)}_{\beta\alpha}(p_\beta,q_\beta;p_\alpha^{(0)};z) \\
&\quad - \sum_{\gamma\neq\beta} \int_0^\infty p'^2_\gamma \, dp'_\gamma \; \hat{\mathscr{V}}^{\mathscr{E}\mathscr{H}}_{\beta\gamma}(p_\beta,q_\beta;p'_\gamma) \frac{1}{\tilde{p}'^2_\gamma - \kappa^2_\gamma - z} \mathscr{H}_{\gamma\alpha}(p'_\gamma;p_\alpha^{(0)};z)
\end{aligned}$$

$$- \sum_{\gamma\neq\beta}\int_0^\infty\int_0^\infty p_\gamma'^2\, dp_\gamma'\, q_\gamma'^2\, dq_\gamma'\; \hat{\mathscr{V}}^{\mathscr{E}\mathscr{E}}_{\beta\gamma}(p_\beta, q_\beta; p_\gamma', q_\gamma')$$

$$\frac{\dfrac{1}{|\mathscr{L}_+(q_\gamma')|^2}}{\tilde{p}_\gamma'^2 + \tilde{q}_\gamma'^2 - z}\; \hat{\mathscr{E}}_{\gamma\alpha}(p_\gamma' q_\gamma'; p_\alpha^{(0)}; z) \tag{4.3}$$

The partial wave components of the effective potentials of the original equations are redefined accordingly, and the resulting potentials in (4.3) are:

$$\mathscr{V}^{\mathscr{K}\mathscr{K}}_{\beta\gamma}(p_\beta; p_\gamma') = -\frac{1}{2}\int_{-1}^{1} d(\cos\vartheta_{\beta\gamma})\, \frac{\Phi^\beta_\kappa(q_\beta^{(1)})\, \Phi^\gamma_\kappa(q_\gamma^{(2)})}{\tilde{q}_\gamma^{(2)2} + \kappa_\gamma^2}$$

$$\hat{\mathscr{V}}^{\mathscr{K}\mathscr{E}}_{\beta\gamma}(p_\beta; p_\gamma', q_\gamma') = \frac{1}{2}\left\{\int_{-1}^{1} d(\cos\vartheta_{\beta\gamma})\, \Phi^\beta_\kappa(q_\beta^{(1)})\, \psi^-_{q_\gamma'}(q_\gamma^{(2)})\right\} \mathscr{L}_-(q_\gamma')$$

$$\hat{\mathscr{V}}^{\mathscr{E}\mathscr{K}}_{\beta\gamma}(p_\beta, q_\beta; p_\gamma') = -\frac{1}{2}\, \mathscr{L}_+(q_\beta)\left\{\int_{-1}^{1} d(\cos\vartheta_{\beta\gamma})\, t_\beta(q_\beta, q_\beta^{(1)}; \tilde{q}_\beta^2 + i0)\, \frac{\Phi^\gamma_\kappa(q_\gamma^{(2)})}{\tilde{q}_\gamma^{(2)2} + \kappa_\gamma^2}\right\}$$

$$\hat{\mathscr{V}}^{\mathscr{E}\mathscr{E}}_{\beta\gamma}(p_\beta, q_\beta; p_\gamma' q_\gamma') = \frac{1}{2}\, \mathscr{L}_+(q_\beta)\left\{\int_{-1}^{1} d(\cos\vartheta_{\beta\gamma})\, t_\beta(q_\beta, q_\beta^{(1)}; \tilde{q}_\beta^2 + i0)\, \psi^-_{q_\gamma'}(q_\gamma^{(2)})\right\} \mathscr{L}_-(q_\gamma') \tag{4.4}$$

where

$$q_\beta^{(1)} = \sqrt{\left(\frac{\mu_\beta}{m_\alpha}\, p_\beta\right)^2 + p_\gamma'^2 + 2\,\frac{\mu_\beta}{m_\alpha}\, p_\beta p_\gamma' \cos\vartheta_{\beta\gamma}}$$

$$q_\gamma^{(2)} = \sqrt{p_\beta^2 + \left(\frac{\mu_\gamma}{m_\alpha}\, p_\gamma'\right)^2 + 2\,\frac{\mu_\gamma}{m_\alpha}\, p_\beta p_\gamma' \cos\vartheta_{\beta\gamma}}\;. \tag{4.5}$$

This redefinition of the $\mathscr{E}$-amplitudes has the following advantages: first, the phase of the Jost function is precisely the two-body phase-

shift, i.e.,

$$\mathscr{L}_{\pm}(q_\beta) = |\mathscr{L}_{\pm}(q_\beta)| e^{\mp i\delta(q_\beta)} \quad . \tag{4.6}$$

Since the same phase is carried by the two-body half-on-shell t-matrix and two-body scattered wave function, we see that all these phases cancel out in the expression for the potentials. That is to say, the potentials (4.4) in the equations for $\hat{\mathscr{E}}$ and $\mathscr{H}$ are not only z-independent, but also <u>real</u>. In addition to the computational simplifications entailed by such a situation, problems related to unitarity (such as the construction of unitary approximation schemes) become easier to handle.

Obviously, to obtain real potentials it is only necessary to factor out the phase of the Jost function from the original $\mathscr{E}$ amplitude. However, we believe it is useful to factor out also the modulus of the Jost function, as we have done above. The reason is that the regular solution $\phi_{q_\beta}(r)$ of (4.1) is analytic everywhere in the complex q_β plane, i.e., it has no bound state or resonance poles, nor any branch points. Instead, this structure of the two-body scattering wave function is carried by the Jost function denominator. Thus, the amplitudes $\hat{\mathscr{E}}_{\beta\alpha}$ are more smoothly-varying functions of q_β than the corresponding $\mathscr{E}_{\beta\alpha}$-amplitudes.

The same two-body structure is also absent from the potentials in (4.4), since they carry factors $\mathscr{L}_+ t$ and $\psi^- \mathscr{L}_-$. In this manner, the two-body bound state and resonance singularities are predominantly carried out by the factor $\frac{1}{|\mathscr{L}|^2}$ in Eq. (4.3).

We conclude by writing the expression for the breakup amplitude in terms of the new amplitudes $\hat{\mathscr{E}}_{\beta\alpha}$ in the S-wave case:

$$\mathscr{B}_{0\alpha} = \sum_{\beta} \frac{1}{\mathscr{L}_{\beta+}} \hat{\mathscr{E}}_{\beta\alpha} \tag{4.7}$$

We see in (4.7) that $\hat{\mathscr{E}}_{\beta\alpha}$ differs from the corresponding breakup amplitude component by a Watson final state interaction factor.[30]

V. The 3-3 and 3-2 Amplitudes

For the sake of completeness, we consider in this section the amplitudes for processes starting from three free particles. For this purpose we recall expression (2.8) for the amplitudes corresponding to processes starting from a bound state and a third free particle, i.e.,

$$2 \to 2: \quad \mathscr{H}_{\beta\alpha} = \langle \vec{p}_\beta \phi_\kappa^\beta | V_\beta G_0 U_{\beta\alpha} G_0 V_\alpha | \vec{p}_\alpha^{(0)} \phi_\kappa^\alpha \rangle \tag{5.1}$$

$$2 \to 3: \quad \mathscr{E}_{\beta\alpha} = \langle \vec{p}_\beta \psi^-_{\vec{q}_\beta} | V_\beta G_0 U_{\beta\alpha} G_0 V_\alpha | \vec{p}_\alpha^{(0)} \phi_\kappa^\alpha \rangle$$

The remaining amplitudes are now defined as

$$3 \to 2: \quad \tilde{\mathscr{E}}_{\beta\alpha} = \langle \vec{p}_\beta \phi_\kappa^\beta | V_\beta G_0 U_{\beta\alpha} G_0 V_\alpha | \vec{p}_\alpha^{(0)} \psi^+_{\vec{q}_\alpha^{(0)}} \rangle$$

$$3 \to 3: \quad \mathscr{T}_{\beta\alpha} = \langle \vec{p}_\beta \psi^-_{\vec{q}_\beta} | V_\beta G_0 U_{\beta\alpha} G_0 V_\alpha | \vec{p}_\alpha^{(0)} \psi^+_{\vec{q}_\alpha^{(0)}} \rangle \tag{5.2}$$

That the 3-3 amplitude of (5.2) directly yields the connected part of the 3-3 transition amplitude can be seen as follows: in Faddeev's treatment, this 3-3 amplitude is obtained by taking the fully-on-shell plane-wave matrix elements of the operator $M_{\beta\alpha}$, i.e.,

$$T = \sum_{\beta\alpha} \langle \vec{p}_\beta \vec{q}_\beta | M_{\beta\alpha}(E+i0) | \vec{p}_\alpha^{(0)} \vec{q}_\alpha^{(0)} \rangle \tag{5.3}$$

where $\tilde{p}_\beta^2 + \tilde{q}_\beta^2 = \tilde{p}_\alpha^{(0)2} + \tilde{q}_\alpha^{(0)2}$.

Since $M_{\beta\alpha} = \delta_{\beta\alpha} t_\beta + W_{\beta\alpha}$, where $W_{\beta\alpha}$ is the connected three-body Faddeev operator, related to $U_{\beta\alpha}$ through $W_{\beta\alpha} = t_\beta G_0 U_{\beta\alpha} G_0 t_\alpha$, we see that

$$T = \sum_{\beta} \langle \vec{p}_\beta \vec{q}_\beta | t_\beta | \vec{p}_\alpha^{(0)} \vec{q}_\alpha^{(0)} \rangle + \sum_{\beta\alpha} \langle \vec{p}_\beta \vec{q}_\beta | t_\beta G_0 U_{\beta\alpha} G_0 t_\alpha | \vec{p}_\alpha^{(0)} \vec{q}_\alpha^{(0)} \rangle \quad (5.4)$$

However, since (5.4) is fully-on-shell, we can write the second term as

$$\sum_{\beta\alpha} \langle \vec{p}_\beta \psi^-_{\vec{q}_\beta} | V_\beta G_0 U_{\beta\alpha} G_0 V_\alpha | \vec{p}_\alpha^{(0)} \psi^+_{\vec{q}_\alpha^{(0)}} \rangle \quad (5.5)$$

so that the 3-3 amplitude is simply given by

$$T = \sum_{\beta\alpha} \mathscr{T}_{\beta\alpha} + \sum_{\beta} t_\beta \quad (5.6)$$

Off-shell, of course, again $\mathscr{T}_{\beta\alpha}$ and the plane wave matrix elements of $W_{\beta\alpha}$ differ.

The Faddeev equations for $\tilde{\mathscr{E}}_{\beta\alpha}$ and $\mathscr{T}_{\beta\alpha}$ can be obtained from (3.2) by replacing $\mathscr{H}_{\beta\alpha}$ by $\tilde{\mathscr{E}}_{\beta\alpha}$ and $\mathscr{E}_{\beta\alpha}$ by $\mathscr{T}_{\beta\alpha}$. In addition, the driving terms must also be replaced: for example, the driving term in the $\mathscr{T}_{\beta\alpha}$-equations is given by

$$\mathscr{T}^{(0)}_{\beta\alpha}(\vec{p}_\beta \vec{q}_\beta; \vec{p}_\alpha^{(0)} \vec{q}_\alpha^{(0)}; z) = -\bar{\delta}_{\beta\alpha}\, t_\beta(\vec{q}_\beta, \vec{q}_\beta^{(1)}; \tilde{q}_\beta^2 + i0) \frac{1}{\tilde{p}_\beta^2 + \tilde{q}_\beta^{(1)2} - z}$$

$$\times\, t_\alpha(\vec{q}_\alpha^{(2)}, \vec{q}_\alpha^{(0)}; \tilde{q}_\alpha^{(0)2} + i0) \quad (5.7)$$

(note that $\tilde{p}_\beta^2 + \tilde{q}_\beta^{(1)2} = \tilde{p}_\alpha^{(0)2} + \tilde{q}_\alpha^{(2)2}$).

Similarly, it can be shown that the amplitude $\tilde{\mathscr{E}}_{\beta\alpha}$ of (5.2), when taken on-shell, is a component of the 3-2 transition amplitude.

Returning to Eqs. (5.1) and (5.2), we observe that the amplitudes for all possible three-body processes are obtained by taking matrix elements of the operator $V_\beta G_0 U_{\beta\alpha} G_0 V_\alpha$ between channel eigenstates appropriate to the initial and final states. Since we have at our disposal both incoming and outgoing scattering states, it should be noted that it is also possible to define amplitudes with a choice of ψ^- and ψ^+ states that is different from the choice used in (5.1) and

(5.2). However, such amplitudes are not as simply related to the physical transition amplitudes. The physical reason for this is that the three-body S-matrix involves inner products of incoming and outgoing three-body scattering states in the same order as they are expanded in (5.1) and (5.2).

APPENDIX

Here we give an outline of the proof of the fact that the new amplitudes $\mathscr{E}_{\beta\alpha}$ are free from primary singularities.[19] Similar proofs can be obtained for the remaining new amplitudes $\tilde{\mathscr{E}}_{\beta\alpha}$ and $\mathscr{T}_{\beta\alpha}$.

We start by noting that the amplitudes $\mathscr{H}$ and $\mathscr{G}$ in terms of which Faddeev carries out the singularity analysis of the three-body wavefunction are defined by the first of Eq. (2.8) and

$$\mathscr{G}_{\beta\alpha}(\vec{p}_\beta \vec{q}_\beta; \vec{p}_\alpha^{\,(0)}; z) = -\langle \vec{p}_\beta \vec{q}_\beta | \hat{t}_\beta(z)\, G_0(z)\, U_{\beta\alpha}(z)\, G_0(z)\, V_\alpha | \vec{p}_\alpha^{\,(0)} \phi_\kappa^\alpha \rangle \qquad \text{(A.1)}$$

where $\hat{t}_\beta$ is obtained by splitting the two-body transition operator t_β into a term t_β^p containing the bound state pole and a remainder $\hat{t}_\beta$. The representation of the three-body wavefunction component in terms of $\mathscr{H}$ and $\mathscr{G}$ can be obtained from (2.5) and the relations $G_0 K_{\beta\alpha} = -G_0 t_\beta G_0 U_{\beta\alpha}$, $t_\beta = t_\beta^p + \hat{t}_\beta$, with the result

$$\begin{aligned}\langle \vec{p}_\beta \vec{q}_\beta | \Psi_\beta^{(\alpha)} \rangle = {} & \delta_{\beta\alpha} \delta^3(\vec{p}_\alpha - \vec{p}_\alpha^{\,(0)})\, \phi_\kappa^\alpha(\vec{q}_\alpha) \\ & - \frac{1}{\tilde{p}_\beta^2 + \tilde{q}_\beta^2 - E_\alpha - i0} \frac{(\tilde{q}_\beta^2 + \kappa_\beta^2)\, \phi_\kappa^\beta(\vec{q}_\beta)}{\tilde{p}_\beta^2 - \kappa_\beta^2 - E - i0} \mathscr{H}_{\beta\alpha}(\vec{p}_\beta; \vec{p}_\alpha^{\,(0)}; E_\alpha + i0) \\ & + \frac{1}{\tilde{p}_\beta^2 + \tilde{q}_\beta^2 - E_\alpha - i0} \mathscr{G}_{\beta\alpha}(\vec{p}_\beta \vec{q}_\beta; \vec{p}_\alpha^{\,(0)}; E_\alpha + i0) \qquad \text{(A.2)}\end{aligned}$$

By comparing (A.2) with (2.4), it can be seen how the choice of the new set $\mathscr{H}$ and $\mathscr{E}$ instead of the set $\mathscr{H}$ and $\mathscr{G}$ simplifies the singularity structure of the expansion.

Now, recalling Eqs. (2.8), (3.1) and (A.1), we find that

$$\mathscr{E}_{\beta\alpha}(\vec{p}_\beta, \vec{q}_\beta; \vec{p}_\alpha^{(0)}; z) = -\bar{\delta}_{\beta\alpha}\, t_\beta(\vec{q}_\beta, \vec{q}_\beta^{(1)}; \tilde{q}_\beta^2 + i0)\, \frac{\Phi_\kappa^\alpha(\vec{q}_\alpha^{(2)})}{\tilde{p}_\alpha^{(0)2} + \tilde{q}_\alpha^{(2)2} - z}$$

$$+ \sum_{\gamma\neq\beta} \iint d^3\vec{p}_\beta' d^3\vec{q}_\beta'\, t_\beta(\vec{q}_\beta, \vec{q}_\beta'; \tilde{q}_\beta^2 + i0)\, \frac{\delta^3(\vec{p}_\beta - \vec{p}_\beta')}{\tilde{p}_\beta'^2 + \tilde{q}_\beta'^2 - z}$$

$$\times \left[\mathscr{G}_{\gamma\alpha}(\vec{p}_\gamma', \vec{q}_\gamma'; \vec{p}_\alpha^{(0)}; z) + \frac{\Phi_\kappa^\gamma(\vec{q}_\gamma')}{\tilde{p}_\gamma'^2 - \kappa_\gamma^2 - z}\, \mathscr{H}_{\gamma\alpha}(\vec{p}_\gamma'; \vec{p}_\alpha^{(0)}; z) \right] \tag{A.3}$$

Consider the first term in (A.3). Since neither the vertex function Φ_κ^α nor the half-off-shell t_β have any real singularities as functions of the momenta, only secondary singularities occur in this term. Turning to the second term in (A.3), we note that it is identical to the expression (6.26) of Ref. 8, with the half-off-shell t_β instead of the off-shell $\hat{t}_\beta$, and the functions $\mathscr{F}^{(3)}$, $\mathscr{F}^{(2)}$ and $\mathscr{G}^{(2)}$ replaced by the functions $\mathscr{E}$, $\mathscr{G}$ and $\mathscr{H}$, respectively. These changes do not affect the character of the estimates used in the subsequent discussion of (6.26); therefore, arguments similar to those of Faddeev enable us to conclude that the singularity structure of $\mathscr{E}$ is similar to that of the Faddeev amplitudes $\mathscr{H}$ and $\mathscr{G}$: In particular, $\mathscr{E}$ is free from primary singularities.

Chapter Five

THREE-BODY UNITARITY

I. Introduction

It will be remembered from Chap. 3 that the Faddeev operator $M_{\beta\alpha}(z)$ depends on the three-body energy parameter z only through the three-body Green's function G(z), i.e.

$$M_{\beta\alpha}(z) = \delta_{\beta\alpha}V_\beta - V_\beta G(z)V_\alpha. \tag{1.1}$$

It is thus straightforward to show that $M_{\beta\alpha}$, as well as the amplitudes arising from taking plane-wave matrix elements of $M_{\beta\alpha}$, obey the appropriate three-body unitarity relations.[31] In addition, because of the symmetry involved in the definition (1.1), the plane-wave matrix elements of $M_{\beta\alpha}$ are simply related to the time-reversed amplitudes; i.e., (recall that $G^+(z) = G(z^*)$),

$$\mathcal{M}^*_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}'_\alpha\vec{q}'_\alpha;z) = \mathcal{M}_{\alpha\beta}(\vec{p}'_\alpha\vec{q}'_\alpha;\vec{p}_\beta\vec{q}_\alpha;z^*). \tag{1.2}$$

In this chapter we proceed to investigate the corresponding symmetry properties of the new amplitudes of Chap. 4, as well as the three-body unitarity relations they satisfy.

II. Symmetry Properties

In the theory we presented in Chap. 4, the situation is different from that encountered in the Faddeev theory. Even though the definition of our three-body operator $T_{\beta\alpha}$, i.e.

$$T_{\beta\alpha}(z) = V_\beta G_o(z)U_{\beta\alpha}(z)\, G_o(z)\, V_\alpha\ , \tag{2.1}$$

is still symmetric, the matrix elements that are taken to define the scattering amplitudes are not: As will be recalled from Chap. 4, these amplitudes are (Im z > 0)

$$2\to 2:\ \mathscr{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}_\alpha^{(o)};z) = \langle\vec{p}\phi_\kappa^\beta|T_{\beta\alpha}(z)|\vec{p}_\alpha^{(o)}\phi_\kappa^\alpha\rangle$$

$$2\to 3:\ \mathscr{E}_{\beta\alpha}^{(+)}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha^{(o)};z) = \langle\vec{p}_\beta\psi_{\vec{q}_\beta}^-|T_{\beta\alpha}(z)|\vec{p}_\alpha^{(o)}\phi_\kappa^\alpha\rangle$$

$$3\to 2:\ \tilde{\mathscr{E}}_{\beta\alpha}^{(+)}(\vec{p}_\beta;\vec{p}_\alpha^{(o)}\vec{q}_\alpha^{(o)};z) = \langle\vec{p}_\beta\phi_\kappa^\beta|T_{\beta\alpha}(z)|\vec{p}_\alpha^{(o)}\psi_{\vec{q}_\alpha^{(o)}}^+\rangle$$

$$3\to 3:\ \mathscr{T}_{\beta\alpha}^{(+)}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha^{(o)}\vec{q}_\alpha^{(o)};z) = \langle\vec{p}_\beta\psi_{\vec{q}_\beta}^-|T_{\beta\alpha}(z)|\vec{p}_\alpha^{(o)}\psi_{\vec{q}_\alpha^{(o)}}^+\rangle$$

where the asymmetric nature of the amplitudes is evident. The reasons for using the $(^+)$ superscript in (2.2) will soon become clear.

From (2.2) we immediately see that, for instance,

$$\mathscr{E}_{\beta\alpha}^{(+)*}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha^{(o)};z) = \langle\vec{p}_\alpha^{(o)}\phi_\kappa^\alpha|T_{\alpha\beta}(z)|\vec{p}_\beta\psi_{\vec{q}_\beta}^-\rangle \neq$$

$$\neq\mathscr{E}_{\alpha\beta}^{(+)}(\vec{p}_\alpha^{(o)};\vec{p}_\beta\vec{q}_\beta;z^*) = \langle\vec{p}_\alpha^{(o)}\phi_\kappa^\alpha|T_{\alpha\beta}(z^*)|p_\beta\psi_{\vec{q}_\beta}^+\rangle\ . \tag{2.3}$$

That is, the time-reversed amplitude $\mathscr{E}_{\beta\alpha}^{(+)*}(z)$ is not related to $\mathscr{E}_{\alpha\beta}^{(+)}(z^*)$, but to a different amplitude not yet defined. A similar situation holds for the 3→2 and 3→3 amplitudes in (2.2).

This situation is however to be expected in the kind of theory we have presented here: As will be recalled, we obtain our amplitudes by expanding the full three-body wavefunctions $|\Psi_{(\alpha)}^\pm\rangle$ or $|\Psi_{(o)}^\pm\rangle$ in terms of the complete sets of two-body spectator solutions. For an outgoing-wave three-body scattering solution $|\Psi^+\rangle$, we showed in Chap. 4 that in order to obtain amplitudes that are simply related to the physical scattering amplitudes, it was necessary to choose incoming-wave two-body spectator solutions; it is thus quite natural that, when analyzing the <u>incoming-</u>wave three-body scattering solution $|\Psi^-\rangle$, we must use instead the two-body <u>outgoing</u>-wave solutions.

In this way, a time-reversed version of our theory gives rise to the new amplitudes (Im $z<0$)

$$2\to 2: \mathscr{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}_\alpha^{(o)};z) = \langle \vec{p}_\beta \phi_\kappa^\beta | T_{\beta\alpha}(z) | \vec{p}_\alpha^{(o)} \phi_\kappa^\alpha \rangle$$

$$2\to 3: \mathscr{E}_{\beta\alpha}^{(-)}(\vec{p}_\beta \vec{q}_\beta;\vec{p}_\alpha^{(o)};z) = \langle p_\beta \psi_{\vec{q}_\beta}^{+} | T_{\beta\alpha}(z) | \vec{p}_\alpha^{(o)} \phi_\kappa^\alpha \rangle$$

$$3\ 2: \tilde{\mathscr{E}}_{\beta\alpha}^{(-)}(\vec{p}_\alpha^{(o)};\vec{p}_\beta \vec{q}_\beta;z) = \langle \vec{p}_\beta \phi_\kappa^\beta | T_{\beta\alpha}(z) | \vec{p}_\alpha^{(o)} \psi_{\vec{q}_\alpha^{(o)}}^{-} \rangle$$

$$3\to 3: \mathscr{T}_{\beta\alpha}^{(-)}(\vec{p}_\beta \vec{q}_\beta;\vec{p}_\alpha^{(o)} \vec{q}_\alpha^{(o)};z) = \langle \vec{p}_\beta \psi_{\vec{q}_\beta}^{+} | T_{\beta\alpha}(z) | \vec{p}_\alpha^{(o)} \psi_{\vec{q}_\alpha^{(0)}}^{-} \rangle \ . \tag{2.4}$$

In terms of both kinds of amplitudes (2.2) and (2.4), we obtain as expected the relations (Im z>0)

$$\mathscr{H}_{\beta\alpha}^{*}(\vec{p}_\beta;\vec{p}_\alpha^{(o)};z) = \mathscr{H}_{\alpha\beta}(\vec{p}_\alpha^{(o)};\vec{p}_\beta;z^*)$$

$$\mathscr{E}_{\beta\alpha}^{(+)*}(\vec{p}_\beta \vec{q}_\beta;\vec{p}_\alpha^{(o)};z) = \tilde{\mathscr{E}}_{\alpha\beta}^{(-)}(\vec{p}_\alpha^{(o)};\vec{p}_\beta \vec{q}_\beta;z^*)$$

$$\tilde{\mathscr{E}}_{\beta\alpha}^{(+)*}(\vec{p}_\beta;\vec{p}_\alpha^{(o)} \vec{q}_\alpha^{(o)};z) = \mathscr{E}_{\alpha\beta}^{(-)}(\vec{p}_\alpha^{(o)} \vec{q}_\alpha^{(o)};\vec{p}_\beta;z^*)$$

$$\mathscr{T}_{\beta\alpha}^{(+)*}(\vec{p}_\beta \vec{q}_\beta;\vec{p}_\alpha^{(o)} \vec{q}_\alpha^{(o)};z) = \mathscr{T}_{\alpha\beta}^{(-)}(\vec{p}_\alpha^{(o)} \vec{q}_\alpha^{(o)};\vec{p}_\beta \vec{q}_\beta;z^*). \tag{2.5}$$

That is, when considering the time-reversed version of our theory, we are led to amplitudes that are different from the ones used in Chap. 4 (with the exception of the elastic/rearrangement amplitude $\mathscr{H}_{\beta\alpha}$, which is the same in both).

In the next section we will obtain our three-body unitarity relations by defining the three-body S-matrix components in terms of the amplitudes of Chap. 4, and then requiring that such S-matrix components be unitary. Because of the reasons outlined above, however, both kinds of amplitudes will be involved in the expressions we obtain.

Before proceeding, however, we will simplify our task by redefining one of our amplitudes. As will be remembered, we proved in Chap. 4 that all our amplitudes are free from primary singularities. For this to be true also in the case of the 3→3 amplitude, it was necessary to define

such an amplitude with the single-scattering term subtracted; that is, the 3→3 physical scattering amplitude T was given as (Eq. (5.6) of Chap. 4)

$$T = \sum_{\beta\alpha} \mathcal{T}_{\beta\alpha} + \sum_{\beta} t_{\beta} \quad . \tag{2.6}$$

The advantages of writing equations for amplitudes free from primary singularities have already been discussed in the previous chapter; nevertheless, since the amplitude that actually occurs in the S-matrix is the physical 3→3 amplitude T of (2.6), it is simpler for the purposes of the present chapter to redefine our 3→3 amplitude to be

$$\bar{\mathcal{M}}^{(+)}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}^{(o)}_\alpha\vec{q}^{(o)}_\alpha;E+io) = \delta_{\beta\alpha}\delta(\vec{p}_\beta-\vec{p}^{(o)}_\alpha)t_\alpha(\vec{q}_\alpha;\vec{q}^{(o)}_\alpha;\tilde{q}^{(o)2}_\alpha+io)$$
$$+\mathcal{T}^{(+)}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}^{(o)}_\alpha\vec{q}^{(o)}_\alpha;E+io) \quad . \tag{2.7}$$

On-shell, of course, $\bar{\mathcal{M}}_{\beta\alpha}$ and the Faddeev amplitude $\mathcal{M}_{\beta\alpha}$ of (1.2) are identical, since they both yield the physical 3→3 scattering amplitudes. $\bar{\mathcal{M}}_{\beta\alpha}$ constitutes therefore a different off-shell extension.

III. The Three-Body S-matrix Components

For the three-body problem Faddeev defines[25] the S-matrix components S_{oo}, $S_{o\alpha}$ and $S_{\beta\alpha}$, corresponding to 3→3, 2→3, and 2→2 transitions respectively, in terms of the Faddeev amplitudes $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{M}$. Following steps analogous to those of Faddeev, it is straightforward to show that, in terms of the amplitudes $\mathcal{H}$, $\mathcal{E}$ and $\mathcal{T}$, the S-matrix amplitude components can be written as

$$S_{oo}(\vec{p}\vec{q};\vec{p}^{(o)}\vec{q}^{(o)}) = \delta(\vec{p}-\vec{p}^{(o)})\delta(\vec{q}-\vec{q}^{(o)})-$$
$$-2\pi i\delta(\tilde{p}^2+\tilde{q}^2-\tilde{p}^{(o)2}-\tilde{q}^{(o)2})\sum_{\beta\alpha}\bar{\mathcal{M}}^{(+)}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}^{(o)}_\alpha\vec{q}^{(o)}_\alpha;E+io)$$

$$S_{o\alpha}(\vec{p}\vec{q};\vec{p}_\alpha^{(o)})=-2\pi i\delta(\tilde{p}^2+\tilde{q}^2-\tilde{p}^{(o)^2}+\kappa_\alpha^2)\sum_\beta \varepsilon_{\beta\alpha}^{(+)}(\vec{p}_\beta\vec{q}_\beta;\vec{p}_\alpha^{(o)};E+io)$$

$$S_{\beta\alpha}(\vec{p}_\beta;\vec{p}_\alpha^{(o)})=\delta_{\beta\alpha}\delta(\vec{p}_\alpha-\vec{p}_\alpha^{(o)}) - 2\pi i\delta(\tilde{p}_\beta^2-\kappa_\beta^2-\tilde{p}_\alpha^{(o)^2}+\kappa_\alpha^2)\mathscr{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}_\alpha^{(o)};E+io), \tag{3.1}$$

with $E_o = \tilde{p}^{(o)^2}+\tilde{q}^{(o)^2}$, $E_\alpha = \tilde{p}_\alpha^{(o)^2}-\kappa_\alpha^2$.

The requirement that the three-body S-matrix be unitary is of course that

$$S^+S = SS^+ = 1, \tag{3.2}$$

which in terms of S-matrix components takes the form[25]

$$\sum_{\gamma=0}^{3} S_{\beta\gamma}^+ S_{\gamma\alpha} = \delta_{\beta\alpha} . \tag{3.3}$$

When the S-matrix components are expressed in terms of the amplitudes $\mathscr{H}$, $\mathscr{G}$ and $\mathscr{M}$, Faddeev proves that the unitarity requirement (3.3) does indeed hold, so that the Faddeev amplitudes are shown to yield the correct scattering amplitudes for all three-body processes.[25] In the next section we proceed to show that the S-matrix components (3.1) defined in terms of our amplitudes $\mathscr{H}$, $\mathscr{E}$ and $\mathscr{T}$ are also unitary.

We conclude this section by writing down the condition our amplitudes must satisfy to fulfill the requirements (3.3). For simplicity, we consider first the case of two-body interactions that support no two-body states. As we will see in the next section, the operator relations we are led to in this way are also valid for the case of interactions with two-body states, so this restriction produces no loss of generality.

For such a situation, (3.3) reduces to the condition $S_{oo}^+S_{oo}=1$, or in terms of the amplitudes (3.1),

$$\int d^3p'd^3q' \; S_{oo}^*(\vec{p}'\vec{q}';\vec{p}\vec{q}) \; S_{oo}(\vec{p}'\vec{q}';\vec{p}^{(o)}\vec{q}^{(o)})= \delta(\vec{p}-\vec{p}^{(o)})\delta(\vec{q}-\vec{q}^{(o)}). \tag{3.4}$$

Replacing (3.1) into (3.4), we see that for the case of no two-body bound states, the unitarity requirement implies that

$$\bar{\mathcal{M}}^{(+)}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}^{(o)}_\alpha\vec{q}^{(o)}_\alpha;E+io)-\bar{\mathcal{M}}^{(-)}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}^{(o)}_\alpha\vec{q}^{(o)}_\alpha;E-io)=$$

$$=-2\pi i\sum_{\gamma\gamma'}\int d^3p'd^3q'\bar{\mathcal{M}}^{(-)}_{\beta\gamma}(\vec{p}_\beta\vec{q}_\beta;\vec{p}'_\gamma\vec{q}'_\gamma;E-io)\delta(\tilde{p}'^2+\tilde{q}'^2-E)\times$$

$$\times\bar{\mathcal{M}}^{(+)}_{\gamma'\alpha}(\vec{p}'_{\gamma'},\vec{q}'_{\gamma'};\ \vec{p}^{(o)}_\alpha\vec{q}^{(o)}_\alpha;\ E+io). \tag{3.5}$$

Note that in (3.5), consistent with the structure outlined in Sec. II, all amplitudes with $\mathrm{Im}\,z>0$ ($\mathrm{Im}\,z<0$) have the superscript $^{(+)}$ ($^{(-)}$).

IV. Operator Unitarity Relations

It is best to prove (3.5) by first showing that our operators $T_{\beta\alpha}(z)$ are themselves unitary. This is done most simply by defining an operator $\bar{M}_{\beta\alpha}(z)$, whose matrix elements give the $3\to3$ amplitudes of (2.7).

Recalling from Chap. 2 that $(1+G_o(E+io)V_\alpha)|\vec{p}^{(o)}_\alpha\psi^+_{\vec{q}^{(o)}_\alpha}\rangle=|\vec{p}^{(o)}_\alpha\vec{q}^{(o)}_\alpha\rangle$ and that $\langle\vec{p}_\beta\psi^-_{\vec{q}_\beta}|V_\beta=\langle\vec{p}_\beta\vec{q}_\beta|t_\beta(\tilde{q}^2_\beta+io)$, we see that such an operator can be defined as

$$\bar{M}_{\beta\alpha}(z)=\delta_{\beta\alpha}V_\beta(1+G_o(z)V_\alpha)+T_{\beta\alpha}(z), \tag{4.1}$$

since then

$$\langle\vec{p}_\beta\psi^-_{\vec{q}_\beta}|\bar{M}_{\beta\alpha}(z)|\vec{p}^{(o)}_\alpha\psi^+_{\vec{q}^{(o)}_\alpha}\rangle=\bar{\mathcal{M}}^{(+)}_{\beta\alpha}(\vec{p}_\beta\vec{q}_\beta;\vec{p}^{(o)}_\alpha\vec{q}^{(o)}_\alpha;z) \tag{4.2}$$

as was desired. The equations the operators $\bar{M}_{\beta\alpha}$ satisfy are

$$\bar{M}_{\beta\alpha}=\delta_{\beta\alpha}V_\beta(1+G_oV_\alpha)-\sum_\gamma\bar{\delta}_{\beta\alpha}V_\beta G_\gamma\bar{M}_{\gamma\alpha}$$

$$\bar{M}_{\beta\alpha}=\delta_{\beta\alpha}V_\beta(1+G_oV_\alpha)-\sum_\gamma\bar{\delta}_{\gamma\alpha}\bar{M}_{\beta\gamma}G_\gamma V_\alpha\ . \tag{4.3}$$

Eqs. (4.3) and the corresponding equations for the operator $T_{\beta\alpha}$ differ only in the driving terms, the kernels being the same in both. Therefore, even though the operator $\bar{M}_{\beta\alpha}$ defines amplitudes $\bar{\mathcal{M}}_{\beta\alpha}$ that

have primary singularites (as opposed to $\mathscr{T}_{\beta\alpha}$), the equations arising from (4.3) still retain all other convenient features of the equations of Chap. 4, and we have introduced no formal complications.

From (4.3) we see that we can factor out V_β to the left or V_α to the right from our operators $\bar{M}_{\beta\alpha}$; i.e., we can define operators $R_{\beta\alpha}$ and $L_{\beta\alpha}$ through

$$V_\beta R_{\beta\alpha} = \bar{M}_{\beta\alpha} = L_{\beta\alpha} V_\alpha, \tag{4.4}$$

where

$$L_{\beta\alpha} = \delta_{\beta\alpha}(1+V_\beta G_o) - \sum_\gamma \bar{\delta}_{\gamma\alpha} \bar{M}_{\beta\gamma} G_\gamma \tag{4.5a}$$

$$R_{\beta\alpha} = \delta_{\beta\alpha}(1+G_o V_\alpha) - \sum_\gamma \bar{\delta}_{\beta\gamma} G_\gamma \bar{M}_{\gamma\alpha} \tag{4.5b}$$

The operators $L_{\beta\alpha}$ and $R_{\beta\alpha}$ will be extremely useful in our proof of operator unitarity, as will be seen below.

Returning to the operator $\bar{M}_{\beta\alpha}$, with the definitions (4.1) and (4.2) we can now write an operator version of the unitarity relation (3.5), i.e.

$$(1+V_\beta G_o^-)(1-t_\beta^+ G_o^+)\ \bar{M}_{\beta\alpha}(+) - \bar{M}_{\beta\alpha}(-)(1-G_o^- t_\alpha^-)(1+G_o^+ V_\alpha) =$$

$$= -\sum_{\gamma\gamma'} \bar{M}_{\beta\gamma}(-)(1-G_o^- t_\gamma^-)(G_o^+ - G_o^-)(1-t_{\gamma'}^+ G_o^+)\bar{M}_{\gamma'\alpha}(+) \quad , \tag{4.6}$$

where for convenience we have used the shorthand notation $\bar{M}_{\beta\alpha}(\pm) = \bar{M}_{\beta\alpha}(E \pm io)$, $G_o^\pm = G_o(E \pm io)$, etc.

Taking matrix elements of (4.6) between <u>outgoing-wave</u> scattering solutions $\langle \vec{p}_\beta \psi^+_{\vec{q}_\beta}|$ and $|\vec{p}_\alpha^{(o)} \psi^+_{\vec{q}_\alpha}(o)\rangle$, with $\tilde{p}_\beta^2 + \tilde{q}_\beta^2 = E = \tilde{p}_\alpha^{(o)2} + \tilde{q}_\alpha^{(o)2}$, we obtain expression (3.5) directly. That is, Eq. (3.5) is the <u>on-shell</u> amplitude version of the general operator unitarity relation (4.6). This result can be seen as follows: It will be recalled that

$$\frac{1}{\tilde{p}^2 + \tilde{q}^2 - E \mp io} = \mathscr{P} \frac{1}{\tilde{p}^2 + \tilde{q}^2 - E} \pm i\pi\delta(\tilde{p}^2 + \tilde{q}^2 - E) \tag{4.7}$$

where $\mathscr{P}$ denotes a principal-value integral, so we can write

$$G_o^+ - G_o^- = 2\pi i \int d^3p' d^3q' \,|\vec{p}'\vec{q}'\rangle \delta(\tilde{p}^2+\tilde{q}^2-E) \langle\vec{p}'\vec{q}'| \tag{4.8}$$

and thus,

$$(1-G_o^- t_\gamma^-)(G_o^+ - G_o^-)(1-t_{\gamma'}^+ G_o^+) = 2\pi i \int d^3p' d^3q' \,|\vec{p}'_\gamma \psi^-_{\vec{q}'_\gamma}\rangle \delta(\tilde{p}^2+\tilde{q}^2-E) \langle\vec{p}'_{\gamma'} \psi^-_{\vec{q}'_{\gamma'}}| \tag{4.9}$$

so the matrix elements of the right-hand side of (4.6) reduce immediately to the right-hand side of (3.5). Furthermore, we note that, since the bras and kets we use are on-shell, we have

$$\langle\vec{p}_\beta \psi^+_{\vec{q}_\beta}| (1+V_\beta G_o^-)(1-t_\beta^+ G_o^+) = \langle\vec{p}_\beta \psi^-_{\vec{q}_\beta}| \tag{4.10a}$$

$$(1-G_o^- t_\alpha^-)(1+G_o^+ V_\alpha)\,|\vec{p}_\alpha^{(o)} \psi^+_{\vec{q}_\alpha^{(o)}}\rangle = |\vec{p}_\alpha^{(o)} \psi^-_{\vec{q}^{(o)}}\rangle . \tag{4.10b}$$

Therefore, also the matrix elements of the left-hand side of (4.6) reduce to the left-hand side of (3.5).

We now proceed to prove the operator unitarity relation (4.6). To do this we separate the diagonal ($\gamma=\gamma'$) and non-diagonal ($\gamma\neq\gamma'$) terms of the right-hand side of (4.6), and note that

$$(1-G_o^- t_\gamma^-)(G_o^+ - G_o^-)(1-t_\gamma^+ G_o^+) = G_\gamma^+ - G_\gamma^- \quad , \tag{4.11}$$

so we can write the diagonal piece of the right hand side as

$$\text{RHS})_d = -\sum_\gamma \bar{M}_{\beta\gamma}(-)(G_\gamma^+ - G_\gamma^-)\bar{M}_{\gamma\alpha}(+), \tag{4.12}$$

which using (4.4) can be written as

$$\text{RHS})_d = -\sum_\gamma L_{\beta\gamma}(-) V_\gamma G_\gamma^+ \bar{M}_{\gamma\alpha}(+) + \sum_\gamma \bar{M}_{\beta\gamma}(-) G_\gamma^- V_\gamma R_{\gamma\alpha}(+). \tag{4.13}$$

We now add and subtract the term

$$\sum_\gamma \bar{M}_{\beta\gamma}(-) R_{\gamma\alpha}(+) \equiv \sum_\gamma L_{\beta\gamma}(-)\bar{M}_{\gamma\alpha}(+) \quad , \tag{4.14}$$

obtaining

$$\text{RHS})_d = -\sum_\gamma \bar{M}_{\beta\gamma}(-)(1-G_\gamma^- V_\gamma) R_{\gamma\alpha}(+) + \sum_\gamma L_{\beta\gamma}(-)(1-V_\gamma G_\gamma^+)\bar{M}_{\gamma\alpha}(+) \tag{4.15}$$

We now use (4.5a) for $L_{\beta\gamma}(-)$ in the 2nd term in (4.15) and (4.5b) for $R_{\gamma\alpha}(+)$ in the first, obtaining

$$\text{RHS})_d = -\bar{M}_{\beta\alpha}(-)(1-G_\alpha^- V_\alpha)(1+G_o^+ V_\alpha)+(1+V_\beta G_o^-)(1-V_\beta G_\beta^+)\bar{M}_{\beta\alpha}(+)$$

$$+\sum_{\gamma\gamma'} \bar{\delta}_{\gamma\gamma'}\bar{M}_{\beta\gamma}\left[(1-G_\gamma^- V_\gamma)G_{\gamma'}^+ - G_\gamma^-(1-V_{\gamma'}G_{\gamma'}^+)\right]\bar{M}_{\gamma'\alpha}(+) \tag{4.16}$$

But the factor in square brackets in the 2nd term in (4.16) is equal to $(1-G_o^- t_\gamma^-)(G_o^+ - G_o^-)(1-t_{\gamma'}^+ G_o^+)$, so that we recognize this 2nd term to be exactly equal to the non-diagonal piece of (4.6), with opposite sign. Thus the surviving terms become identical to the left-hand side of (4.6), and we have completed the proof.

V. Unitarity Relations for the Amplitudes

We now complete our discussion by allowing two-body bound states to be present. We first note that the proof the operator unitarity relation (4.6) given in the last section is completely general, and holds whether the two-body interactions can support bound states or not.

However, when taking matrix elements of (4.6) when there <u>are</u> two-body bound states, we are not led to (3.5), but to a different expression. The reason for this is that when there are bound states, the two-body t-matrices in the factors $(1-G_o^- t_\gamma^-)$ and $(1-t_{\gamma'}^+ G_o^+)$ have a pole for each bound state (Cf. Eq. (5.8) of Chap. 2), and when $\gamma=\gamma'$ these poles combine with the factor $(G_o^+ - G_o^-)$ to yield a finite extra term involving the two-body bound-state wavefunctions.

A direct way to see this explicitly is to first write, using (4.11),

$$(1-G_o^- t_\gamma^-)(G_o^+ - G_o^-)(1-t_{\gamma'}^+ G_o^+) = (G_\gamma^+ - G_\gamma^-)\delta_{\gamma\gamma'}+$$

$$+(1-G_o^- t_\gamma^-)(G_o^+ - G_o^-)(1-t_{\gamma'}^+ G_o^+)\bar{\delta}_{\gamma\gamma'}. \tag{5.1}$$

Using the spectral decomposition of G_γ (Cf. Eq. (3.11) of Chap. 2), and

recalling (4.7), we can write in (5.1)

$$G_\gamma^+ - G_\gamma^- = 2\pi i \int d^3p' \,|\vec{p}'_\gamma \phi_\kappa^\gamma\rangle \delta(\tilde{p}'^2_\gamma - \kappa_\gamma^2 - E) \langle \vec{p}'_\gamma \phi_\kappa^\gamma| \; +$$

$$+ \; 2\pi i \int d^3p' d^3q' \,|\vec{p}'_\gamma \psi^-_{\vec{q}'_\gamma}\rangle \delta(\tilde{p}'^2 + \tilde{q}'^2 - E) \langle \vec{p}'_\gamma \psi^-_{\vec{q}'_\gamma}| \tag{5.2}$$

so that replacing (5.2) into (5.1) yields

$$(1 - G_o^- t_\gamma^-)(G_o^+ - G_o^-)(1 - t_{\gamma'}^+ G_o^+) = \delta_{\gamma\gamma'} 2\pi i \int d^3p'_\gamma \,|\vec{p}'_\gamma \phi_\kappa^\gamma\rangle \delta(\tilde{p}'^2 - \kappa_\gamma^2 - E) \langle \vec{p}'_\gamma \phi_\kappa^\gamma|$$

$$+ \, 2\pi i \int d^3p' d^3q' \,|\vec{p}'_\gamma \psi^-_{\vec{q}'_\gamma}\rangle \delta(\tilde{p}'^2 + \tilde{q}'^2 - E) \langle \vec{p}'_{\gamma'} \psi^-_{\vec{q}'_{\gamma'}}| \tag{5.3}$$

when two-body bound states are present. In (5.3), the (diagonal) extra term that occurs compared to expression (4.9) (which is valid when there are no two-body bound states) is explicitly exhibited.

As an example of the on-shell unitarity relations for amplitudes we can obtain from (4.6) when there are bound states, we take matrix elements between states $\langle \vec{p}_\beta \psi^+_{\vec{q}_\beta}|$ and $|\vec{p}_\alpha^{(o)} \phi_\kappa^\alpha\rangle$, with $\tilde{p}_\beta^2 + \tilde{q}_\beta^2 = E = \tilde{p}_\alpha^{(o)2} - \kappa_\alpha^2$.

To do this we note that when acting on a bound-state wavefunction $|\phi_\kappa^\alpha\rangle$, the operator $(1 - G_o^- t_\alpha^-)(1 + G_o^+ V_\alpha)$ is equal to the identity operator, i.e.

$$(1 - G_o^- t_\alpha^-)(1 + G_o^+ V_\alpha)|\phi_\kappa^\alpha\rangle = |\phi_\kappa^\alpha\rangle \; . \tag{5.4}$$

(This can be easily seen by recalling that the operator $(1 - G_o^+ t_\alpha^+)(1 + G_o^+ V_\alpha) \equiv$ $\equiv 1$, and that the bound-state solutions do not distinguish between incoming and outgoing states). With the aid of (5.4) we obtain

$$\mathscr{E}^{(+)}_{\beta\alpha}(\vec{p}_\beta \vec{q}_\beta; \vec{p}_\alpha^{(o)}; E + io) - \mathscr{E}^{(-)}_{\beta\alpha}(\vec{p}_\beta \vec{q}_\beta; \vec{p}_\alpha^{(o)}; E - io) =$$

$$= -2\pi i \sum_\gamma \int d^3p'_\gamma \mathscr{E}^{(-)}_{\beta\gamma}(\vec{p}_\beta \vec{q}_\beta; \vec{p}'_\gamma; E - io) \delta(\tilde{p}'^2_\gamma - \kappa_\gamma^2 - E) \mathscr{H}_{\gamma\alpha}(\vec{p}'_\gamma; \vec{p}_\alpha^{(o)}; E + io)$$

$$- \, 2\pi i \sum_{\gamma\gamma'} \int d^3p'_\gamma d^3q'_\gamma \bar{\mathscr{U}}^{(-)}_{\beta\gamma}(\vec{p}_\beta \vec{q}_\beta; \vec{p}'_\gamma \vec{q}'_\gamma; E - io) \delta(\tilde{p}'^2_\gamma + \tilde{q}'^2_\gamma - E) \times$$

$$\times \, \mathscr{E}^{(+)}_{\gamma'\alpha}(\vec{p}'_\gamma, \vec{q}'_\gamma; \vec{p}_\alpha^{(o)}; E + io) \; . \tag{5.5}$$

As the on-shell amplitudes $\mathscr{E}_{\beta\alpha}$ directly yield the breakup scattering amplitudes, Eq. (5.5), when summed over β, has the form one would expect from physical grounds for the breakup case.[32]

Chapter Six

FOUR-BODY EQUATIONS WITH HALF-ON-SHELL INPUT

In this chapter we present the second main result of this work, namely the generalization of the half-on-shell three-body equations we presented in Chap. 4 to the four-body case.

For this purpose we follow a method suggested by our three-body theory, in which a thorough singularity analysis of the Faddeev three-body kernel led us to singularity-free physical amplitudes that obey dynamical equations with a considerably simplified input.

With these results in mind, we carry out a similar singularity analysis of the four-body kernel. As in the three-body case, this task will be considerably simplified by using the complete sets of eigenstates of the channel Hamiltonians.

Before we do this, however, we shall very briefly review the four-body theory whose kernel we will analyze, namely the approach due to Faddeev and Yakubovskii.[17,33]

II. The Faddeev-Yakubovskii Equations for Four-Body Scattering

We present here a direct method[34] for obtaining the FY equation for the FY wavefunction components, starting from the four-body Schrödinger equation for the full four-body wavefunction,

$$(H_0 + V - E)\,\Psi = 0 \quad , \tag{2.1}$$

where $V = \sum_\gamma V_\gamma$ is the sum of all potentials between pairs of particles. We will now carry out steps analogous to those of Sec. V, Chap. 3. We are now interested in outgoing wave scattering solutions corresponding to two-cluster initial states. Inverting (2.1), we obtain

$$\Psi = C\phi - G_0(E+io)V\Psi \quad , \tag{2.2}$$

where ϕ is a solution of $(H_0 - E)\phi = 0$, and C is a constant. Since we have chosen to consider only two-cluster configurations as initial states, C = 0. As indicated in (2.2), all operators of this section are to be taken at an energy corresponding to that of the initial state.

As in Chap. 3, Faddeev components of the four-body wavefunctions are defined through

$$\Psi_\beta = - G_0 V_\beta \Psi \tag{2.3}$$

so that

$$\Psi = \sum_\beta \Psi_\beta \quad , \tag{2.4}$$

where β is a label corresponding to a splitting of the four particles into three groups, i.e. so that only a single pair (β) is interacting. Again as in Sec. V, Chap. 3, we apply the Faddeev procedure of removing two-body disconnected pieces from the kernel of (2.3), obtaining

$$(1 + G_0 V_\beta)\Psi_\beta = - G_0 V_\beta \sum_\gamma \bar{\delta}_{\beta\gamma} \Psi_\gamma \, . \tag{2.5}$$

Operating on (2.5) with $(1-G_0 t_\beta)$, we get

$$\Psi_\beta = C\,\phi_\beta - G_0 t_\beta \sum_\gamma \bar{\delta}_{\beta\gamma} \Psi_\gamma \quad , \tag{2.6}$$

where ϕ_β satisfies $(1+G_0 V_\beta)\phi_\beta = 0$. Again, due to our choice of initial state, C=0.

For the four-body case, the kernel of (2.6) must be further modified, since it still contains disconnected pieces corresponding to two non-interacting clusters - i.e. to clusters of the type 1+3 and 2+2 - labeled by the index σ. We now proceed to remove those pieces. From (2.6) we define the components

$$\Psi_\beta^\sigma = - G_0 t_\beta \sum_{\gamma \subset \sigma} \bar{\delta}_{\beta\gamma} \Psi_\gamma \quad , \tag{2.7}$$

where $\sigma \supset \beta$, and such that

$$\Psi_\beta = \sum_{\sigma \supset \beta} \Psi_\beta^\sigma \quad . \tag{2.8}$$

As before, we take the diagonal piece in (2.7) to the left, and get

$$\sum_{\gamma\subset\sigma}(\delta_{\beta\gamma}+G_0t_\beta\bar{\delta}_{\beta\gamma})\Psi^{\sigma}_{\gamma} = -G_0t_\beta\sum_{\gamma\subset\sigma}\bar{\delta}_{\beta\gamma}\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}\Psi^{\rho}_{\gamma} \quad . \tag{2.9}$$

Consider now an operator $K^{\sigma}_{\beta\alpha}$ [33] defined in the subsystem σ through the equation

$$K^{\sigma}_{\beta\alpha} = \bar{\delta}_{\beta\gamma}t_\beta - \sum_{\gamma'\subset\sigma}K^{\sigma}_{\beta\gamma'}G_0t_{\gamma'}\bar{\delta}_{\gamma'\gamma} \quad . \tag{2.10}$$

Operating on (2.10) with the expression $(\delta_{\beta\beta'} - G_0K^{\sigma}_{\beta\beta'})$, β, $\beta'\subset\sigma$, we get

$$\Psi^{\sigma}_{\beta} = C\phi^{(\sigma)}_{\beta} - \sum_{\gamma\subset\sigma}G_0K^{\sigma}_{\beta\gamma}\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}\Psi^{\rho}_{\gamma} \quad , \quad \beta\subset\sigma \tag{2.11}$$

where $\sum_{\gamma\subset\sigma}(\delta_{\beta\gamma}+G_0t_\beta\bar{\delta}_{\beta\gamma})\phi^{(\sigma)}_{\gamma} = 0$, i.e. $\phi^{(\sigma)}_{\beta}$ is the Faddeev component of the wavefunction corresponding to a bound state in the subsystem σ. With our choice of initial state, $C = \delta^{\sigma\tau}$ and we obtain the four-body Faddeev-Yakubovskii (FY) equations.[33]

The kernel of (2.11) is now more connected than that of (2.6), in the sense that its third (or higher) power does not contain disconnected pieces corresponding to two non-interacting clusters. The kernel of (2.11) is said to be <u>four-body connected</u> (after two iterations).

III. Generalization to the Four-Body Case: Preliminary Considerations

Let us now rewrite (2.11) in more detail, i.e.

$$|\Psi^{\sigma(\tau)}_{\beta}\rangle = \delta^{\sigma\tau}|\Phi^{(\tau)}_{\beta}\rangle - \sum_{\gamma\subset\sigma}G_0(E+i0)K^{\sigma}_{\beta\gamma}(E+i0)\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}|\Psi^{\rho(\tau)}_{\gamma}\rangle \quad . \tag{3.1}$$

and review the meaning of our notation. The FY wavefunction components are labeled <u>both</u> by two-cluster indices σ,ρ,τ, etc. (i.e., of the type (123)(4) or (12)(34)), and by three-cluster indices α,β,γ, etc. (of the type (12)(3)(4), i.e., pair indices). The decomposition is such that $\sum_{\sigma}\sum_{\beta\subset\sigma}\Psi^{\sigma(\tau)}_{\beta}$ is the full four-body wavefunction. A three-cluster index below a two-cluster index (as in $\Psi^{\sigma(\tau)}_{\beta}$) indicates that the three clusters have been obtained by further splitting one of the two clusters (as

in $\sigma=(123)(4)\to(12)(3)(4)=\beta$).[35] This is also described by writing $\beta\subset\sigma$.

In Eq. (3.1), $\Phi_\beta^{(\tau)}$ denotes the β-component of the initial state wavefunction;[36] the operator $K^\sigma_{\beta\gamma}$ is the three-body kernel operator of subsystem σ (more precisely, it is the two-cluster subsystem kernel operator, since σ can be either of the 3+1 or the 2+2 type), defined as

$$K^\sigma_{\beta\gamma} = \sum_{\lambda\subset\sigma} \left\{ V_\beta \delta_{\beta\lambda} - V_\beta G^\sigma V_\lambda \right\} \bar{\delta}_{\lambda\gamma} \ , \tag{3.2}$$

where $G^\sigma = (H^\sigma - E - i0)^{-1} = \left(H_0 + \sum_{\gamma\subset\sigma} V_\gamma - E - i0 \right)^{-1}$.

In order to proceed with our treatment of the four-body case, we need to define the appropriate complete sets of eigenstates of the channel Hamiltonians H^σ. For σ of the 3+1 type, the complete set of eigenstates of the <u>three-body</u> Hamiltonian $\tilde{p}^2+\tilde{q}^2+\sum_{\gamma\subset\sigma} V_\gamma$ is given by Faddeev[37] as being

$$\left\{ |\Phi\rangle,\ |\Psi^\pm_{(\delta)\vec{p}}\rangle\ ,\ |\Psi^\pm_{\vec{p}\,\vec{q}}\rangle \right\} \ , \quad \text{all } \delta\subset\sigma \tag{3.3}$$

where $|\Phi\rangle$ is a three-body bound state (we only consider one three-body bound state per channel) of energy $-\kappa_\sigma^2$; $|\Psi^+_{(\delta)\vec{p}}\rangle$ is the (outgoing wave) scattering state corresponding to an initial state of a bound pair δ and a third free particle with relative momentum $\vec{p}$, and $|\Psi^+_{\vec{p}\vec{q}}\rangle$ is the (outgoing wave) scattering state corresponding to an initial state of three free particles of relative momenta $\vec{p},\vec{q}$.

Therefore, in the 3+1 case, the complete set of four-body channel eigenstates can be written as

$$\left\{ |\vec{r}\,\Phi^{(\sigma)}\rangle\ ,\ |\vec{r}\Psi^{(\sigma)\pm}_{(\delta)\vec{p}}\rangle\ ,\ |\vec{r}\,\Psi^{(\sigma)\pm}_{\vec{p}\,\vec{q}}\rangle \right\} \ , \quad \text{all } \delta\subset\sigma \tag{3.4}$$

where if, say, $\sigma=(123)(4)$, $\vec{r}_\sigma$ is the momentum of the fourth particle relative to the center-of-mass of the other three. (Note that we suppress the channel indices of all variables.)

On the other hand, if σ is of the 2+2 type, the complete set of channel eigenstates is given by

$$\left\{ |\vec{s}\,\Phi^{(\sigma)}\rangle\,, \quad |\vec{s}\,\Psi^{(\sigma)\pm}_{(\delta)\vec{q}}\rangle\,, \quad |\vec{s}\,\Psi^{(\sigma)\pm}_{\vec{q}\,\vec{q}'}\rangle \right\} . \tag{3.5}$$

In (3.5), if we let δ,γ label the two subsystems of σ (i.e., if $\sigma=(12)(34)$ and $\delta=(12)$, then $\gamma=(34)$), $|\vec{s}\Phi^{(\sigma)}\rangle = |\vec{s}\phi^{\delta}_{\kappa}\phi^{\gamma}_{\kappa}\rangle$ represents a state of two bound pairs moving with relative momentum $\vec{s}$ and corresponding to a total energy $E=\tilde{s}^2-\kappa^2_{\delta}-\kappa^2_{\gamma}$, where $\tilde{s}^2_{\sigma}=s^2_{\sigma}/2\eta_{\sigma}$, with $\eta_{\sigma}=[(m_1+m_2)(m_3+m_4)]/(m_1+m_2+m_3+m_4)$ if $\sigma=(12(34)$. Similarly, $|\vec{s}\,\Psi^{(\sigma)\pm}_{(\delta)\vec{q}}\rangle = |\vec{s}\,\phi^{\delta}_{\kappa}\,\psi^{\pm}_{\vec{q}_{\gamma}}\rangle$ represents a state where the δ-pair is bound, while the γ-pair is in a scattering state of initial momentum $\vec{q}_{\gamma}$, and so forth.

In what follows, we will in general not treat the two kinds of indices σ separately, but use only the set (3.4), with the understanding that when σ is of the 2+2 type, the labels $\vec{r},\vec{p},\vec{q}$ of (3.4) should be replaced by the labels $\vec{s},\vec{q},\vec{q}'$ of (3.5).

IV. Singularity Analysis of $\Psi^{\sigma(\tau)}$: The Scattering Amplitudes

The most natural generalization of our three-body formalism would be to consider four-body wavefunction components labeled only by a two-cluster index σ. As we have seen in Eq. (3.1), however, the FY components $\psi^{\sigma(\tau)}_{\beta}$ represent a more detailed splitting of the full wavefunction, since in them not only the last interacting subsystem is specified (labeled by σ), but also the last interacting pair (within the subsystem labeled by σ).

Therefore, we first consider the singularity structure of the "partially summed" wavefunction component $\Psi^{\sigma(\tau)}=\sum_{\beta\subset\sigma}\Psi^{\sigma(\tau)}_{\beta}$. Using Eq. (3.1), we find

$$|\Psi^{\sigma(\tau)}\rangle = \delta^{\sigma\tau}\,|\vec{r}^{(0)}\Phi^{(\tau)}\rangle - G^{\sigma}(E+i0)\sum_{\gamma\subset\sigma}\sum_{\rho\supset\gamma}\bar{V}^{(\sigma)}_{\gamma}\,\bar{\delta}^{\sigma\rho}\,|\Psi^{\rho(\tau)}_{\gamma}\rangle\,, \tag{4.1}$$

where $\bar{V}_\gamma^{(\sigma)} = \sum_{\lambda\subset\sigma} V_\lambda \bar{\delta}_{\lambda\gamma}$ (it is understood that $\gamma\subset\sigma$), and we have used the relation

$$G_0 \sum_\beta K^\sigma_{\beta\gamma} = G^\sigma \bar{V}^{(\sigma)}_\gamma \quad , \tag{4.2}$$

which follows from (3.2).

With the explicit appearance of the channel Green's function G^σ in (4.1), the singularity analysis of $\Psi^{\sigma(\tau)}$ becomes straightforward. Using the complete set of channel eigenstates (3.4) or (3.5), we obtain

$$\begin{aligned} G^\sigma(E+i0) = & \int |\vec{r}\,\Phi^{(\sigma)}\rangle \frac{d\vec{r}}{\tilde{r}^2-\kappa_\sigma^2-E-i0} \langle\vec{r}\,\Phi^{(\sigma)}| \\ & + \sum_{\delta\subset\sigma} \int |\vec{r}\,\Psi^{(\sigma)-}_{(\delta)\vec{p}}\rangle \frac{d\vec{r}\,d\vec{p}}{\tilde{r}^2+\tilde{p}^2-\kappa_\delta^2-E-i0} \langle\vec{r}\,\Psi^{(\sigma)-}_{(\delta)\vec{p}}| \\ & + \int |\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}\rangle \frac{d\vec{r}\,d\vec{p}\,d\vec{q}}{\tilde{r}^2+\tilde{p}^2+\tilde{q}^2-E-i0} \langle\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}| \quad , \end{aligned} \tag{4.3}$$

where $\tilde{p}^2$ and $\tilde{q}^2$ are defined in Section II, and $\tilde{r}^2_\sigma = \tilde{r}^2_\sigma/2\eta_\sigma$, with $\eta_\sigma = [m_4(m_1+m_2+m_3)]/(m_1+m_2+m_3+m_4)$ if $\sigma=(123)(4)$. With the aid of Eq. (4.3), (4.1) can now be written as

$$\begin{aligned} \langle\vec{r}\,\vec{p}\,\vec{q}\,|\Psi^{\sigma(\tau)}\rangle = & \delta^{\sigma\tau}\delta(\vec{r}-\vec{r}^{(0)})\,\Phi^{(\tau)}(\vec{p},\vec{q}) - \Phi^{(\sigma)}(\vec{p},\vec{q}) \frac{\mathscr{H}^{\sigma\tau}(\vec{r},\vec{r}^{(0)},\ E+i0)}{\tilde{r}^2-\kappa_\sigma^2-E-i0} \\ & - \sum_{\delta\subset\sigma} \int \Psi^{(\sigma)-}_{(\delta)\vec{p}'}(\vec{p}\,\vec{q}) \frac{d\vec{p}'}{\tilde{r}^2+\tilde{p}'^2-\kappa_\delta^2-E-i0} \mathscr{F}^{\sigma\tau}_{(\delta)}(\vec{r},\vec{p}';\vec{r}^{(0)};E+i0) \\ & - \int \Psi^{(\sigma)-}_{\vec{p}'\vec{q}'}(\vec{p}\,\vec{q}) \frac{d\vec{p}'\,d\vec{q}'}{\tilde{r}^2+\tilde{p}'^2+\tilde{q}'^2-E-i0} \mathscr{E}^{\sigma\tau}(\vec{r},\vec{p}',\vec{q}';\vec{r}^{(0)};E+i0) \ , \end{aligned} \tag{4.4}$$

where $$\mathscr{H}^{\sigma\tau}(\vec{r},\vec{r}^{(0)};E+i0) = \langle\vec{r}\,\Phi^{(\sigma)}|\sum_{\gamma\subset\sigma}\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}\,\bar{V}_{\gamma}^{(\sigma)}\,|\Psi_{\gamma}^{\rho(\tau)}\rangle$$

$$\mathscr{F}_{(\delta)}^{\sigma\tau}(\vec{r},\vec{p}\,';\vec{r}^{(0)};E+i0) = \langle\vec{r}\,\Psi_{(\delta)\vec{p}\,'}^{(\sigma)-}|\sum_{\gamma\subset\sigma}\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}\,\bar{V}_{\gamma}^{(\sigma)}|\Psi_{\gamma}^{\rho(\tau)}\rangle \tag{4.5}$$

$$\mathscr{E}^{\sigma\tau}(\vec{r},\vec{p}\,'\vec{q}\,';\vec{r}^{(0)};E+i0) = \langle\vec{r}\,\Psi_{\vec{p}\,'\vec{q}\,'}^{(\sigma)-}|\sum_{\gamma\subset\sigma}\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}\bar{V}_{\gamma}^{(\sigma)}|\Psi_{\gamma}^{\rho(\tau)}\rangle\ .$$

Equation (4.4) constitutes a four-body analog of Eqs. (2.4) of Chap. IV; i.e., it explicitly exhibits all the physical poles of the wavefunction components $\Psi^{\sigma(\tau)}$ in separate terms. The residues at these poles - i.e., the amplitudes (4.5) - are free from primary singularities (just as in our three-body formalism), and are the components of the physical scattering amplitudes: As is shown in the Appendix, the on-shell values of $\mathscr{H}^{\sigma\tau}$, $\sum_{\sigma\supset\delta}\mathscr{F}_{(\delta)}^{\sigma\tau}$ and $\sum_{\sigma}\mathscr{E}^{\sigma\tau}$ are the amplitudes for elastic/rearrangement, partial breakup and full breakup, respectively.[38]

The remaining step in the generalization would now be to find equations for these amplitudes. Unfortunately, as can be seen from Eqs. (4.1), $\Psi^{\sigma(\tau)}$ is coupled to <u>all</u> the FY components $\Psi_{\beta}^{\sigma(\tau)}$, and not simply to the remaining $\Psi^{\rho(\tau)}$. As a result, no equations for the wavefunction components $\Psi^{\sigma(\tau)}$ are available within the FY formalism, and it is therefore not possible to obtain dynamical equations for the amplitudes (4.5) at this stage.

To proceed within the FY formalism, it is also necessary to perform a singularity analysis of the FY components $\Psi_{\beta}^{\sigma(\tau)}$ (for which, of course, Eqs. (3.1) are available). This however is not straightforward, as will be seen in the next sections, and is certain to lead to a larger number of amplitude components (this being the weak point of the FY formalism in general).

At this point one could therefore abandon the FY formalism and use other dynamical equations for the components $\Psi^{\sigma(\tau)}$, for example those discussed in Refs. 18 and 39. However, all such alternatives we are aware of lead to dynamical equations with effective potentials that are not only energy dependent, but also require fully-off-shell subsystem input. In addition, these alternative equations may possibly admit spurious solutions. For these reasons, we choose to remain within the FY formalism for the present work.

V. Singularity Analysis of the FY Components $\Psi_\beta^{\sigma(\tau)}$

Recalling Eqs. (3.1) and (3.2), we see that the kernel that must be now analyzed for singularities is $G_0 K^\sigma_{\beta\gamma}$. In analogy with (4.2), we write

$$G_0 K^\sigma_{\beta\gamma} = \sum_{\lambda\subset\sigma} G^\sigma_{\beta\lambda} V_\lambda \bar{\delta}_{\lambda\gamma} \quad , \tag{5.1}$$

where

$$G^\sigma_{\beta\lambda} = \delta_{\beta\lambda} G_0 - G_0 V_\beta G^\sigma \tag{5.2}$$

is the Faddeev component of the Green's function G^σ, with the property that $\sum_\beta G^\sigma_{\beta\lambda} = G^\sigma$. Therefore, we see that for the pole decomposition of $\Psi_\beta^{\sigma(\tau)}$ it is necessary to analyze the Green's function <u>components</u> $G^\sigma_{\beta\lambda}$, rather than G^σ itself. As is evident from (5.2), use of the spectral decomposition of G^σ (Eq. (4.3)) is not sufficient, since there is also a pole in G_0. This pole is accounted for in the following way: In each term that results from applying the spectral decomposition (4.3) to the product $G_0(E+io)V_\beta G^\sigma(E+io)$ of (5.2) we use the resolvent identity

$$G_0(E+i\epsilon) = G_0(z') + (E+i\epsilon - z')\, G_0(E+i\epsilon)\, G_0(z') \quad , \tag{5.3}$$

with z' equal to the energy of the corresponding channel eigenstate (with an imaginary part ε' that is always understood to go zero before ε).

Then, the $G_0(z')V_\beta$ factors in (5.2) can be eliminated using the three-body relations

$$G_0(\tilde{r}^2 - \kappa_\sigma^2)\, V_\beta\, |\vec{r}\, \Phi^{(\sigma)}\rangle = -\, |\vec{r}\, \Phi_\beta^{(\sigma)}\rangle$$

$$G_0(\tilde{r}^2 + \tilde{p}^2 - \kappa_\delta^2 - i\epsilon')\, V_\beta\, |\vec{r}\, \Psi^{(\sigma)-}_{(\delta)\vec{p}}\rangle = -\, |\vec{r}\, \Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}}\rangle \quad , \tag{5.4}$$

$$G_0(\tilde{r}^2 + \tilde{p}^2 + \tilde{q}^2 - i\epsilon')\, V_\beta\, |\vec{r}\, \Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}\rangle = -\, |\vec{r}\, \chi^{(\sigma)-}_{\beta;\vec{p}\,\vec{q}}\rangle$$

where $\chi_\beta^{(\sigma)-}$ is what remains of $\Psi_\beta^{(\sigma)-}$ once the initial-state plane wave has been subtracted.

As a result, we obtain a "pole decomposition" of the Green's function components given by

$$\begin{aligned}
G^\sigma_{\beta\lambda}(E+i0) = {} & \int |\vec{r}\, \Phi_\beta^{(\sigma)}\rangle \frac{d\vec{r}}{\tilde{r}^2 - \kappa_\sigma^2 - E - i0} \langle \vec{r}\, \Phi^{(\sigma)}| \\
& + \sum_{\delta\subset\sigma} \int |\vec{r}\, \Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}}\rangle \frac{d\vec{r}\, d\vec{p}}{\tilde{r}^2 + \tilde{p}^2 - \kappa_\delta^2 - E - i0} \langle \vec{r}\, \Psi^{(\sigma)-}_{(\delta)\vec{p}}| \\
& + \int |\vec{r}\, \Psi^{(\sigma)-}_{\beta(\lambda);\vec{p}\,\vec{q}}\rangle \frac{d\vec{r}\, d\vec{p}\, d\vec{q}}{\tilde{r}^2 + \tilde{p}^2 + \tilde{q}^2 - E - i0} \langle \vec{r}\, \Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}| \\
& + G_0(E+i0) \Big\{ \delta_{\beta\lambda} - \int |\vec{r}\, \Phi_\beta^{(\sigma)}\rangle\, d\vec{r}\, \langle \vec{r}\, \Phi^{(\sigma)}| - \\
& \quad - \sum_{\delta\subset\sigma} \int |\vec{r}\, \Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}}\rangle\, d\vec{r}\, d\vec{p}\, \langle \vec{r}\, \Psi^{(\sigma)-}_{(\delta)\vec{p}}| - \\
& \quad - \int |\vec{r}\, \Psi^{(\sigma)-}_{\beta(\lambda);\vec{p}\,\vec{q}}\rangle\, d\vec{r}\, d\vec{p}\, d\vec{q}\, \langle \vec{r}\, \Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}| \Big\},
\end{aligned} \tag{5.5}$$

where we have also replaced $|\vec{r}\, \chi^{(\sigma)}_{\beta;\vec{p}\,\vec{q}}\rangle$ by $|\vec{r}\, \Psi^{(\sigma)-}_{\beta(\lambda);\vec{p}\,\vec{q}}\rangle - \delta_{\beta\lambda}|\vec{r}\, \vec{p}\, \vec{q}\rangle$ and made use of the fact that G_0 is diagonal in an $|\vec{r}\, \vec{p}\, \vec{q}\rangle$ representation.

In (5.5) we see that upon summation over $\beta\subset\sigma$, the factor multiplying $G_0(E+io)$ vanishes identically. In addition, the first three

terms become equal to the expression (4.3) for G^{σ}, since the Faddeev components of (5.5) add up to the full channel eigenstates.

Using (5.5), we finally obtain the sought-for pole decomposition of the FY kernel (5.1), and also of the FY wavefunction components (3.1):

$$
\begin{aligned}
\langle \vec{r}\,\vec{p}\,\vec{q}\,|\Psi_{\beta}^{\sigma(\tau)}\rangle = {} & \delta^{\sigma\tau}\,\delta(\vec{r}-\vec{r}^{(0)})\,\Phi_{\beta}^{(\tau)}(\vec{p}\,\vec{q}) \\
& - \frac{\Phi_{\beta}^{(\sigma)}(\vec{p}\,\vec{q})}{\tilde{r}^2-\kappa_{\sigma}^2-E-i0}\,\mathscr{H}^{\sigma\tau}(\vec{r},\vec{r}^{(0)};E+i0) \\
& - \sum_{\delta\subset\sigma}\int \Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}'}(\vec{p}\,\vec{q})\,\frac{d\vec{p}'}{\tilde{r}^2+\tilde{p}'^2-\kappa_{\delta}^2-E-i0}\,\mathscr{F}^{\sigma\tau}_{(\delta)}(\vec{r}\,\vec{p}';\vec{r}^{(0)};E+i0) \\
& - \sum_{\lambda\subset\sigma}\int \Psi^{(\sigma)-}_{\beta(\lambda);\vec{p}'\vec{q}'}(\vec{p}\,\vec{q})\,\frac{d\vec{p}'\,d\vec{q}'}{\tilde{r}^2+\tilde{p}'^2+\tilde{q}'^2-E-i0}\,\mathscr{E}^{\sigma\tau}_{\lambda}(\vec{r}\,\vec{p}'\vec{q}';\vec{r}^{(0)};E+i0) \\
& - \frac{1}{\tilde{r}^2+\tilde{p}^2+\tilde{q}^2-E-i0}\,\mathscr{Y}^{\sigma\tau}_{\beta}(\vec{r}\,\vec{p}\,\vec{q};\vec{r}^{(0)};E+i0)\quad, \qquad (5.6)
\end{aligned}
$$

where $\mathscr{H}^{\sigma\tau}$ and $\mathscr{F}^{\sigma\tau}_{(\delta)}$ have already been defined in (4.5), and $\mathscr{E}^{\sigma\tau}_{\lambda}$ is a decomposition of the amplitude $\mathscr{E}^{\sigma\tau}$ of (4.5), i.e.,

$$
\mathscr{E}^{\sigma\tau}_{\lambda}(\vec{r}\,\vec{p}'\vec{q}';\vec{r}^{(0)};E+i0) = \langle \vec{r}\,\Psi^{(\sigma)-}_{\vec{p}'\vec{q}'}|V_{\lambda}\sum_{\gamma\subset\sigma}\bar{\delta}_{\lambda\gamma}\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}|\Psi^{\rho(\tau)}_{\gamma}\rangle\quad, \qquad (5.7)
$$

with $\sum_{\lambda\subset\sigma}\mathscr{E}^{\sigma\tau}_{\lambda} = \mathscr{E}^{\sigma\tau}$. The remaining amplitude $\mathscr{Y}^{\sigma\tau}_{\rho}$ is given by

$$
\begin{aligned}
\mathscr{Y}^{\sigma\tau}_{\beta}|\vec{r}\,\vec{p}\,\vec{q};\vec{r}^{(0)};E+i0) = {} & \langle \vec{r}\,\vec{p}\,\vec{q}\,|\sum_{\lambda\subset\sigma}\Big\{\delta_{\beta\lambda} - \\
& - \int |\vec{r}'\,\Phi^{(\sigma)}_{\beta}\rangle\, d\vec{r}'\langle \vec{r}'\,\Phi^{(\sigma)}| - \sum_{\delta\subset\sigma}\int |\vec{r}'\,\Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}'}\rangle\, d\vec{r}'\,d\vec{p}'\langle \vec{r}'\,\Psi^{(\sigma)-}_{(\delta)\vec{p}'}| \\
& - \int |\vec{r}'\,\Psi^{(\sigma)-}_{\beta(\lambda);\vec{p}'\vec{q}'}\rangle\, d\vec{r}'\,d\vec{p}'\,d\vec{q}'\langle \vec{r}'\,\Psi^{(\sigma)-}_{\vec{p}'\vec{q}'}|\Big\}V_{\lambda}\sum_{\gamma\subset\sigma}\bar{\delta}_{\lambda\gamma}\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}|\Psi^{\rho(\tau)}_{\gamma}\rangle\ .
\end{aligned}
$$

Equation (5.6) constitutes a further generalization of our previous decomposition (4.4), where now all physical singularities of the FY

component $\Psi_\beta^{\sigma(\tau)}$ are explicitly exhibited in separate terms. It is a remarkable fact that in (5.6) the β-dependence in the terms containing $\mathscr{H}$ and $\mathscr{F}$ factorizes, so that these scattering amplitudes still depend only on the two-cluster index σ of the wavefunction. In other words, further splitting of $\Psi^{\sigma(\tau)}$ in (4.4) into $\Psi_\beta^{\sigma(\tau)}$ in (5.6) only produces a splitting of the amplitude $\mathscr{E}^{\sigma\tau}$.

In addition, the amplitude $\mathscr{Y}_\beta^{\sigma\tau}$ must now be introduced. Just as in (5.5), this amplitude vanishes identically upon summation of $\Psi_\beta^{\sigma(\tau)}$ over all $\beta\subset\sigma$. (as did the last term in (5.5)), and is therefore also absent from the full wavefunction. Consequently, $\mathscr{Y}_\beta^{\sigma\tau}$ is not a physical scattering amplitude.

VI. Equations for the Scattering Amplitudes

Let us now derive the equations that our amplitudes $\mathscr{H}^{\sigma\tau}$, $\mathscr{F}_{(\delta)}^{\sigma\tau}$ and $\mathscr{E}_\beta^{\sigma\tau}$ satisfy.

Replacing the pole decomposition (5.6) for $|\Psi_\gamma^{\rho(\tau)}>$ in the definitions (4.5) and (5.7) for these amplitudes, and in the definition (5.8) for $\mathscr{Y}_\beta^{\sigma\tau}$, the following (half-on-shell) equations are immediately obtained:

$$\mathscr{H}^{\sigma\tau}(\vec{r};\vec{r}^{(0)};E+i0) = \bar{\delta}^{\sigma\tau}\mathscr{V}^{(\mathscr{H}\mathscr{H})\sigma\tau}(\vec{r};\vec{r}^{(0)}) \tag{6.1}$$

$$-\sum_{\rho\neq\sigma}\int \mathscr{V}^{(\mathscr{H}\mathscr{H})\sigma\rho}(\vec{r};\vec{r}')\frac{d\vec{r}'}{\tilde{r}'^2-\kappa_\rho^2-E-i0}\mathscr{H}^{\rho\tau}(\vec{r}';\vec{r}^{(0)};E+i0)$$

$$-\sum_{\rho\neq\sigma}\sum_{\delta\subset\rho}\int \mathscr{V}^{(\mathscr{H}\mathscr{F})\sigma\rho}_{\delta}(\vec{r};\vec{r}'\ \vec{p}')\frac{d\vec{r}'\,d\vec{p}'}{\tilde{r}'^2+\tilde{p}'^2-\kappa_\delta^2-E-i0}\mathscr{F}_{(\delta)}^{\rho\tau}(\vec{r}'\ \vec{p}';\vec{r}^{(0)};E+i0)$$

$$-\sum_{\rho\neq\sigma}\sum_{\lambda\subset\rho}\int \mathscr{V}^{(\mathscr{H}\mathscr{E})\sigma\rho}_{\lambda}(\vec{r};\vec{r}'\ \vec{p}'\ \vec{q}')\frac{d\vec{r}'\ d\vec{p}'\ d\vec{q}'}{\tilde{r}'^2+\tilde{p}'^2+\tilde{q}'^2-E-i0}\mathscr{E}_\lambda^{\rho\tau}(\vec{r}'\ \vec{p}'\ \vec{q}';\vec{r}^{(0)};E+i0)$$

$$-\sum_{\rho\neq\sigma}\sum_{\lambda\subset\rho}\int \mathscr{V}^{(\mathscr{H}\mathscr{Y})\sigma\rho}_{\lambda}(\vec{r};\vec{r}'\ \vec{p}'\ \vec{q}')\frac{d\vec{r}'\ d\vec{p}'\ d\vec{q}'}{\tilde{r}'^2+\tilde{p}'^2+\tilde{q}'^2-E-i0}\mathscr{Y}_\lambda^{\rho\tau}(\vec{r}'\ \vec{p}'\ \vec{q}';\vec{r}^{(0)};E+i0)$$

where $E=\tilde{r}^{(0)2}-\kappa_\tau^2$. The corresponding equations for $\mathscr{F}^{\sigma\tau}_{(\beta)}$, $\mathscr{E}^{\sigma\tau}_{\beta}$ and $\mathscr{Y}^{\sigma\tau}_{\beta}$ are obtained from (6.1) by replacing, respectively, $\mathscr{V}^{(\mathscr{H}\mathscr{H})}$ by $\mathscr{V}^{(\mathscr{F}\mathscr{H})}$, $\mathscr{V}^{(\mathscr{E}\mathscr{H})}$ and $\mathscr{V}^{(\mathscr{Y}\mathscr{H})}$, and so on.

Examples of the potentials appearing in (6.1) are,

$$\mathscr{V}^{(\mathscr{H}\mathscr{H})\sigma\rho}(\vec{r};\vec{r}\,') = \sum_{\beta\subset\sigma} \langle \vec{r}\,\Phi^{(\sigma)}|V_\beta\,\bar{\delta}_{\beta\gamma}|\vec{r}\,'\,\Phi^{(\rho)}_\gamma\rangle$$

$$\mathscr{V}^{(\mathscr{H}\mathscr{F})\sigma\rho}_{\lambda}(\vec{r};\vec{r}\,'\,\vec{p}\,') = \sum_{\beta\subset\sigma} \langle \vec{r}\,\Phi^{(\sigma)}|V_\beta\,\bar{\delta}_{\beta\gamma}|\vec{r}\,'\Psi^{(\rho)-}_{\gamma(\lambda);\vec{p}\,'}\rangle$$

$$\mathscr{V}^{(\mathscr{F}\mathscr{F})\sigma\rho}_{\delta\lambda}(\vec{r}\,\vec{p}\,;\vec{r}\,'\,\vec{p}\,') = \sum_{\beta\subset\sigma} \langle \vec{r}\,\Psi^{(\sigma)-}_{(\delta);\vec{p}}|V_\beta\,\bar{\delta}_{\beta\gamma}|\vec{r}\,'\,\Psi^{(\rho)-}_{\gamma(\lambda);\vec{p}\,'}\rangle \tag{6.2}$$

where the index γ is uniquely determined by the conditions $\gamma\subset\sigma$ and $\gamma\subset\rho$, $(\sigma\neq\rho)$. (Note that when both σ and ρ are of the 2+2 type, $\sigma\cap\rho\equiv 0$, so the corresponding potentials vanish.)

In spite of the fact that two-body potentials appear in (6.2), all effective potentials in (6.1) can be expressed in terms of half-on-shell subsystem scattering amplitudes and bound state wavefunctions, with no two-body potentials remaining explicitly. For example, $\mathscr{V}^{(\mathscr{H}\mathscr{F})}$ in (6.2) can be written as

$$\begin{aligned}\mathscr{V}^{(\mathscr{H}\mathscr{F})\sigma\rho}_{\lambda}(\vec{r}_\sigma;\vec{r}\,'_\rho\,\vec{p}\,'_\lambda) = &-\langle\Phi^{(\sigma)}|\vec{p}^{(1)}_\gamma\,\phi^\gamma_\kappa\rangle(\tilde{p}^{(1)2}_\gamma-\kappa^2_\gamma+\kappa^2_\sigma)\times \\ &\times\left\{\delta_{\gamma\lambda}\,\delta(\vec{p}^{(2)}_\lambda-\vec{p}\,'_\lambda)-\frac{\mathscr{H}^\rho_{\gamma\lambda}(\vec{p}^{(2)}_\gamma,\vec{p}\,'_\lambda;\tilde{p}\,'^2_\lambda-\kappa^2_\lambda-i0)}{\tilde{p}^{(2)2}_\gamma-\kappa^2_\gamma-\tilde{p}\,'^2_\lambda+\kappa^2_\lambda+i0}\right\} \\ &-\int d\vec{q}_\gamma\,\langle\Phi^\sigma|\vec{p}^{(1)}_\gamma\,\psi^+_{\vec{q}_\gamma}\rangle(\tilde{p}^{(1)2}_\gamma+\tilde{q}^2_\gamma+\kappa^2_\sigma)\frac{1}{\tilde{p}^{(2)2}_\gamma+\tilde{q}^2_\gamma-\tilde{p}\,'^2_\lambda+\kappa^2_\lambda+i0}\,\mathscr{E}^\rho_{\gamma\lambda}(\vec{p}^{(2)}_\gamma,\vec{q}_\gamma;\vec{p}\,'_\lambda;\tilde{p}\,'^2_\lambda-\kappa^2_\lambda-i0)\end{aligned} \tag{6.3}$$

where, as in (6.2), $\gamma=\sigma\cap\rho$ is uniquely determined by σ and $\rho\,(\sigma\neq\rho)$. Also,

$$\vec{p}^{(1)}_\gamma = \frac{M_\rho}{M_\gamma+M_\rho}\,\vec{r}_\sigma+\vec{r}\,'_\rho \qquad \text{and} \qquad \vec{p}^{(2)}_\gamma = \vec{r}_\sigma+\frac{M_\sigma}{M_\gamma+M_\sigma}\,\vec{r}\,'_\rho\ ,$$

where if, say $\sigma=(123)(4)$ and $\rho=(124)(3)$, $\gamma=12$, $M_\gamma=M_{12}=m_1+m_2$, $M_\sigma=m_4$ and $M_\rho=m_3$.

The factors appearing to the left in (6.3) are projections of the three-body bound state wavefunction onto the complete set of two-body channel eigenstates. The amplitudes $\mathscr{H}$ and $\mathscr{E}$ are the scattering amplitudes of our three-body formalism, taken half-on-shell.

The potentials coupling $\mathscr{Y}_\beta^{\sigma\tau}$ to the physical amplitudes differ somewhat from those in (6.2); e.g.,

$$\mathscr{V}^{(\mathscr{Y}\mathscr{Y})\sigma\rho}_{\beta\lambda}(\vec{r}\,\vec{p}\,\vec{q}\,,\vec{r}'\vec{p}'\vec{q}') = \langle\vec{r}\,\vec{p}\,\vec{q}\,|\Big\{\delta_{\beta\gamma} - \int|\vec{r}''\,\Phi^{(\sigma)}_\beta\rangle d\vec{r}''\langle\vec{r}''\,\Phi^{(\sigma)}|$$
$$-\sum_{\delta\subset\sigma}\int|\vec{r}''\Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}''}\rangle d\vec{r}''\,d\vec{p}''\langle\vec{r}''\,\Psi^{(\sigma)-}_{(\delta)\vec{p}''}|$$
$$-\int|\vec{r}''\,\Psi^{(\sigma)-}_{\beta(\lambda);\vec{p}''\vec{q}''}\rangle d\vec{r}''\,d\vec{p}''\,d\vec{q}''\langle\vec{r}''\,\Psi^{(\sigma)-}_{\vec{p}''\vec{q}''}|\Big\}V_\gamma\,\bar{\delta}_{\gamma\lambda}|\vec{r}'\vec{p}'\vec{q}'\rangle\ . \qquad (6.4)$$

As expected, all these potentials vanish upon summation over $\beta\subset\sigma$ Again, all two-body potentials that appear explicitly in (6.4) can be eliminated in favor of half-on-shell subsystem amplitudes and bound state wavefunctions (the first term $\delta_{\beta\gamma}$ in (6.4) is actually cancelled by a piece of the fourth term).

The coupled integral equations (6.1) constitute a generalization of our three-body equations to the four-body case. We obtain in this way a formalism with advantages similar to those present in our three-body theory, namely:

(i) The dynamical equations are expressed in terms of components of the physical scattering amplitudes;

(ii) The amplitude components defined in the formalism are free from primary singularities, i.e., from poles (in the off-shell variables);

(iii) The equations have the structure of a multichannel Lippmann-Schwinger formulation, with effective potentials that are independent of the four-body energy;

(iv) The equations require as input only half-on-shell subsystem transition amplitudes and bound state wavefunctions.

As pointed out before, however, the equations also include a non-physical amplitude $\mathscr{Y}_{\beta}^{\sigma\tau}$ and our goal is therefore not fully achieved. The presence of this nonphysical amplitude can be understood as follows:

The FY equations are obtained from the four-body Lippmann-Schwinger equations by means of a two-step procedure[34]: the two-body disconnected pieces are first removed from the kernel, and only then are three-body disconnected pieces removed. (This is done in such a way that the resulting FY kernel connects three particles after one iteration and all four particles after two iterations.[34]) As a consequence, the full wavefunction is split <u>first</u> according to three-cluster indices, and <u>then</u> split further according to two-cluster indices.

On the other hand, as we have seen, the singularity structure of the full wavefunction is most naturally exhibited by considering the wavefunction components $\Psi^{\sigma(\tau)}$, split only according to the two-cluster index σ. The (prior) additional splitting according to three-cluster indices required by the FY formalism (in order to achieve connectedness of the kernel) appears thus far less natural from the point of view of the singularities of the kernel (or from the point of view of asymptotic channels).

The FY formalism nevertheless requires that we perform the more complicated singularity analysis of the fully-split wavefunction components $\Psi_{\beta}^{\sigma(\tau)}$, i.e., that we retain the full index context of the FY

equations. In choosing to remain within the FY formalism, and insisting on energy-dependent half-on-shell input, we are not only required to split the breakup amplitude $\mathscr{E}^{\sigma\tau}$ further into components $\mathscr{E}_{\beta}^{\sigma\tau}$ (an expected complication) but also to introduce the nonphysical amplitudes $\mathscr{Y}_{\beta}^{\sigma\tau}$.

VII. Generalization to the Fully-Off-Shell Case

In the previous sections we constructed our four-body formalism keeping the use of four-body operators and operator relations to a minimum; i.e., staying essentially within the wavefunction approach. It is illustrative however to consider how our formulation relates to the four-body transition operators, and how a fully-off-shell version of our amplitudes can be obtained from these operators.

To do so, we first recall from Chap. 5 that in our three-body formalism the fully-off-shell amplitudes are defined using the three-body operator[15]

$$T_{\beta\alpha}(z) = V_{\beta} G_0(z)\, U_{\beta\alpha}(z)\, G_0(z)\, V_{\alpha} \quad , \tag{7.1}$$

where $U_{\beta\alpha}(z)$ is the three-body AGS transition operator.[10] The on-shell matrix elements of the operator (7.1) between appropriate channel eigenstates give the various three-body physical transition amplitude components.

In order to obtain the corresponding four-body operators, it is convenient to make use of the matrix formalism[33]: We first define a matrix version of (7.1) by means of the four-body matrix of operators $\hat{\mathbf{T}}^{\sigma\tau} = \left\{\hat{T}_{\beta\alpha}^{\sigma\tau}\right\}$, according to

$$\hat{\mathbf{T}}^{\sigma\tau} = \mathbf{V}^{(\sigma)} \mathbf{G}_0^{(\sigma)} \mathbf{T}^{\sigma\tau} \mathbf{G}_0^{(\tau)} \mathbf{V}^{(\tau)} \quad , \tag{7.2}$$

where $\mathbf{V}^{(\sigma)} = \left\{-\bar{\delta}_{\beta\alpha} G_0^{-1}\right\}$, $\mathbf{G}_0^{(\sigma)} = \left\{-\delta_{\beta\alpha} G_0 t_{\beta} G_0\right\}$, etc., (with $\beta, \alpha \subset \sigma$), and $\mathbf{T}^{\sigma\tau} = \left\{U_{\beta\alpha}^{\sigma\tau}\right\}$ stands for the matrix of four-body AGS operators.[33]

Next, as in (7.1), we define

$$\dot{T}^{\sigma\tau}_{\beta\alpha} = V_\beta G_0 \hat{T}^{\sigma\tau}_{\beta\alpha} G_0 V_\alpha \quad , \tag{7.2}$$

or, more explicitly,

$$T^{\sigma\tau}_{\beta\alpha} = V_\beta G_0 \Big(\sum_{\gamma\subset\sigma} \sum_{\lambda\subset\tau} \bar{\delta}_{\beta\gamma} t_\gamma G_0 U^{\sigma\tau}_{\gamma\lambda} G_0 t_\lambda \bar{\delta}_{\lambda\alpha} \Big) G_0 V_\alpha \quad . \tag{7.3}$$

The equations these operators satisfy are easily obtained using the four-body equations for $U^{\sigma\tau}_{\beta\alpha}$:[33]

$$T^{\sigma\tau}_{\beta\alpha}(z) = \bar{\delta}^{\sigma\tau} \bar{\delta}_{\beta\gamma} V_\beta G_0(z)\, t_\gamma(z)\, G_0(z)\, V_\alpha \bar{\delta}_{\gamma\alpha} \tag{7.4}$$

$$- \sum_{\rho\neq\sigma} \sum_{\lambda\subset\rho} V_\beta \bar{\delta}_{\beta\gamma'} G^\rho_{\gamma'\lambda}(z)\, T^{\rho\tau}_{\lambda\alpha}(z) \quad ,$$

where $G^\rho_{\gamma'\lambda}$ has been defined in (5.2) (recall also (5.1)), and γ,γ' are determined by the conditions $\gamma=\sigma\cap\tau$ and $\gamma'=\sigma\cap\rho$.

By analogy with the three-body case, we expect matrix elements of the operators (7.3) (rather than matrix elements of just $U^{\sigma\tau}_{\beta\alpha}$) to be closely related to the amplitudes of the previous sections. Indeed, by applying the pole decomposition (5.5) of $G^\rho_{\gamma'\lambda}$ (with $E+io\to z$) to (7.4), and projecting onto channel eigenstates, we easily verify that the resulting kernels are identical to the kernels of Eqs. (6.1). Moreover, when z is chosen to be the energy of the initial state, also the resulting driving terms become identical to the driving terms of (6.1).

We can therefore identify the half-on-shell matrix elements of $T^{\sigma\tau}_{\beta\alpha}$ between appropriate initial and final states with our previously defined scattering amplitudes $\mathscr{H}^{\sigma\tau}$, $\mathscr{F}^{\sigma\tau}_{(\delta)}$ of (4.5) and $\mathscr{E}^{\sigma\tau}_\beta$ of (5.7) (recall that $\mathscr{E}^{\sigma\tau} = \sum_{\beta\subset\sigma} \mathscr{E}^{\sigma\tau}_\beta$).

With this identification it is straightforward to define the corresponding fully-off-shell versions of our amplitudes as

$$\mathscr{H}^{\sigma\tau}(\vec{r};\vec{r}^{(0)};z) = \langle \vec{r}\,\Phi^{(\sigma)}|T^{\sigma\tau}(z)|\vec{r}^{(0)}\Phi^{(\tau)}\rangle$$

$$\mathscr{F}^{\sigma\tau}_{(\delta)}(\vec{r}\;\vec{p};\vec{r}^{(0)};z) = \langle \vec{r}\,\Psi^{(\sigma)-}_{(\delta)\vec{p}}|T^{\sigma\tau}(z)|\vec{r}^{(0)}\Phi^{(\tau)}\rangle$$

$$\mathscr{E}^{\sigma\tau}_{\beta}(\vec{r}\;\vec{p}\;\vec{q};\vec{r}^{(0)};z) = \langle \vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}|\sum_{\alpha\subset\tau} T^{\sigma\tau}_{\beta\alpha}(z)|\vec{r}^{(0)}\Phi^{(\tau)}\rangle \tag{7.5}$$

where

$$T^{\sigma\tau} = \sum_{\beta\subset\sigma}\sum_{\alpha\subset\tau} T^{\sigma\tau}_{\beta\alpha} \quad . \tag{7.6}$$

It is important to note that it is from the appropriately "dressed" operator (7.3) that we can obtain singularity-free scattering amplitudes. This is in analogy with the three-body case, where the factor $V_\beta G_0$ in $T_{\beta\alpha}$ (Eq. (7.1)) is present to eliminate the primary singularities of the matrix elements of $U_{\beta\alpha}$. In the four-body case, the factor $V_\beta G_0 \bar{\delta}_{\beta\lambda} t_\lambda G_0$ in (7.3) performs a similar function.

The equations satisfied by the amplitudes (7.5) can be directly obtained from the operator equation (7.4), using (5.5) with E+io replaced by z. The effective potentials in the resulting equations are identical to those of Eqs. (6.1), but the driving terms are slightly different.

At this point, in view of the complications we have encountered in generalizing our three-body formalism (in particular the appearance of the nonphysical amplitude $\mathscr{Y}^{\sigma\tau}_{\beta}$), one may ask whether the off-shell four-body amplitudes have really been chosen properly. We therefore conclude this section by giving another argument in favor of our choice.

For this we turn to the full four-body Green's function G, and note that in terms of the transition operators we have defined, it is straightforward to write

$$G = G_0 - G_0 T G_0 =$$

$$= G_0 - \sum_{\gamma} G_0 t_\gamma G_0 - \sum_{\sigma} \sum_{\substack{\beta\subset\sigma \\ \alpha\subset\sigma}} G_\beta T^{(\sigma)}_{\beta\alpha} G_\alpha - \sum_{\sigma,\tau} G^\sigma T^{\sigma\tau} G^\tau \quad , \tag{7.7}$$

where $T^{(\sigma)}_{\beta\alpha}$ is the three-body (i.e., two-cluster) transition operator Eq. (7.1)), and $T^{\sigma\tau}$ has been defined in (7.6).

In (7.7) we observe that the four-, three- and two-cluster disconnected pieces of G have been separated from the true one-cluster (i.e., four-body connected) piece in a very natural manner. In addition, it is easy to verify that the four-body connected pieces of G can be written as

$$G^\sigma T^{\sigma\tau} G^\tau = \sum_{\substack{\beta\subset\sigma \\ \gamma\subset\sigma}} G^\sigma_{\beta\gamma} T^{\sigma\tau}_{\gamma\lambda} \tilde{G}^\tau_{\lambda\alpha} \quad , \tag{7.8}$$

where $G^\sigma_{\beta\gamma}$ is the "left-hand" splitting of G^σ as defined in (5.2), and $\tilde{G}^\tau_{\lambda\alpha} = \delta_{\lambda\alpha} G_0 - G^\tau V_\alpha G_0$ is the corresponding "right-hand" splitting of G^τ.

We thus see that both the operators $T^{\sigma\tau}$ of (7.6) and $T^{\sigma\tau}_{\beta\alpha}$ of (7.3) appear in the cluster decomposition of the four-body Green's function in a very natural manner, suggesting that they are indeed the proper choice of transition operators in this formalism.

APPENDIX

We show here that the on-shell values of our amplitudes $\mathscr{H}^{\sigma\tau}$, $\mathscr{F}^{\sigma\tau}_{(\delta)}$ and $\mathscr{E}^{\sigma\tau}$ yield the transition amplitudes for all physical processes starting from an initial state of the 3+1 type.

In order to do so we first establish some intermediate results, such as the relationship between the three-body initial state wavefunction and its Faddeev components. Combining the relations $|\Phi^{(\tau)}_{\lambda}\rangle = -G_0 V_\lambda |\Phi^{(\tau)}\rangle$ with the Faddeev equations $|\Phi^{(\tau)}_{\gamma}\rangle = -G_0 t_\gamma \sum_{\lambda\subset\tau} \bar{\delta}_{\gamma\lambda} |\Phi^{(\tau)}_{\lambda}\rangle$, we get

$$|\Phi^{(\tau)}_{\gamma}\rangle = G_0 t_\gamma \sum_{\lambda\subset\tau} \bar{\delta}_{\gamma\lambda} G_0 V_\lambda |\Phi^{(\tau)}\rangle \quad , \tag{A.1}$$

where it is understood that all operators are to be taken on-shell.

Combining now relations (7.5), (7.3) and (A.1), we get for the half-on-shell amplitude $\mathscr{H}^{\sigma\tau}$ the expression

$$\mathscr{H}^{\sigma\tau}(\vec{r};\vec{r}^{(0)};E+i0) = \langle\vec{r}\,\Phi^{(\sigma)}|\sum_{\gamma\subset\sigma}\sum_{\lambda\subset\sigma} V_\lambda \bar{\delta}_{\lambda\gamma} \sum_{\alpha\subset\tau} G_0 t_\gamma G_0 U^{\sigma\tau}_{\gamma\alpha} |\vec{r}^{(0)}\Phi^{(\tau)}_{\alpha}\rangle \quad . \tag{A.2}$$

If we now take (A.2) fully-on-shell, we can again use (A.1) to obtain

$$\mathscr{H}^{\sigma\tau}(\vec{r};\vec{r}^{(0)};E+i0) = \sum_{\gamma\subset\sigma}\sum_{\alpha\subset\tau} \langle\vec{r}\,\Phi^{(\sigma)}_{\gamma}|U^{\sigma\tau}_{\gamma\alpha}|\vec{r}^{(0)}\Phi^{(\tau)}_{\alpha}\rangle \quad , \tag{A.3}$$

which is known to be the expression for the elastic and rearrangement scattering amplitudes.[34]

Next we turn to the full breakup amplitude. Taking the expression for $\mathscr{E}^{\sigma\tau}$ in (7.5) fully-on-shell, and applying (A.1), we get

$$\mathscr{E}^{\sigma\tau}(\vec{r}\,\vec{p}\,\vec{q};\vec{r}^{(0)};E+i0) = \langle\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}|\sum_{\gamma\subset\sigma}\sum_{\lambda\subset\sigma} V_\lambda \bar{\delta}_{\lambda\gamma} \sum_{\alpha\subset\tau} G_0 t_\gamma G_0 U^{\sigma\tau}_{\gamma\alpha} |\vec{r}^{(0)}\Phi^{(\tau)}_{\alpha}\rangle \quad . \tag{A.4}$$

In order to proceed we need the expression for $|\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}\rangle$ in terms of the initial state $|\vec{r}\,\vec{p}\,\vec{q}\rangle$. This is obtained from three-body theory by recalling that

$$|\chi^{(\sigma)-}_{\beta;\vec{p}\,\vec{q}}\rangle = -G_0(E-i0)\sum_{\lambda\subset\sigma} M^{\sigma}_{\beta\lambda}(E-i0)\,|\vec{p}\,\vec{q}\rangle \quad , \tag{A.5}$$

where $M^{\sigma}_{\beta\lambda} = V_\beta \delta_{\beta\lambda} - V_\beta G^{\sigma} V_\lambda$ is the three-body Faddeev operator in subsystem σ. Combining (A.5) with the last of Eqs. (5.4) we obtain

$$G_0(E-i0)\,V_\gamma\,|\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}\rangle = G_0(E-i0)\sum_{\lambda\subset\sigma} M^{\sigma}_{\gamma\lambda}(E-i0)|\vec{p}\,\vec{q}\rangle \ . \tag{A.6}$$

With (A.6), the on-shell amplitude $\mathscr{E}^{\sigma\tau}$ can be written (recall that $G_0^{+}(E-io) = G_0(E+io)$, etc.),

$$\mathscr{E}^{\sigma\tau} = \sum_{\beta\subset\sigma}\sum_{\gamma\subset\sigma}\sum_{\lambda\subset\sigma}\sum_{\alpha\subset\tau} \langle\vec{r}\,\vec{p}\,\vec{q}|M^{\sigma}_{\beta\lambda}\bar{\delta}_{\lambda\alpha}G_0 t_\gamma G_0 U^{\sigma\tau}_{\gamma\alpha}|\vec{r}^{\,(0)}\Phi^{(\tau)}_{\alpha}\rangle \ . \tag{A.7}$$

To simplify this expression we recall from the matrix notation[33] that $\mathbf{G}_0^{(4)} = \left\{-\delta^{\sigma\tau} G_0 W^{\sigma}_{\beta\alpha} G_0\right\}$, where $W^{\sigma}_{\beta\alpha}$ is the connected part $M^{\sigma}_{\beta\alpha}$; i.e.,

$$W^{\sigma}_{\beta\alpha} = M^{\sigma}_{\beta\alpha} - \delta_{\beta\alpha} t_\beta = -\sum_{\gamma\subset\sigma} M^{\sigma}_{\beta\gamma}\bar{\delta}_{\gamma\alpha} G_0 t_\alpha \ . \tag{A.8}$$

Using the fact that $\mathbf{G}_0^{(4)}\mathbf{T}^{(4)} = \mathbf{N}^{(4)} = \left\{G_0 K^{\sigma\tau}_{\beta\alpha}\right\}$, where $K^{\sigma\tau}_{\beta\alpha}$ is the four-body kernel operator, we can now write instead of (A.7),

$$\mathscr{E}^{\sigma\tau} = \sum_{\lambda\subset\sigma}\sum_{\alpha\subset\tau} \langle\vec{r}\,\vec{p}\,\vec{q}|K^{\sigma\tau}_{\lambda\alpha}|\vec{r}^{\,(0)}\Phi^{(\tau)}_{\alpha}\rangle \ . \tag{A.9}$$

When summed over σ, (A.9) becomes identical to the expression for the full breakup scattering amplitude given in Ref. 34.

We conclude by considering the partial breakup amplitude. We proceed as before, and take expression (7.5) for $\mathscr{F}^{\sigma\tau}_{(\delta)}$ fully on-shell,

$$\mathscr{F}^{\sigma\tau}_{(\delta)} = \sum_{\gamma\subset\sigma}\sum_{\lambda\subset\sigma}\sum_{\alpha\subset\tau} \langle\vec{r}\,\Psi^{(\sigma)-}_{(\delta)\vec{p}}|V_\lambda\bar{\delta}_{\lambda\gamma}G_0 t_\gamma G_0 U^{\sigma\tau}_{\gamma\alpha}|\vec{r}^{\,(0)}\Phi^{(\tau)}_{\alpha}\rangle \tag{A.10}$$

again using (A.1). Further, we recall from three-body theory that

$$|\Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}}\rangle = \left(\delta_{\beta\delta} - G_0(E-i0)\,K^{\sigma}_{\beta\delta}(E-i0)\right)|\vec{p}\,\phi^{\delta}_{\kappa}\rangle \ . \tag{A.11}$$

If this expression is multiplied by $t_\beta(E-io)\bar{\delta}_{\beta\lambda}$, the Faddeev equation for $K^{\sigma}_{\beta\delta}$ can be used to simplify the right-hand side. Using in addition the second of Eqs. (5.4) on the left-hand side, we get

$$-\sum_{\lambda\subset\sigma} t_\beta(E-i0)\,\bar{\delta}_{\beta\lambda} G_0(E-i0)\, V_\lambda \,|\, \Psi^{(\sigma)-}_{(\delta)\vec{p}}> = K^\sigma_{\beta\delta}(E-i0)\,|\,\vec{p}\,\phi^\delta_\kappa> \quad . \tag{A.12}$$

Finally, with the relation $K^\sigma_{\beta\delta} = -t_\beta G_0 U^\sigma_{\beta\delta}$ we get for the on-shell value of (A.10),

$$\mathscr{F}^{\sigma\tau}_{(\delta)} = \sum_{\gamma\subset\sigma} \sum_{\alpha\subset\tau} <\vec{r}\,\vec{p}\,\phi^\delta_\kappa \,|\, \sum_{\lambda\subset\sigma} U^\sigma_{\gamma\lambda} G_0 t_\lambda G_0 U^{\sigma\tau}_{\lambda\alpha} \,|\, \vec{r}^{\,(0)} \Phi^{(\tau)}_\alpha> \quad . \tag{A.13}$$

We compare this with the expression obtained in Ref. 34 for the partial breakup amplitude, i.e., with

$$\sum_{b_2} <\boldsymbol{\Phi}^{[b_3]} \,|\, \mathbf{B}^{(3,2)} \,|\, \boldsymbol{\Phi}^{[a_2]}> \; ; \tag{A.14}$$

with the definitions $\boldsymbol{\Phi}^{[b_3]} = \left\{\delta_{\beta\gamma}\Phi_{(\beta)}\right\}$, $\boldsymbol{\Phi}^{[a_2]} = \left\{\delta^{\sigma\tau}\Phi^{(\tau)}_\gamma\right\}$ and $\mathbf{B}^{(3,2)} = \left\{B^{\sigma\tau}_{\beta\alpha}\right\} = \left\{\sum_{\gamma\subset\sigma} U^\sigma_{\beta\gamma} G_0 t_\gamma G_0 U^{\sigma\tau}_{\gamma\alpha}\right\}$, (A.14) becomes identical to (A.13) when the latter is summed over all $\sigma\subset\delta$.

Chapter Seven

CONCLUSIONS

Within the context of non-relativistic quantum mechanics and the framework of a Hamiltonian scattering theory, we have presented in this work new sets of dynamical equations for three- and four-body scattering, expressed in terms of components of the physical scattering amplitudes.

For the three-body case, we have seen how the use of the complete sets of eigenstates of the channel Hamiltonians significantly simplifies the formulation of three-body scattering theory. By using this representation we have obtained a new set of amplitudes for all three-body processes that coincide on-shell with the physical transition amplitudes. We have further shown how these amplitudes satisfy integral equations that are simpler than the usual Faddeev equations:

(i) The effective potentials are all independent of the three-body energy;

(ii) The input consists solely of two-body bound state wave-functions and half-off-shell transition amplitudes;

(iii) Our choice of partial wave components of the three-body amplitudes satisfy equations with <u>real</u> effective potentials. In addition, the breakup amplitudes explicitly exhibit a Watson fsi factor.

Finally, we expect that by the nature of the input to these equations, they will be particularly useful in understanding the dependence of three-body observables on the off-shell two-body input. In addition, the simplified structure of our equations suggests that the problem of constructing approximation schemes should now be reconsidered.

We have also carried out a generalization of this method to the

four-body case, by performing an analogous singularity analysis of the Faddeev-Yakubovskii four-body kernel. When performing such an analysis on the wavefunction components $\Psi^{\sigma(\tau)}$ - where σ is a two-cluster-index - we find, as expected, a natural expansion of $\Psi^{\sigma(\tau)}$ in terms of singularity-free scattering amplitudes that exhibits all the physical singularities of the full wavefunction. In addition, we also find a corresponding natural separation of the four-body Green's function into pieces of increasing degree of connectedness.

However, since this analysis is carried out on objects that are labeled only by two-cluster indices, while the FY formalism involves objects labeled by both two- and three-cluster indices, no dynamical equations within the FY formalism can be obtained in this manner; it becomes necessary to carry out a more detailed and much less transparent singularity analysis of the FY components $\Psi_{\beta}^{\sigma(\tau)}$.

Such an analysis does yield dynamical equations that exhibit advantages analogous to those obtained in our three-body formalism, namely,

(i) The equations are expressed in terms of components of the physical amplitudes;

(ii) The amplitude components defined are free from primary singularities, i.e., from poles (in the off-shell variables) that correspond to physical singularities;

(iii) The equations have the structure of a multichannel Lippmann-Schwinger formulation, with effective potentials that are independent of the four-body energy;

(iv) The equations require as input only half-on-shell subsystem transition amplitudes and bound state wavefunctions.

However, the equations also include a nonphysical amplitude $\mathscr{Y}_{\beta}^{\sigma\tau}$, which is an unexpected complication. This additional amplitude is the result of a lack of correspondence between the singularity structure of the FY equations and their detailed index structure: In fact, to our present understanding, the connectedness of the (twice iterated) FY kernel has been obtained through a procedure that is incompatible with a straightforward singularity analysis. The nonphysical amplitude serves to compensate for this incompatibility, in a way that allows the desired features (i) to (iv) to be carried over directly from the three-body case.

Whether or not to remain within the FY formalism becomes therefore a matter of deciding which characteristics of the four-body equations one chooses to emphasize. As was pointed out, we could have chosen to consider formalisms other than that of FY to obtain equations for the components $\Psi^{\sigma(\tau)}$. None of these formalisms, however, are clearly free from spurious solutions; and, more importantly for our present treatment, all the alternative formalisms we are aware of lead to equations with an input that is not only energy-dependent, but also fully-off-shell. In keeping with our aim of obtaining a theory without such features, we have chosen for the present work to remain within the FY formalism. Nevertheless, further work on alternative formulations of the four-body theory is clearly called for.

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