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Topics in Theories of Quantum Gravity

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TOPICS IN THEORIES OF QUANTUM GRAVITY

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I certify that I have read this dissertation and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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Abstract

In this thesis, we address several issues involving gravity. The first half of the thesis is devoted to studying quantum properties of Einstein gravity and its supersymmetric extensions in the perturbative regime. String theory suggests that perturbative scattering amplitudes in the theories of gravity are related to the amplitudes in gauge theories. This connection has been studied at classical (tree) level by Kawai, Lewellen and Tye. Here, we will explore the relationship between gravity and gauge theory at quantum (loop) level. This relationship, together with the cut-based approach to computing loop amplitudes, will allow us to obtain new non-trivial results for quantum gravity. In particular, we will present two infinite sequences of one-loop n-graviton scattering amplitudes: the maximally helicity violating amplitudes in $N = 8$ supergravity, and the "all-plus" helicity amplitudes in Einstein gravity with any minimally coupled massless matter content. The results for $n \leq 6$ will be obtained by an explicit calculation, while those for $n > 6$ will be inferred from the soft and collinear properties of the amplitudes. We will also present an explicit expression for the two-loop contribution to the four-particle scattering amplitude in $N = 8$ supergravity, and observe a simple relation between this result and its counterpart in $N = 4$ super-Yang-Mills theory. Furthermore, the simple structure of the two-particle unitarity cuts in these theories suggests that similar relations exist to all loop orders. If this is the case, the first ultraviolet divergence in $N = 8$ supergravity should appear at five loops, contrary to the earlier expectation of a three-loop counterterm.

The second part of this thesis deals with the solution to the gauge hierarchy problem suggested by Arkani-Hamed, Dimopoulos and Dvali. In their proposal, the fundamental gravitational scale M is assumed to be around a TeV, thus eliminating the hierarchy. The observed weakness of gravitational interactions is then explained by the presence of extra compact dimensions, with characteristic sizes much larger than the "natural" size of an inverse TeV. The Standard Model fields are localized on a four-dimensional hypersurface inside the full space-time, while gravitons are free to propagate in extra dimensions. Here we discuss experimental signatures of this scenario. In particular, we point out that the emission of a higher-dimensional graviton in a particle collision would lead to a missing energy signature in a collider detector. Non-observation of such missing energy events allows us to put relevant, model-independent constraints on the parameters of the model. While a wide range of values of M relevant for the hierarchy problem is currently allowed, next-generation colliders, such as LHC, will most likely be able to either rule out or confirm this idea. We also discuss collider signatures which would arise if the theory at the TeV scale is string theory.

Preface and Acknowledgements

The material in this thesis is based on several published papers. Chapter 1 is based on the papers [1] and [2]; chapter 2 is based on the paper [3]. This work has been done in collaboration with Z. Bern, L. Dixon, D. Durban, and J. Rozowsky. The third chapter is based on the paper [4] and the preprint [5], written in collaboration with S. Cullen, E. Mirabelli and M. Peskin. During the work on these papers, I have been supported by a research assistantship at SLAC theory group, made possible by DOE Grant DE-AC03-76SF00515.

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Introduction

Today, three out of four known fundamental forces of Nature — strong, weak and electromagnetic interactions — are reasonably well understood. The Standard Model of Glashow, Salam and Weinberg describes these interactions in the language of quantum field theory. The model has been remarkably successful in explaining the phenomena observed in particle physics experiments up to the highest available energy scales (at present, around a few hundred GeV). Detailed quantitative predictions can be made in the electroweak sector of the model, as well as in the strong sector at sufficiently large momentum transfers. Precision experiments at colliders have verified some of these predictions with remarkable accuracy. In many cases, the calculations of non-leading terms in the quantum loop expansion are required to match the experimental precision, so the agreement with experiment verifies the validity of the theory at the quantum level. Predicting the behaviour of strongly interacting systems at low energies, such as hadrons, is more difficult, since the perturbation theory techniques cannot be used. However, the model gives a qualitative understanding of such systems, and continuing advances in lattice gauge theory indicate that the situation will improve in the future.

In contrast to this impressive success, our understanding of the fourth fundamental force of Nature, gravitational force, is rather limited. There exists a classical theory of gravity, general relativity, some aspects of which have been verified expreimentally (primarily in the weak field limit). However, many questions, both experimental and theoretical, remain unanswered. In this thesis we will address several issues involving gravity.

On the theoretical side, the major challenge is to construct a self-consistent quantum theory of gravity. In general relativity, the gravitational fields are described by the Einstein-Hilbert action,

$$S_{\rm EH} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\mathcal{R}, \tag{0.1}$$

where $g_{\mu\nu}$ is the space-time metric, $\mathcal{R}$ is the corresponding Riemann tensor, and $G_N = 6.7 \times 10^{-39}$ GeV^{-2} is the Newton's gravitational constant. In the weak field approximation, the metric $g_{\mu\nu}$ can be expanded as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{0.2}$$

where $h_{\mu\nu}$ is a spin-2 field, the graviton, and we have introduced the gravitational coupling constant $\kappa = (32\pi G_N)^{1/2}$. The strength of graviton self-couplings, which can be obtained from the action (0.1), is governed by κ. The couplings of the gravitons to scalar, fermion and vector particles of the Standard Model ("matter fields") are also proportional to this constant. The fact that κ is dimensionful means that a naive quantization of general relativity would lead to a theory which is non-renormalizable at the level of power-counting. Such theories require an infinite number of counterterms to cancel their ultraviolet divergences, which means that no predictions can be made at the quantum level. Therefore, general relativity should be regarded as an effective low-energy limit of some more fundamental theory of gravity with a softer ultraviolet behaviour. For example, the supersymmetric generalizations of general relativity (supergravity theories) have been proposed as candidates for such a fundamental theory. While these theories are still non-renormalizable by naive power counting, some of the divergences are absent due to cancellations between fermion and boson loops. Unfortunately, it is now believed that these cancellations are not sufficient, and supergravity theories are still non-renormalizable. The best candidate for a quantum theory of gravity at present is string theory, which is completely finite in the ultraviolet. Nevertheless, it is interesting to study in more detail the quantum behaviour of general relativity and supergravity theories. The first two chapters of this thesis will be devoted to such an exploration.

At tree (classical) level, there exist certain relations between gravity and gauge theory amplitudes. At first sight, the existence of such relations comes as a surprise: The Lagrangians and Feynman rules of the two theories are very different. These relations become natural, however, if we recall that both theories can be thought of as low-energy limits of a unifying string theory. Gauge interactions arise in this limit from the open-string sector of the theory, while the gravitational interactions come from the closed-string sector. The relationship between the open and closed string amplitudes stems from the fact that any closed string vertex operator can be written as a product of open string vertex operators,

$$V^{\text{closed}} = V^{\text{open}}_{\text{left}} \overline{V}^{\text{open}}_{\text{right}} . \tag{0.3}$$

The string amplitudes can be represented as integrals over world-sheet variables — complex integrals for closed strings and real integrals for open strings — and the closed-string integrand factoriozes into two copies of the open-string integrand due to (0.3). The connection between open and closed string amplitudes at tree level was studied in detail by Kawai, Lewellen and Tye [7]. Taking the low-energy limit of their results, one can obtain the corresponding relations between gravity and gauge theory tree amplitudes (KLT relations), which are of the general form

$$\text{gravity} \sim (\text{gauge theory}) \times (\text{gauge theory}) . \tag{0.4}$$

We will review these relations in section 1.2.

The general structure of string theory amplitudes suggests that some relations reminiscent of KLT should also hold at the loop level. For example, at one loop level, an open string scattering amplitude for vector states is of the form,

$$\int \frac{d^D p}{(2\pi)^D} \, \frac{\exp[\varepsilon_i \cdot p_i] \times (\text{oscillator contributions})|_{\text{multi-linear}}}{p_1^2 p_2^2 \cdots p_n^2} , \tag{0.0.1}$$

where

$$p_i = p - k_1 - k_2 - \cdots - k_{i-1} = p + k_i + \cdots + k_n . \tag{0.0.2}$$

The k_i are the momenta of the external vectors and p is the loop momentum. The oscillator contributions depend on the external momenta, but are independent of p. The closed string graviton scattering amplitude is of the form

$$\int \frac{d^D p}{(2\pi)^D} \frac{\exp[(\varepsilon_i + \bar{\varepsilon}_i) \cdot p_i] \times (\text{left oscillator cont.}) \times (\text{right oscillator cont.})|_{\text{multi-lin.}}}{p_1^2 p_2^2 \cdots p_n^2}, \tag{0.0.3}$$

where the numerator in the loop integral again factorizes due to (0.3). While taking field-theory limit of string theory loop amplitudes can be highly non-trivial, it is reasonable to expect that "squaring" relations of the general form (0.4) will also exist between the loop diagrams of gravity and gauge theory. In the first two chapters of this thesis, we will demonstrate that in many cases this is indeed so. We will also see that this connection between gauge theory and gravity is a useful practical tool in studying the quantum properties of the latter theory.

One way to make the "squaring" relations between field theory loop amplitudes manifest is to obtain these amplitudes by taking the low-energy limit of the corresponding string theory results. However, this procedure is technically involved, especially beyond one-loop level. Here, we will take a different approach and reconstruct the field theory loop amplitudes from their unitarity cuts. This technique has been developed and used extensively for gauge theory loop calculations, and is reviewed in [8]. It relies on the well-known fact that perturbative unitarity allows one to obtain the imaginary, or absorptive, parts of an L-loop amplitude directly from the products of amplitudes with up to $L-1$ loops, using the "Cutkosky rules". To obtain the complete expression for the L-loop amplitude, one then has to reconstruct its real (dispersive) part. Such reconstruction can be performed using dispersion relations, but it is incomplete: there remain additive rational function ambiguities. These ambiguities can be avoided, however, if Cutkosky rules are used together with dimensional regularization. For example, let us consider a one-loop four-point amplitude in which all the internal and external legs are massless. In dimensional regularization, the momentum running in the loop is taken to be $(4-2\epsilon)$-dimensional. By dimensional analysis, our amplitude is then given by

$$\begin{aligned} A_4^{D=4-2\epsilon} &= (-s)^{-\epsilon} f_1 + (-t)^{-\epsilon} f_2 \\ &= (1-\epsilon \ln(-s)) f_1 + (1-\epsilon \ln(-t)) f_2 + \ldots \end{aligned} \qquad (0.5)$$

where f_1 and f_2 are dimensionless functions of the kinematic variables. This expression now contains branch cuts at $\mathcal{O}(\epsilon)$ even if f_1 and f_2 are cut-free. Thus, if one keeps the terms of this order in the calculation, the real parts of the amplitude can be unambiguously reconstructed. This argument can be easily extended to higher-point and higher-loop amplitudes[1].

The cut-based method of evaluating the loop amplitudes has significant calculational advantages over the traditional diagrammatic approach. In particular, individual Feynman diagrams are not gauge-invariant, and are often much more complicated than the final sum over diagrams. In the cut-based approach, one only manipulates gauge-invariant expressions, so the unnecessary complexity in the intermediate steps of the calculation can be avoided. This technique is especially powerful in quantum gravity calculations when it is combined with the KLT relations. Using these relations to express tree-level gravity amplitudes in terms of their gauge theory counterparts, the gravity cuts can often be written as "double copies" of the corresponding gauge theory cuts, allowing one to effectively "recycle" the old gauge theory calculations to obtain new, non-trivial results for gravity.

Despite their non-renormalizability, Einstein gravity (and its supersymmetric generalizations) can still be used to reliably calculate certain loop amplitudes which are free of ultraviolet divergences. These amplitudes should match the low-energy limit of any fundamental theory of gravity. An example of ultraviolet finite gravity amplitudes is provided by the one-loop amplitudes with n external gravitons of the same helicity. In the first chapter of this thesis, we will evaluate these amplitudes for $n = 4, 5, 6$, using the KLT decomposition and the cutting method described above. Then, we will use soft and collinear properties of these amplitudes to conjecture their form for

[1]The technique can also be extended to amplitudes with massive legs, even though a number of technical complications arise in those cases[9].

arbitrary n. We will also present an ansatz for another infinite sequence of ultraviolet finite one-loop graviton scattering amplitudes, the "maximally helicity-violating" (MHV) amplitudes in the maximally supersymmetric generalization of general relativity, $N = 8$ supergravity. MHV configurations are those where exactly two particles have a helicity opposite to that of the remaining $n - 2$ particles. (In supersymmetric theories, the amplitudes where all of the external particles have the same helicity, or just one has the opposite helicity, vanish to all loop orders by a supersymmetry Ward identity.) We will see that, in many cases, relations reminiscent of (0.4) hold between one-loop gravity and gauge theory amplitudes.

In the second chapter of this thesis, we will extend this connection between gravity and gauge theory beyond one-loop level. In particular, we will obtain an exact expression for a two-loop, four-point scattering amplitude in $N = 8$ supergravity, and show how it is related to the corresponding result in $N = 4$ super-Yang-Mills theory. We will also present evidence that such "squaring" relations between $N = 8$ supergravity and $N = 4$ super-Yang-Mills exist at any loop order.

The most interesting question about the higher-loop gravity amplitudes is their ultraviolet divergences. Non-supersymmetric theories of gravity with matter generically diverge at one-loop [10, 11, 12], and pure gravity diverges at two loops [13, 14]. As we have already mentioned, supersymmetric theories of gravity are less divergent: in $D = 4$, all possible one-loop and two-loop counterterms are incompatible with supersymmetry. At three loops, however, one can construct a counterterm [15] which is allowed even in the maximally supersymmetric version of the theory, $N = 8$ supergravity. It is the existence of this counterterm that led most physicists to believe that supergravity is non-renormalizable. In principle, however, the theory may contain some unknown additional symmetry which would forbid this counterterm. This possibility can only be excluded by an explicit the three-loop calculation. To date, no such calculation has been done. While we have no explicit results beyond the two-loop level, our work suggests a natural conjecture for the divergences appearing at L loops. The conjecture would imply that the first divergence in $N = 8$ supergravity in four dimensions occurs at five loops, not at three loops. Clearly, a lot of work remains to be done to settle the issue.

While the puzzles related to the quantum properties of gravitational theories are of great interest theoretically, there is also a variety of more phenomenological questions involving gravity. In particular, gravity may play a crucial role in the solution to the hierarchy problem of the Standard Model, as we will explain below.

The appearance of the dimensionful parameter, G_N, in the Einstein-Hilbert action (0.1) indicates the presence of a fundamental energy scale, the Planck scale, at which the gravitational coupling becomes of order one and quantum gravity effects become important. Naively, one can estimate this scale to be $M_{\mathrm{P}l} = \sqrt{\hbar c/G_N} \sim 1.2 \times 10^{19}$ GeV. If one now assumes that the Standard Model itself is valid up to the Planck scale and calculates radiative correction to the Higgs boson mass parameter of the Standard Model, one finds that it is naturally of order $M_{\mathrm{P}l}$, unless some highly fine-tuned cancellations occur in the theory at the quantum level. Depending on the sign of this correction, it would either lead to a theory in which the electroweak symmetry is unbroken, or drive the scale of its breaking up to $M_{\mathrm{P}l}$. In the first case, the W and Z bosons would be massless; in the second case, their mass would be of order $M_{\mathrm{P}l}$. Since these particles have been observed experimentally with masses around 100 GeV, both possibilities are excluded. This conflict between theory and observation is referred to as the hierarchy problem of the Standard Model.

Traditionally, physicists have attempted to solve the hierarchy problem by questioning the assumption of the validity of the Standard Model up to $M_{\mathrm{P}l}$. It has been shown that large quantum corrections to the Higgs mass can be avoided if certain kinds of new physics appear at the TeV scale. One popular solution to the hierarchy problem is to embed the Standard Model into a supersymmetric theory, with supersymmetry softly broken at the TeV scale. Another idea is to postulate that the Higgs field of the Standard Model is not elementary, but rather is composed of more fundamental fermions. These fermions are assumed to be bound by new strong interaction, the so-called "technicolor" force, which becomes confining at the TeV scale.

Recently, Arkani-Hamed, Dimopoulos and Dvali (ADD) have pointed out that gravity itself can play a significant role in solving the hierarchy problem [6, 16]. Indeed, in predicting that quantum gravity effects first become important at the Planck scale, we have assumed that conventional general relativity is valid up to that scale, or, equivalently, down to distances as short as 10^{-33} cm. However, the theory has currently been tested experimentally only down to distances of about 1 mm; therefore, we have extrapolated it over 32 orders of magnitude to derive the value of M_{Pl}. It is entirely possible that this extrapolation is invalid; if this is the case, the actual scale at which quantum gravity becomes important may turn out to be much lower than our naive expectation. If this scale is around a TeV, the hierarchy problem does not arise: There are no abnormally large corrections to the Higgs mass. ADD have proposed a model in which this possibility is realized.

The model assumes that the full space-time is $(4+n)$-dimensional, with $n \geq 1$. The extra n dimensions are taken to be compact, with characteristic size R. Let us denote the fundamental scale of the theory, the scale at which higher-dimensional gravity becomes strong, by M. At distances large compared to R, gravity obeys the usual 4-dimensional Newton's Law, with the gravitational constant given by [2]

$$G_N \sim M^{-n-2} R^{-n}. \tag{0.6}$$

In order to reproduce the measured value $G_N = 6.7 \times 10^{-39}$ GeV^{-2} with the fundamental scale $M \sim 1$ TeV, we require

$$R \sim 2 \times 10^{31/n-17} \text{cm}. \tag{0.7}$$

The most obvious experimental consequence of this scenario is the violation of Newton's Law at distances of order R, where the higher-dimensional nature of gravity is revealed. As we have mentioned above, macroscopic measurements of gravity constrain R to be less than about a millimeter [17]. From (0.7) we see that for $n = 1$, $R \sim 10^{13}$ cm, so this case is clearly excluded. For $n \geq 2$, however, this constraint is satisfied. In particular, in the case $n = 2$, (0.7) predicts $R \sim 0.1 - 1$ mm; these distances will be probed by the precision table-top gravitational experiments in the near future [17, 18].

[2]The exact numerical coefficient in this equation depends on the conventions used and will be specified in chapter 3.

Since the Standard Model provides an accurate description of the strong, electromagnetic and weak processes up to energies of order 100 GeV, we have to assume that the quarks, leptons and gauge bosons cannot propagate in the extra dimensions. That is, they must be localized to a 4-dimensional hypersurface within the full space-time. While such a localization may be achieved by purely field-theoretic mechanisms [6], the most attractive possibility is provided by D-branes of string theory [16], since they naturally appear with gauge degrees of freedom confined to them. In fact, it is possible to construct string vacua [19, 20] in which the low energy theory of the degrees of freedom living on a D-brane closely resembles the Standard Model, while gravitons are free to propagate in the full space-time. Because of this connection to string theory, the hypersurface on which the Standard Model fields live is commonly referred to as "the brane", while the full higher dimensional space-time is termed "the bulk".

Notice that if n is not too high, the radii of the extra dimensions predicted by (0.7) are much larger than their "natural" size, M^{-1}. In a sense, the ADD model has reformulated, rather than solved, the hierarchy problem: Instead of the original hierarchy between the Higgs mass and the Planck scale, one now has to explain the hierarchy between R and M^{-1}. One possible solution is to assume that the bulk degrees of freedom are supersymmetric, while on our brane supersymmetry is broken at the TeV scale. It is worth emphasizing that while the ultimate cause of stability in this case is again supersymmetry, the physics at TeV scale is drastically different from the usual four-dimensional supersymmetric models.

The last chapter of this thesis will be concerned with the experimental signatures of the ADD scenario. If indeed gravity becomes strong at TeV scales, the higher-dimensional gravitons should have significant couplings to the Standard Model particles at energies accessible to current experiments and observations. After a graviton is emitted, it effectively "escapes" into the extra dimensions, and does not interact with the states on the brane. If the emission rate is high enough, this can lead to significant deviations of various observables from the Standard Model predictions, in contradiction with data. This places constraints on the parameters of the model, such as the fundamental scale M and the radii of extra dimensions R. The authors of the model have performed order-of-magnitude estimates of the bounds coming from a variety of sources[21], and concluded that the model was not grossly phenomenologically excluded. Here, we will study in detail the bounds which can be obtained from missing-energy searches at high-energy colliders.

Higher-dimensional nature of gravity and a low quantum gravity scale are generic predictions of the ADD model. The missing-energy signatures associated with graviton emission are therefore independent of the nature of the fundamental theory of quantum gravity, as long as the energies reached in particle collisions are well below the fundamental scale. Another class of experimental signatures of the ADD scenario are the signatures specific to a particular theory at scale M. In chapter 3, we will also study some signatures of this class, assuming that quantum gravity is described by some version of string theory. String theory predicts the appearance of string Regge resonances of the Standard Model particles and gravitons, with masses quantized in units of the string scale. These particles can appear as virtual states in Standard Model scattering processes at low energies. We will use a simple stringy toy model to illustrate their effects in two QED scattering processes, Bhabha scattering and $e^+e^- \to \gamma\gamma$. We will derive the experimental bounds on the string scale in the context of our toy model.

As we will see in chapter 3, the current experimental data do not exclude the possibility that the scale M is sufficiently low to play a role in a solution to the hierarchy problem, as suggested by ADD. Future colliders, such as the Large Hadron Collider (LHC) at CERN, will probe most of the interesting parameter range. If the ADD scenario is realized in Nature, it is almost certain that new phenomena such as the graviton emission discussed here will be discovered at LHC.

Chapter 1

Multi-Leg One-Loop Gravity Amplitudes from Gauge Theory

1.1 Introduction

While the quantum-mechanical properties of gauge theories are reasonably well understood, the same cannot be said for general relativity. In particular, the latter contains a dimensionful coupling parameter, $G_N = 1/M^2_{\text{Planck}}$, and therefore it is non-renormalizable at the level of power-counting. On the other hand, in perturbation theory certain relations have been found expressing gravity amplitudes in terms of gauge theory amplitudes. Such relations may lead to a better understanding of the properties of quantum gravity. At the classical (tree) level, Kawai, Lewellen and Tye (KLT) [7] found explicit general relations between closed- and open-string amplitudes. In the infinite-string-tension limit the KLT relations provide a representation of tree-level gravity amplitudes as the 'square' of tree-level gauge-theory amplitudes (where appropriate permutation sums and kinematic prefactors also have to be applied to the latter). These relationships, however, are not obvious from the point of view of the Lagrangian and the associated Feynman diagrams.

As an example of how such relations can lead to more explicit information about gravity amplitudes, one can consider special helicity assignments for the external gravitons and gauge bosons ('gluons'). The 'maximally helicity-violating' (MHV) configurations are those where exactly two particles have a helicity opposite to that of the remaining $n-2$ particles. (Tree amplitudes where all of the external particles have the same helicity, or just one has the opposite helicity, vanish trivially by a supersymmetry Ward identity (SWI) [22].) In gauge theory, the MHV tree amplitudes are known for all n, and are remarkably compact [23]. As we review in section 1.2, Berends, Giele and Kuijf (BGK) [24] used these gauge theory results together with the KLT relations to find a closed-form expression for the tree-level MHV gravity amplitudes. They also verified the universal behavior of these amplitudes as one of the graviton momenta becomes soft [25], providing a non-trivial consistency check on their results.

In this chapter we exploit the relationship between gravity and gauge theory amplitudes to obtain non-trivial results for gravity at loop level. We obtain one-loop amplitudes with an arbitrary number n of external gravitons. We consider two special types of helicity configurations: (1) the same MHV configurations that were considered at tree level by BGK, for the maximally supersymmetric theories of $N = 8$ supergravity and $N = 4$ super-Yang-Mills; and (2) the 'all-plus' configurations where all the gravitons (or gauge bosons) have the same helicity, for non-supersymmetric theories with arbitrary massless matter content. (The all-plus amplitudes vanish for all supersymmetric theories.) For gauge theory, both infinite sequences (1) and (2) are known [26, 27, 28], and are reviewed in section 1.3.

These two sequences of gravity (gauge theory) amplitudes are not as different as one might expect, even though they involve different helicity states and different matter content. Indeed, in ref. [29] a 'dimension-shifting' relation was exhibited between the gauge theory amplitudes of types (1) and (2): The all-plus gauge amplitudes can be obtained from the $N = 4$ MHV amplitudes by shifting the dimension of the loop integration upward by 4 units, $\int d^D L \to \int d^{D+4} L$, and multiplying by an overall prefactor. This relation was explicitly verified for $n = 4, 5, 6$. In section 1.4.2 we use a combination of the KLT relations and unitarity to extend this relation from gauge theory to gravity, where it implies that the all-plus gravity amplitudes can be obtained from one-loop MHV $N = 8$ supergravity amplitudes by shifting the dimension upward by 8 units, $\int d^D L \to \int d^{D+8} L$. (Ref. [29] also speculated on this relation for gravity, but only provided evidence for $n = 4$.)

The dimension-shifting relation can be applied in either direction. Here we shall explicitly determine the all-plus n-graviton amplitudes for $n = 4, 5, 6$, by calculating their unitarity cuts in an arbitrary dimension D (see section 1.4). The cut calculations may be effectively performed by recycling the analogous gauge theory cut calculations. By working in an arbitrary dimension, we can determine the complete amplitudes from the cuts, free of the subtraction ambiguities frequently associated with dispersion relations [8]. Then (in section 1.4.4) we use the dimension-shifting relation of section 1.4.2 to obtain the MHV $N = 8$ supergravity amplitudes. This provides a non-trivial example of how gauge theory properties may be used to derive analogous results for gravity theories.

The one-loop MHV $N = 4$ amplitudes can be written as linear combinations of a restricted class of scalar box (four-point) integrals. The one-loop MHV $N = 8$ supergravity amplitudes can also be expressed in terms of the same class of box integrals. The relation between the coefficients of these integrals for the $N = 8$ and $N = 4$ cases is reminiscent of the tree-level KLT relations.

The unitarity-based results for gravity for $n \leq 6$ provide the starting point for constructing an ansatz for all n which satisfies all known analytic properties. In order to go beyond the explicit graviton amplitude calculations for $n \leq 6$, we use the analytic properties of the two series of amplitudes as the n-point kinematics approaches special regions. In particular, we study the soft limits noted above, as well as the limits where two gravitons become collinear. Both of these limits for gravity can be understood at tree-level from the corresponding limits in gauge theory, by exploiting the KLT relations, as we show in section 1.5. We also show that there are no loop corrections to the structure of these limits in the case of gravity (unlike gauge theory). In section 1.6 we obtain ansatze that satisfy the appropriate limits for both the all-plus helicity (self-dual) gravity and $N = 8$ supergravity series of amplitudes. Although we do not have a proof that the $n > 6$ amplitudes are the unique expressions with the proper limits, we know of no counterexample with six- or higher-point kinematics where this method of obtaining amplitudes has failed to produce the correct expression.

The all-plus gravity and gauge amplitudes in $D = 4$ are of interest in part because of their connection with self-dual gravity (SDG) [30] and self-dual Yang-Mills theory (SDYM) [31], i.e. gravity and gauge theory restricted to self-dual configurations of the respective field strengths, $R_{\mu\nu\rho\sigma} = \frac{i}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta\rho\sigma}$ and $F_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}F_{\alpha\beta}$, with $\epsilon_{0123} = +1$. This connection is simple to see at the linearized (free) level of superpositions of plane waves of identical helicity. It has been further studied at tree level [32, 33, 34] and at the one-loop level [33, 35]. Chalmers and Siegel [35, 36] have presented self-dual actions for gravity and gauge theory which reproduce the all-plus scattering amplitudes at both tree level and one loop. Their actions have no amplitudes beyond one loop, and the tree-level amplitudes vanish on-shell. Thus the one-loop all-plus amplitudes constitute a complete perturbative solution to the theories defined by the Chalmers-Siegel self-dual actions. (See also ref. [2].)

In fact the one-loop gravity-gauge-theory relations can be stated in terms of just the components of the gauge amplitudes that dominate in the limit of a large number of colors, N_c. (These components are the $A_{n;1}$ partial amplitudes defined in section 1.3.) The large-N_c limit of $N = 4$ super-Yang-Mills theory has recently attracted much attention through its connection to superstring configurations in anti-de-Sitter space [37]. In this context, gauge theories with $N < 4$ supersymmetry have been constructed by an orbifold-style [38] truncation of the $N = 4$ spectrum, and it has been argued that at large-N_c their amplitudes actually coincide (up to overall constants) with those of the $N = 4$ theory [39]. These results may provide some additional motivation for studying the relation between large-N_c $N = 4$ amplitudes and supergravity amplitudes, although we know of no direct connection between the purely perturbative relations found here and the anti-de-Sitter-space results, which are non-perturbative, involving a weak $\leftrightarrow$ strong coupling duality.

1.2 Review of Tree-Level Properties

In this section we review the tree-level KLT relations [7], and the known analytic properties of tree-level amplitudes in gauge theory and in gravity. These properties, as well as additional ones derived in section 1.5, will be used in section 1.6 to obtain an ansatz for the all-plus gravity and MHV $N = 8$ supergravity amplitudes with an arbitrary number of external legs.

1.2.1 KLT Relations

The KLT relations are between tree-level amplitudes in closed and open string theories, and arise from the representation of any closed-string vertex operator as a product of open-string vertex operators,

$$V^{\rm closed}(z_i, \bar{z}_i) = V^{\rm open}_{\rm left}(z_i)\, \overline{V}^{\rm open}_{\rm right}(\bar{z}_i)\,. \tag{1.2.1}$$

The left and right string oscillators appearing in $V_{\rm left}$ and $\overline{V}_{\rm right}$ are distinct, but the zero mode momentum is shared. In the open-string tree amplitude, the z_i are real variables, to be integrated over the boundary of the disk, while in the closed-string tree amplitude the z_i are complex and integrated over the sphere. The closed-string integrand is thus a product of two open-string integrands. This statement holds for any set of closed-string states, since they can all be written as tensor products of open-string states. KLT evaluated the $(n-3)$ two-dimensional closed-string world-sheet integrals, via a set of contour-integral deformations, in terms of the $(n-3)$ open-string integrals, and thereby related the two sets of string amplitudes.

After taking the field-theory limit [40, 41], $\alpha' k_i \cdot k_j \to 0$, the KLT relations for four-, five- and six-point amplitudes are [24],

$$\begin{aligned}
M_4^{\rm tree}(1,2,3,4) &= -is_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,\\
M_5^{\rm tree}(1,2,3,4,5) &= is_{12}s_{34} A_5^{\rm tree}(1,2,3,4,5)\, A_5^{\rm tree}(2,1,4,3,5)\\
&\quad + is_{13}s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)\,,\\
M_6^{\rm tree}(1,2,3,4,5,6) &= -is_{12}s_{45} A_6^{\rm tree}(1,2,3,4,5,6)[s_{35} A_6^{\rm tree}(2,1,5,3,4,6)\\
&\quad + (s_{34}+s_{35}) A_6^{\rm tree}(2,1,5,4,3,6)]\\
&\quad + \mathcal{P}(2,3,4)\,.
\end{aligned} \tag{1.2.2}$$

Here the M_n's are the amplitudes in a gravity theory stripped of couplings, the A_n's are the color-ordered amplitudes in a gauge theory [42, 43], $s_{ij} \equiv (k_i + k_j)^2$, and

$\mathcal{P}(2,3,4)$ instructs one to sum over all permutations of the labels 2, 3 and 4. The n arguments of M_n and A_n are the external states j, which have momentum k_j. The n-point generalization of eq. (1.2.2) [7, 24] is presented in appendix A.

Each gravity state j appearing in M_n is the tensor product of the corresponding two gauge theory states appearing in the A_n's on the right-hand side of the equation. In particular, each of the 256 states of the $N = 8$ supergravity multiplet, consisting of 1 graviton, 8 gravitinos, 28 gauge bosons, 56 gauginos, and 70 real scalars, can be interpreted as a tensor product of two sets of the 16 states of the $N = 4$ super-Yang-Mills multiplet, consisting of 1 gluon, 4 gluinos and 6 real scalars. (In string theory, this correspondence may be understood in terms of the factorization of the closed string vertex operator for each $N = 8$ state into a product of $N = 4$ open string vertex operators.) Thus a sum over the $N = 8$ supergravity states can be interpreted as a double sum over a tensor product of $N = 4$ super-Yang-Mills states.

Full amplitudes are obtained from $M_n^{\rm tree}$ and $A_n^{\rm tree}$ via,

$$\begin{aligned}\mathcal{M}_n^{\rm tree}(1,2,\ldots,n) &= \left(\frac{\kappa}{2}\right)^{(n-2)} M_n^{\rm tree}(1,2,\ldots,n)\,,\\ \mathcal{A}_n^{\rm tree}(1,2,\ldots,n) &= g^{(n-2)} \sum_{\sigma\in S_n/Z_n} {\rm Tr}\left(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\cdots T^{a_{\sigma(n)}}\right) A_n^{\rm tree}(\sigma(1),\sigma(2),\ldots,\sigma(n))\,,\end{aligned} \tag{1.2.3}$$

where $\kappa^2 = 32\pi G_N$, and S_n/Z_n is the set of all permutations, but with cyclic rotations removed. The T^{a_i} are fundamental representation matrices for the Yang-Mills gauge group $SU(N_c)$, normalized so that ${\rm Tr}(T^aT^b) = \delta^{ab}$.

The relations (1.2.2) hold for arbitrary external states. For external gravitons and gluons it is convenient to quote the results in a helicity basis, using the spinor helicity formalism [44]. At tree-level — and to all orders for supersymmetric theories — helicity amplitudes where all, or all but one, of the external particles have the same helicity vanish by a SWI [22],

$$M_n(\pm,+,+,\ldots,+) = A_n(\pm,+,+,\ldots,+) = 0\,, \tag{1.2.4}$$

where the helicity assignments are for outgoing particles. These relations hold for any states in the respective $N = 8$ and $N = 4$ multiplets. (For scalar states, one interprets 'helicity' as particle vs. anti-particle.)

For a given number of external legs n, the simplest non-vanishing tree amplitudes — and supersymmetric loop amplitudes — are the maximally helicity-violating (MHV) amplitudes, where exactly two helicities are opposite to the majority. At tree-level, and to all loop orders in $N=4$ super-Yang-Mills theory, the MHV n-gluon amplitudes are all related to each other by the $N=4$ SWI [29],

$$\frac{1}{\langle i\,j\rangle^4} A_n(1^+,2^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = \frac{1}{\langle a\,b\rangle^4} A_n(1^+,2^+,\ldots,a^-,\ldots,b^-,\ldots,n^+)\,, \tag{1.2.5}$$

where i and j are the only negative helicity legs on the left-hand side and a and b are the only negative helicities on the right-hand side. (We will generally indicate the type of external particle by a subscript; e.g. 1_g^- for a negative-helicity gluon. However, for gluons in gauge theory and gravitons in gravity, we will usually omit the subscript.) Thus at tree-level it suffices to give the formula [23]

$$A_n^{\rm tree}(1^-,2^-,3^+,4^+,\ldots,n^+) = i\,\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\cdots\langle n\,1\rangle}\,. \tag{1.2.6}$$

We use the notation $\langle k_i^-|k_j^+\rangle = \langle i\,j\rangle$ and $\langle k_i^+|k_j^-\rangle = [i\,j]$, where $|k_i^\pm\rangle$ are massless Weyl spinors, labeled with the sign of the helicity and normalized by $\langle i\,j\rangle\,[j\,i] = s_{ij} = 2k_i{\cdot}k_j$. For the case where both energies are positive the spinor inner products are given by

$$\langle i\,j\rangle = \sqrt{|s_{ij}|}e^{i\phi_{ij}}\,, \qquad\qquad [i\,j] = \sqrt{|s_{ij}|}e^{-i(\phi_{ij}+\pi)}\,, \tag{1.2.7}$$

where

$$\cos\phi_{ij} \;=\; \frac{k_i^1 k_j^+ - k_j^1 k_i^+}{\sqrt{|s_{ij}|k_i^+ k_j^+}}\,, \qquad\qquad \sin\phi_{ij} \;=\; \frac{k_i^2 k_j^+ - k_j^2 k_i^+}{\sqrt{|s_{ij}|k_i^+ k_j^+}}\,, \tag{1.2.8}$$

and $k_i^+ = k_i^0 + k_i^3$. (The cases where one or both of the energies are negative are similar, except for additional overall phases.) For later use, we also define the spinor strings

$$\begin{aligned} \langle i^-|(l+m)|j^-\rangle &\equiv \langle k_i^- \,|\, (\not{k}_l + \not{k}_m) \,|\, k_j^- \rangle\,, \\ \langle i^+|lm\cdots|j^-\rangle &\equiv \langle k_i^+ \,|\, \not{k}_l \not{k}_m \cdots \,|\, k_j^- \rangle\,, \\ \langle i^-|\ell_m|j^-\rangle &\equiv \langle k_i^- \,|\, \not{\ell}_m \,|\, k_j^- \rangle\,, \end{aligned} \tag{1.2.9}$$

etc., where ℓ_m is a loop momentum.

The MHV graviton tree (and $N = 8$ loop) amplitudes satisfy an $N = 8$ SWI analogous to eq. (1.2.5),

$$\frac{1}{\langle i\, j\rangle^8} \mathcal{M}_n(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \frac{1}{\langle a\, b\rangle^8} \mathcal{M}_n(1^+, 2^+, \ldots, a^-, \ldots, b^-, \ldots, n^+). \tag{1.2.10}$$

The MHV four- and five-graviton tree amplitudes,[1] which satisfy eq. (1.2.10) as well as the appropriate KLT relations (1.2.2), are [24]

$$\begin{aligned} M_4^{\rm tree}(1^-, 2^-, 3^+, 4^+) &= i\, \langle 1\, 2\rangle^8 \, \frac{[1\, 2]}{\langle 3\, 4\rangle\, N(4)}\,, \\ M_5^{\rm tree}(1^-, 2^-, 3^+, 4^+, 5^+) &= i\, \langle 1\, 2\rangle^8 \, \frac{\varepsilon(1,2,3,4)}{N(5)}\,, \end{aligned} \tag{1.2.11}$$

where

$$\varepsilon(i,j,m,n) \equiv 4i\varepsilon_{\mu\nu\rho\sigma} k_i^\mu k_j^\nu k_m^\rho k_n^\sigma \;=\; [i\, j]\, \langle j\, m\rangle\, [m\, n]\, \langle n\, i\rangle - \langle i\, j\rangle\, [j\, m]\, \langle m\, n\rangle\, [n\, i]\;, \tag{1.2.12}$$

and

$$N(n) \equiv \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \langle i\, j\rangle\;. \tag{1.2.13}$$

For $n > 4$, Berends, Giele and Kuijf [24] presented the expression,

$$\begin{aligned} &M_n^{\rm tree}(1^-, 2^-, 3^+, \ldots, n^+) = -i\, \langle 1\, 2\rangle^8 \\ &\times \left[\frac{[1\, 2]\, [n-2\;\; n-1]}{\langle 1\;\; n-1\rangle\, N(n)} \left(\prod_{i=1}^{n-3} \prod_{j=i+2}^{n-1} \langle i\, j\rangle \right) \prod_{l=3}^{n-3} \left(-\langle n^-| \not{K}_{l+1,n-1} | l^-\rangle \right) + \mathcal{P}(2, 3, \ldots, n-2) \right], \end{aligned} \tag{1.2.14}$$

where $K_{i,j}^\mu \equiv \sum_{s=i}^{j} k_s^\mu$, and $+\mathcal{P}(M)$ instructs one to sum the quantity inside the

---

[1] Our overall phase conventions differ from those of ref. [24] by a '$-i$'.

brackets over all permutations of the set M. They numerically verified its correctness for $n \leq 11$. The expression in brackets is totally symmetric (although this is not manifest), as is required for consistency with eq. (1.2.10).

1.2.2 Soft and Collinear Properties at Tree Level

There are two important universal limits of color-ordered n-gluon tree amplitudes. In the limit that the gluon s becomes soft, A_n^{tree} has the universal behavior [45],

$$A_n^{\text{tree}}(\ldots, a, s^\pm, b, \ldots) \stackrel{k_s\to 0}{\longrightarrow} \mathcal{S}^{\text{tree}}(a, s^\pm, b) \times A_{n-1}^{\text{tree}}(\ldots, a, b, \ldots)\,, \tag{1.2.15}$$

where the soft (eikonal) factors are

$$\begin{aligned} \mathcal{S}^{\text{tree}}(a, s^+, b) &= \frac{\langle a\, b\rangle}{\langle a\, s\rangle \langle s\, b\rangle}\,, \\ \mathcal{S}^{\text{tree}}(a, s^-, b) &= \frac{-\,[a\, b]}{[a\, s]\,[s\, b]}\,. \end{aligned} \tag{1.2.16}$$

In the collinear limit where two gluon momenta k_a and k_b become parallel (denoted by $a \parallel b$), we have $k_a \approx z k_P$ and $k_b \approx (1-z) k_P$ for some $z \in [0,1]$, where $k_P \equiv k_a + k_b$. The behavior of a tree amplitude in this limit is [46]

$$A_n^{\text{tree}}(\ldots, a^{\lambda_a}, b^{\lambda_b}, \ldots) \stackrel{a\parallel b}{\longrightarrow} \sum_{\lambda=\pm} \text{Split}_{-\lambda}^{\text{tree}}(z, a^{\lambda_a}, b^{\lambda_b}) \times A_{n-1}^{\text{tree}}(\ldots, P^\lambda, \ldots)\,, \tag{1.2.17}$$

where the gluon splitting amplitudes are

$$\begin{aligned} \text{Split}_+^{\text{tree}}(z, a^+, b^+) &= 0\,, \\ \text{Split}_-^{\text{tree}}(z, a^+, b^+) &= \frac{1}{\sqrt{z(1-z)}\,\langle a\, b\rangle}\,, \\ \text{Split}_+^{\text{tree}}(z, a^-, b^+) &= \sqrt{\frac{z^3}{1-z}}\frac{1}{\langle a\, b\rangle}\,; \end{aligned} \tag{1.2.18}$$

the remaining ones may be obtained by parity. Similar expressions exist including fermions. For example, for a gluon splitting into two fermions, the color-ordered splitting amplitudes are

$$\text{Split}^{\text{tree}}_{+}(z, a_{\bar{q}}^{-}, b_{q}^{+}) = \frac{z}{\langle a\, b\rangle}\,, \qquad\qquad \text{Split}^{\text{tree}}_{-}(z, a_{\bar{q}}^{-}, b_{q}^{+}) = \frac{1-z}{[a\, b]}\,. \tag{1.2.19}$$

A more complete discussion of splitting amplitudes may be found in reviews [46, 47, 8]. (Our sign conventions in the splitting functions are the ones used in ref. [48].)

Gravity amplitudes, like gauge amplitudes, are known to satisfy universal soft limits[25, 24]. The gravitational soft limits have the form,

$$M_n^{\text{tree}}(\ldots, a, s^{\pm}, b, \ldots) \stackrel{k_s\to 0}{\longrightarrow} \mathcal{S}^{\text{gravity}}(s^{\pm}) \times M_{n-1}^{\text{tree}}(\ldots, a, b, \ldots)\,. \tag{1.2.20}$$

For the limit $k_n \to 0$ in $M_n^{\text{tree}}(1, 2, \ldots, n)$, the gravitational soft factor (for positive helicity) is

$$\mathcal{S}_n \;\equiv\; \mathcal{S}^{\text{gravity}}(n^+) \;=\; \frac{-1}{\langle 1\, n\rangle \langle n,\, n-1\rangle} \sum_{i=2}^{n-2} \frac{\langle 1\, i\rangle \langle i,\, n-1\rangle\, [i\, n]}{\langle i\, n\rangle}\,. \tag{1.2.21}$$

Although it is not manifest, $\mathcal{S}_n$ is also symmetric under the interchange of legs 1 and $n-1$ with the others. For all n, BGK verified that the MHV amplitudes (1.2.14) have the correct behavior as an external momentum becomes soft.

In section 1.5 we will show that gravity tree amplitudes also have universal behavior as two external momenta become collinear, and that the splitting amplitudes are composed of products of pairs of the ones for gauge theory. We shall further demonstrate that the tree-level soft factors and splitting amplitudes for gravity do not incur any higher loop corrections, in contrast to the situation for gauge theory.

1.3 One-Loop MHV Amplitudes in Gauge Theory

Before proceeding to gravity, it is useful to review the structure of the one-loop maximally helicity-violating (MHV) amplitudes in gauge theory [27, 28, 26, 49]. These amplitudes were constructed by techniques similar to those used in the following sections for the corresponding gravity amplitudes. In particular we shall use unitarity or cutting techniques [50, 26, 49, 9], as well as the factorization bootstrap approach [23, 27] of finding ansatze for amplitudes based on their known kinematic poles.

The unitarity cuts in the $N = 4$ MHV gauge case are simple enough that the direct computation can be performed for all n simultaneously. We have not been able to do that yet for the analogous $N = 8$ MHV supergravity computation, and so we shall resort to an ansatz for $n > 6$, based on soft and collinear limits.

1.3.1 General Properties of One-Loop Amplitudes

We first define one-loop amplitudes M_n for gravity and $A_{n;j}$ for gauge theory, from which all couplings have been removed. Color has also been removed from the $A_{n;j}$, according to the one-loop color decomposition [51]. For the case where all states are in the adjoint representation, the full amplitudes are given by,

$$\begin{aligned} \mathcal{M}_n^{\text{1-loop}}(1,2,\ldots,n) &= \left(\frac{\kappa}{2}\right)^n M_n(1,2,\ldots,n)\,, \\ \mathcal{A}_n^{\text{1-loop}}(1,2,\ldots,n) &= g^n \sum_{j=1}^{\lfloor n/2\rfloor+1} \sum_{\sigma\in S_n/S_{n;j}} \mathrm{Gr}_{n;j}(\sigma)\, A_{n;j}(\sigma(1),\ldots,\sigma(n))\,, \end{aligned} \tag{1.3.1}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, the (unpermuted) color trace structures are $\mathrm{Gr}_{n;1}(1) \equiv N_c\,\mathrm{Tr}(T^{a_1}\cdots T^{a_n})$ and $\mathrm{Gr}_{n;j}(1) = \mathrm{Tr}(T^{a_1}\cdots T^{a_{j-1}}) \times \mathrm{Tr}(T^{a_j}\cdots T^{a_n})$ for $j > 1$, and $S_{n;j}$ is the subset of permutations S_n that leaves the trace structure $\mathrm{Gr}_{n;j}$ invariant. Similar color decompositions exist for the cases with fundamental representation particles in the loop. In fact, the partial amplitudes $A_{n;j}$ for $j > 1$ can be expressed in terms of the $A_{n;1}$ through the formula [26],

$$A_{n;j}(1,2,\ldots,j-1;j,j+1,\ldots,n) \;=\; (-1)^{j-1} \sum_{\sigma\in COP\{\alpha\}\{\beta\}} A_{n;1}(\sigma)\,. \tag{1.3.2}$$

Here $\alpha_i \in \{\alpha\} \equiv \{j-1, j-2, \ldots, 2, 1\}$, $\beta_i \in \{\beta\} \equiv \{j, j+1, \ldots, n-1, n\}$, and $COP\{\alpha\}\{\beta\}$ is the set of all permutations of $\{1, 2, \ldots, n\}$ with n held fixed that preserve the cyclic ordering of the α_i within $\{\alpha\}$ and of the β_i within $\{\beta\}$, while allowing for all possible relative orderings of the α_i with respect to the β_i.

Thus the full gauge amplitude can be constructed just from the $A_{n;1}$, which are *color-ordered* (i.e, they only receive contributions from planar graphs with a fixed ordering of the external legs), and therefore have simpler analytic properties than the remaining $A_{n;j}$. For this reason we need only explicitly discuss the case of $A_{n;1}$. The $A_{n;1}$ contributions are the ones which dominate the amplitude for a large number of colors N_c.

We consider one-loop amplitudes where the external momenta are taken to lie in four dimensions, but the number of dimensions D appearing in the loop-momentum integration measure d^DL remains arbitrary (for the time being). (To maintain supersymmetry we leave the number of states at their four-dimensional values.) In general, the m-point loop integrals with $m \geq 5$ which appear in such one-loop amplitudes can be reduced down to at most box (four-point) integrals and pentagon (five-point) integrals, where the pentagon integrals are scalar integrals (i.e., they contain no loop momenta in the numerator of the integrand) and are evaluated in $D+2$ dimensions [52]. Furthermore, if we now set $D = 4 - 2\epsilon$, then to $\mathcal{O}(\epsilon^0)$ the pentagon contributions may be neglected, because the scalar pentagon integrals in $D = 6 - 2\epsilon$ have no poles as $\epsilon \to 0$ (they are infrared and ultraviolet finite in $D = 6$), and because they are generated in the integral reduction procedure with a manifest ϵ prefactor [52].

For an amplitude in a generic theory, after applying these reductions the box integrals may have powers of the loop momentum L inserted in the numerator of the integrand, in addition to the four scalar propagators which make up the denominator. The amplitude may also contain triangle and bubble integrals arising from the corresponding Feynman diagrams. The all-plus helicity and $N = 4$ supersymmetric cases which we discuss below are special, however, and do not contain the full set of possible scalar integrals.

1.3.2 Review of One-Loop Soft and Collinear Properties in Gauge Theory

The collinear limits for the leading color-ordered one-loop amplitudes, $A_{n;1}$, are similar to the tree-level case and have the form

$$A_{n;1}(\ldots, a^{\lambda_a}, b^{\lambda_b}, \ldots) \xrightarrow{a \parallel b} \sum_{\lambda=\pm} \Big(\text{Split}^{\text{tree}}_{-\lambda}(z, a^{\lambda_a}, b^{\lambda_b})\, A_{n-1;1}(\ldots, P^{\lambda}, \ldots) + \text{Split}^{\text{1-loop}}_{-\lambda}(z, a^{\lambda_a}, b^{\lambda_b})\, A^{\text{tree}}_{n-1}(\ldots, P^{\lambda}, \ldots) \Big) . \tag{1.3.3}$$

In addition to the tree-level splitting amplitudes, $\text{Split}^{\text{tree}}_{-\lambda}(z, a^{\lambda_a}, b^{\lambda_b})$, one-loop corrections now also appear, $\text{Split}^{\text{1-loop}}_{-\lambda}(z, a^{\lambda_a}, b^{\lambda_b})$. Both quantities are universal, depending only on the two momenta becoming collinear, and not upon the specific amplitude under consideration [53]. The explicit values of the $\text{Split}^{\text{1-loop}}_{-\lambda}(z, a^{\lambda_a}, b^{\lambda_b})$ (which we shall not need here) were originally determined [26] from the four- [54] and five-point [55, 48] one-loop gauge amplitudes. Their universality for an arbitrary number of external legs was demonstrated in ref. [53].

Similarly, as the momentum of an external leg becomes soft the color-ordered one-loop amplitudes behave as,

$$A_{n;1}(\ldots, a, s^{\pm}, b, \ldots) \xrightarrow{k_s \to 0} \mathcal{S}^{\text{tree}}(a, s^{\pm}, b)\, A_{n-1;1}(\ldots, a, b, \ldots) + \mathcal{S}^{\text{1-loop}}(a, s^{\pm}, b)\, A^{\text{tree}}_{n-1}(\ldots, a, b, \ldots) , \tag{1.3.4}$$

where $\mathcal{S}^{\text{1-loop}}(a, s^{\pm}, b)$ is universal.

In the application of eqs. (1.3.3) and (1.3.4) to the collinear and soft limits of the one-loop all-plus gauge amplitudes, the second term always drops out, because of the vanishing of the tree-level amplitudes with all plus helicities, or all but one plus. In the $N = 4$ super-Yang-Mills MHV case, however, the contributions of $\text{Split}^{\text{1-loop}}_{-\lambda}(z, a^{\lambda_a}, b^{\lambda_b})$ and $\mathcal{S}^{\text{1-loop}}(a, s^{\pm}, b)$ survive.

1.3.3 All-Plus Amplitudes

The analytic properties of the one-loop all-plus amplitudes in gauge theory [27, 28] are remarkably simple. First of all, the unitarity cuts vanish in four dimensions, since eq. (1.2.4) implies that at least one of the two tree amplitudes on either side of a unitarity cut vanishes, for every possible helicity assignment for the two gluons crossing the cut. Similarly, by considering their factorization on particle poles, one finds that the one-loop all-plus amplitudes cannot contain multi-particle poles, i.e., factors of the form $1/(k_{i_1}+k_{i_2}+\cdots+k_{i_m})^2$ with $m>2$. The only permitted kinematic singularities are the ones where one external momentum becomes soft, or two external momenta become collinear. Finally, the loop-momentum integration does not generate any infrared nor ultraviolet divergences or associated logarithms. In summary, $A_{n;1}(1^+,2^+,\ldots,n^+)$ is a finite rational function of the momenta, cyclically symmetric in its n arguments, with singularities only in the regions where a momentum is soft or two cyclically (color) adjacent momenta are collinear.

These analytic properties were crucial in obtaining an ansatz for the explicit form of the amplitudes [27], which was verified by Mahlon via recursive techniques [28]. The one-loop amplitudes for n identical-helicity gluons in pure Yang-Mills theory are,

$$A_{n;1}(1^+,2^+,\ldots,n^+) = -\frac{i}{48\pi^2}\sum_{1\le i_1<i_2<i_3<i_4\le n}\frac{\mathrm{tr}_-[i_1i_2i_3i_4]}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,, \tag{1.3.5}$$

where $\mathrm{tr}_\pm[i_1i_2i_3i_4]\equiv\frac{1}{2}\mathrm{tr}[(1\pm\gamma_5)\not{k}_{i_1}\not{k}_{i_2}\not{k}_{i_3}\not{k}_{i_4}]$. These amplitudes are generated by actions for self-dual Yang-Mills theory [33, 35] as well as ordinary gauge theory.

The all-plus amplitudes vanish in any supersymmetric theory by the SWI (1.2.4). Thus the contribution from a gluon circulating around the loop is the negative of that from an adjoint fermion in the loop, and equal to that from an adjoint scalar. For a fundamental representation fermion one must divide the contribution to $A_{n;1}$ by an additional factor of N_c. In particular, for QCD with n_f flavors of quarks we have

$$A_{n;1}^{\mathrm{QCD}}(1^+,2^+,\ldots,n^+) = \Big(1-\frac{n_f}{N_c}\Big)A_{n;1}(1^+,2^+,\ldots,n^+)\,. \tag{1.3.6}$$

For $n\le 6$, the all-plus amplitudes have also been computed to all orders in the dimensional regularization parameter ϵ (but with four-dimensional external momenta) via their unitarity cuts [29]. These amplitudes may be compactly expressed as

$$
\begin{aligned}
A_{4;1}(1^+,2^+,3^+,4^+) &= \frac{-2s_{12}s_{23}}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\mathcal{I}_4^{1234}[\mu^4]\,,\\
A_{5;1}(1^+,2^+,3^+,4^+,5^+) &= \frac{1}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,1\rangle}\Big\{\Big[-s_{12}s_{23}\mathcal{I}_4^{123(45)}[\mu^4]+\text{cyclic}\Big]\\
&\qquad +2\,\varepsilon(1,2,3,4)\mathcal{I}_5^{12345}[\mu^6]\Big\}\,,\\
A_{6;1}(1^+,2^+,3^+,4^+,5^+,6^+) &= \frac{1}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,1\rangle}\Bigg\{\Bigg[-s_{12}s_{23}\,\mathcal{I}_4^{123(456)}[\mu^4]\\
&\quad -\frac{1}{2}(t_{123}t_{234}-s_{23}s_{56})\,\mathcal{I}_4^{1(23)4(56)}[\mu^4]+\varepsilon(1,2,3,4)\,\mathcal{I}_5^{1234(56)}[\mu^6]+\text{cyclic}\Bigg]\\
&\quad -\text{tr}[123456]\,\mathcal{I}_6^{123456}[\mu^6]\Bigg\}\,,
\end{aligned}
\tag{1.3.7}
$$

where $s_{ij}=(k_i+k_j)^2$, $t_{ijl}=(k_i+k_j+k_l)^2$, and '+ cyclic' implies a sum over the n cyclic permutations (for $A_{n;1}$) of the quantity within the brackets ([ ]) in which the phrase appears.

In eq. (1.3.7), $\mathcal{I}_5^{12345}$ and $\mathcal{I}_6^{123456}$ are pentagon and hexagon integrals where all the external legs are massless. The integral $\mathcal{I}_5^{1234(56)}$ is a one-mass pentagon integral, where legs 5 and 6 form the one external mass. The parentheses in the arguments of the one- and two-mass box integrals $\mathcal{I}_4$ similarly indicate the grouping of massless external legs for the amplitude into massive legs for the integral. (See appendix B for further exposition of our notation for the integrals.) In the integration measure d^DL for the integrals, the ($D=4-2\epsilon$)-dimensional loop momentum L can be decomposed into a 4-dimensional part ℓ and a (-2ϵ)-dimensional part μ, as $L=\ell+\mu$. Following the prescriptions of 't Hooft and Veltman [56], we take the four- and (-2ϵ)-dimensional parts of the loop momenta to be orthogonal, so that $L^2=\ell^2-\mu^2$. The symbol '$[\mu^{2r}]$' instructs one to insert an extra factor of $\mu^{2r}\equiv(\mu^2)^r$ into the loop integrand before performing the integral, i.e.,

$$
\mathcal{I}_m[\mu^{2r}]\equiv\int\frac{d^4\ell}{(2\pi)^4}\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\frac{(\mu^2)^r}{(\ell^2-\mu^2)((\ell-K_1)^2-\mu^2)\dots((\ell-\sum_{i=1}^{m-1}K_i)^2-\mu^2)}\,.
\tag{1.3.8}
$$

Carrying out the μ integration explicitly leads to the formula [28, 9, 29]

$$\mathcal{I}_m[\mu^{2r}] = -\epsilon(1-\epsilon)\cdots(r-1-\epsilon)\,(4\pi)^r\,\mathcal{I}_m^{D=4+2r-2\epsilon}\,, \tag{1.3.9}$$

where $\mathcal{I}_m^{D=4+2r-2\epsilon}$ is $\mathcal{I}_m$ with the number of dimensions in the loop-momentum integration shifted upward by $2r$; i.e., one replaces $D \to D+2r$ in eq. (B.1).

The fact that the all-orders-in-ϵ formulas (1.3.7) for the all-plus amplitudes contain insertions of the (-2ϵ)-dimensional components of the loop-momentum is just a reflection of the vanishing of the amplitudes' unitarity cuts for $D=4$ ($\epsilon \to 0$). It is straightforward to show that eq. (1.3.7) reduces to the $n \leq 6$ cases of eq. (1.3.5) as $\epsilon \to 0$. The explicit ϵ in the prefactor of $\mathcal{I}_m^{D=4+2r-2\epsilon}$ in eq. (1.3.9) means that only the $1/\epsilon$ pole coming from the ultraviolet divergence of $\mathcal{I}_m^{D=4+2r-2\epsilon}$ will contribute. These contributions, which are pure numbers, are given in appendix B.

1.3.4 $N=4$ Super Yang-Mills Amplitudes

The one-loop MHV amplitudes of $N=4$ super-Yang-Mills theory provide another example of amplitudes that may be evaluated for an arbitrary number of external legs. The higher degree of supersymmetry present in $N=4$ super-Yang-Mills theory considerably simplifies the analytic properties of its loop amplitudes. (Infinite sequences of MHV amplitudes have also been determined for $N=1$ supersymmetric theories, but their analytic structure is more complicated [49].)

In particular, supersymmetry cancellations forbid all triangle and bubble integrals, and only scalar box integrals (with no loop momenta in the numerator) may appear [26]. These supersymmetry cancellations, which may be seen in $N=1$ superspace [57], or in components by using a string-based approach [58, 59], imply a maximum of $m-4$ powers of loop momentum in the numerator of an m-point integral. The integral reduction procedure mentioned in section 1.3.1 uses equations such as $L \cdot k_i = -\frac{1}{2}((L-k_i)^2 - L^2 - k_i^2)$, where L is the loop momentum and k_i is an external momentum. The factors L^2 and $(L-k_i)^2$ cancel denominator factors from scalar propagators and reduce the number of external legs for the integral by one [60]. Thus the degree of the loop-momentum polynomial in the numerator of the integral is reduced by one whenever the number of legs for the integral is reduced by one. As a consequence, the m-point integrals in $N=4$ super-Yang-Mills theory can lead, after reduction, to at most scalar box integrals. Later, in section 1.4.5, we shall compare this loop-momentum power counting to what we find from inspecting the $N=8$ MHV supergravity amplitudes.

Although the $N=4$ power-counting allows any scalar box integral to appear at one loop, for the MHV helicity configurations one finds only the two-mass scalar box integrals where the two massive legs are diagonally opposite [26]. (Massless legs of the integral correspond directly to external momenta of the amplitude, while massive legs correspond to sums of external momenta.) Denoting the massless legs by a and b, we define

$$\mathcal{I}_4^{aK_1bK_2} \equiv \mathcal{I}_4^{a\,(a+1,\ldots,b-1)\,b\,(b+1,\ldots,a-1)} = \int \frac{d^DL}{(2\pi)^D} \frac{1}{L^2(L-k_a)^2(L-k_a-K_1)^2(L+K_2)^2}, \tag{1.3.10}$$

where $K_1 = k_{a+1} + k_{a+2} + \cdots + k_{b-1}$ is the sum of the adjacent momenta between a and b (in the cyclic sense) and $K_2 = -k_a - K_1 - k_b$. (See fig. 1.1.) In general we label the one-loop integrals by their external legs, following the cyclic ordering around the loop. The parentheses group together those legs of the amplitude which combine together to form a massive leg of the integral. As a more compact notation, we sometimes label the massive legs just by their total momentum (e.g., K_1 or K_2). See appendix B for more details. We use the same labeling for the coefficients of the integrals.

The explicit form for the integrals (1.3.10) near $D=4$ [52] is given in eq. (B.4). In terms of these integrals, the $N=4$ MHV amplitudes are given by [26]

$$A_{n;1}(1^-,2^-,3^+,\ldots,n^+) = \frac{1}{2}\langle 1\,2\rangle^4 \sum_{\substack{a,b\\ \text{cyclic}}} \alpha_{aK_1bK_2}\mathcal{I}_4^{aK_1bK_2} + \mathcal{O}(\epsilon), \tag{1.3.11}$$

where the sum is over all integrals with the *standard* $123\ldots n$ cyclic ordering of external legs, and over all distinct non-adjacent pairs of massless legs a, b. The coefficients

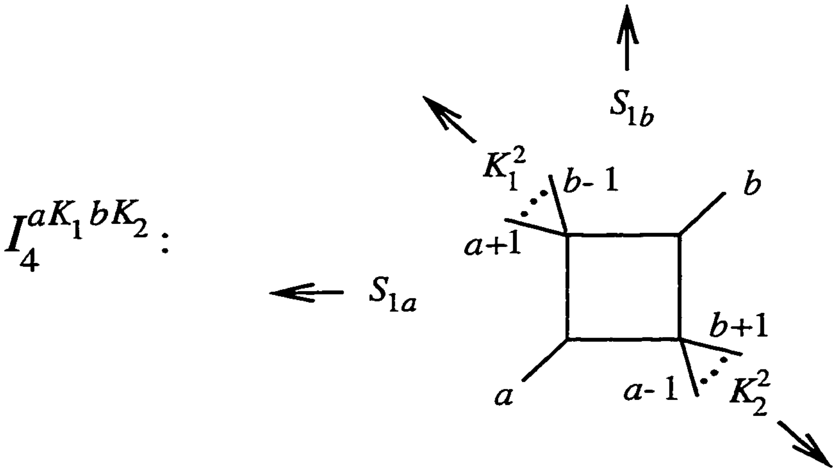


Figure 1.1: Kinematics of the two-mass box integrals $\mathcal{I}_4^{aK_1bK_2}$ that enter n-point MHV amplitudes in both $N = 4$ super-Yang-Mills theory and $N = 8$ supergravity. Here a and b label the external massless legs for the integral, which coincide with two external momenta for the amplitude, k_a and k_b. The massive legs carry momenta K_1 and K_2, which are sums of the remaining external momenta. The four Lorentz invariants are the masses K_1^2 and K_2^2, and the Mandelstam invariants $S_{1a} = (K_1+k_a)^2$ and $S_{1b} = (K_1 + k_b)^2$.

of the box integrals are

$$\begin{aligned} \alpha_{aK_1bK_2} &= -\frac{(K_1+k_a)^2(K_1+k_b)^2 - K_1^2K_2^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\cdots\langle n\,1\rangle} = \frac{\langle a^-|K_1|b^-\rangle\langle b^-|K_2|a^-\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\cdots\langle n\,1\rangle} \\ &= \frac{1}{2}\frac{\mathrm{tr}[a\,K_1\,b\,K_2]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\cdots\langle n\,1\rangle}\,, \end{aligned} \tag{1.3.12}$$

where $\langle i^-|K|j^-\rangle \equiv \langle i^-|\not{K}|j^-\rangle$ and $\mathrm{tr}[a\,K_1\,b\,K_2] \equiv \mathrm{tr}[\not{k}_a\,\not{K}_1\,\not{k}_b\,\not{K}_2]$.

For the purpose of facilitating comparisons to the gravity results, an alternative representation for the coefficients is

$$\alpha_{aK_1bK_2} = \frac{1}{2}\,g(a,K_1,b)\,g(b,K_2,a)\ \mathrm{tr}[a\,K_1\,b\,K_2]\,, \tag{1.3.13}$$

where the functions g are

$$g(a,K_1,b) \equiv g(a,\{a+1,a+2,\ldots,b-1\},b) = \frac{1}{\langle a,\,a+1\rangle\,\langle a+1,\,a+2\rangle\cdots\langle b-1,\,b\rangle}\,. \tag{1.3.14}$$

For $n = 4$, eq. (1.3.11) is the exact answer, to all orders in ϵ. For $n > 4$, the $\mathcal{O}(\epsilon)$ terms contain pentagon and higher-point integrals evaluated in $D = 6 - 2\epsilon$, with a manifest ϵ prefactor. These terms are currently known only for $n = 5, 6$ [29] — see below.

1.3.5 'Dimension-Shifting' Relations

The one-loop MHV amplitudes (1.3.11) in $N = 4$ super-Yang-Mills theory bear a curious 'dimension-shifting' relation to the sequence of one-loop 'all-plus' amplitudes (1.3.5). The relation between $N = 4$ MHV and all-plus amplitudes may be expressed in two different ways [29],

$$\begin{aligned} A_{n;1}(1^+, 2^+, \ldots, n^+) &= -2\epsilon(1-\epsilon)(4\pi)^2 \left[\frac{A_{n;1}^{N=4}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)}{\langle i\, j\rangle^4} \Bigg|_{D \to D+4} \right] \\ &= 2 \frac{A_{n;1}^{N=4}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)[\mu^4]}{\langle i\, j\rangle^4} . \end{aligned} \tag{1.3.15}$$

Here '$D \to D + 4$' instructs one to replace the D appearing in the loop-momentum integration measure $d^D L$ by $D + 4$ (where $D = 4 - 2\epsilon$) in all integrals appearing in $A_{n;1}$. The notation '$[\mu^4]$' means that an extra factor of μ^4 should be inserted into the numerator of every loop integral in the amplitude. The equivalence of the two forms follows from eq. (1.3.9). This relation (1.3.15) has been established for $n = 4, 5, 6$ but remains a conjecture for $n \geq 7$.

A few comments about the dimension-shifting relation (1.3.15) are in order. First of all, the manifest symmetry of the all-plus amplitudes on the left-hand side of the relation under the cyclic substitution $1 \to 2 \to \ldots \to n \to 1$ is also present on the right-hand side, as a consequence of the SWI (1.2.5). Secondly, in four dimensions the all-plus amplitudes have no unitarity cuts, and hence are pure rational functions [27, 28], while the $N = 4$ MHV amplitudes have cuts in all channels. These facts are consistent with eq. (1.3.15) because of the manifest ϵ on the right-hand side (in the first form of the relation): Only the ultraviolet $1/\epsilon$ poles in the higher-dimensional box, pentagon, etc., integrals for the $N = 4$ amplitudes contribute as $D \to 4$ ($\epsilon \to 0$), and these have rational-function coefficients.

The dimension-shifting relation (1.3.15) is a statement about all orders in ϵ, so its complete verification requires all-orders evaluation of both sides. The all-n formulae (1.3.11) and (1.3.5), which are only valid through $\mathcal{O}(\epsilon^0)$, are not sufficient for this purpose. However, for $n = 4, 5, 6$ the relation was verified in ref. [29], with the all-plus amplitudes given in eq. (1.3.7). The $N = 4$ amplitudes may be obtained from this equation by applying the dimension-shifting formula; the net effect on the integrals is to remove four powers of μ from their arguments.

For the all-plus amplitudes, the pentagon terms in eq. (1.3.7) do contribute as $\epsilon \to 0$, since these integrals are ultraviolet divergent at $D = 10$, canceling the ϵ prefactor in eq. (1.3.9). On the other hand, for the $N = 4$ amplitudes the pentagon and hexagon terms are finite and do not cancel the overall ϵ implied by the remaining μ^2 arguments. This leaves only box integrals in the expression (1.3.11) for the $N = 4$ MHV amplitudes in $D = 4 - 2\epsilon$ as $\epsilon \to 0$.

Eqs. (1.3.7) — or rather the cuts of eqs. (1.3.7) in various channels — will play a role in section 1.4 as we construct the analogous amplitudes in $N = 8$ supergravity and pure (or self-dual) gravity, using in part the KLT relations between tree amplitudes on either side of the cuts.

In direct analogy to eq. (1.3.15), one may conjecture a relation between one-loop MHV amplitudes in $N = 8$ supergravity, and all-plus amplitudes in pure gravity [29]:

$$\begin{aligned} &M_n(1^+, 2^+, \ldots, n^+) \\ &= -2\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)(4\pi)^4 \left[\frac{M_n^{N=8}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)}{\langle i\, j\rangle^8} \Bigg|_{D\to D+8} \right] \\ &= 2\frac{M_n^{N=8}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)[\mu^8]}{\langle i\, j\rangle^8} . \end{aligned} \tag{1.3.16}$$

This equation respects the heuristic relation 'gravity $\sim$ (gauge theory)^2', since the μ^4 gauge theory factor becomes a μ^8 factor in the gravity case. Here $M_n^{N=8}$ refers to an entire $N = 8$ multiplet circulating in the loop, while $M_n(1^+, 2^+, \ldots, n^+)$ gives the contribution of a graviton in the loop. For the same reasons as in the gauge case, this M_n could equally well be calculated with a massless scalar in the loop instead [61].

In ref. [29] eq. (1.3.16) was only verified for the simplest case, $n = 4$. In the following section we shall see that it holds for $n = 5$ and 6 as well, thus strengthening the all-n conjecture.

1.4 Cut Construction of One-Loop MHV $N = 8$ Supergravity and All-Plus Gravity Amplitudes

In this section we construct the one-loop n-point all-plus gravity amplitudes from their unitarity cuts, for $n \leq 6$. (The case of $n = 4$ has been computed previously in refs. [61, 62].) Then we exploit the gauge dimension-shifting relations of section 1.3.5 to obtain the $N = 8$ MHV supergravity amplitudes. These calculations will provide a firm basis from which we shall construct ansatze for an arbitrary number of external legs in section 1.6, using the soft and collinear behavior of gravity amplitudes to be discussed in section 1.5.

1.4.1 Brief Review of Cutting Method

The cutting method that we use has been discussed extensively for the case of gauge theory amplitudes, and reviewed in ref. [8], so we only briefly describe it. This is a proven technology. It has been used, for example, in the calculation of analytic expressions for the QCD one-loop helicity amplitudes for $Z \to 4$ partons [63] and in the construction of infinite sequences [27, 26, 49] of one-loop MHV amplitudes[2]. This technique allows for a complete reconstruction of the amplitudes from the cuts, provided that all cuts are known in arbitrary dimension. Because on-shell expressions are used throughout, gauge invariance, Lorentz covariance and unitarity are manifest.

The unitarity cuts of one-loop amplitudes are given simply by phase-space integrals of products of tree amplitudes, summing over all intermediate states that can cross the cut. For example, the cut in the channel carrying momentum $k_{m_1} + \cdots + k_{m_2}$

[2]The technique can also be applied at multi-loop level, as we will discuss in chapter 2.

for $M_n(1,2,\ldots,n)$ is given by

$$C_{m_1\ldots m_2} = i\sum_{\lambda_1,\lambda_2}\int d\text{LIPS}(-L_1,L_2)\; M^{\text{tree}}_{m_2-m_1+3}((-L_1)^{-\lambda_1},m_1,\ldots,m_2,L_2^{\lambda_2}) \times M^{\text{tree}}_{n+m_1-m_2+1}((-L_2)^{-\lambda_2},m_2+1,\ldots,m_1-1,L_1^{\lambda_1}), \tag{1.4.1}$$

where the integration is over the two-particle D-dimensional Lorentz-invariant phase-space, and $\lambda_{1,2}$ denote the helicity/particle-type of the states crossing the cut. (Polarization labels for the external graviton states have been suppressed.) One can replace [8] the phase-space integral with an unrestricted loop momentum integral $\int d^DL$, yet continue to apply the on-shell conditions $L_1^2 = L_2^2 = 0$, so long as one remembers that only functions with a cut in the given channel are reliably computed in this way. (The positive energy conditions are automatically imposed with the use of Feynman propagators.)

A principal advantage of the cutting approach for gauge theory calculations is that the tree amplitudes on either side of the cut can be simplified *before* attempting to evaluate the cut integral [8]. In the case of gravity, the KLT relations provide convenient representations of the tree amplitudes. In the supersymmetric case, on-shell supersymmetry Ward identities can also be used to reduce the amount of work required. (To maintain the supersymmetry cancellations the dimensional regularization scheme should not alter the number of states from their four-dimensional values [64].)

For the non-supersymmetric all-plus calculation, the SWI (1.2.4) allow us to replace gravitons or any other massless particles in the loop with massless scalars [61]. That is, at one loop we have,

$$M_n^{\text{any states}}(1^+,2^+,\ldots,n^+) = N_s M_n^{\text{scalar}}(1^+,2^+,\ldots,n^+), \tag{1.4.2}$$

where N_s is the number of bosonic states minus fermionic states circulating in the loop in $M_n^{\text{any states}}$. (We have taken the normalization of the 'scalar' amplitude to be that for a single real scalar state.) For scalars (or fermions) crossing the cuts, a detailed study of the effect of the D-dimensional loop momentum has previously been presented [9, 8, 29], and the requisite gauge theory tree amplitudes, where the scalar carries non-zero momenta in the extra (-2ϵ) dimensions, have been computed for $n \leq 6$ [29]. We shall obtain the scalar+graviton tree amplitudes from the scalar+gluon tree amplitudes using the KLT relations. Thus we shall be able to directly evaluate the cuts for the all-plus amplitudes to all orders in ϵ, which in turn gives the full amplitudes to all orders in ϵ [65, 8, 9].

One good way to obtain the $N = 8$ MHV amplitudes with up to six legs, through $\mathcal{O}(\epsilon^0)$, is to make use of a 'cut constructible' criterion that allows one to use four-dimensional momenta in the cuts without introducing any errors in $\mathcal{O}(\epsilon^0)$ rational functions, assuming that certain power counting criteria are satisfied [49]. In the present case, however, we can avoid explicit computations of the $N = 8$ MHV cuts by instead obtaining the $N = 8$ amplitudes from the all-plus amplitudes using the gravitational version (1.3.16) of the gauge theory 'dimension-shifting' results presented in section 1.3.5. We now explain how the gravitational relation can be derived from the gauge theory one.

1.4.2 Dimension-Shifting Relations between Gravity Cuts

In ref. [29] the dimension-shifting formula (1.3.16) was shown to hold for the four-graviton amplitudes, and was conjectured to hold for n-point amplitudes. Here we demonstrate that it does hold at n-points, if the gauge theory relation (1.3.15) holds at n-points. Since the latter relation has been proven for $n \leq 6$, this establishes the gravity dimension-shifting relation (1.3.16) up to six points, but leaves the $n \geq 7$ cases as a conjecture.

We begin with the cuts of the gauge theory dimension-shifting formula (1.3.15),

$$\begin{aligned}
&A^{\text{tree}}_{m+2}(-L^s_1, i^+_1, i^+_2, \ldots, L^s_2, \ldots, i^+_m) \times A^{\text{tree}}_{n-m+2}(-L^s_2, i^+_{m+1}, i^+_{m+2}, \ldots, L^s_1, \ldots, i^+_n) \\
&= \frac{\mu^4}{\langle i\, j\rangle^4} \sum_{N=4 \text{ states}} A^{\text{tree}}_{m+2}(-L_1, i^+_1, i^+_2, \ldots, L_2, \ldots, i^+_m) \\
&\qquad\qquad \times A^{\text{tree}}_{n-m+2}(-L_2, i^+_{m+1}, i^+_{m+2}, \ldots, L_1, \ldots, i^+_n)\,,
\end{aligned} \tag{1.4.3}$$

where the superscript s denotes a scalar line, the sum on the right-hand side runs over all $N = 4$ super-Yang-Mills states that can cross the cut, and we have suppressed the $\lambda_{1,2}$ helicity/state labels for these states. (The reason the '2' in eq. (1.3.15) has disappeared from eq. (1.4.3) is that $A_{n;1}$ in the former equation corresponds to 2 real scalars circulating in the loop.) Eq. (1.3.15) was originally derived in its loop-momentum integrated version. Nevertheless, it turns out that the manipulations used in ref. [29] to verify the relations between the amplitudes can be arranged so as not to introduce any total derivatives. This means that eq. (1.3.15) holds point-by-point in the integrands.

Also, the cuts of eq. (1.3.15) (a leading-in-N_c equation) only correspond directly to the configurations where the cut loop momenta L_1 and L_2 are adjacent, i.e. $A^{\rm tree}_{m+2}(-L_1^s, i_1^+, \ldots, i_m^+, L_2^s) \times A^{\rm tree}_{n-m+2}(-L_2^s, i_{m+1}^+, \ldots, i_n^+, L_1^s)$ on the left-hand side of eq. (1.4.3). However, it is possible to obtain all the other permutations of this equation, by using the following relation among color-ordered tree amplitudes [66],

$$A^{\rm tree}(1, \{\alpha\}, 2, \{\beta\}) = (-1)^{n_\beta} \sum_{\sigma \in OP\{\alpha\}\{\beta^T\}} A^{\rm tree}(1, \sigma(\{\alpha\}\{\beta^T\}), 2)\,, \tag{1.4.4}$$

where n_β is the number of elements in $\{\beta\}$, the set β^T is β with the ordering reversed, and $OP\{\alpha\}\{\beta^T\}$ is the set of all permutations of $\{\alpha\} \cup \{\beta^T\}$ that preserve the ordering of elements within each of the two sets. Eq. (1.4.4) can be inserted twice each into the left- and right-hand sides of eq. (1.4.3), in order to reduce the general case to the case where L_1 and L_2 are adjacent.

We may now use the n-point KLT equation (A.1) to rewrite the cuts of the all-plus gravity amplitudes in terms of gauge theory cuts,

$$\begin{aligned}
&M^{\rm tree}_{m+2}(-L_1^s, 1^+, 2^+, \ldots, m^+, L_2^s) \times M^{\rm tree}_{n-m+2}(-L_2^s, (m+1)^+, \ldots, n^+, L_1^s) \\
&\quad = \Big(\sum_{\rm perms} f \bar{f}\, A^{\rm tree}_{m+2}(-L_1^s, 1^+, 2^+, \ldots, m^+, L_2^s)\; A^{\rm tree}_{m+2}(i_1^+, \ldots, -L_1^s, m^+, i_{m-1}^+, \ldots, L_2^s) \Big) \\
&\qquad \times \Big(\sum_{\rm perms'} f' \bar{f}'\, A^{\rm tree}_{n-m+2}(-L_2^s, (m+1)^+, \ldots, n^+, L_1^s) \\
&\qquad\qquad \times\; A^{\rm tree}_{n-m+2}(i_{m+1}^+, \ldots, -L_2^s, n^+, i_{n-1}^+, \ldots, L_1^s) \Big)\,,
\end{aligned} \tag{1.4.5}$$

where 'perms' and 'perms′' stand for the full sum over KLT permutations in eq. (A.1), and f, $\bar{f}$, f' and $\bar{f}'$ are the functions f, $\bar{f}$ defined in eq. (A.2), for the appropriate sets of arguments. (The KLT equations hold in any dimension $D \leq 10$ where string constructions exist and may be analytically continued to arbitrary dimensions.) After rearranging the right-hand side of eq. (1.4.5) and applying the cut version of the gauge theory dimension-shifting formula (1.4.3), we obtain

$$\begin{aligned}
&M_{m+2}^{\text{tree}}(-L_1^s, 1^+, 2^+, \ldots, m^+, L_2^s) \times M_{n-m+2}^{\text{tree}}(-L_2^s, (m+1)^+, \ldots, n^+, L_1^s) \\
&\quad = \sum_{\text{perms}} \sum_{\text{perms}'} f\bar{f} f'\bar{f}' \\
&\quad \times [A_{m+2}^{\text{tree}}(-L_1^s, 1^+, 2^+, \ldots, m^+, L_2^s)\ A_{n-m+2}^{\text{tree}}(-L_2^s, (m+1)^+, \ldots, n^+, L_1^s)] \\
&\quad \times [A_{m+2}^{\text{tree}}(i_1^+, \ldots, -L_1^s, m^+, i_{m-1}^+, \ldots, L_2^s)\ A_{n-m+2}^{\text{tree}}(i_{m+1}^+, \ldots, -L_2^s, n^+, i_{n-1}^+, \ldots, L_1^s)] \\
&\quad = \sum_{\text{perms}} \sum_{\text{perms}'} f\bar{f} f'\bar{f}' \\
&\quad \times \sum_{N=4\ \text{states}} \left[\frac{\mu^4}{\langle i\, j\rangle^4} A_{m+2}^{\text{tree}}(-L_1, 1^+, 2^+, \ldots, m^+, L_2) \right. \\
&\qquad\qquad \left. \times\ A_{n-m+2}^{\text{tree}}(-L_2, (m+1)^+, \ldots, n^+, L_1)\right] \\
&\quad \times \sum_{N=4\ \text{states}} \left[\frac{\mu^4}{\langle i\, j\rangle^4} A_{m+2}^{\text{tree}}(i_1^+, \ldots, -L_1, m^+, i_{m-1}^+, \ldots, L_2) \right. \\
&\qquad\qquad \left. \times\ A_{n-m+2}^{\text{tree}}(i_{m+1}^+, \ldots, -L_2, n^+, i_{n-1}^+, \ldots, L_1)\right] .
\end{aligned} \tag{1.4.6}$$

As discussed in section 1.2.1, in terms of a $D = 4$ decomposition of states the double sum over the 16 $N = 4$ super-Yang-Mills states may be reassembled as a single sum over the 256 $N = 8$ supergravity states. (In higher dimensions up to $D = 10$, the sum over $N = 4$ states can be reassembled into a sum over the appropriate multiplet in the higher dimensional theory.) This yields,

$$\begin{aligned}
&M_{m+2}^{\text{tree}}(-L_1^s, 1^+, \ldots, m^+, L_2^s) \times M_{n-m+2}^{\text{tree}}(-L_2^s, (m+1)^+, \ldots, n^+, L_1^s) \\
&\quad = \frac{\mu^8}{\langle i\, j\rangle^8} \sum_{N=8\ \text{states}} M_{m+2}^{\text{tree}}(-L_1, 1^+, 2^+, \ldots, m^+, L_2) \\
&\qquad\qquad \times M_{n-m+2}^{\text{tree}}(-L_2, (m+1)^+, \ldots, n^+, L_1) .
\end{aligned} \tag{1.4.7}$$

Note that the precise details of which permutation sums are included, or what exactly the f functions are, are unimportant in the derivation of eq. (1.4.7), because the same permutations and f functions appear in the KLT expressions for any of the tree amplitudes appearing in the cuts, regardless of the particle type of the states crossing the cut. Since eq. (1.4.7) can be used to obtain all cuts of the amplitudes and is valid to all orders in ϵ, the gravity dimension-shifting relation (1.3.16) is established for all values of n for which eq. (1.4.3) has been proven (currently this is for $n \leq 6$).

The relationships between the infinite sequences of one-loop all-plus (or self-dual) amplitudes and MHV amplitudes in maximally supersymmetric theories, and between gravity and gauge theory, are summarized in fig. 1.2. The horizontal arrows correspond to the gauge-gravity relations that follow from the KLT equations, and the vertical arrows represent the dimension-shifting relations.

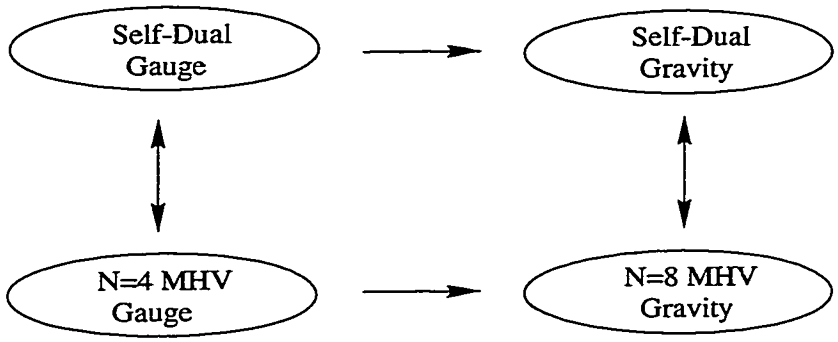


Figure 1.2: Relations between infinite sequences of one-loop amplitudes in four different theories. The vertical arrows correspond to the 'dimension-shifting' relations of ref. [29], within (super-) Yang-Mills theory and (super-) gravity. (These remain a conjecture for $n \geq 7$ legs.) The horizontal arrows correspond to the gauge-gravity relations which follow from the KLT equations.

Using eq. (1.3.16) it is then sufficient to calculate the all-plus graviton amplitudes as a function of ϵ (i.e., for arbitrary $D = 4 - 2\epsilon$), in order to obtain the $N = 8$ supergravity MHV amplitudes.

1.4.3 All-Plus Amplitudes for $n \leq 6$

As a warmup we first calculate the four-graviton all-plus amplitude, before proceeding to the five- and six-graviton cases. Using eq. (1.4.2) we can replace the graviton in the loop with two real scalars. Thus, the cut in the s_{12} channel is

$$M_4(1^+, 2^+, 3^+, 4^+)\Big|_{s_{12}-\text{cut}} = \int \frac{d^D L_1}{(2\pi)^D} \frac{i}{L_1^2} M_4^{\text{tree}}(-L_1^s, 1^+, 2^+, L_3^s) \frac{i}{L_3^2} M_4^{\text{tree}}(-L_3^s, 3^+, 4^+, L_1^s)\Big|_{s_{12}-\text{cut}}, \tag{1.4.8}$$

where the superscript s denotes that the cut lines are scalars, and $L_3 = L_1 - k_1 - k_2$. An overall 2, from two real scalars propagating in the loop, is canceled by an identical-particle phase-space factor of 1/2. Using the KLT expressions (1.2.2) we may replace the gravity tree amplitudes appearing in the cuts with products of gauge theory amplitudes. The required gauge theory tree amplitudes, with two external scalar legs and two gluons, are relatively simple to obtain [9, 8],

$$\begin{aligned} A_4^{\text{tree}}(-L_1^s, 1^+, 2^+, L_3^s) &= i \frac{\mu^2 [1\,2]}{\langle 1\,2\rangle \left[(\ell_1 - k_1)^2 - \mu^2\right]}, \\ A_4^{\text{tree}}(-L_1^s, 1^+, L_3^s, 2^+) &= -i \frac{\mu^2 [1\,2]}{\langle 1\,2\rangle} \left[\frac{1}{(\ell_1 - k_1)^2 - \mu^2} + \frac{1}{(\ell_1 - k_2)^2 - \mu^2}\right], \end{aligned} \tag{1.4.9}$$

where the gluon momenta are four-dimensional, but the scalar momenta are allowed to have a (-2ϵ)-dimensional component $\vec{\mu}$, with $\vec{\mu} \cdot \vec{\mu} = \mu^2 > 0$. The overall factor of μ^2 appearing in these tree amplitudes means that they vanish in the four-dimensional limit, in accord with the SWI (1.2.4). In the KLT relation (1.2.2), one of the propagators cancels, leaving

$$M_4(-L_1^s, 1^+, 2^+, L_3^s) = -i \left(\frac{\mu^2 [1\,2]}{\langle 1\,2\rangle}\right)^2 \left[\frac{1}{(\ell_1 - k_1)^2 - \mu^2} + \frac{1}{(\ell_1 - k_2)^2 - \mu^2}\right]. \tag{1.4.10}$$

Inserting eq. (1.4.10) and the permuted formula for $M_4(-L_3^s, 3^+, 4^+, L_1^s)$ into the cut (1.4.8) yields

$$
\begin{aligned}
&M_4(1^+,2^+,3^+,4^+)|_{s_{12}-\text{cut}} \\
&= \frac{[1\,2]^2\,[3\,4]^2}{\langle 1\,2\rangle^2\,\langle 3\,4\rangle^2} \int \frac{d^4\ell}{(2\pi)^4} \int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\, \mu^8\, \frac{1}{\ell^2-\mu^2} \left[\frac{1}{(\ell-k_1)^2-\mu^2} + \frac{1}{(\ell-k_2)^2-\mu^2}\right] \\
&\qquad \times \frac{1}{(\ell-k_1-k_2)^2-\mu^2} \left[\frac{1}{(\ell+k_4)^2-\mu^2} + \frac{1}{(\ell+k_3)^2-\mu^2}\right]\Bigg|_{s_{12}-\text{cut}},
\end{aligned}
\tag{1.4.11}
$$

which corresponds to a sum of $(12-2\epsilon)$-dimensional scalar integrals, using (1.3.9). By symmetry, the other cuts are the same up to relabelings. Combining all three cuts into a single function that has the correct cuts in all channels yields

$$
M_4(1^+,2^+,3^+,4^+) = 2\frac{[1\,2]^2\,[3\,4]^2}{\langle 1\,2\rangle^2\,\langle 3\,4\rangle^2}\left(\mathcal{I}_4^{1234}[\mu^8] + \mathcal{I}_4^{3124}[\mu^8] + \mathcal{I}_4^{2314}[\mu^8]\right), \tag{1.4.12}
$$

where

$$
\begin{aligned}
&\mathcal{I}_4^{1234}[\mu^8] = \\
&\int \frac{d^4\ell}{(2\pi)^4} \int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\, \mu^8\, \frac{1}{[\ell^2-\mu^2][(\ell-k_1)^2-\mu^2][(\ell-k_1-k_2)^2-\mu^2][(\ell+k_4)^2-\mu^2]}
\end{aligned}
\tag{1.4.13}
$$

is the scalar box integral with the external legs arranged in the order 1234. The two other scalar integrals that appear correspond to the two other distinct orderings of the four external legs. (See appendix B for our notation for general one-loop integrals.) Using eq. (1.4.2), this result can be applied to any set of massless fields circulating in the loop.

The spinor factor $[1\,2]^2\,[3\,4]^2/(\langle 1\,2\rangle^2\,\langle 3\,4\rangle^2)$ in eq. (1.4.12) is actually completely symmetric, although not manifestly so. By rewriting this factor and using eq. (B.9) for the box integral, the final one-loop result in $D=4$ is

$$
M_4(1^+,2^+,3^+,4^+) = -\frac{i}{(4\pi)^2}\left(\frac{st}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\right)^2 \frac{s^2+t^2+u^2}{120} + \mathcal{O}(\epsilon), \tag{1.4.14}
$$

in agreement with a previous calculation [62].

For the purpose of constructing an ansatz for $n \geq 7$, it is useful to write the $n = 4$ amplitude as

$$M_4(1^+,2^+,3^+,4^+) = -\frac{i}{(4\pi)^2}\frac{1}{480}\Big[h(1,\{3\},2)h(2,\{4\},1)\ \mathrm{tr}^3[1324] + h(1,\{2\},3)h(3,\{4\},1)\ \mathrm{tr}^3[1234] + h(1,\{2\},4)h(4,\{3\},1)\ \mathrm{tr}^3[1243]\Big] + \mathcal{O}(\epsilon)\,, \tag{1.4.15}$$

where $\mathrm{tr}[i_1 i_2 i_3 i_4] \equiv \mathrm{tr}[\not{k}_{i_1}\not{k}_{i_2}\not{k}_{i_3}\not{k}_{i_4}]$ and

$$h(a,\{1\},b) = \frac{1}{\langle a\,1\rangle^2\,\langle 1\,b\rangle^2}\,. \tag{1.4.16}$$

Next we compute $M_5(1^+,2^+,3^+,4^+,5^+)$ to all orders in ϵ. Its total symmetry implies that the s_{12} cut again suffices for its complete reconstruction. Thus we require the tree amplitudes for two scalars and either two or three gravitons, $M_4^{\rm tree}(-L_1^s,1^+,2^+,L_3^s)$ from eq. (1.4.10), and $M_5^{\rm tree}(-L_3^s,3^+,4^+,5^+,L_1^s)$, which may be constructed from the gauge amplitudes for two scalars and three gluons [29],

$$\begin{aligned} A_5^{\rm tree}(L_1^s,-L_3^s,3^+,4^+,5^+) &= i\,\mu^2\,\frac{\langle 5^+|(3+4)\ell_3|3^-\rangle}{(L_3-k_3)^2\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,(L_1+k_5)^2}\,,\\ A_5^{\rm tree}(L_1^s,3^+,-L_3^s,4^+,5^+) &= -A_5^{\rm tree}(L_1^s,-L_3^s,3^+,4^+,5^+) - A_5^{\rm tree}(L_1^s,-L_3^s,4^+,3^+,5^+)\\ &\quad - A_5^{\rm tree}(L_1^s,-L_3^s,4^+,5^+,3^+)\,, \end{aligned} \tag{1.4.17}$$

using the five-point KLT relation (1.2.2). The second equation (a special case of eq. (1.4.4)) follows from the $U(1)$ decoupling identity [43, 67, 46].

After applying several spinor-product identities to the right-hand side of the five-point KLT relation, we obtain the manifestly symmetric form

$$\begin{aligned} M_5^{\rm tree}(-L_3^s,3^+,4^+,5^+,L_1^s) = -i\frac{\mu^4}{\langle 3\,4\rangle^2\,\langle 3\,5\rangle^2\,\langle 4\,5\rangle^2}\Bigg[&\frac{s_{34}s_{45}}{(L_3-k_3)^2(L_1+k_5)^2}\\ &\times\Big(\langle 5^-|3(4+5)\ell_1|5^-\rangle + \mu^2 s_{35}\Big) + \mathcal{P}(3,4,5)\Bigg]\,. \end{aligned} \tag{1.4.18}$$

Inserting eqs. (1.4.10) and (1.4.18) into the s_{12}-channel cut of the five-point all-plus amplitude gives

$$M_5(1^+,2^+,\ldots,5^+)|_{s_{12}\text{-cut}} = \frac{[1\,2]^2}{\langle 1\,2\rangle^2\,\langle 3\,4\rangle^2\,\langle 3\,5\rangle^2\,\langle 4\,5\rangle^2}\int \frac{d^DL}{(2\pi)^D}\,\mu^8 \times\left\{\frac{1}{L_1^2L_2^2L_3^2}\left[\frac{s_{34}s_{45}}{L_4^2L_5^2}\Big(\langle 5^-|3(4+5)\ell_1|5^-\rangle+\mu^2 s_{35}\Big) + \mathcal{P}(3,4,5)\right] + \mathcal{P}(1,2)\right\}\Bigg|_{s_{12}\text{-cut}}, \tag{1.4.19}$$

where $L_i = L - k_1 - \cdots - k_{i-1}$ and $L_i^2 = \ell_i^2 - \mu^2$. The $L_1^2L_2^2L_3^2L_4^2L_5^2$ denominator factors signal the presence of a pentagon integral with a 12345 cyclic ordering of external legs. All other orderings appearing in the s_{12} cut are generated by the permutation sums.

By using standard integration formulas, and combining all cuts into a single function, we find that the five-point all-plus gravity amplitude may be put into a form similar to the all-plus gauge amplitude (eq. (1.3.5)),

$$M_5(1^+,2^+,\ldots,5^+) = \beta_{123(45)}\,\mathcal{I}_4^{123(45)}[\mu^8] - 2\,\frac{[1\,2]\,[2\,3]\,[3\,4]\,[4\,5]\,[5\,1]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\mathcal{I}_5^{12345}[\mu^{10}] + \text{perms}, \tag{1.4.20}$$

where the permutation sum is over all distinct one-mass box integrals and massless pentagon integrals (no cyclic ordering is imposed, in contrast to the gauge case). There are 30 different box-integral terms and 12 pentagons in the sum. The box coefficient is

$$\beta_{123(45)} = -\frac{[1\,2]^2\,[2\,3]^2\,[4\,5]}{\langle 1\,4\rangle\,\langle 1\,5\rangle\,\langle 3\,4\rangle\,\langle 3\,5\rangle\,\langle 4\,5\rangle}\,. \tag{1.4.21}$$

We can rewrite this coefficient in terms of the one given for the five-point all-plus gauge amplitude in eq. (1.3.7) (or equivalently, for the MHV amplitude in $N = 4$ super-Yang-Mills theory in eq. (1.3.12)):

$$\begin{aligned}\beta_{123(45)} &= -s_{45}\,\frac{s_{12}\,s_{23}}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\,\frac{s_{12}\,s_{23}}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,5\rangle\,\langle 5\,4\rangle\,\langle 4\,1\rangle}\\ &= -s_{45}\,\alpha_{123(45)}\alpha_{123(54)}\,.\end{aligned} \tag{1.4.22}$$

This relation is reminiscent of the tree-level four-point KLT relation in eq. (1.2.2), in that

(1) a quantity in gravity is expressed as a product of gauge quantities,

(2) one s_{ij} appears as a prefactor on the gauge side of the relation, and

(3) the indices on the s_{ij} coincide with the arguments which are permuted between the two gauge-theory factors.

Through $\mathcal{O}(\epsilon^0)$ the expression for the amplitude can be simplified considerably by non-trivial rearrangements to yield,

$$\begin{aligned} M_5(1^+,2^+,3^+,4^+,5^+) \\ &= \frac{i}{(4\pi)^2}\frac{1}{960}\, h(1,\{2\},3)\, h(3,\{4,5\},1)\, \mathrm{tr}^3[123(4+5)] \;+\; \mathrm{perms} + \mathcal{O}(\epsilon), \end{aligned} \tag{1.4.23}$$

where $\mathrm{tr}[\cdots(i+j)\cdots] \equiv \mathrm{tr}[\cdots(\not{k}_i+\not{k}_j)\cdots]$, $h(a,\{1\},b)$ is defined in eq. (1.4.16),

$$h(a,\{1,2\},b) = \frac{[1\,2]}{\langle 1\,2\rangle\,\langle a\,1\rangle\,\langle 1\,b\rangle\,\langle a\,2\rangle\,\langle 2\,b\rangle}\,, \tag{1.4.24}$$

and the sum is over $10\times 3 = 30$ distinct permutations. (There are $\binom{5}{2} = 10$ possible choices for the pair of arguments in braces in the second h function, and for each of these there are 3 more choices for the argument in braces in the first h function.)

We have also obtained the six-point all-plus amplitude to all orders in ϵ from the cuts. The result of this computation is

$$\begin{aligned} M_6(1^+,\ldots,6^+) = \beta_{1(23)4(56)}\,\mathcal{I}_4^{1(23)4(56)}[\mu^8] + \beta_{123(456)}\,\mathcal{I}_4^{123(456)}[\mu^8] \\ + \rho_{1234(56)}\,\mathcal{I}_5^{1234(56)}[\mu^{10}] + \mathrm{perms}, \end{aligned} \tag{1.4.25}$$

where the sum over permutations again runs over all distinct integral functions. The coefficients of the integrals are

$$\begin{aligned}
\beta_{1(23)4(56)} &= \frac{[2\,3]}{\langle 2\,3\rangle}\frac{[5\,6]}{\langle 5\,6\rangle}\frac{\langle 1^-|(2+3)|4^-\rangle^2\,\langle 4^-|(5+6)|1^-\rangle^2}{\langle 1\,2\rangle\,\langle 2\,4\rangle\,\langle 1\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle\,\langle 4\,6\rangle\,\langle 6\,1\rangle}\,,\\
\beta_{123(456)} &= \frac{[1\,2]^2\,[2\,3]^2}{\langle 1\,4\rangle\,\langle 4\,3\rangle\,\langle 1\,5\rangle\,\langle 5\,3\rangle\,\langle 1\,6\rangle\,\langle 6\,3\rangle}\\
&\quad\times\left(\langle 1\,4\rangle\,\langle 4\,3\rangle\,\frac{[4\,5]\,[4\,6]}{\langle 4\,5\rangle\,\langle 4\,6\rangle}+\langle 1\,5\rangle\,\langle 5\,3\rangle\,\frac{[4\,5]\,[5\,6]}{\langle 4\,5\rangle\,\langle 5\,6\rangle}+\langle 1\,6\rangle\,\langle 6\,3\rangle\,\frac{[4\,6]\,[5\,6]}{\langle 4\,6\rangle\,\langle 5\,6\rangle}\right)\,,\\
\rho_{1234(56)} &= 2\,\frac{s_{12}\,s_{23}\,s_{34}\langle 1^-|(2+3)|4^-\rangle\langle 4^-|(5+6)|1^-\rangle}{\mathrm{tr}_5[1234]}\Big[c_1+c_2+c_2|_{5\leftrightarrow 6}\Big]\,,
\end{aligned} \tag{1.4.26}$$

where $\mathrm{tr}_5[\cdots]\equiv\mathrm{tr}[\gamma_5\cdots]$ and

$$\begin{aligned}
c_1 &= \frac{\mathrm{tr}_+[1234]\;\;[5\,6]}{\langle 1\,2\rangle^2\,\langle 2\,3\rangle^2\,\langle 3\,4\rangle^2\,\langle 4\,5\rangle\,\langle 5\,1\rangle\,\langle 4\,6\rangle\,\langle 6\,1\rangle\,\langle 5\,6\rangle}\,,\\
c_2 &= \frac{1}{\mathrm{tr}_5[123456]}\frac{[1\,2]\,[2\,3]\,[3\,4]\,[4\,5]\,[5\,6]\,[6\,1]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle}\,.
\end{aligned} \tag{1.4.27}$$

Again the expression for the amplitude through $\mathcal{O}(\epsilon^0)$ can be simplified, to

$$\begin{aligned}
M_6(1^+,2^+,3^+,4^+,5^+,6^+) &= -\frac{i}{(4\pi)^2}\frac{1}{960}\Big\{h(1,\{2,3\},4)h(4,\{5,6\},1)\\
&\times\mathrm{tr}^3[1(2+3)4(5+6)]+h(1,\{2\},3)h(3,\{4,5,6\},1)\,\mathrm{tr}^3[123(4+5+6)]+\text{perms}\Big\}\,,
\end{aligned} \tag{1.4.28}$$

where the permutation sum is over distinct terms and the new h function that appears is

$$\begin{aligned}
h(a,\{1,2,3\},b) &= \frac{[1\,2]\,[2\,3]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle a\,1\rangle\,\langle 1\,b\rangle\,\langle a\,3\rangle\,\langle 3\,b\rangle}+\frac{[2\,3]\,[3\,1]}{\langle 2\,3\rangle\,\langle 3\,1\rangle\,\langle a\,2\rangle\,\langle 2\,b\rangle\,\langle a\,1\rangle\,\langle 1\,b\rangle}\\
&\quad+\frac{[3\,1]\,[1\,2]}{\langle 3\,1\rangle\,\langle 1\,2\rangle\,\langle a\,3\rangle\,\langle 3\,b\rangle\,\langle a\,2\rangle\,\langle 2\,b\rangle}\,.
\end{aligned} \tag{1.4.29}$$

1.4.4 $N = 8$ MHV Amplitudes from Dimension-Shifting

By using the gravitational dimension-shifting relation (1.3.16), we may obtain the $N = 8$ four-, five-, and six-point MHV amplitudes from the all-plus (self-dual) amplitudes (1.4.12), (1.4.20), and (1.4.25), by dividing out a factor of μ^8 from each integrand and multiplying by an overall factor of $\langle i\,j\rangle^8/2$, where i and j are the two negative helicity legs. After removing a factor of μ^8, the pentagon integrals are no longer ultraviolet divergent and are suppressed by an overall power of ϵ near $D = 4$, since a power of μ^2 remains. Hence, the $N = 8$ amplitudes through $\mathcal{O}(\epsilon^0)$ are given just by the box integral contributions.

A representation of the four-, five- and six-point amplitudes which is convenient for extending the result to an arbitrary number of external legs is

$$
\begin{aligned}
&M_4^{N=8}(1^-,2^-,3^+,4^+) = \\
&\qquad \frac{1}{4}\langle 1\,2\rangle^8 \Big[h(1,\{2\},3)h(3,\{4\},1)\,\mathrm{tr}^2[1234]\,\mathcal{I}_4^{1234} + \text{perms}\Big] + \mathcal{O}(\epsilon)\,, \\
&M_5^{N=8}(1^-,2^-,3^+,4^+,5^+) \\
&\qquad = -\frac{1}{8}\langle 1\,2\rangle^8 \Big[h(1,\{2\},3)h(3,\{4,5\},1)\,\mathrm{tr}^2[123(4+5)]\,\mathcal{I}_4^{123(45)} \\
&\qquad\qquad + \text{perms}\Big] + \mathcal{O}(\epsilon)\,, \\
&M_6^{N=8}(1^-,2^-,3^+,4^+,5^+,6^+) = \\
&\qquad \frac{1}{8}\langle 1\,2\rangle^8 \Big[h(1,\{2\},3)h(3,\{4,5,6\},1)\,\mathrm{tr}^2[123(4+5+6)]\,\mathcal{I}_4^{123(456)} \\
&\qquad\qquad + h(1,\{2,3\},4)h(4,\{5,6\},1)\,\mathrm{tr}^2[1(2+3)4(5+6)]\,\mathcal{I}_4^{1(23)4(56)} \\
&\qquad\qquad + \text{perms}\Big] + \mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{1.4.30}
$$

where the permutation sums are over all distinct permutations.

As a check, we have explicitly calculated the cuts of the $N = 8$ MHV supergravity amplitudes up to six legs in $D = 4$ (i.e., through $\mathcal{O}(\epsilon^0)$). We find complete agreement with the results (1.4.30) obtained via eq. (1.3.16). This cut calculation is similar to the one performed for the all-plus amplitudes in section 1.4.3, and makes use of the KLT relations (1.2.2) to express the the gravity tree amplitudes appearing in the cuts in terms of gauge theory amplitudes.

1.4.5 Power-Counting for $N = 8$ MHV Amplitudes vs. $N = 4$

Let us compare the structure of the $N = 8$ MHV results (1.4.30) with general expectations from loop-momentum power-counting. First recall from section 1.3.4 that in a one-loop amplitude in $N = 4$ super-Yang-Mills theory, a maximum of $m - 4$ powers of loop momentum can appear in the numerator of each m-point integral. In a string-based approach [68, 62], the loop-momentum integrand for $N = 8$ supergravity is just the product of two $N = 4$ integrands. Therefore one expects a maximum of $2(m-4)$ powers of loop momentum to appear in the numerator of an m-point integral for $N = 8$ supergravity. After carrying out the same integral reductions sketched in section 1.3.4, this power-counting allows for box integrals with up to $n - 4$ powers of loop momentum in the numerator [69], for an n-point amplitude. Such integrals can be reduced to scalar box integrals, but (for $n \geq 5$) only at the expense of introducing scalar triangle and perhaps bubble integrals. On the other hand, we find no such integrals in eq. (1.4.30), only scalar box integrals. Nor will we find any need for integrals besides scalar boxes in the all-n ansatz in section 1.6.

In other words, all the $N = 8$ MHV amplitudes are consistent with having at most $(m - 4)$ powers of loop momentum for each m-point integral, instead of the $2(m - 4)$ powers expected from 'squaring' gauge theory. The additional cancellations we find in one-loop MHV amplitudes in $N = 8$ supergravity amplitudes for $n \geq 5$ legs presumably arise from sums over different orderings of external legs. It would be interesting to know whether they can be understood at the Lagrangian level, or in a string-based framework, and whether they might extend to non-MHV helicity configurations as well, or to theories with less supersymmetry.

1.5 Soft and Collinear Behavior of Gravity Amplitudes

In order to extend the results of the previous section beyond the six-point level we will make use of the analytic behavior of gravity amplitudes as momenta become soft or collinear. A feature that the all-plus gravity and $N = 8$ MHV amplitudes have in common with the all-plus gauge and $N = 4$ MHV amplitudes is the absence of multi-particle kinematic poles. (This may be demonstrated using the SWI (1.2.4), which implies the vanishing of each product of amplitudes, tree×loop, that forms the residue of a multi-particle pole.) It is therefore sufficient to focus on the soft and collinear limits, which determine the two-particle poles. We perform our analysis in Minkowski space-time with signature (1,3).

The behavior of tree-level gravity amplitudes as momenta become soft is well known [25, 24], and was reviewed in section 1.2.2. However, the behavior as momenta become collinear is more subtle.[3] In the following subsection we obtain the graviton collinear splitting amplitudes from the gauge theory ones using the KLT expressions (1.2.2). We then show that these splitting amplitudes are universal: they apply to collinear limits of amplitudes with an arbitrary number of external legs. Furthermore, we shall argue in section 1.5.2 that the tree-level soft functions and collinear splitting amplitudes suffer no higher loop corrections. That is, we shall show that at *any* loop order (including tree level) a gravity amplitude behaves as

$$M_n^{\rm loop}(\ldots, a^{\lambda_a}, b^{\lambda_b}, \ldots) \stackrel{a \parallel b}{\longrightarrow} \sum_{\lambda} {\rm Split}^{\rm gravity}_{-\lambda}(z, a^{\lambda_a}, b^{\lambda_b}) \times M_{n-1}^{\rm loop}(\ldots, P^{\lambda}, \ldots)\,, \quad (1.5.1)$$

when k_a and k_b are collinear, and as

$$M_n^{\rm loop}(\ldots, a, s^{\pm}, b, \ldots) \stackrel{k_s \to 0}{\longrightarrow} \mathcal{S}^{\rm gravity}(s^{\pm}) \times M_{n-1}^{\rm loop}(\ldots, a, b, \ldots)\,, \quad (1.5.2)$$

when k_s becomes soft.

[3]The suggestion that collinear limits in gravity are universal was made by Chalmers and Siegel [36].

1.5.1 Collinear Behavior of Gravity from Gauge Theory

Assuming that the collinear behavior of graviton amplitudes is universal, the splitting amplitudes in eq. (1.5.1) can be computed in terms of the gauge splitting amplitudes, using the four- and five-point KLT relations. Taking 1 || 2 in the five-point gravity amplitude (1.2.2) and applying eq. (1.2.17) we have,

$$\begin{aligned}\mathrm{Split}^{\mathrm{gravity}}_{-(\lambda+\tilde{\lambda})}(z, 1^{\lambda_1+\tilde{\lambda}_1}, 2^{\lambda_2+\tilde{\lambda}_2}) = -s_{12} \times \mathrm{Split}^{\mathrm{tree}}_{-\lambda}(z, 1^{\lambda_1}, 2^{\lambda_2}) \\ \times \mathrm{Split}^{\mathrm{tree}}_{-\tilde{\lambda}}(z, 2^{\tilde{\lambda}_2}, 1^{\tilde{\lambda}_1}) ,\end{aligned} \tag{1.5.3}$$

where $\mathrm{Split}^{\mathrm{gravity}}_{-\lambda}$ is a tree-level gravity splitting amplitude and the $\mathrm{Split}^{\mathrm{tree}}_{-\lambda}$ are gauge theory splitting amplitudes, such as those given in eqs. (1.2.18) and (1.2.19).

Equation (1.5.3) may be applied to arbitrary $N = 8$ supergravity states by factorizing them into products of states in $N = 4$ gauge theory, as discussed in section 1.2.1; the addition of helicities in the equation, $\lambda_i + \tilde{\lambda}_i$, corresponds to this factorization. For example, the pure graviton splitting amplitudes are obtained by substituting the values of the pure gluon splitting amplitudes (1.2.18) into eq. (1.5.3), yielding

$$\begin{aligned}\mathrm{Split}^{\mathrm{gravity}}_{+}(z, a^+, b^+) &= 0 , \\ \mathrm{Split}^{\mathrm{gravity}}_{-}(z, a^+, b^+) &= -\frac{1}{z(1-z)} \frac{[a\, b]}{\langle a\, b\rangle} , \\ \mathrm{Split}^{\mathrm{gravity}}_{+}(z, a^-, b^+) &= -\frac{z^3}{1-z} \frac{[a\, b]}{\langle a\, b\rangle} .\end{aligned} \tag{1.5.4}$$

As a second example, the splitting amplitudes for a graviton into two gravitinos ($h^+ \to \tilde{h}^- \tilde{h}^+$) follow from eqs. (1.5.3), (1.2.18) and (1.2.19),

$$\mathrm{Split}^{\mathrm{gravity}}_{+}(z, a^-_{\tilde{h}}, b^+_{\tilde{h}}) = -s_{ab} \times \mathrm{Split}^{\mathrm{tree}}_{+}(z, a^-, b^+) \times \mathrm{Split}^{\mathrm{tree}}_{+}(z, b^+_q, a^-_{\bar{q}}) = -\sqrt{\frac{z^5}{1-z}} \frac{[a\, b]}{\langle a\, b\rangle} , \tag{1.5.5}$$

and so forth.

In terms of its implication for subleading terms, eq. (1.5.1) has a slightly different meaning from the corresponding equations for the collinear limits in gauge theory, or for the soft limits in either gauge theory or gravity. In these other limits, the leading power-law behavior is determined; subleading, non-universal behavior is down by a power of either $\sqrt{k_s}$ or $\sqrt{s_{ab}}$. In the case of eq. (1.5.1), there are other terms of the same order as $[a\,b]\,/\,\langle a\,b\rangle$ as $s_{ab} \to 0$, namely any term that does not vanish as $s_{ab} \to 0$. However, these terms do not acquire any phase as $\vec{k}_a$ and $\vec{k}_b$ are rotated around their sum $\vec{P}$, as depicted in fig. 1.3, and thus they can be meaningfully separated from the terms described by eq. (1.5.1). (In space-time signature $(2,2)$, the spinor products $\langle a\,b\rangle$ and $[a\,b]$ are not complex conjugates of each other, so that $\langle a\,b\rangle$ can be taken to zero independently of $[a\,b]$, in order to separate out the $[a\,b]\,/\,\langle a\,b\rangle$ terms [36].)

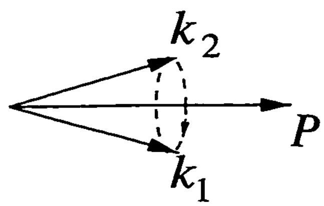


Figure 1.3: As two momenta become collinear, the gravity S-matrix element develops a phase singularity which can be detected by rotating the two momenta about the axis formed by their sum.

For example, consider the two factors,

$$\text{(a)}\quad \frac{[1\,2]}{\langle 1\,2\rangle}, \qquad\qquad \text{(b)}\quad \frac{[1\,3]}{\langle 1\,3\rangle}. \tag{1.5.6}$$

If we take $\vec{k}_1$ to be nearly collinear with $\vec{k}_2$ and rotate $\vec{k}_1$ and $\vec{k}_2$ around the vector $\vec{P} = \vec{k}_1 + \vec{k}_2$ the factor (b) undergoes only a slight numerical variation. On the other hand, from eq. (1.2.7), the factor (a) undergoes a large phase variation, proportional to the angle of rotation. Thus a Fourier analysis in this azimuthal rotation angle will extract the universal terms in eq. (1.5.1) from the (approximately) constant non-universal terms, giving meaning to this equation.

The universality of the tree-level splitting amplitudes for gravity amplitudes with any number of external legs may be understood in terms of Feynman diagrams in any gauge which does not introduce extra singularities into the vertices or propagators, besides the usual $1/p^2$ propagator factor (for example, de Donder gauge [70]). Although use of Feynman diagrams generally obscures the relationships between gravity and gauge theory scattering amplitudes, here we only require the diagrams' factorization properties. Terms with a phase singularity for $a \parallel b$ must contain a factor of the form $[a\,b]^2/s_{ab}$. The only tree-level Feynman diagrams that contain a pole in s_{ab} (from a propagator) are of the type shown in fig. 1.4; they all contain the same three-point vertex.[4] The splitting amplitudes are given by a straightforward evaluation of the three-vertex (multiplied by the s_{ab} pole) in a helicity basis in the collinear limit. (For the analogous gauge theory computation, see refs. [46, 53].)

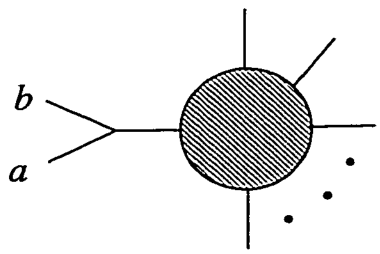


Figure 1.4: The class of tree diagrams in a gravity theory that can have a phase singularity factorizes in the collinear limit $a \parallel b$. The appearance of the same three-vertex for any number of external legs implies the universality of the tree-level splitting amplitudes. The splitting amplitudes are given by evaluating the three-vertex in the collinear limit in a helicity basis.

Similarly, the validity of the soft factor (1.2.21) for an arbitrary number of external legs also follows from the factorization properties of Feynman diagrams. In this case the tree diagrams that contribute to the soft factors are of the form shown in fig. 1.5, and the complete soft factor is given by summing over all three-point vertices with a soft leg (multiplied by the respective propagator).

One may also prove the universality of the tree-level splitting amplitudes using the n-point version of the KLT relations given in appendix A. Alternatively, one may obtain the tree-level soft and collinear splitting functions from string theory by extending the gauge theory discussion given by Mangano and Parke [46] to the case of gravity, using a closed string instead of an open string. The factorization of the closed string integrands into products of open string integrands ensures that the

[4] In the helicity formalism [44], a reference momentum entering the polarization vector or tensor could produce a pole in s_{ab} in other diagrams, but this is easily avoided by choosing the reference momenta to be neither k_a nor k_b.

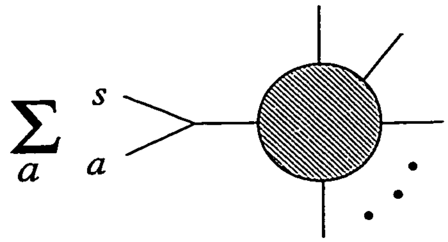


Figure 1.5: The class of tree diagrams in a gravity theory that contribute in the soft limit, where leg s is soft. The soft functions are found by summing over all three-vertices containing a soft leg.

gravity splitting functions are given in terms of products of the corresponding gauge theory splitting functions, as given in eq. (1.5.3).

1.5.2 Absence of Loop Corrections

We now show that the soft and collinear splitting amplitudes for gravity — in contrast to those for gauge theory — do not have any higher loop corrections. In general, in covariant gauges the splitting and soft functions may be classified into two categories: factorizing and non-factorizing contributions [53]. Diagrams for the factorizing one-loop corrections to the splitting and soft functions are shown in fig. 1.6. Non-factorizing contributions can arise whenever infrared divergences do not behave smoothly in the soft or collinear limits, as discussed in ref. [53].

First consider the factorizing contributions. Since each Feynman diagram in fig. 1.6 has a power of κ^2 as compared to the tree-level functions, dimensional analysis requires that the diagram carry an extra power of $|s_{ab}|$ in the collinear case or an extra power of $|s_{as}|$ in the soft limit. This suppresses potential one-loop corrections to either the collinear ($s_{ab} \to 0$) or soft ($s_{as} \to 0$) limits.

Now consider the non-factorizing contributions. In one-loop gauge theory amplitudes the infrared divergences, for e.g. a pure gluon amplitude, are of the form

$$-A_n^{\rm tree} \sum_{i\neq j}^{n} \Big[\frac{1}{\epsilon^2} - \frac{\ln(-s_{ij})}{\epsilon}\Big] . \tag{1.5.7}$$

The mismatch between the infrared divergence of the n- and $(n-1)$-point one-loop

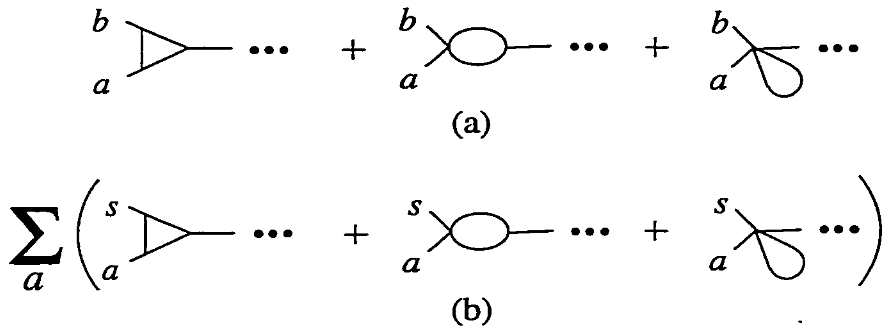


Figure 1.6: One-loop factorizing corrections to (a) the collinear splitting amplitudes and (b) the soft functions.

amplitudes on the left- and right-hand-sides of the collinear limit (1.3.3) implies that there must be a non-trivial contribution to the one-loop gauge splitting amplitude. This may be contrasted with the case of gravity: a pure graviton amplitude has infrared divergences of the form [71]

$$M_n^{\text{tree}} \sum_{i \neq j}^{n} \Big[s_{ij} \frac{\ln(-s_{ij})}{\epsilon} \Big] . \tag{1.5.8}$$

In this case the infrared divergences exhibit smooth behavior in soft or collinear limits, because of the extra power of s_{ij} in each term; as any kinematic variable vanishes, the infrared divergent term containing that variable goes smoothly to zero. Thus there are no one-loop contributions to soft or collinear splitting amplitudes arising from non-factorizing contributions. Again this difference in behavior between the gauge and the gravity case is due to the dimensionful coupling in gravity theories.

More generally, the appearance of a dimensionful coupling in gravity implies that the contributions of the form that appear in gauge theory splitting amplitudes and soft functions will be suppressed by additional powers of vanishing s_{ij} at all loop orders. For the factorizing contributions the argument is the same as for the one-loop case. For non-factorizing contributions, which involve infrared divergences, e.g. $(-s_{ij})^{-\epsilon}/\epsilon^2$, a closer inspection is required.

In particular, in the gauge theory case it is possible for the loop integration to generate a pole in s_{ij}, leading to a non-factorizing contribution to soft factors or splitting amplitudes. A one-loop example of a diagram where this can happen is shown in fig. 1.7a. In the soft limit where $k_1 \to 0$, the region of loop integration that can produce a kinematic pole in k_1 is where an extra propagator diverges.

As an example, before taking $k_1 \to 0$, in the region $L_1 \approx 0$ only the three propagators with momenta L_n, L_1 and L_2 in fig. 1.7a diverge. As $k_1 \to 0$, the propagator with momentum $L_3 = L_1 - k_1 - k_2$ also diverges since $L_3^2 \approx 2k_1 \cdot k_2 - 2L_1 \cdot (k_1 + k_2)$. Since we are interested only in the leading behavior as $k_1 \to 0$ we may set $L_1 = 0$ in the remaining part of the diagram, effectively leaving only a box diagram to be analyzed.

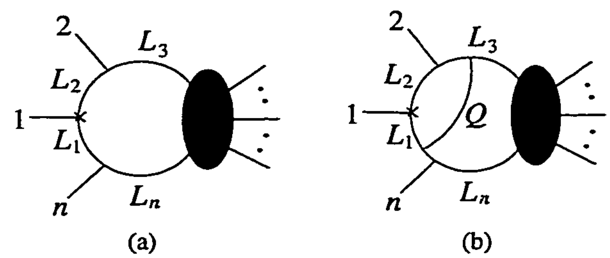


Figure 1.7: One- and two-loop examples of diagrams that can produce a pole in k_1 from the loop integration in the gauge theory case, but not in the gravity case.

For the case of gravity, the graviton vertex attached to leg 1 (and marked by a cross in the figure) contains one extra power of k_1 or L_1, as compared to the gauge theory vertex, and the contribution is therefore suppressed compared to gauge theory. Since one can obtain at most a single power of $1/k_1$ in the gauge theory case, the gravity case cannot have a pole in k_1 and is therefore suppressed compared to the tree gravity soft $\mathcal{S}$ function which does contain a single pole in k_1. The cases where three propagators diverge can also be analyzed by observing that these cases effectively reduce to triangle integrals.

These arguments extend to the multi-loop case. Consider, for example, the two-loop diagram in fig. 1.7b. In the gauge theory case, in order to obtain a contribution analogous to the one-loop one discussed above we must also take $Q \approx 0$. Again the extra powers of k_1, L_1 or Q in the vertices suppress any potential gravity contribution.

Similarly, for other potential non-factorizing soft contributions, and also in the case of collinear limits, one may show that the extra powers of momenta in the vertices suppress all potential loop contributions. Thus the appearance of a dimensionful coupling in gravity theories implies that the tree-level soft and collinear functions are exact to all orders of perturbation theory, so that eqs. (1.5.1) and (1.5.2) hold at any loop order.

1.6 Ansatze for an Arbitrary Number of External Legs

In this section we make use of the soft and collinear limits to construct ansatze for both the all-plus and $N = 8$ MHV amplitudes for an arbitrary number of external legs. In a previous letter we presented the ansatz for the all-plus (self-dual) case [2]; here we provide some of the details of the derivation as well as an alternate representation of the amplitude that has manifest symmetry under relabelings of external legs. We also present a new ansatz for the $N = 8$ supergravity amplitudes. The constraints that the amplitudes have the correct poles in all channels are rather restrictive and very likely determine the unique form of the amplitude, although we do not have a proof that this is so. Analogous constructions in the gauge theory case have been proven to lead to the correct results [26, 27, 28].

1.6.1 Functions with Simple Soft Properties

The first step in constructing ansatze for the amplitudes is to find a set of functions which have simple behavior in the soft limits. A good starting point is the three h functions defined in equations (1.4.16), (1.4.24) and (1.4.29), which appear in the coefficients of the box integrals in the explicit expressions for the four-, five- and six-point all-plus and $N=8$ MHV amplitudes. We collect them again here,

$$\begin{aligned} h(a,\{1\},b) &= \frac{1}{\langle a\,1\rangle^2\,\langle 1\,b\rangle^2}\,, \\ h(a,\{1,2\},b) &= \frac{[1\,2]}{\langle 1\,2\rangle\,\langle a\,1\rangle\,\langle 1\,b\rangle\,\langle a\,2\rangle\,\langle 2\,b\rangle}\,, \\ h(a,\{1,2,3\},b) &= \frac{[1\,2]\,[2\,3]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle a\,1\rangle\,\langle 1\,b\rangle\,\langle a\,3\rangle\,\langle 3\,b\rangle} + \frac{[2\,3]\,[3\,1]}{\langle 2\,3\rangle\,\langle 3\,1\rangle\,\langle a\,2\rangle\,\langle 2\,b\rangle\,\langle a\,1\rangle\,\langle 1\,b\rangle} \\ &\quad + \frac{[3\,1]\,[1\,2]}{\langle 3\,1\rangle\,\langle 1\,2\rangle\,\langle a\,3\rangle\,\langle 3\,b\rangle\,\langle a\,2\rangle\,\langle 2\,b\rangle}\,. \end{aligned} \tag{1.6.1}$$

It is easy to verify that these functions satisfy the following soft limits,

$$h(a,M,b) \stackrel{k_m\to 0}{\longrightarrow} -\mathcal{S}_m(a,M,b)\times h(a,M-m,b), \qquad \text{for } m\in M. \tag{1.6.2}$$

Here the 'half-soft' factor,

$$\mathcal{S}_m(a,M,b) \equiv \frac{-1}{\langle a\,m\rangle\,\langle m\,b\rangle} \sum_{j\in M} \langle a\,j\rangle\,\langle j\,b\rangle\,\frac{[j\,m]}{\langle j\,m\rangle}\,, \tag{1.6.3}$$

is closely related to the gravity soft function $\mathcal{S}_n \equiv \mathcal{S}^{\rm gravity}(n^+)$ defined in eq. (1.2.21), except that the sum in eq. (1.6.2) is over only a subset of the legs in the amplitude. Thus h obeys soft limits very similar to the tree-level gravity amplitudes, except that there is no momentum conservation constraint on h or $\mathcal{S}_m(a,M,b)$, so the h functions may be thought of as off-shell extensions of the tree amplitudes.

Here we wish to find explicit forms for 'half-soft' functions $h(a,M,b)$, which satisfy eq. (1.6.2) for an arbitrary number of external legs. This is accomplished by using eq. (1.6.1) to motivate a guess for the general form of h, and then using the soft properties to fix its components. From the form of $\mathcal{S}_m(a,M,b)$, we see that $h(a,M,b)$ should be symmetric in $a \leftrightarrow b$, and in the exchange of any members of M. Also, eq. (1.6.1) suggests that it can be written as sums of products of spinor phase factors $[j_1\,j_2]\,/\,\langle j_1\,j_2\rangle$, where $j_1, j_2 \in M$, multiplied by appropriate powers of $\langle a\,j_l\rangle\,\langle j_l\,b\rangle$. We write $h(a, M, b)$ as

$$h(a, \{1, 2, \ldots, m\}, b) = \sum_{i_1, i_2, \ldots, i_m = 0}^{m-2} \phi(i_1, i_2, \ldots, i_m) \prod_{j=1}^{m} (\langle a\, j\rangle \langle j\, b\rangle)^{i_j - 1}, \qquad (1.6.4)$$

where $\phi(i_1, i_2, \ldots, i_m)$ is defined to be a symmetric function of its arguments, and nonzero only for $\sum_{j=1}^{m} i_j = m - 2$. Thus at least one of the arguments i_j must equal zero, and using the symmetry we can choose this to be the last argument. Then, to incorporate the soft limits (1.6.2), we define $\phi(i_1, i_2, \ldots, i_m)$ recursively by

$$\begin{aligned} \phi(0,0) &= \frac{[1\,2]}{\langle 1\,2\rangle}, \\ \phi(i_1, i_2, \ldots, i_{m-1}, 0) &= \sum_{j=1}^{m-1} \phi(i_1, i_2, \ldots, i_j - 1, \ldots, i_{m-1}) \times \frac{[j\, m]}{\langle j\, m\rangle}, \end{aligned} \qquad (1.6.5)$$

where ϕ is also defined to be zero if any of its arguments is negative.

We give a few examples of the factors ϕ:

$$\begin{aligned} \phi(0,0) &= \frac{[1\,2]}{\langle 1\,2\rangle}, \\ \phi(1,0,0) &= \frac{[1\,2]\,[1\,3]}{\langle 1\,2\rangle \langle 1\,3\rangle}, \\ \phi(2,0,0,0) &= \frac{[1\,2]\,[1\,3]\,[1\,4]}{\langle 1\,2\rangle \langle 1\,3\rangle \langle 1\,4\rangle}, \\ \phi(1,1,0,0) &= \frac{[1\,2]\,[2\,3]\,[1\,4]}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 1\,4\rangle} + \frac{[1\,2]\,[1\,3]\,[2\,4]}{\langle 1\,2\rangle \langle 1\,3\rangle \langle 2\,4\rangle}, \\ \phi(2,1,0,0,0) &= \frac{[1\,2]\,[2\,3]\,[1\,4]\,[1\,5]}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 1\,4\rangle \langle 1\,5\rangle} + \frac{[1\,2]\,[1\,3]\,[2\,4]\,[1\,5]}{\langle 1\,2\rangle \langle 1\,3\rangle \langle 2\,4\rangle \langle 1\,5\rangle} + \frac{[1\,2]\,[1\,3]\,[1\,4]\,[2\,5]}{\langle 1\,2\rangle \langle 1\,3\rangle \langle 1\,4\rangle \langle 2\,5\rangle}. \end{aligned} \qquad (1.6.6)$$

An interesting property of the ϕ functions is that they can be generated from group theory Young tableaux, as an alternative to eq. (1.6.5). We can restrict our attention to the $\phi(i_1, i_2, \ldots, i_m)$ with $i_1 \geq i_2 \geq \cdots \geq i_m = 0$, since all other ϕ's can be obtained by simple relabelings. The formula for ϕ can be schematically represented as

$$\phi(i_1, i_2, \ldots, i_m) = \sum_{\text{NSYT}} \underbrace{\left(\frac{[\;]}{\langle\;\rangle}\right) \cdots \left(\frac{[\;]}{\langle\;\rangle}\right)}_{m-1 \text{ terms}}, \tag{1.6.7}$$

where each term in the sum corresponds to a non-standard Young tableaux (NSYT). The NSYT are defined as all possible labelings from 1 to $m-2$ of the tableaux (with i_1 boxes in the first row, i_2 boxes in the second row, etc...), with the restriction of ascending order along rows. The standard requirement of descending order along columns is relaxed. The rule for constructing each phase factor in eq. (1.6.7) from the corresponding NSYT is best illustrated by an example. For $\phi(2,1,0,0,0)$, the three terms in eq. (1.6.6) correspond to these NSYT:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$$

To obtain the phase factors, one first extends the YT vertically with 'empty' boxes until it has m rows, in order to represent all m arguments in ϕ. Then one removes both the last empty box (in row j, say) and the full box containing the highest number (in row i), writing a factor of $[i\,j]\,/\,\langle i\,j\rangle$ for this step. Repeating the step until all boxes are gone yields the phase factor. For example,

$$\begin{array}{r|c|c|}\cline{2-3} 1 & 1 & 2 \\ \cline{2-3} 2 & 3 \\ \cline{2-2} 3 & \\ 4 & \\ 5 & \end{array} = \frac{[2\,5]}{\langle 2\,5\rangle} \times \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \\ \\ \\ \end{array} = \frac{[1\,4]}{\langle 1\,4\rangle}\frac{[2\,5]}{\langle 2\,5\rangle} \times \begin{array}{|c|}\hline 1 \\ \hline \\ \\ \end{array} = \frac{[1\,3]}{\langle 1\,3\rangle}\frac{[1\,4]}{\langle 1\,4\rangle}\frac{[2\,5]}{\langle 2\,5\rangle} \times \begin{array}{|c}\\ \\ \end{array} = \frac{[1\,2]}{\langle 1\,2\rangle}\frac{[1\,3]}{\langle 1\,3\rangle}\frac{[1\,4]}{\langle 1\,4\rangle}\frac{[2\,5]}{\langle 2\,5\rangle} \tag{1.6.8}$$

gives the last term in $\phi(2,1,0,0,0)$ in eq. (1.6.6).

The NSYT approach gives a simple formula for the number of terms in each ϕ,

$$\left[\#\text{ of terms in } \phi(i_1, \ldots, i_m)\right] = \frac{(m-2)!}{\prod_{j=1}^{m} i_j!}, \tag{1.6.9}$$

which is the analog of the 'hook' formula for standard Young tableaux.

We have also found an explicit non-recursive form for the h functions,

$$h(a,\{1,2,\ldots,n\},b) \equiv \frac{[1\,2]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle \cdots \langle n-1,\,n\rangle} \frac{\langle a^-|\not{K}_{1,2}|3^-\rangle\langle a^-|\not{K}_{1,3}|4^-\rangle\cdots\langle a^-|\not{K}_{1,n-1}|n^-\rangle}{\langle a\,1\rangle\,\langle a\,2\rangle\,\langle a\,3\rangle\cdots\langle a\,n\rangle\;\langle 1\,b\rangle\,\langle n\,b\rangle} + \mathcal{P}(2,3,\ldots,n), \tag{1.6.10}$$

where $K_{i,j} = k_i + k_{i+1} + \cdots + k_j$. In the form (1.6.10) the symmetry properties of h under the interchange of $a \leftrightarrow b$ and $1 \leftrightarrow j \in \{2,\ldots,n\}$ are not manifest. Nevertheless, in appendix C we show that the forms in eqs. (1.6.4) and (1.6.10) are in fact equal.

As mentioned above, the h functions can be thought of as off-shell extensions of gravity tree amplitudes. Using the non-recursive form (1.6.10), it is not hard to show that they are related to the BGK expressions for the MHV tree amplitudes (1.2.14) via,

$$\left.\frac{h(n,\{n-1,n-2,\ldots,2\},1)}{\langle n\,1\rangle^2}\right|_{k_1+k_2+\cdots+k_n=0} = (-1)^n \frac{M_n^{\rm tree}(1^-,2^-,3^+,\ldots,n^+)}{i\;\langle 1\,2\rangle^8}\,. \tag{1.6.11}$$

In this form, momentum conservation only has to be used in one factor in $M_n^{\rm tree}$, in order to convert it into h.

In light of eq. (1.6.11), it is perhaps not too surprising that the h functions satisfy a squaring relation to the g functions (1.3.14) appearing in the gauge theory amplitudes, analogous to the KLT relations for tree amplitudes. For example,

$$\begin{aligned} h(a,\{1\},b) &= [g(a,\{1\},b)]^2\,, \\ h(a,\{1,2\},b) &= s_{12}\, g(a,\{1,2\},b)\, g(a,\{2,1\},b)\,, \\ h(a,\{1,2,3\},b) &= s_{12}\, s_{23}\, g(a,\{1,2,3\},b)\, g(a,\{3,2,1\},b) + \text{perms}\,, \end{aligned} \tag{1.6.12}$$

and so forth. These relations are analogous to the KLT relations, in eq. (1.2.2), except that they hold for functions that appear at one loop (and the s_{ij} factors and permutations appearing are not precisely the same).

1.6.2 Ansatz for All-Plus Amplitudes

The forms of the four-, five- and six-point amplitudes in eqs. (1.4.15), (1.4.23) and (1.4.28), and the soft properties (1.6.2) of the h functions have led us to the following ansatz (see also ref. [2]) for the one-loop all-plus (self-dual) amplitudes in $D = 4$,

$$M_n(1^+, 2^+, \dots, n^+) = -\frac{i\,(-1)^n}{(4\pi)^2 \cdot 960} \sum_{\substack{1 \le a < b \le n \\ M,N}} h(a, M, b) h(b, N, a) \, \mathrm{tr}^3[a\, M\, b\, N] \; + \; \mathcal{O}(\epsilon), \tag{1.6.13}$$

where a and b are massless legs, and M and N are two sets forming a 'distinct non-trivial partition' of the remaining $n-2$ legs; i.e., M and N should both be non-empty, and the partition (M, N) is not considered distinct from (N, M). This configuration of external legs is depicted in fig. 1.8. We do not have an ansatz that works to all orders in ϵ. The $D = 4$ amplitudes (1.6.13) are also generated by a self-dual gravity action [72, 35, 36].

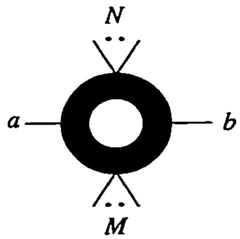


Figure 1.8: The configurations of external legs that are summed over in eq. (1.6.13).

The fact that the amplitudes (1.6.13) have the correct soft limits is a consequence of the soft properties of the h functions, eq. (1.6.2). As $k_n \to 0$, the term labeled by (a, M, b, N) in M_{n-1} gets contributions from two terms in M_n, those labeled by $(a, M{+}n, b, N)$ and $(a, M, b, N{+}n)$. Each of the factors $h(a, M{+}n, b)$ and $h(b, N{+}n, a)$ in eq. (1.6.13) supplies 'half' of the soft factor in this limit, since

$$\mathcal{S}_n = \mathcal{S}_n(a, M, b) + \mathcal{S}_n(b, N, a). \tag{1.6.14}$$

The trace factors behave smoothly in the soft limit, serving only to prevent the unwanted terms where a or b becomes soft.

The collinear properties are slightly more difficult to establish. They rely on the two non-trivial collinear limits of the half-soft function h (after taking into account its symmetries),

$$\begin{aligned} h(a,\{1,2,3,\ldots,n\},b) &\xrightarrow{1\|2} \frac{1}{z(1-z)}\frac{[1\,2]}{\langle 1\,2\rangle} \times h(a,\{P,3,\ldots,n\},b)\,, \\ h(1,\{2,3,\ldots,n\},b) &\xrightarrow{1\|2} \frac{1}{\langle 1\,2\rangle}\frac{\langle 1\,b\rangle\,\langle b^-|\not{K}_{3,n}|2^-\rangle}{\langle 2\,b\rangle^2} \times h(1,\{3,\ldots,n\},b)\,, \end{aligned} \qquad (1.6.15)$$

where we have used the Schouten identity, eq. (C.5), to derive the second limit, dropping terms without phase singularities, in accordance with the discussion in section 1.5.1.

Consider the collinear limit $1 \parallel 2$ of $M_n(1^+,2^+,\ldots,n^+)$. For a term labeled by (a,M,b,N) in the ansatz (1.6.13), there are three independent possibilities:
(1) 1 and 2 both belong to the same set, say M,
(2) $a=1$ and $b\neq 2$, so that 2 belongs to a massive set,
(3) $a=1$ and $b=2$.

Case (3) is trivial; eq. (1.6.10) has no $1/\langle a\,b\rangle$ factor, and hence there is no contribution to the collinear (phase) singularity. Case (1) is also simple. The first eq. (1.6.15) shows that these terms precisely account for all the terms in the expression (1.6.13) for the target amplitude $M_{n-1}(P^+,\ldots,n^+)$ in which $P\in M$.

The only remaining task is to show that the case (2) terms correctly give rise to the terms in the expression for the target amplitude in which P does not belong to M or N. The second eq. (1.6.15) shows that an individual h function has a 'too singular' $1/\langle 1\,2\rangle$ behavior in case (2). However, the combination of two different terms, labeled by $(1,M+2,b,N)$ and $(1,M,b,N+2)$, cancels the singularity down to the desired level of $[1\,2]\,/\,\langle 1\,2\rangle$. More concretely, by using momentum conservation, $K_M+K_N+k_1+k_2+k_b=0$, where K_M is the sum of the massless momenta in the set M, we have $\langle 1^+|\not{K}_M|b^+\rangle = -\langle 1^+|\not{K}_N|b^+\rangle - [1\,2]\,\langle 2\,b\rangle$. Also expanding the traces to first order in $[1\,2]$, we obtain

$$h(1,M+2,b)h(b,N,1)\,\mathrm{tr}^3[1(M+2)bN]+h(1,M,b)h(b,N+2,1)\,\mathrm{tr}^3[1Mb(N+2)]$$
$$\xrightarrow{1\|2}\ \frac{z^3+3z^2(1-z)}{z(1-z)}\frac{[1\,2]}{\langle 1\,2\rangle}h(P,M,b)h(b,N,P)\,\mathrm{tr}^3[PMbN]\,. \tag{1.6.16}$$

Adding the analogous equation with the roles of 1 and 2 (and z and $1-z$) exchanged gives

$$h(1,M+2,b)h(b,N,1)\,\mathrm{tr}^3[1(M+2)bN]+h(1,M,b)h(b,N+2,1)\,\mathrm{tr}^3[1Mb(N+2)]$$
$$+\ (1\leftrightarrow 2)\ \xrightarrow{1\|2}\ \frac{1}{z(1-z)}\frac{[1\,2]}{\langle 1\,2\rangle}h(P,M,b)h(b,N,P)\,\mathrm{tr}^3[PMbN]\,, \tag{1.6.17}$$

which accounts properly for the terms in the target expression M_{n-1} in which $P=a$ (or similarly, $P=b$). Thus $M_n(1^+,2^+,\ldots,n^+)$ does obey the required collinear limits.

1.6.3 Ansatz for $N=8$ MHV Amplitudes

In section 1.4.4 we used the dimension-shifting relation (1.3.16) to obtain the four-, five- and six-point one-loop MHV amplitudes in $N=8$ supergravity, given in eq. (1.4.30), from the corresponding all-plus amplitudes. These results suggest the following ansatz for the n-point one-loop MHV $N=8$ amplitudes:

$$M_n^{N=8}(1^-,2^-,3^+,\ldots,n^+)=$$
$$\frac{(-1)^n}{8}\langle 1\,2\rangle^8\sum_{\substack{1\le a<b\le n\\ M,N}}h(a,M,b)h(b,N,a)\,\mathrm{tr}^2[a\,M\,b\,N]\,\mathcal{I}_4^{aMbN}\ +\ \mathcal{O}(\epsilon), \tag{1.6.18}$$

where the notation is identical to that of eq. (1.6.13). The scalar box integral functions, $\mathcal{I}_4^{aMbN}$, are given through $\mathcal{O}(\epsilon^0)$ by eq. (B.4). The sum in eq. (1.6.18) includes all inequivalent two-mass scalar box integrals with diagonally-opposite massive legs, as shown in fig. 1.1, as well as all one-mass scalar box integrals arising from the terms in eq. (1.6.18) where either M or N consists of a single massless leg. (In the four-point case the sum is over 6 boxes with all massless legs, of which only

3 are inequivalent because of an extra degeneracy, leading to the extra factor of 2 in the first line of eq. (1.4.30).) The $N = 8$ SWI, eq. (1.2.10), requires that $M_n^{N=8}(1^-, 2^-, 3^+, \ldots, n^+)/\langle 1\,2\rangle^8$ is totally symmetric with respect to permutations of its arguments; this symmetry is manifest in eq. (1.6.18).

Just as in the case of the all-plus amplitudes, the soft behavior of eq. (1.6.18) follows from the soft properties of the h functions. The only real difference is that one of the powers of the trace $\mathrm{tr}[a\, M\, b\, N]$ is replaced by $\mathcal{I}_4^{aMbN}$. But these integrals also transform smoothly in the limit $k_n \to 0$,

$$\mathcal{I}_4^{aMb(N+n)} \quad \text{and} \quad \mathcal{I}_4^{a(M+n)bN} \quad \xrightarrow{k_n \to 0} \quad \mathcal{I}_4^{aMbN}\,, \tag{1.6.19}$$

and two powers of the traces suffice to kill the unwanted terms where either a or b becomes soft ($\mathcal{I}_4^{aMbN}$ does not develop a singularity as $k_a \to 0$).

The ansatz (1.6.18) also must have universal collinear behavior. In the limit $1 \parallel 2$, the analysis is again quite similar to that presented in section 1.6.2 for the all-plus ansatz. The same three cases are encountered; the only subtle case is case (2). In this case, using the same labeling as in the all-plus discussion, the coefficients of the scalar box integrals $\mathcal{I}_4^{1(M+2)bN}$, $\mathcal{I}_4^{1Mb(N+2)}$, $\mathcal{I}_4^{2(M+1)bN}$ and $\mathcal{I}_4^{2Mb(N+1)}$ each have a $1/\langle 1\,2\rangle$ singularity. This singularity should cancel, since we only expect the phase singularity $[1\,2]\,/\,\langle 1\,2\rangle$. The cancellation can be demonstrated with the use of integral relations of the type,

$$z\,\mathcal{I}_4^{1(M+2)bN} + (1-z)\,\mathcal{I}_4^{2Mb(N+1)} \xrightarrow{1\parallel 2} \mathcal{I}_4^{PMbN} \;+\; \mathcal{O}(\sqrt{s_{12}})\,, \tag{1.6.20}$$

where $k_1 = zk_P$ and $k_2 = (1-z)k_P$ in the collinear limit. However, to verify that phase singularity of eq. (1.6.18) matches that of eq. (1.5.4) requires collinear analysis of the integrals to one higher order in $\sqrt{s_{12}}$. The required integral relation appears to be more subtle and involves a larger combination of integrals. We have, however, verified numerically that the ansatz (1.6.18) has the correct collinear limits for $n \leq 7$. (We have also checked numerically through $n = 8$ that the infrared singularities of $M_n^{N=8}$ are correctly given by eq. (1.5.8).)

Comparing the all-plus and $N = 8$ MHV ansatze, eqs. (1.6.13) and (1.6.18), it might appear that the former is obtained from the latter (up to an overall factor) simply by substituting the higher-dimensional values (B.9) for the box integrals. However, the dimension-shifting relation does not work that simply, for $n = 5$ or 6. The quadratic polynomial for $\mathcal{I}_4^{aMbN}[\mu^8]$ in eq. (B.9) is not the same as the extra trace factor $\mathrm{tr}[a\, M\, b\, N]$ in the all-plus expression, and the pentagon and hexagon contributions to eq. (1.4.30) give a non-vanishing contribution, which somehow compensates for this discrepancy. Presumably the required rearrangements become yet more complicated for $n > 6$.

It is instructive to compare the one-loop MHV $N = 8$ supergravity amplitudes, eq. (1.6.18), with the corresponding amplitudes in $N = 4$ super-Yang-Mills theory, eq. (1.3.11). Note that both amplitudes are expressed just in terms of scalar box integral functions. This result, though expected for $N = 4$ super-Yang-Mills theory based on power-counting grounds, is somewhat surprising for $N = 8$ supergravity, since as remarked in section 1.4.5, a naive power-count for $n \geq 5$ does not exclude the appearance of triangle or bubble integrals.

1.7 Discussion

In this chapter we have exploited relations between gauge theory and gravity to calculate the first three members of two infinite sequences of one-loop gravity amplitudes: the all-plus helicity amplitudes of non-supersymmetric gravity, and the maximally helicity-violating amplitudes of $N = 8$ supergravity. From these results, and the analytic properties of n-graviton amplitudes, we obtained ansatze for the remaining members of both sequences.

In approaching any amplitude calculation in gravity we have found it useful to first consider the corresponding gauge theory calculation. Kawai, Lewellen and Tye (KLT) have given precise expressions for closed-string tree amplitudes as (roughly speaking) the squares of open-string tree amplitudes. In the field theory limit, this implies that properties of gauge theory tree amplitudes should be reflected in gravity tree amplitudes. As an example of this, we derived the properties of the gravity amplitudes as two momenta become collinear from the corresponding properties of gauge theory amplitudes.

The methods used to construct gravity amplitudes in this chapter, relying on the analytic properties of the amplitudes, i.e. their (unitarity) cuts and poles, are essentially the same as those developed previously for evaluating multi-leg one-loop amplitudes in QCD [27, 26, 49, 8, 63]. The analytic properties for dimensions away from $D = 4$ are useful since they remove ambiguities normally associated with reconstruction of amplitudes from their cuts. When the KLT relations are combined with this analytic approach, gravity loop amplitudes can be obtained without evaluating even a single gravity Feynman diagram; the entire process relies only on gauge theory tree amplitudes.

Unitarity implies that relationships between tree amplitudes should somehow be reflected as relationships between loop amplitudes. Here we demonstrated that 'dimension-shifting' relations between one-loop all-plus and maximally helicity-violating $N = 4$ supersymmetric gauge theory amplitudes can be transformed into relations between one-loop all-plus and maximally helicity-violating $N = 8$ supersymmetric gravity amplitudes. The gravity relation is of practical value since it allows us to obtain the exact forms for the four-, five- and six-point $N = 8$ amplitudes without performing an explicit calculation; instead we applied a simple 'dimension shift' to the integral functions appearing in the all-plus amplitudes. We have also seen that there are 'squaring relations', eq. (1.6.12), reminiscent of the tree-level KLT relations, between the coefficients of the box integrals for $N = 8$ supergravity and $N = 4$ super-Yang-Mills, which have survived the integral reduction procedure.

The one-loop $N = 8$ amplitudes first develop ultraviolet divergences in $D = 8$. This is immediately apparent in the four-point amplitude first evaluated by Green, Schwarz and Brink [40]. In dimensional regularization, there are no divergences in odd dimensions, because odd-dimensional Lorentz-invariant quantities do not exist. So the next higher dimension one can inspect for one-loop ultraviolet divergences in $N = 8$ supergravity is $D = 10$. In the four-point amplitude, the $D = 10$ divergence cancels, but from the point of view of eq. (1.4.12) the cancellation is again for a relatively trivial reason; the $D = 10$ divergences of the sum of the three integrals has to be proportional to the only totally symmetric dimension-two four-point invariant,

$s+t+u=0$. We have investigated the $D=10$ divergences for the five- and six-point amplitudes as well, using eq. (1.4.30), and find that they also cancel. (Because D is not close to 4, we need to use an expression good to all orders in ϵ.) In the next chapter, however, we will show that this cancellation does not persist at the two-loop level (see section 2.3.2.) Thus, it should presumably be regarded as an artifact of dimensional regularization.

Currently the squaring relations that have been found between gravity and gauge theory are expressed entirely in terms of on-shell S-matrix elements. It would be nice to find an off-shell field-theoretic formulation of gravity that also exhibits a squaring relation with gauge theory, and to investigate the implications of such a formalism for the field equations. Some progress in this direction has been reported in [73].

Chapter 2

Supergravity Amplitudes Beyond One Loop

2.1 Introduction

Although gravity and gauge theory Lagrangians look very different, string theory suggests that their perturbative amplitudes are related. In the first chapter of this Thesis, we have reviewed the relations between tree-level amplitudes of the two theories, found by Kawai, Lewellen and Tye (KLT). We have also shown that similar relations exist between one-loop amplitudes. In this chapter, we will extend this connection further to multi-loop level, and discuss some of its implications.

For reasons of technical simplicity, our analysis in this chapter will concentrate on the theories with maximal degree of supersymmetry, $N = 4$ Yang-Mills theory and $N = 8$ supergravity. (The number of supersymmetries refers to their four-dimensional values; in any dimension we define $N = 4$ Yang-Mills theory to be the dimensional reduction of ten-dimensional $N = 1$ Yang-Mills theory, and $N = 8$ supergravity to be the dimensional reduction of eleven-dimensional $N = 1$ supergravity.) We will also only consider four-point amplitudes since they are the easiest to calculate.

Just like in our one-loop analysis in chapter 1, we will use the KLT relations to generate gravity amplitudes from gauge theory amplitudes, which we will then use as input into cutting rules. In this way one can compute (super) gravity loop amplitudes without any reference to a Lagrangian or to Feynman rules. Indeed, we will obtain the complete four-point two-loop amplitude for $N = 8$ supergravity, in terms of scalar integral functions, without having evaluated a single Feynman diagram.

The multi-loop four-point amplitudes in the $N = 4$ Yang-Mills theory have been discussed in [74]. In particular, the two-loop $N = 4$ amplitude has been calculated exactly. We will observe that a simple relationship holds between this amplitude and the corresponding $N = 8$ amplitude calculated here: The coefficients of the integral functions entering the $N = 8$ amplitude can be obtained by squaring the coefficients of the corresponding integral functions in the $N = 4$ amplitude, after dropping the color factors. We will conjecture that the same relationship holds between the two amplitudes at arbitrary loop level.

An interesting feature of the $N = 4$ and $N = 8$ theories is that the two-particle cuts needed to evaluate the four-point amplitudes can be iterated to an arbitrary loop order. While the two-particle cuts are not sufficient to reproduce an L-loop amplitude completely for $L > 1$, they do provide information about the coefficients of certain "entirely two-particle constructible" integral functions entering the amplitude. The analysis for the $N = 4$ case was presented in [74]; we will briefly review it in section 2.2.4. We will use these results to conjecture the degree of ultraviolet divergence of higher-loop $N = 4$ amplitudes (see section 2.2.5.) Analogous arguments can be used to conjecture the ultraviolet behaviour of $N = 8$ amplitudes, as we will discuss in section 2.3.4.

The ultraviolet divergences of theories of gravity have been under investigation for quite some time. In four dimensions, pure gravity was shown to be finite on-shell at one loop by 't Hooft and Veltman [10, 11], but the addition of scalars [10, 11], fermions or photons [12] renders it non-renormalizable at that order. At two loops, a potential counterterm for pure gravity of the form $R^3 \equiv R^{\lambda\rho}_{\mu\nu} R^{\sigma\tau}_{\lambda\rho} R^{\mu\nu}_{\sigma\tau}$ was identified in refs. [75, 76]. An explicit computation by Goroff and Sagnotti [13], and later by van de Ven [14], verified that the coefficient of this counterterm was indeed nonzero.

On the other hand, in any supergravity theory, supersymmetry Ward identities

(SWI) [22] forbid all possible one-loop [77] and two-loop [78] counterterms. For example, the R^3 operator, when added to the Einstein Lagrangian, produces a non-vanishing four-graviton scattering amplitude for the helicity configuration $(-+++)$, where all gravitons are considered outgoing [76]. But this configuration is forbidden by the SWI, hence R^3 cannot belong to a supersymmetric multiplet of counterterms [78].

At three loops, the square of the Bel-Robinson tensor [79], which we denote by R^4, has been identified as a potential counterterm in supergravity (or more accurately, as a member of a supermultiplet of potential counterterms) [80]. This operator does not suffer from the obvious problem that R^3 did, in that its four-graviton matrix elements populate only the $(--++)$ helicity configuration which is allowed by the $(N=1)$ supersymmetry Ward identities. In fact, the R^4 operator has been shown to belong to a full $N=8$ supermultiplet at the linearized level [15]. (A manifestly $SU(8)$-invariant form for the supermultiplet has also been given [81].) Furthermore, this operator appears in the first set of corrections to the $N=8$ supergravity Lagrangian, in the inverse string-tension expansion of the effective field theory for the type II superstring [82]. Therefore we know it has a completion into an $N=8$ supersymmetric multiplet of operators, even at the non-linear level. However, no explicit counterterm computation has been performed in any supergravity theory beyond one loop (until now), leaving it an open question whether supergravities actually do diverge at three (or more) loops.

The analysis of which counterterms can be generated can often be strengthened when the theory is quantized in a manifestly supersymmetric fashion, using superspace techniques. In particular, ref. [83] used an off-shell covariant $N=2$ superspace formalism to perform a power-counting analysis of divergences in $N=4$ Yang-Mills theory, and ref. [84] similarly used an $N=4$ superspace formalism to study $N=8$ supergravity. However, it is not possible to covariantly quantize either of the maximally extended theories, $N=4$ or $N=8$, while maintaining all of the supersymmetries. For example, in the $N=4$ Yang-Mills analysis of ref. [83], the complete $N=4$ spectrum falls into an $N=2$ gauge multiplet plus an $N=2$ matter multiplet. The 'superspace arguments' consist of applying power-counting rules to manifestly $N=2$ supersymmetric counterterms made out of the $N=2$ gauge multiplet. An important point is that these rules might not fully take into account all the constraints of $N = 4$ supersymmetry, because only the $N = 2$ is manifest [85]; similar remarks apply to the superspace analysis of $N = 8$ supergravity.

Here we shall first investigate the ultraviolet divergences of the $N = 8$ supergravity at two loops, using our exact knowledge of the four-point amplitude. We find that there is no divergence for $D < 7$. This behavior is less divergent than expected based on superspace arguments [84, 83]. Moreover, our conjecture for the higher-loop power-counting, based on the ultraviolet behaviour of the "two-particle constructible" parts of the amplitudes, implies that there are no divergences in $D = 4$ at three or four loops, contrary to expectations from the same types of superspace arguments. Assuming that the contributions to the amplitudes which cannot be detected by two-particle cuts do not alter this power count, we conclude that the potential $D = 4$ three-loop counterterm vanishes.

At five loops, our investigation of the cuts for the four-point $N = 8$ amplitude indeed indicates a non-vanishing counterterm, of the generic form $\partial^4 R^4$. This suggests that $N = 8$ supergravity is a non-renormalizable theory, with a four-point counterterm arising at five loops. Thus our cut calculations represent the first hard evidence that a four-dimensional supergravity theory is non-renormalizable, *albeit at a higher loop order than had been expected.* Since superspace power counting amounts to putting a bound on allowed divergences, our results are compatible with the discussion of ref. [83]. Our results are inconsistent, however, with some earlier work [86, 57] based on the speculated existence of an unconstrained covariant off-shell superspace for $N = 8$ supergravity, which in $D = 4$ would imply finiteness up to seven loops. The non-existence of such a superspace has already been noted [83].

2.2 $N = 4$ Yang-Mills Amplitudes

In ref. [74] the two-loop four-point amplitudes for $N = 4$ Yang-Mills theory were computed in terms of scalar integral functions via cutting methods. Furthermore, from an inspection of the two-particle cuts, a conjecture for the planar parts of the four-point amplitude was presented to all loop orders. In this section we examine the results of ref. [74] in preparation for the analogous construction for $N = 8$ supergravity. We will, in addition, extract the two-loop counterterms in various dimensions implied by the $N = 4$ amplitudes. These counterterms were previously obtained by Marcus and Sagnotti [87] via an explicit Feynman diagram calculation, using a specialized computer program. Comparison to their calculation provides a non-trivial two-loop check on our methods. The results of ref. [74] are more general, being the complete amplitudes and not just divergences. We also comment that a comparison of the two calculations illustrates the computational efficiency of the cutting techniques: The calculation of the complete amplitudes in terms of scalar integrals can easily be performed without computer assistance. Moreover, the technicalities associated with overlapping divergences are alleviated. We will apply the same cutting techniques to obtain new results for $N = 8$ supergravity.

2.2.1 Cut Construction Method at Two Loops and Beyond.

In section 1.4.1, we have already briefly reviewed the application of the cutting technique for constructing complete one-loop amplitudes, using tree-level amplitudes as input. Here, we will discuss how the same approach can be applied to multi-loop calculations.

To reconstruct a one-loop amplitude, one only has to consider a single cut in each channel, with 2 particles crossing the cut ("two-particle cut", as in eq. (1.4.1) of section 1.4.1). The main new feature which arises in the multi-loop calculations is the appearance of multiple possible cuts in the same channel. In general, one must calculate all cuts with up to $(L + 1)$ intermediate states in order to reconstruct an L-loop amplitude. In practice, the various cuts are related to each other, so one can often write down complete expressions for the amplitudes based on a calculation of a small subset of cuts. (When combining the cuts into a single function, care must be exercised not to over-count a particular term.) Once one has a robust ansatz for the form of the amplitude, the remaining cuts become much easier to calculate, since one has a definite final form to compare with. As one calculates additional cuts, one obtains cross-checks on the terms inferred from the earlier cuts; the consistency of the different cuts provides a rather powerful check that one is calculating correctly.

As an example, consider the cut construction method for a two-loop amplitude $\mathcal{M}_4^{\text{2-loop}}(1,2,3,4)$. At two loops one must consider both two- and three-particle cuts. In each channel there can be multiple contributing cuts. For example, in the s channel there are two two-particle cuts, as depicted in fig. 2.1. The first of these has the explicit representation

$$\mathcal{M}_4^{\text{2-loop}}(1,2,3,4)\Big|_{\text{cut(a)}} = \int \frac{d^D\ell_1}{(2\pi)^D} \sum_{S_1,S_2} \frac{i}{\ell_1^2} \mathcal{M}_4^{\text{tree}}(-\ell_1^{S_1},1,2,\ell_2^{S_2}) \frac{i}{\ell_2^2} \mathcal{M}_4^{\text{1-loop}}(-\ell_2^{S_2},3,4,\ell_1^{S_1})\Big|_{\ell_1^2=\ell_2^2=0}, \tag{2.2.1}$$

where ℓ_1 and ℓ_2 are the momenta of the cut legs and the sum runs over all particle states S_1 and S_2 which may propagate across the two cut lines. Following the discussion in ref. [8], it is useful to have replaced the phase-space integrals with cuts of unrestricted loop momentum integrals, even though we use the on-shell conditions on the amplitudes appearing in the integrand. In this way, one may simultaneously construct the imaginary and associated real parts of the cuts, avoiding the need for dispersion relations.

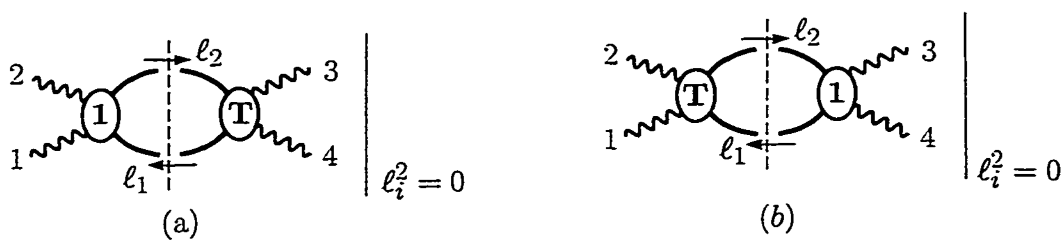


Figure 2.1: The two-particle s-channel cut has two contributions: one with the four-point one-loop amplitude '1' to the left and the tree amplitude 'T' to the right (a) and the other with the reverse assignment (b).

For a two-loop amplitude one must also calculate three-particle cuts. If one does not calculate these cuts one could, in principle, miss functions that have no two-particle cuts. Examples of such functions are the integral functions shown in fig. 2.2.

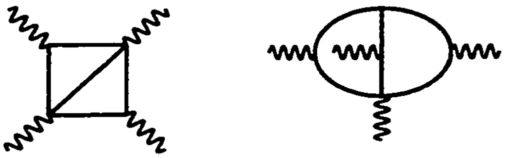


Figure 2.2: Examples of loop integrals with no two-particle cuts.

The three-particle s-channel cut is

$$M_4^{\text{2-loop}}(1,2,3,4)\Big|_{\text{3-cut}} = \int \frac{d^D\ell_1}{(2\pi)^D}\frac{d^D\ell_2}{(2\pi)^D} \sum_{S_1,S_2,S_3} M_5^{\text{tree}}(1,2,\ell_3^{S_3},\ell_2^{S_2},\ell_1^{S_1})\frac{i}{\ell_1^2}\frac{i}{\ell_2^2}\frac{i}{\ell_3^2}M_5^{\text{tree}}(3,4,-\ell_1^{S_1},-\ell_2^{S_2},-\ell_3^{S_3})\Big|_{\ell_i^2=0}, \tag{2.2.2}$$

which is depicted in fig. 2.3. The other channels are, of course, similar.

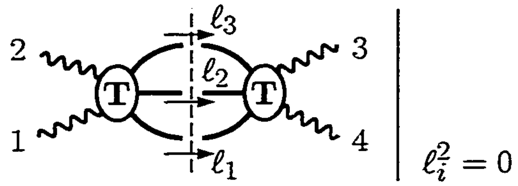


Figure 2.3: The three-particle s-channel cut for a two-loop amplitude.

The complete two-loop amplitude may be constructed by finding a single function which has the correct two- and three-particle cuts in D dimensions. As discussed in refs. [65, 9, 8], by computing all cuts to all orders in the dimensional regularization parameter $\epsilon = (4-D)/2$, one may perform a complete reconstruction of a massless loop amplitude in any dimension. This follows from dimensional analysis, since every term in an L-loop amplitude must carry L prefactors of the form $(-s_{ij})^{-\epsilon}$, which have cuts away from integer dimensions. As with Feynman diagrams, the result is unique for a given dimensional regularization scheme. As mentioned in section 2.1, we define the $N=4$ Yang-Mills amplitudes to be the dimensional reduction of ten-dimensional $N=1$ Yang-Mills amplitudes, and $N=8$ supergravity amplitudes to be the dimensional reduction of eleven-dimensional $N=1$ supergravity amplitudes; these definitions should also include the non-integer dimensions implied by dimensional regularization.

This same technique may be applied at any loop order; at L loops one would need to compute cuts with up to $(L+1)$ intermediate particles. In general, when performing cut calculations, it is convenient to ignore normalizations until the end of the calculation. There are sufficiently many cross-checks between the various cuts so that relative normalizations can usually be fixed.

We find it convenient to perform the cut construction using components instead of superfields. The potential advantage of a superfield formalism would be that one would simultaneously include contributions from all particles in a supersymmetry multiplet. However, for the cases we investigate, the supersymmetry Ward identities are sufficiently powerful that once the contribution from one component is known the others immediately follow. In a sense these identities are equivalent to using an on-shell superspace formulation. A component formulation is also more natural for extensions to non-supersymmetric theories.

2.2.2 Cut Construction of $N = 4$ Yang-Mills Amplitudes

Let us first outline how the one-loop four-point $N = 4$ amplitude can be reconstructed from its cuts. We will concentrate on a single partial amplitude, $A_{4;1}(1,2,3,4)$, since the full amplitude can be reconstructed from it. (See section 1.3.1 for a review of color decomposition technique.) At one loop it is sufficient to consider a single two-particle cut in each channel. For example, the cut in the s_{12} channel is given by

$$A_{4;1}^{\text{one-loop}}(1,2,3,4)\Big|_{s_{12}-\text{cut}} = \int \frac{d^D\ell_1}{(2\pi)^D} \sum_{S_1,S_2\in\{N=4\}} \frac{i}{\ell_1^2} A_4^{\text{tree}}(-\ell_1^{S_1},1,2,\ell_2^{S_2}) \frac{i}{\ell_2^2} A_4^{\text{tree}}(-\ell_2^{S_2},3,4,\ell_1^{S_1})\Big|_{\ell_1^2=\ell_2^2=0}, \tag{2.2.3}$$

where $\ell_2 = \ell_1 - k_1 - k_2$, $S_{1,2}$ label states of the $N = 4$ multiplet, and A_4^{tree} are the (color-ordered) tree amplitudes. The $N = 4$ labels corresponding to the external states with momenta k_i (including their helicity) have been suppressed. The key equation necessary to evaluate the cut is,

$$\sum_{S_1,S_2\in\{N=4\}} A_4^{\rm tree}(-\ell_1^{S_1},1,2,\ell_2^{S_2}) \times A_4^{\rm tree}(-\ell_2^{S_2},3,4,\ell_1^{S_1}) = \\ -i\frac{s_{12}s_{23}}{(\ell_1-k_1)^2\,(\ell_2-k_3)^2}A_4^{\rm tree}(1,2,3,4)\,. \tag{2.2.4}$$

This equation is valid for arbitrary combinations of external states. It is also valid for arbitrary (not just four-dimensional) momenta. One convenient way to verify it for the case of external gluons is by using background field Feynman gauge and second order fermions [88, 89, 90].

Plugging (2.2.4) into the right-hand side of (2.2.3), one immediately obtains

$$A_{4;1}^{\text{one-loop}}(1,2,3,4)\Big|_{s_{12}-\text{cut}} = is_{12}s_{23}A_4^{\rm tree}(1,2,3,4)\mathcal{I}_4^{\text{one-loop}}(s_{12},s_{23})\Big|_{s_{12}-\text{cut}}, \tag{2.2.5}$$

where

$$\mathcal{I}_4^{\text{one-loop}}(s_{12},s_{14}) = \int\frac{d^D\ell_1}{(2\pi)^D}\frac{1}{\ell_1^2(\ell_1-k_1)^2(\ell_1-k_1-k_2)^2(\ell_1+k_4)^2} \tag{2.2.6}$$

is a scalar box integral. Combining this result with the similar cut in the s_{23} channel, one gets

$$A_{4;1}^{\text{one-loop}}(1,2,3,4) = is_{12}s_{23}A_4^{\rm tree}(1,2,3,4)\mathcal{I}_4^{\text{one-loop}}(s_{12},s_{23}). \tag{2.2.7}$$

Relation (2.2.4) shows that when one sews two tree amplitudes and sums over intermediate states, one gets back a tree amplitude, multiplied by scalar functions. This means that we can iterate it to obtain two-particle cuts of higher-loop amplitudes. Consider, for example, a two-particle cut in the s_{12} channel of a two-loop amplitude shown in figure 2.1 (a):

$$A_{4;1}^{\text{two-loop}}(1,2,3,4)\Big|_{s_{12}-\text{cut},(a)} = \\ \int\frac{d^D\ell_1}{(2\pi)^D}\sum_{S_1,S_2\in\{N=4\}}\frac{i}{\ell_1^2}A_4^{\rm tree}(-\ell_1^{S_1},1,2,\ell_2^{S_2})\frac{i}{\ell_2^2}A_{4;1}^{\text{one-loop}}(-\ell_2^{S_2},3,4,\ell_1^{S_1})\Big|_{\ell_1^2=\ell_2^2=0}. \tag{2.2.8}$$

Plugging in the expression (2.2.7), we obtain

$$A_{4;1}^{\text{two-loop}}(1,2,3,4)\Big|_{s_{12}-\text{cut,(a)}} = \\ is_{12}s_{23}\int \frac{d^D\ell_1}{(2\pi)^D} \sum_{S_1,S_2\in\{N=4\}} \frac{i}{\ell_1^2} A_4^{\text{tree}}(-\ell_1^{S_1},1,2,\ell_2^{S_2}) \frac{i}{\ell_1^2} A_4^{\text{tree}}(-\ell_2^{S_2},3,4,\ell_1^{S_1}) \\ \times \mathcal{I}_4((\ell_2-k_3)^2, s_{34})\Big|_{\ell_1^2=\ell_2^2=0}. \tag{2.2.9}$$

Now we can use the "sewing relation" (2.2.4) again to obtain

$$A_{4;1}^{\text{two-loop}}(1,2,3,4)\Big|_{s_{12}-\text{cut,(a)}} = -s_{12}^2 s_{23} A_4^{\text{tree}}(1,2,3,4)\mathcal{I}_4^{\text{2-loop,P}}(s_{12},s_{23})\Big|_{s_{12}-\text{cut,(a)}}, \tag{2.2.10}$$

where

$$\mathcal{I}_4^{\text{2-loop,P}}(s_{12},s_{23}) = \\ \int \frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D} \frac{1}{p^2\,(p-k_1)^2\,(p-k_1-k_2)^2\,(p+q)^2 q^2\,(q-k_4)^2\,(q-k_3-k_4)^2} \tag{2.2.11}$$

is a two-loop planar double-box integral (see figure 2.4).

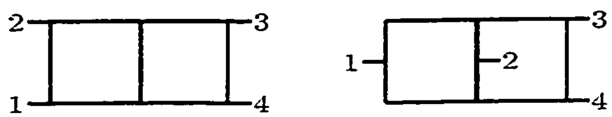


Figure 2.4: The planar and non-planar scalar integrals.

Combining this cut with other two-particle cuts allows one to construct an ansatz for the complete two-loop four-point $N=4$ Yang-Mills amplitude [74]. It turns out that in addition to the planar double-box integral the two-particle cuts also require a non-planar double-box integral, shown in figure 2.4 and given by

$$\mathcal{I}_4^{\text{2-loop,NP}}(s_{12},s_{23}) = \\ \int \frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D} \frac{1}{p^2\,(p-k_2)^2\,(p+q)^2\,(p+q+k_1)^2\,q^2\,(q-k_3)^2\,(q-k_3-k_4)^2}. \tag{2.2.12}$$

To verify that the two-particle cuts do not miss any contributions to the amplitude, one must also calculate the more complicated three-particle cuts. This was done in [74].

Combining all the color-ordered amplitudes, one obtains the full two-loop amplitude,

$$\begin{aligned}\mathcal{A}_4^{\text{2-loop}}(1,2,3,4) = -g^6 s_{12}s_{23}\, A_4^{\text{tree}}(1,2,3,4)\Big(C^{\text{P}}_{1234}\, s_{12}\,\mathcal{I}_4^{\text{2-loop,P}}(s_{12},s_{23}) \\ + C^{\text{P}}_{3421}\, s_{12}\,\mathcal{I}_4^{\text{2-loop,P}}(s_{12},s_{24}) + C^{\text{NP}}_{1234}\, s_{12}\,\mathcal{I}_4^{\text{2-loop,NP}}(s_{12},s_{23}) \\ + C^{\text{NP}}_{3421}\, s_{12}\,\mathcal{I}_4^{\text{2-loop,NP}}(s_{12},s_{24}) + \text{cyclic}\Big)\,,\end{aligned} \tag{2.2.13}$$

where '+ cyclic' instructs one to add the two cyclic permutations of (2,3,4). The group theory factors C^{P}_{1234} and C^{NP}_{1234} are obtained by dressing the diagrams in fig. 2.4 with $\tilde{f}^{abc}$ factors at each vertex. (In ref. [74] the result was presented in a color decomposed form, but to facilitate a comparison to the results of Marcus and Sagnotti we choose not to do so here.) The massless scalar integral functions $\mathcal{I}_4^{\text{2-loop,P}}$ and $\mathcal{I}_4^{\text{2-loop,NP}}$ have recently been calculated analytically in [91, 92]. Here, we will only need their divergences in various dimensions, which are discussed in appendix D.

As discussed in ref. [74], for the case of $N=4$ Yang-Mills theory at two loops all the ambiguities of constructing amplitudes from cuts were resolved and eq. (2.2.13) contains all terms to all orders in the dimensional regulating parameter ϵ. This allows us to continue the amplitude to arbitrary dimension.

2.2.3 Two Loop Ultraviolet Infinities and Counterterms

Although $N=4$ Yang-Mills theory is ultraviolet finite in four dimensions, for $D>4$ the theory is non-renormalizable. We can use eq. (2.2.13) to extract the two-loop $N=4$ counterterms in dimensions $D>4$. Before proceeding to two loops, we recall that the one-loop amplitude, eq. (2.2.7), first diverges at $D=8$, and this divergence is proportional to stA^{tree}. The corresponding gluonic counterterm is fixed by supersymmetry to be

$$\begin{aligned} t_8 F^4 &\equiv t_8^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} F^a_{\mu_1\nu_1} F^b_{\mu_2\nu_2} F^c_{\mu_3\nu_3} F^d_{\mu_4\nu_4} C_{abcd} \\ &= 4! \left(F^a_{\alpha\beta} F^{b\,\beta\gamma} F^c_{\gamma\delta} F^{d\,\delta\alpha} - \frac{1}{4} F^a_{\alpha\beta} F^{b\,\alpha\beta} F^c_{\gamma\delta} F^{d\,\gamma\delta} \right) C_{abcd} \,, \end{aligned} \tag{2.2.14}$$

where C_{abcd} is a group theory factor (which we shall mostly suppress for clarity). In four dimensions we can rewrite this term as

$$t_8 F^4 = \tfrac{3}{2}(F - \tilde{F})^2 (F + \tilde{F})^2 = \tfrac{3}{2}\left(F_{\alpha\beta} - \tilde{F}_{\alpha\beta}\right)\left(F^{\alpha\beta} - \tilde{F}^{\alpha\beta}\right)\left(F_{\gamma\delta} + \tilde{F}_{\gamma\delta}\right)\left(F^{\gamma\delta} + \tilde{F}^{\gamma\delta}\right), \tag{2.2.15}$$

where $\tilde{F}$ is the dual of F. The full counterterm also includes the scalar and fermionic operators obtained by the $N = 4$ completion of the F^4 terms [87]. The two-loop counterterms will be specified in terms of derivatives acting on $t_8 F^4$.

We may extract the coefficient of the counterterm from the ultraviolet divergences in our amplitude. In general, to extract a counterterm from a two-loop amplitude one must take into account sub-divergences and one-loop counterterms. Indeed, the divergences in ref. [87] received contributions from a large number of two-loop graphs with diverse topologies, many of which contained sub-divergences (i.e. $1/\epsilon^2$ poles) which required subtraction before the $1/\epsilon$ poles could be extracted. The cancellation of $1/\epsilon$ poles in the $D = 6$ theory occurred only after summing all diagrams and taking into account one-loop off-shell counterterms.

In our case, however, eq. (2.2.13) is manifestly finite in $D = 6$, since both planar and non-planar double-box integrals first diverge in $D = 7$. The manifest finiteness in $D = 6$ is not an accident and is due to the lack of off-shell sub-divergences when using on-shell cutting rules. It is also simple to extract the $D = 7$ and $D = 9$ counterterms, which are the ones evaluated in ref. [87], by evaluating the ultraviolet poles of the scalar integrals (2.2.11) and (2.2.12). The evaluation of these poles is presented in appendix D.

The counterterm in $D = 7$ is of the form $\partial^2 F^4$, as can be seen from dimensional analysis. The form is again unique (at the on-shell linearized level)

$$t_8^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\partial_\alpha F^a_{\mu_1\nu_1}\partial^\alpha F^b_{\mu_2\nu_2}F^c_{\mu_3\nu_3}F^d_{\mu_4\nu_4}\tilde{C}_{abcd}\,, \tag{2.2.16}$$

where $\tilde{C}_{abcd}$ is another group theory factor. Again, in $D = 4$ such a tensor takes on a simple schematic form,

$$(F-\tilde{F})^2\partial^2(F+\tilde{F})^2\,. \tag{2.2.17}$$

In the notation of Marcus and Sagnotti [87] the counterterm is presented as

$$T_D\Big(F_{\alpha\beta}F^{\beta\gamma}F_{\gamma\delta}F^{\delta\alpha} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} + \cdots\Big)\,. \tag{2.2.18}$$

From the expression (2.2.13) and appendix D we obtain

$$\begin{aligned}
T_7 &= -\frac{g^6\,\pi}{(4\pi)^7\,2\epsilon}\Big[s\Big(\frac{1}{10}(C^{\rm P}_{1234}+C^{\rm P}_{1243})+\frac{2}{15}\,C^{\rm NP}_{1234}\Big)+\text{cyclic}\Big],\\
T_9 &= -\frac{g^6\pi s}{(4\pi)^9\,4\epsilon}\Big[\frac{1}{99792}(-45s^2+18st+2t^2)C^{\rm P}_{1234}+\frac{1}{99792}(-45s^2+18su+2u^2)C^{\rm P}_{1243}\\
&\qquad -\frac{2}{83160}(75s^2+2tu)C^{\rm NP}_{1234}\Big] \;+\; \text{cyclic},
\end{aligned} \tag{2.2.19}$$

corresponding to the $D = 7$ and $D = 9$ counterterms.

To compare eq. (2.2.19) to the results of ref. [87] we must rearrange the group theory factors to coincide with their basis. Performing the necessary rearrangements, we find that all the relative factors agree (up to a typographical error in eq. (4.6) in which the tree group theory factors accompanying the s^3 and t^3 factors were exchanged). After accounting for a different normalization of the operator t_8F^4, as deduced from the one-loop case, the overall factor for T_9 also agrees, while our result for T_7 is larger than that in ref. [87] by a factor of 3/2. Nevertheless, the agreement of the relative factors is rather non-trivial and provides a strong check that the amplitude in eq. (2.2.13) is correct.

2.2.4 Higher Loop Structure

As shown in ref. [74] the two-particle cut sewing equation is the same at any loop order, allowing one to iterate the sewing algebra to all loop orders. The two-particle cuts were performed to all orders in the dimensional regularization parameter ϵ, and are therefore valid in any dimension. However, since this construction is based only on two-particle cuts it is only reliable for integral functions which can be built using such cuts. We call a function which is successively two-particle reducible into a set of four-point trees 'entirely two-particle constructible'. Such contributions can be both planar and non-planar. (Planar topologies give the leading Yang-Mills contributions for a large number of colors.) All two-loop contributions, and the three-loop contributions shown in fig. 2.9, are entirely two-particle constructible. An example of a three-loop non-planar graph which is not entirely two-particle constructible is given in fig. 2.5.

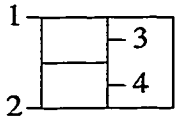


Figure 2.5: This three-loop non-planar graph is not 'entirely two-particle constructible'. In fact, it has no two-particle cuts at all.

The two-particle cut sewing equation leads to a loop-momentum factor insertion rule for planar contributions [74], as shown in fig. 2.6. The pattern is that one takes each L-loop graph in the L-loop amplitude and generates all the possible $(L+1)$-loop graphs by inserting a new leg between each possible pair of internal legs. Diagrams where triangle or bubble subgraphs are created should not be included. The new loop momentum is integrated over, after including an additional factor of $i(\ell_1 + \ell_2)^2$ in the numerator, where ℓ_1 and ℓ_2 are the momenta flowing through each of the legs to which the new line is joined. (This rule is depicted in fig. 2.6.) This procedure does not create any four-point vertices. Each distinct $(L+1)$-loop contribution should be counted once, even though they can be generated in multiple ways. (Contributions which have identical diagrammatic topologies but different numerator factors should be counted as distinct.) The $(L+1)$-loop amplitude is then the sum of all distinct

$(L+1)$-loop graphs. This insertion rule has only been proven for the entirely two-particle constructible contributions.

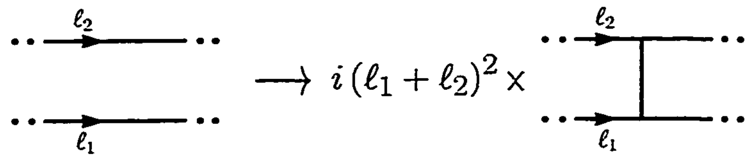


Figure 2.6: Starting from an L-loop planar integral function appearing in an $N=4$ Yang-Mills amplitude we may add an extra line, using this rule. The two-lines on the left represent two lines buried in some L-loop integral.

For both the planar and non-planar entirely two-particle constructible contributions to the Yang-Mills amplitude, the color factors associated with a given diagram are given, as in the two-loop case, by associating a $\tilde{f}^{abc}$ with each vertex and a δ^{ab} with each bond of the $(L+1)$-loop graph. On the other hand, we have investigated the three-particle cuts at three loops and have found non-planar contributions (such as those associated with fig. 2.5) which are not given by the rule of fig. 2.6. Nevertheless, all contributions at three loops have the same power count as the terms obtained from fig. 2.6.

2.2.5 Higher-Loop Divergences in $N=4$ Yang-Mills Theory

The superspace arguments [83] mentioned in section 2.1 lead to predictions for the higher-loop ultraviolet divergences of $N=4$ Yang-Mills theory in $D>4$. We can compare these predictions to the divergences implied by our conjectured form for the $N=4$ four-point amplitudes. The power counting which we will perform assumes that the terms we have are representative of the amplitude.

First consider the planar three-loop case. If we apply the insertion rule of fig. 2.6 to the two-loop amplitudes, we can obtain up to two powers of loop momentum in the numerators of the three-loop integrands. To find the dimension where the amplitude first becomes divergent we focus on the diagrams with two powers of loop momenta in the numerators, since they are the most divergent ones. For these diagrams the superficial degree of divergence is obtained by ignoring distinctions between the momenta of the various loops and dropping all external momenta, thus reducing the integral to

$$\int (d^D p)^3 \frac{(p^2)}{(p^2)^{10}} \,. \tag{2.2.20}$$

This integral is ultraviolet divergent for $D \geq 6$.

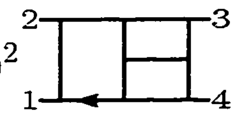
$-ist\, A_4^{\text{tree}}\, s(\ell + k_4)^2$  $+$ cyclic

Figure 2.7: The leading-color diagrams that diverge in $D = 6$. The arrow indicates the line with momentum ℓ.

This analysis easily generalizes to all loop orders. From fig. 2.6 for each additional loop the maximum number of powers of loop momentum in the numerator increases by two. Thus, for $L > 1$ loops we expect that the most divergent integrals have $2L-4$ powers of loop momenta in the numerator. These integrals will reduce to

$$\int (d^D p)^L \frac{(p^2)^{(L-2)}}{(p^2)^{3L+1}} \,. \tag{2.2.21}$$

(The $L = 1$ case is special and must be treated separately.) These integrals are finite for

$$D < \frac{6}{L} + 4 \,, \qquad (L > 1)\,. \tag{2.2.22}$$

This degree of divergence is eight powers less than the maximum for non-supersymmetric Yang-Mills theory.

This $N = 4$ power count has differences with the conventional one based on superspace arguments [83]. Specifically, for dimensions $D = 5, 6$ and 7 the amplitudes first diverge at $L = 6, 3$ and 2 loops. The corresponding superspace arguments indicate that the first divergence may occur at $L = 4, 3$ and 2, respectively. Since the superspace arguments of ref. [83] only place a bound on finiteness, our results at four and five loops are not inconsistent. However, the ultraviolet behavior of the amplitudes seems to indicate that the extra symmetries in $N = 4$ Yang-Mills theory, which are

Dimension D	Loop	Counterterm
8	1	F^4
7	2	$\partial^2 F^4$
6	3	$\partial^2 F^4$
5	6	$\partial^2 F^4$

Table 2.1: For $N = 4$ Yang-Mills theory in D dimensions, the number of loops at which the *first* ultraviolet divergence occurs for four-point amplitudes, and the generic form of the associated counterterm. In each case the degree of divergence is logarithmic, but the specific color factors will differ.

not taken into account by the off-shell $N = 2$ superspace arguments, are important to understanding its divergences in $D > 4$. Curiously, the finiteness condition (2.2.22) agrees with the power count based on the assumption of the existence of an unconstrained off-shell covariant $N = 4$ superspace formalism [86, 57]. This agreement is probably accidental, because it is known that such a formalism does not exist; for example, the two-loop $D = 7$ counterterm has the wrong group-theory structure (although the right dimension) to be written as an $N = 4$ superspace integral [87].

Combining the $N = 4$ finiteness condition (2.2.22) with those for $N = 1, 2$ [57] (for which off-shell superspaces for the full supersymmetry exist) we find that an $L > 1$ loop amplitude is finite for

$$D < \frac{2(N-1)}{L} + 4\,, \qquad\qquad N = 1, 2, 4\,. \tag{2.2.23}$$

2.3 Higher-Loop $N = 8$ Supergravity Amplitudes

In this section we present the exact result for the $N = 8$ two-loop four-point amplitude, in terms of scalar integral functions. We establish a simple "squaring" relationship between this amplitude and the corresponding $N = 4$ amplitude discussed above. We also evaluate the ultraviolet divergences of this amplitude in various dimensions. Moreover, we present a conjecture for a precise relationship between (parts of) the

$N = 4$ Yang-Mills and $N = 8$ supergravity four-point amplitudes to all loop orders, extending the tree and one-loop relationships discussed earlier. Finally, we discuss the supergravity power-counting rules suggested by our conjecture.

2.3.1 Two-Loop $N = 8$ Supergravity Amplitudes

In section 2.2.2 we have outlined the reconstruction of the $N = 4$ four-point amplitudes from their cuts. This procedure can be repeated for the case of $N = 8$ supergravity. Again, let us start with a one-loop amplitude. The cut in the s_{12} channel is given by,

$$M_4^{\text{one-loop}}(1,2,3,4)\Big|_{s_{12}-\text{cut}} = \int \frac{d^D\ell_1}{(2\pi)^D} \sum_{S_1,S_2\in\{N=8\}} \frac{i}{\ell_1^2} M_4^{\text{tree}}(-\ell_1^{S_1},1,2,\ell_2^{S_2}) \frac{i}{\ell_2^2} M_4^{\text{tree}}(-\ell_2^{S_2},3,4,\ell_1^{S_1})\Big|_{\ell_1^2=\ell_2^2=0}, \tag{2.3.1}$$

where $\ell_2 = \ell_1 - k_1 - k_2$, $S_{1,2}$ label states of the $N = 8$ multiplet, M_4^{tree} are the tree amplitudes, and the $N = 8$ labels on the external states (including their helicity) have again been suppressed. As we have discussed in section 2.2.2, the key relation for evaluating the $N = 4$ two-particle cuts is

$$\sum_{S_1,S_2\in\{N=4\}} A_4^{\text{tree}}(-\ell_1^{S_1},1,2,\ell_2^{S_2}) \times A_4^{\text{tree}}(-\ell_2^{S_2},3,4,\ell_1^{S_1}) = -i\frac{st}{(\ell_1-k_1)^2(\ell_2-k_3)^2} A_4^{\text{tree}}(1,2,3,4)\,. \tag{2.3.2}$$

Using the four-point KLT relation (1.2.2), we can obtain the equivalent relation for $N = 8$ supergravity,

$$\begin{aligned}
\sum_{N=8 \text{ states}} & M_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times M_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1) \\
&= -s_{12}^2 \Big(\sum_{N=4 \text{ states}} A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times A_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1) \Big) \\
&\qquad\qquad \times \Big(\sum_{N=4 \text{ states}} A_4^{\text{tree}}(\ell_2, 1, 2, -\ell_1) \times A_4^{\text{tree}}(\ell_1, 3, 4, -\ell_2) \Big) \\
&= s_{12}^2 (s_{12} s_{23})^2 [A_4^{\text{tree}}(1,2,3,4)]^2 \frac{1}{(\ell_1 - k_1)^2 (\ell_2 - k_3)^2 (\ell_2 + k_1)^2 (\ell_1 + k_3)^2} \\
&= i\, s_{12}^2\, s_{12} s_{23} s_{13}\, M_4^{\text{tree}}(1,2,3,4) \frac{1}{(\ell_1 - k_1)^2 (\ell_2 - k_3)^2 (\ell_1 - k_2)^2 (\ell_2 - k_4)^2} \,.
\end{aligned} \tag{2.3.3}$$

In eq. (2.3.3) we have used the decomposition of a state in the $N = 8$ multiplet into a 'left' and a 'right' $N = 4$ state, and the fact that summing over the $N = 8$ multiplet is equivalent to summing over the left and right $N = 4$ multiplets independently. We then perform a partial-fraction decomposition of the denominators (using the on-shell conditions),

$$\begin{aligned}
- \frac{s_{12}}{(\ell_1 - k_1)^2 (\ell_1 - k_2)^2} &= \frac{1}{(\ell_1 - k_1)^2} + \frac{1}{(\ell_1 - k_2)^2} \,, \\
- \frac{s_{12}}{(\ell_2 - k_3)^2 (\ell_2 - k_4)^2} &= \frac{1}{(\ell_2 - k_3)^2} + \frac{1}{(\ell_2 - k_4)^2} \,,
\end{aligned} \tag{2.3.4}$$

to obtain the basic $N = 8$ two-particle on-shell sewing relation,

$$\begin{aligned}
\sum_{N=8 \text{ states}} & M_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times M_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1) \\
&= i\, s_{12} s_{23} s_{13} M_4^{\text{tree}}(1,2,3,4) \left[\frac{1}{(\ell_1 - k_1)^2} + \frac{1}{(\ell_1 - k_2)^2} \right] \\
&\qquad \times \left[\frac{1}{(\ell_2 - k_3)^2} + \frac{1}{(\ell_2 - k_4)^2} \right] \,,
\end{aligned} \tag{2.3.5}$$

where the ℓ_i are on-shell. Using the relation (2.3.5), we can evaluate the cut in (2.3.1):

$$M_4^{\text{one-loop}}(1,2,3,4)\Big|_{s_{12}-\text{cut}} = \\ -is_{12}s_{23}s_{13}M_4^{\text{tree}}(1,2,3,4)\Big(\mathcal{I}_4(s_{12},s_{23})+\mathcal{I}_4(s_{12},s_{13})\Big)\Big|_{s_{12}-\text{cut}}. \tag{2.3.6}$$

Combining this result with the analogous cuts in s_{23} and s_{13} channels, we can obtain the full one-loop amplitude,

$$M_4^{\text{one-loop}}(1,2,3,4) = \\ -is_{12}s_{23}s_{13}M_4^{\text{tree}}(1,2,3,4)\Big(\mathcal{I}_4(s_{12},s_{23})+\mathcal{I}_4(s_{12},s_{13})+\mathcal{I}_4(s_{23},s_{13})\Big). \tag{2.3.7}$$

Using the four-point KLT relation (1.2.2), one can prove that

$$s_{12}s_{23}s_{13}M_4^{\text{tree}}(1,2,3,4) = -i\Big(s_{12}s_{23}A_4^{\text{tree}}(1,2,3,4)\Big)^2. \tag{2.3.8}$$

This identity implies a simple relationship between the $N=8$ amplitude (2.3.7) and its $N=4$ counterpart, (2.2.7): the prefactor multiplying the scalar box integrals in (2.3.7) is just the square of the corresponding prefactor in (2.2.7).

Just like in the $N=4$ case, the sewing relation (2.3.5) has a "self-replicating" structure: the sewing of two tree amplitudes results in another tree amlitude, multiplied by simple scalar factors. This means that it can be iterated to obtain the two-particle cuts of higher-loop amplitudes. For example, let us again consider the two-particle s_{12}-cut with a tree amplitude on the left and a one-loop amplitude on the right,

$$M_4^{\text{2-loop}}(1,2,3,4)\Big|_{s_{12}\text{-cut}} = \\ \int\frac{d^D\ell_1}{(2\pi)^D}\sum_{N=8\text{ states}}\frac{i}{\ell_1^2}M_4^{\text{tree}}(-\ell_1,1,2,\ell_2)\,\frac{i}{\ell_2^2}\,M_4^{\text{1-loop}}(-\ell_2,3,4,\ell_1)\Big|_{\ell_1^2=\ell_2^2=0}. \tag{2.3.9}$$

Inserting eq. (2.3.7) for $M_4^{\text{1-loop}}$ and applying the sewing relation (2.3.5), we have

$$\begin{aligned} M_4^{\text{2-loop}}(1,2,3,4)\Big|_{s_{12}\text{-cut}} &= -s_{12}s_{23}s_{13}M_4^{\text{tree}}\int \frac{d^D\ell_1}{(2\pi)^D} s_{12}(\ell_2-k_3)^2(\ell_2-k_4)^2 \\ &\times \Big[\frac{1}{(\ell_1-k_1)^2}+\frac{1}{(\ell_1-k_2)^2}\Big]\frac{i}{\ell_1^2}\Big[\frac{s_{12}}{(\ell_2-k_3)^2(\ell_2-k_4)^2}\Big]\frac{i}{\ell_2^2} \\ &\times \big[\mathcal{I}_4^{\text{1-loop}}(s_{12},(\ell_2-k_3)^2)+\mathcal{I}_4^{\text{1-loop}}((\ell_2-k_3)^2,(\ell_2-k_4)^2) \\ &\qquad + \mathcal{I}_4^{\text{1-loop}}((\ell_2-k_4)^2,s_{12})\big]\Big|_{\ell_1^2=\ell_2^2=0}, \end{aligned} \tag{2.3.10}$$

where we have combined the two propagators on a common denominator in the second brackets in order to make a cancellation manifest. The unwanted propagators cancel and our final result is remarkably simple,

$$\begin{aligned} M_4^{\text{2-loop}}(1,2,3,4)\Big|_{s_{12}\text{-cut}} &= s_{12}s_{23}s_{13}M_4^{\text{tree}}s_{12}^2\Big(\mathcal{I}_4^{\text{2-loop,P}}(s_{12},s_{23})+\mathcal{I}_4^{\text{2-loop,P}}(s_{12},s_{13}) \\ &\quad + \mathcal{I}_4^{\text{2-loop,NP}}(s_{12},s_{23})+\mathcal{I}_4^{\text{2-loop,NP}}(s_{12},s_{13})\Big)\Big|_{s_{12}\text{-cut}}, \end{aligned} \tag{2.3.11}$$

where the scalar integrals $\mathcal{I}_4^{\text{2-loop,P}}$ and $\mathcal{I}_4^{\text{2-loop,NP}}$ are defined in (2.2.11) and (2.2.12), respectively. The analysis of the two-particle s_{13}- and s_{23}-channel cuts is identical. We emphasize that the results for the two-particle cuts are valid in any dimension and do not rely on any four-dimensional properties.

It is simple to find a single function with the correct two-particle cuts in all three channels; the answer is

$$\begin{aligned} \mathcal{M}_4^{\text{2-loop}}(1,2,3,4) &= \left(\frac{\kappa}{2}\right)^6 s_{12}s_{23}s_{13}\, M_4^{\text{tree}}(1,2,3,4)\Big(s_{12}^2\,\mathcal{I}_4^{\text{2-loop,P}}(s_{12},s_{23}) \\ &+ s_{12}^2\,\mathcal{I}_4^{\text{2-loop,P}}(s_{12},s_{24}) + s_{12}^2\,\mathcal{I}_4^{\text{2-loop,NP}}(s_{12},s_{23}) + s_{12}^2\,\mathcal{I}_4^{\text{2-loop,NP}}(s_{12},s_{24}) \quad + \text{cyclic}\Big). \end{aligned} \tag{2.3.12}$$

Here we have reintroduced the gravitational coupling constant κ according to (1.2.3), and '+ cyclic' instructs one to add the two cyclic permutations of (2,3,4), just as in eq. (2.2.13).

In order to confirm that (2.3.12) is the complete answer for the two-loop amplitude, one must also evaluate three-particle cuts. This calculation can also be simplified considerably by using the KLT relations (1.2.2). We will present it in appendix E.

As we will see there, no new terms are necessary to satisfy the three-particle cutting equations, showing that (2.3.12) is indeed the full answer.

It is instructive to compare the $N = 8$ two-loop amplitude (2.3.12) with the corresponding $N = 4$ result (2.2.13). We immediately observe that the scalar integrals which enter the two amplitudes are identical. Moreover, we can use the identity (2.3.8) to rewrite (2.3.12) as

$$\mathcal{M}_4^{\text{2-loop}}(1,2,3,4) = -i\left(\frac{\kappa}{2}\right)^6 [s_{12}s_{23}\, A_4^{\text{tree}}(1,2,3,4)]^2 \Big(s_{12}^2 \mathcal{I}_4^{\text{2-loop,P}}(s_{12}, s_{23}) + s_{12}^2 \mathcal{I}_4^{\text{2-loop,P}}(s_{12}, s_{24}) + s_{12}^2 \mathcal{I}_4^{\text{2-loop,NP}}(s_{12}, s_{23}) + s_{12}^2 \mathcal{I}_4^{\text{2-loop,NP}}(s_{12}, s_{24}) \quad + \text{cyclic}\Big)\,, \tag{2.3.13}$$

where A_4^{tree} is the $N = 4$ Yang-Mills four-gluon tree amplitude. This shows that the coefficients of the scalar box integrals in the $N = 8$ amplitude can be obtained by squaring the coefficients of the corresponding integrals in the $N = 4$ amplitude, after dropping the group theory factors (see figure 2.8.) This relationship is reminiscent of the tree-level KLT relations, as well as the relations between one-loop amplitudes discussed above.

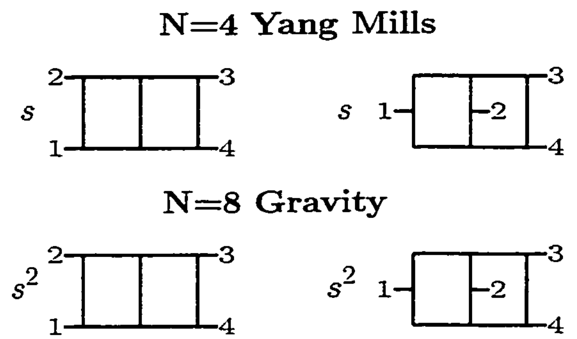


Figure 2.8: The expected relationship between two-loop contributions to $N = 8$ four-graviton amplitudes and $N = 4$ four-gluon amplitudes: the graviton coefficients are squares of the gluon coefficients. The $N = 4$ and $N = 8$ contributions depicted here are to be multiplied respectively by factors of $-g^6 st A_4^{\text{tree}}$ (dropping the group theory factor) and $-i(\kappa/2)^6[st A_4^{\text{tree}}]^2$.

2.3.2 Two Loop Ultraviolet Divergences of $N = 8$ Supergravity

The amplitude (2.3.12) is ultraviolet finite in $D = 4$. This is consistent with the supersymmetric non-renormalization theorem [78]. However, just like in the $N = 4$ case, the amplitudes do have ultraviolet divergences in $D \geq 7$. Using the results of appendix D, we can evaluate these divergences in various dimensions. In particular, we obtain

$$\begin{aligned}
\mathcal{M}_4^{\text{2-loop},\ D=7-2\epsilon}\big|_{\text{pole}} &= \frac{1}{2\epsilon}\frac{1}{(4\pi)^7}\frac{\pi}{3}(s^2+t^2+u^2)\times\left(\frac{\kappa}{2}\right)^6 \times stuM_4^{\text{tree}},\\
\mathcal{M}_4^{\text{2-loop},\ D=9-2\epsilon}\big|_{\text{pole}} &= \frac{1}{4\epsilon}\frac{1}{(4\pi)^9}\frac{-13\pi}{9072}(s^2+t^2+u^2)^2\times\left(\frac{\kappa}{2}\right)^6 \times stuM_4^{\text{tree}},\\
\mathcal{M}_4^{\text{2-loop},\ D=11-2\epsilon}\big|_{\text{pole}} &= \frac{1}{48\epsilon}\frac{1}{(4\pi)^{11}}\frac{\pi}{5791500}\left(438(s^6+t^6+u^6)-53s^2t^2u^2\right)\times\left(\frac{\kappa}{2}\right)^6\\
&\qquad\qquad \times stuM_4^{\text{tree}}.
\end{aligned} \tag{2.3.14}$$

There are no sub-divergences because one-loop divergences are absent in odd dimensions when using dimensional regularization.

In both $D = 8$ and $D = 10$, the box integrals encountered in the one-loop $N = 4$ Yang-Mills amplitude (2.2.7) and $N = 8$ supergravity amplitude (2.3.7) do have ultraviolet poles. Curiously, for the $N = 8$ case in $D = 10$ the coefficient of the pole cancels in dimensional regularization. (The pole in the box integral $\mathcal{I}_4^{\text{1-loop}}(s_{12}, s_{23})$ is proportional to $s_{12} + s_{23}$, and the sum over boxes in eq. (2.3.7) cancels using $s_{12} + s_{23} + s_{13} = 0$.) This cancellation between quadratically divergent integrals may well be an artifact of dimensional regularization. In any case, we can investigate whether the cancellation persists to two loops. To do so, we have calculated the ultraviolet divergences of subtracted planar and non-planar double-box integrals in $D = 10$. For completeness we have also calculated the divergence in $D = 8$. A subtraction is required because the individual integrals have one-loop sub-divergences. The results for the integrals are presented in appendix D. The sum over double-box integrals in eq. (2.3.12) then yields

$$\mathcal{M}_4^{\text{2-loop, } D=8-2\epsilon}\Big|_{\text{pole}} = \frac{1}{2\,(4\pi)^8}\Big(-\frac{1}{24\,\epsilon^2}+\frac{1}{144\epsilon}\Big)(s^3+t^3+u^3)\times\Big(\frac{\kappa}{2}\Big)^6 \times stu M_4^{\text{tree}}\,,$$
$$\mathcal{M}_4^{\text{2-loop, } D=10-2\epsilon}\Big|_{\text{pole}} = \frac{1}{12\epsilon\,(4\pi)^{10}}\frac{-13}{25920}\,stu\,(s^2+t^2+u^2)\times\Big(\frac{\kappa}{2}\Big)^6 \times stu M_4^{\text{tree}}\,,$$
(2.3.15)

so there is indeed a two-loop divergence in $D = 10$.

In all dimensions $D \leq 11$, for four graviton external states, the linearized counterterms take the form of derivatives acting on

$$t_8 t_8 R^4 \equiv t_8^{\mu_1\mu_2\cdots\mu_8}\, t_8^{\nu_1\nu_2\cdots\nu_8}\, R_{\mu_1\mu_2\nu_1\nu_2}\, R_{\mu_3\mu_4\nu_3\nu_4}\, R_{\mu_5\mu_6\nu_5\nu_6}\, R_{\mu_7\mu_8\nu_7\nu_8}\,, \qquad (2.3.16)$$

plus the appropriate $N = 8$ completion. As mentioned in the section 2.1, the operator (2.3.16) appears in the tree-level superstring effective action. It also appears as the one-loop counterterm for $N = 8$ supergravity in $D = 8$. Finally, it is thought to appear in the M-theory one-loop effective action [93].

2.3.3 Higher-loop Conjecture

We conjecture that to all orders in the perturbative expansion the four-point $N = 8$ supergravity amplitude may be found by squaring the coefficients and numerator factors of all the loop integrals that appear in the $N = 4$ Yang-Mills amplitude at the same order, after stripping away the color factors. We have seen already that this statement is true for one- and two-loop amplitudes, which can be calculated completely.

In order to verify the conjecture at L loops one would need to investigate cuts with up to $(L + 1)$ intermediate particles. Nevertheless, as we have discussed in section 2.2.4, some of the integral coefficients and numerators can be obtained just from the two-particle cuts. Due to a simple "self-replicating" structure of the sewing relation (2.3.5), these cuts can be iterated for any number of loops. This calculation supports our conjecture. For example, fig. 2.9 contains a few sample three-loop integrals which are entirely two-particle constructible, and their associated coefficients for the case of $N = 4$ Yang-Mills theory [74]; the $N = 8$ supergravity coefficients are given by squaring the super-Yang-Mills coefficients. This provides an explicit example of three-loop relationships between contributions to the Yang-Mills and gravity amplitudes for the cases of $N = 4$ and $N = 8$ supersymmetry. It would be interesting to determine whether squaring relations do indeed continue to hold for all remaining three-loop diagram topologies.

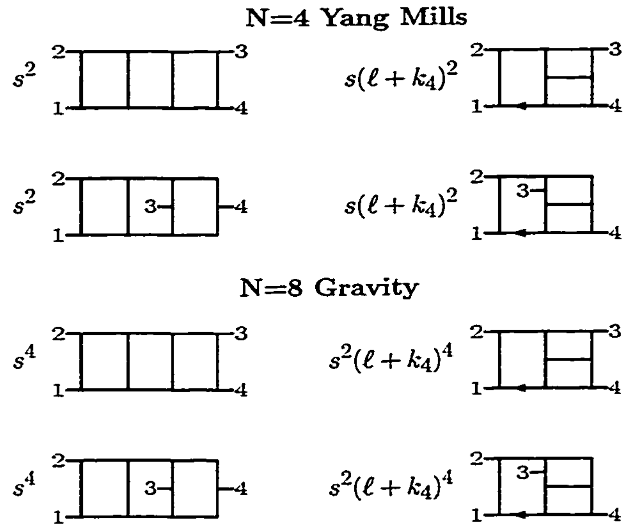


Figure 2.9: Some sample three-loop integrals and their coefficients for $N = 4$ Yang-Mills theory and for $N = 8$ supergravity. The coefficients for $N = 8$ supergravity are just the squares of those for $N = 4$ Yang-Mills theory. The $N = 4$ and $N = 8$ contributions depicted here are to be multiplied respectively by overall factors of $-ig^8 st A_4^{\rm tree}$ and $(\kappa/2)^8 [st A_4^{\rm tree}]^2$.

2.3.4 Higher-Loop Divergences in $N = 8$ Supergravity

In this subsection we discuss the ultraviolet behavior of $N = 8$ supergravity arising from our all-loop conjecture for the form of the four-point amplitude. This 'squaring' conjecture gives twice as many powers of loop momentum in the numerator of the integrand as for Yang-Mills theory. The power-counting equation that describes the leading divergent behavior for $N = 4$ Yang-Mills theory, eq. (2.2.21), becomes for $N = 8$ supergravity at L loops,

$$\int (d^D p)^L \frac{(p^2)^{2(L-2)}}{(p^2)^{3L+1}} \,. \tag{2.3.17}$$

This integral will be finite when

$$D < \frac{10}{L} + 2 \,, \qquad\qquad (L > 1) \,. \tag{2.3.18}$$

The results of this analysis are summarized in table 2.2. In particular, in $D = 4$ no three-loop divergence appears — contrary to expectations from a superspace analysis [84, 83] — and the first R^4-type counterterm occurs at five loops. The divergence will have the same kinematical structure as the $D = 7$ divergence in eq. (2.3.14), but with a different non-vanishing numerical coefficient.

Dimension	**Loop**	**Degree of Divergence**	**Counterterm**
8	1	logarithmic	R^4
7	2	logarithmic	$\partial^4 R^4$
6	3	quadratic	$\partial^6 R^4$
5	4	quadratic	$\partial^6 R^4$
4	5	logarithmic	$\partial^4 R^4$

Table 2.2: The relationship between dimensionality and the number of loops at which the *first* ultraviolet divergence should occur in the $N = 8$ supergravity four-point amplitude. The form of the associated counterterm assumes the use of dimensional regularization.

We again emphasize that while the one- and two-loop entries in table 2.2 are based on complete calculations of the amplitudes, the results beyond two-loop level remain conjectural. A complete proof of the ultraviolet behavior of the L-loop amplitude would require an analysis of all contributions to the cuts.

2.4 Discussion

In this chapter we have continued our study of the relationships between gravity and gauge theory S-matrix elements. We have chosen to investigate the maximally supersymmetric versions of these theories, $N = 4$ super-Yang-Mills theory and $N = 8$ supergravity; such theories are heavily constrained and their amplitudes are therefore relatively simple.

At tree and one-loop level, the four-point $N = 4$ and $N = 8$ amplitudes exhibit a 'squaring' relationship. In this chapter, we have calculated the complete two-loop four-point $N = 8$ amplitude. Comparing this result with its $N = 4$ counterpart, we have shown that the squaring relationship persists at the two-loop level. Motivated by these results, we have conjectured that a similar relationship between the amplitudes will extend to all loop orders. Evidence for this conjecture is provided by two-particle cuts, which can be iterated for any number of loops. Further checks require calculating the cuts with more intermediate states, though this becomes increasingly difficult. Nevertheless, the cutting method is overwhelmingly simpler than a corresponding Feynman diagram calculation.

It is well known that $N = 8$ supergravity is free of ultraviolet divergences in four dimensions at one- and two-loop levels. The existence of a three-loop counterterm compatible with $N = 8$ supersymmetry [15] leads one to expect that a divergence will occur there. So far, however, no explicit calculation has been presented to verify this expectation. Here, we have been able to evaluate the coefficients of certain "entirely two-particle constructible" contributions to the $N = 8$ three-loop four-point function. These contributions *do not* have an ultraviolet divergence in four dimensions. To reconstruct the three-loop amplitude completely, one would have to evaluate more complicated three- and four-particle cuts. If such a calculation does not detect an ultraviolet divergence, we would have to conclude that the coefficient of the candidate counterterm vanishes, presumably because of some unknown symmetry of the theory.

Another example of "better-than-expected" ultraviolet behaviour in $N = 8$ supergravity is given by our exact two-loop amplitude: We find that it does not diverge in $D < 7$ dimensions, in contrast with expectations from superspace power counting [84, 83]. Note, however, that since no $N = 8$ superspace formalism exists, the superspace arguments only take into account the constraints from one-half of the full algebra. Our results indicate that this approach misses some of the supersymmetric cancellations.

At five loops, we have found an ultraviolet divergence in the "two-particle constructible" contributions to the amplitude which we have evaluated. Again, this does not constitute a proof that the full five-loop amplitude has such a divergence, since it could in principle be cancelled by other contributions. Cuts with up to six intermediate particles would be necessary for the complete reconstruction of the amplitude in this case. If no cancellations occur, this divergence would mean that $N = 8$ supergravty is indeed non-renormalizable, and cannot be thought of as the fundamental theory of gravity.

Chapter 3

Collider Signatures of Large Extra Dimensions

3.1 Introduction

Recently, Arakni-Hamed, Dimopoulos and Dvali (ADD) have proposed a novel solution to the hierarchy problem [6, 16], which does not rely either on supersymmetry or technicolor. Instead, they have postulated that the fundamental gravitational scale M, the scale at which gravity becomes strong, is not much higher than the scale of electroweak symmetry breaking. The hierarchy problem is thus nullified. The observed weakness of gravitational interactions at large distances is then explained by the presence of n extra dimensions, which are much larger than the fundamental length scale $\hbar/M$. The size of the extra dimensions R is related to the scale M and the four-dimenisonal Newton's constant G_N. The usual 4-dimensional laws of gravity, such as Newton's law, are valid at distances larger than R, but break down at shorter scales. For $n = 1$, the model requires R to be around 10^{14} cm; this is clearly inconsistent with astronomical observations. However, already for $n = 2$ the required values of R are sub-millimeter; macroscopic measuremets of gravity do not yet provide information about the behaviour of gravity at those scales. Thus, the cases of $n = 2$ and higher are not excluded by gravitational experiments.

Unlike gravity, Standard Model interactions have been probed at very short length scales, down to about $\hbar/(100 \text{ GeV}) \sim 10^{-16}$ cm. No evidence has been found that quarks, leptons and gauge bosons of the Standard Model can feel extra dimensions. Therefore, in the ADD model, these particles are assumed to be confined to a hypersurface within the full space-time whose thickness is at most around $\hbar/(1 \text{ TeV})$. An example of such localization is provided by D branes of string theory [94]. In the minimal version of the model, gravity is the only field which can freely propagate in the extra dimensions.

The coupling of the Standard Model states to higher-dimensional gravitons is suppressed by powers of the fundamental scale M. But this scale is around 1 TeV; this means that the gravitons can be emitted at substantial rates in particle collisions at typical collider energies. This could lead to large deviations of cross sections from the Standard Model predictions; the fact that such deviations are not observed experimentally can be used to put constraints on the parameters of the model, such as M and R. In the first part of this chapter, we will study graviton emission processes at e^+e^- and $\bar{p}p$ colliders.

The fact that the gravitons can propagate in the extra dimensions is a generic prediction of the ADD model, which does not depend on which theory describes quantum gravity at the TeV scale. Moreover, as we discuss in section 3.3, graviton emission at energies below the fundamental scale can be described by an effective Lagrangian. The coefficient of the leading operator contributing to the process is also independent of the details of physics at the fundamental scale. This will allow us to use low-energy data to put model-independent constraints on the parameters of the model.

Other experimental signatures of large extra dimensions do depend on the structure of the fundamental theory of gravity. The only known candidate for a consistent description of quantum gravity is string theory, so it is natural to assume that some version of this theory will desribe the TeV-scale physics in the ADD model. We will make this assumption in the second part of this chapter (section 3.4). A generic prediction of string theory is the appearance of String Regge (SR) excitations of the Standard Model particles, as well as gravitons, at the string scale, which in this case is around 1 TeV. These particles have the same Standard Model quantum numbers as their massless counterparts, but can have higher spin. It is clear that the presence of these extra states at the TeV scale can affect particle physics processes at current collider energies. We will use a simple stringy toy model to study their effects in sample QED processes. In the context of this model, we will derive current experimental bounds on the string scale, which can be converted into bounds on M. Unlike the bounds obtained in the first part of this chapter, these results are highly model-dependent. We will also discuss the effect of SR exchanges on the graviton emission processes discussed in section 3.3.

3.2 Framework and Assumptions

In the first part of our analysis, we will consider processes for which the typical center-of-mass energies E are well below the fundamental scale of the theory M. Such processes are not expected to be sensitive to the details of the physics at the fundamental scale, so our analysis will be rather model-independent. The only assumptions we make concern the low-energy spectrum of the model, and the properties of the background geometry. Once we specify these features, we can write down the most general action compatible with the symmetries of the model. This action will contain operators of various mass dimensions. At energies well below M, the operators of the lowest dimension will be most relevant, since the operators of higher dimensions are suppressed by powers of E/M. The coefficients of the operators that will be important for our analysis turn out to be completely independent of the physics at the scale M.

We assume that the full space-time (referred to as "the bulk") is d-dimensional, and has the topology of $\mathbf{R}^4 \times \mathbf{T}^n$, where $n = d-4$ is the number of extra dimensions. For simplicity, we will take the torus $\mathbf{T}^n$ to have the same periodicity, $2\pi R$, in all directions. Let us denote the bulk coordinates by $X^M, M = 0 \ldots d-1$, and the bulk metric by $G^{MN}(X)$. The bulk gravitational action is,

$$S_{\text{bulk}} = \int d^d X \sqrt{-G} \hat{M}^{d-2} \mathcal{R} + \ldots, \tag{3.2.1}$$

where $\hat{M}$ is the d-dimensional reduced Planck scale. We define the *fundamental gravitational scale* of the theory, M, via

$$M^{n+2} = (2\pi)^n \hat{M}^{n+2}. \quad (3.2.2)$$

The advantage of using M is that this scale is directly linked to the hierarchy problem [21]. We will present our results in this chapter in terms of constraints on this scale.

The ellipsis in (3.2.1) indicates the terms involving higher-dimensional gravitational invariants, suppressed by powers of M. Note that we have assumed here that the gravitational field is the only light field which can propagate in the extra dimensions. It is likely that in realistic models extra light degrees of freedom will appear in the bulk, coupled to the Standard Model fields with gravitational strength. We will discuss the effect of such fields on the results of our analysis in section 3.5.

We further assume that the Standard Model fields are confined to a four-dimensional hypersurface, which we will refer to as "the brane", which spans the $\mathbf{R}^4$ part of the full space-time. The coordinates on the brane will be denoted by x^μ, $\mu = 0 \ldots 3$, and the bulk coordinate of a point x on the brane is denoted by $Y^M(x)$. Then, the metric induced on the brane is given by

$$g_{\mu\nu}(x) = G_{MN}(Y(x))\partial_\mu Y^M \partial_\nu Y^N. \quad (3.2.3)$$

The leading-order couplings of the Standard Model degrees of freedom to gravity are described by the usual four-dimensional general relativity Lagrangian, with the metric given by (3.2.3). For example, the Lagrangian for scalar and gauge fields is given by

$$S_{\text{brane}} = \int d^4x\sqrt{-g}\left(-f^4 + \frac{g^{\mu\nu}}{2}D_\mu\phi D_\nu\phi - V(\phi) - \frac{g^{\mu\nu}g^{\rho\sigma}}{4}F_{\mu\rho}F_{\nu\sigma} + \ldots\right), \quad (3.2.4)$$

where f^4 denotes the tension of the brane. We will assume that this tension is of the same order as M, which is usually the case in string theories (see eq. (3.4.3)). With this assumption, the low-energy processes will not be sensitive to the structure of the brane, and it can effectively be considered as a rigid wall. The ellipsis in (3.2.4) again indicates higher-dimensional invariants which are subdominant at low energies.

Now, we have to expand the actions (3.2.1) and (3.2.4) around the vacuum state. Expanding the metric and the brane coordinate Y gives,

$$\begin{aligned} G_{MN}(X) &= \eta_{MN} + \sqrt{2}\frac{H_{MN}}{\hat{M}^{d/2-1}} + \dots\ , \\ Y^M(X) &= \delta^M_\mu x^\mu + \frac{y^M}{f^2} + \dots\ , \end{aligned} \tag{3.2.5}$$

where H is the higher-dimensional graviton, and y is the field associated with the motion of the brane. The dimensionful coefficients of the fields H_{MN} and y^M have been chosen such that these fields have canonical kinetic terms. (The kinetic term for y can be found by expanding (3.2.3) up to quadratic terms, and plugging it into (3.2.4).) Due to reparametrization invariance, the fields $y^\mu, \mu = 0 \dots 3$, can be gauged away [95], so the only physical components of the y field are four-dimensional scalars.

Plugging (3.2.5) into (3.2.4) (and the analogous expression for fermions), we find that at the leading order in an expansion in $1/M$ the Standard Model fields only couple to the $\mu\nu$-components of the higher-dimensional graviton. The coupling is given by

$$S_{\text{l.o.}} = -\frac{\sqrt{2}}{\hat{M}^{d/2-1}} \int d^4x\, (H^{\mu\nu}(x; y=0) T_{\mu\nu}(x) + \dots)\,, \tag{3.2.6}$$

where $T_{\mu\nu}$ is the conserved energy-momentum tensor of the matter. The other components of the graviton, as well as the y fields, only couple at higher order in the $1/M$ expansion. This is due to the fact that they carry Lorentz indices in the directions transverse to the brane. Any Lorentz invariant operator has to involve at least two of these fields, and is therefore suppressed by higher powers of M than the operators in (3.2.6). In writing (3.2.6), we have assumed that the energy-momentum tensor $T_{\mu\nu}$ is traceless. In this chapter, we will only consider processes involving (effectively) massless Standard Model degrees of freedom, such as photons and electrons, for which this assumption is true. If the trace of the energy-momentum tensor does not vanish, there will be an extra term in the Lagrangian, proportional to $H^M_M T^\nu_\nu$.

In practice, it is convenient to decompose the higher-dimensional graviton field into its Kaluza-Klein (KK) modes, according to

$$H^{\mu\nu}(x;y) = \frac{1}{(2\pi R)^{n/2}} \sum_k e^{ik\cdot y/R} h_k^{\mu\nu}(x), \tag{3.2.7}$$

where the sum is over all possible integer-valued n-dimensional vectors k. The KK modes $h_k^{\mu\nu}$ are four-dimensional spin-2 fields. Their kinetic terms are canonically normalized by virtue of the choice of normalization in (3.2.7), and their mass is given by

$$m_k^2 = \frac{\sum_{i=1}^n k_i^2}{R^2}.$$

Since the k_i are integers, this mass is quantized in units of $1/R$. Substituting (3.2.7) into (3.2.6) and dropping the trace terms, we find

$$S_{\text{l.o.}} = -\frac{\kappa}{2} \int d^4x \sum_k h_k^{\mu\nu}(x) T_{\mu\nu}(x) + \dots , \tag{3.2.8}$$

where $\kappa = 2^{3/2} M^{-n/2-1} R^{-n/2}$, with M defined in (3.2.2). We see that at leading order the couplings of the KK modes to the Standard Model fields are universal and depend on physics at the fundamental scale only through the constant κ. We will show in the next paragraph that κ is equal to the usual four-dimensional gravitational coupling, $(32\pi G_N)^{1/2}$. One has to remember, however, that the Lagrangian (3.2.8) is only valid for modes with masses well below M. The heavier modes correspond to gravitons with high momenta in the extra dimensions for which the operators of higher dimensions become unsuppressed. Their couplings depend on the structure of the theory at the fundamental scale, as we will discuss in section 3.4.

In section 3.3 we will use the Lagrangian (3.2.8) to obtain the rates of graviton emission in particle collision processes. Before presenting this analysis, however, we would like to discuss the physics of the low-lying KK modes. The first mode is massless, corresponding to the null vector k. This mode, referred to as "zero mode", is the four-dimensional graviton. It mediates the gravitational force obeying the usual Newton's law. The potential between two bodies with masses m_1 and m_2 separated by distance r is given by

$$V_0 = \frac{-m_1 m_2 G_N}{r}, \tag{3.2.9}$$

where the gravitational constant G_N is

$$G_N = \frac{1}{4\pi} M^{-n-2} R^{-n}. \tag{3.2.10}$$

The parameters M, R and n have to be chosen to reproduce the measured value $G_N = 6.7 \times 10^{-39}$ GeV^{-2}. If $M \sim 1$ TeV, this implies

$$R \sim 2 \times 10^{31/n-17} \text{cm} . \tag{3.2.11}$$

With this choice of parameters, κ in (3.2.8) is just equal to $(32\pi G_N)^{1/2}$, the usual four-dimensional gravitational coupling.

On the first excited level there are $2n$ KK modes, corresponding to vector k with a single non-zero entry. These modes mediate a Yukawa-type correction to Newton's law of the form

$$V_1 = 2n V_0 e^{-r/R}. \tag{3.2.12}$$

For distances $r > R$, this is the leading correction. (The corrections from higher KK modes are suppressed by higher powers of the exponential.) For large r, this correction is exponentially small, so Newton's law is valid at large distances. Macroscopic measurements of gravity have searched for corrections of the form (3.2.12), and constrain R to be less than about a millimeter; for example, for $n = 2$, $R < 0.77$ mm at 95% confidence [96, 17]. This is in contradiction with (3.2.11) for $n = 1$, so in this case the ADD model is phenomenologically excluded. For $n > 2$, however, the radii predicted by (3.2.11) are below this bound. As we will see below, in the context of ADD model the collider experiments put further, stronger constraints on the radius R.

3.3 Missing Energy Signatures of Large Extra Dimensions

According to (3.2.8), each KK mode of the graviton is coupled to the Standard Model fields with the usual four-dimensional strength, $\kappa = (32\pi G_N)^{1/2}$. This is a very weak coupling. Still, the KK modes can have significant effects in high-energy particle collisions due to their enormous multiplicity. Indeed, using (3.2.11), one finds that the splittings between KK levels for $n = 2$ are as small as 10^{-4} eV. For higher n the splittings are larger, but even in the case $n = 6$ they only reach about 10 MeV. (We will not consider cases $n > 6$ here.) This means that a very large number of KK modes is kinematically accessible at typical collider energies. When one calculates a probability of emitting a graviton in a collision, one has to sum over all the accessible modes, and the resulting cross sections are substantial. Once a particular mode is emitted, however, it interacts with the states on the brane only very weakly, with the usual four-dimensional gravitational strength. This means that it will not be observed by collider detectors, resulting in a missing-energy signature in which the transverse momentum in the detector is not balanced. In this section we will compute the rates of two such missing-energy processes,

$$e^+e^- \to \gamma + (\text{missing}) \; , \qquad p\bar{p} \to \text{jet} + (\text{missing}). \tag{3.3.1}$$

and discuss the corresponding experimental constraints.

The experimental constraints on the fundamental gravitational scale M and the radius of the extra dimensions R are summarized in Table 3.1. The constraint in the first line comes from the consistency of the observed neutrino flux from the supernova SN1987A with the predictions of the stellar collapse models [97]. The other constraints come from collider searches for missing-energy processes (3.3.1), and will be discussed below.

Collider		R / M ($n=2$)	R / M ($n=4$)	R / M ($n=6$)
Present:	SN1987A	$3\times10^{-5}/50000$	$1\times10^{-9}/1000$	$6\times10^{-11}/100$
	LEP 2	4.1×10^{-2} / 1300	1.6×10^{-9} / 800	6.2×10^{-12} / 580
	Tevatron	5.5×10^{-2} / 1140	1.4×10^{-9} / 860	4.1×10^{-12} / 780
Future:	LC	1.2×10^{-3} / 7700	1.2×10^{-10} / 4500	6.5×10^{-13} / 3100
	LHC	4.5×10^{-4} /12500	5.6×10^{-11} / 7500	2.7×10^{-13} / 6000

Table 3.1: Current and future sensitivities to large extra dimensions from missing-energy experiments. All values for colliders are expressed as 95% confidence exclusion limits on the size of extra dimensions R (in cm) and the effective Planck scale M (in GeV). For the analysis of SN1987A, we give probable-confidence limits. Details of collider searches are discussed in the text.

3.3.1 Electron-Positron Collisions.

Using the effective Lagrangian (3.2.8), it is straightforward to evaluate the rate of e^+e^- annihilation into an anomalous single photon recoiling against an unobserved KK graviton. This reaction could potentially be observed at the CERN e^+e^- collider LEP 2, or at a higher-energy e^+e^- collider.

First, we compute the differential cross section for the reaction $e^-_L e^+_R \to \gamma G_k$, where G_k is a single KK mode of the graviton. In the center of mass system, it is given by[1] [98, 4]

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha G_N}{1-m^2/s}\left[(1+\cos^2\theta)\left(1+(\frac{m^2}{s})^4\right) + \left(\frac{1-3\cos^2\theta+4\cos^4\theta}{1-\cos^2\theta}\right)\frac{m^2}{s}\left(1+(\frac{m^2}{s})^2\right)+6\cos^2\theta(\frac{m^2}{s})^2\right], \tag{3.3.2}$$

where m is the mass of the emitted KK mode. The same formula holds for $e^-_R e^+_L$; the helicity-violating cross sections are zero.

To obtain the total emission rate, we have to sum (3.3.2) over all kinematically accessible KK modes. Since the splittings between the modes are much smaller than the typical energies of the processes we consider here, this sum can be replaced by an

[1]As $m^2 \to 0$, this expression has a smooth limit in which only the helicity states $\gamma(-)G(+2)$ and $\gamma(+)G(-2)$ are produced.

integral according to

$$\sum_k \to R^n \int d^n m = \frac{1}{2}\Omega_n R^n \int (m^2)^{(n-2)/2} dm^2 = \frac{\Omega_n}{8\pi} M^{-(n+2)} G_N^{-1} \int (m^2)^{(n-2)/2} dm^2, \tag{3.3.3}$$

where Ω_n is the surface area of the unit sphere in n dimensions. We see that the G_N in the prefactor of (3.3.2) is cancelled in the integral, so the cross section is not suppressed by powers of the four-dimensional Planck scale. Instead, it behaves as $\sigma \sim s^{n/2}/M^{n+2}$, becoming unsuppressed as the center of mass energy approaches the TeV scale. It is clear that this behaviour violates the unitarity bound as s grows. Of course, we do not expect it to hold beyond the fundamental scale, since the formula (3.3.2) was derived from the effective Lagrangian (3.2.8). In section 3.4, we will calculate this cross-section in the context of a simple string-based toy model, and see how unitarity is restored.

Searches for events with a single photon and missing energy have been performed at LEP 2. The main source of background for this search is the Standard Model reaction $e^+e^- \to \gamma\nu\bar{\nu}$, which can proceed through s-channel Z^0 exchange or (for the case of ν_e) through t-channel W exchange [99]. In Figure 3.1, we show the energy distribution of single photons recoiling against KK gravitons, for the cases $n = 2, 6$ and representative values of the scale M, compared to the single-photon distribution from the background. The peak in the SM cross section results from the process in which the γ recoils against an on-shell Z^0 which decays invisibly. Some additional advantage can be gained, then, in applying a cut which excludes this peak.

For example, the OPAL collaboration has recently presented the analysis of the data collected in 1998, with an intergrated luminosity of 177.3 pb^{-1} and center-of-mass energy of 188.6 GeV [100]. Events with $E_\gamma \sin\theta/E_{\text{beam}} > 0.05$ and $15° < \theta < 165°$ have been accepted by the detector. The kinematic cut $E_\gamma < 60$ GeV has been applied to get rid of the Z peak. The search has not found any events above the Standard Model background. The resulting constraints on the fundamental scale M and the radii of the extra dimensions R can be found in the second line of Table 3.1. In particular, note that for $n = 2$ this search constrains $R < 0.41$ mm; this constraint

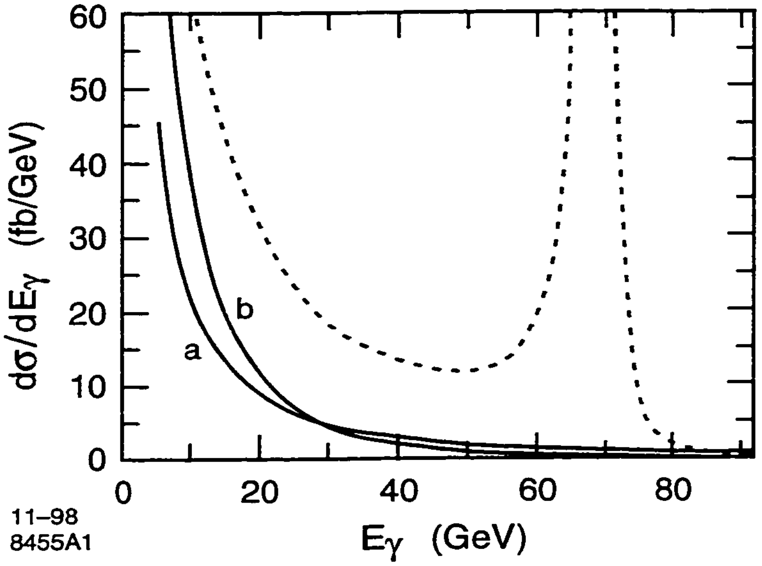


Figure 3.1: Energy spectrum of single photons recoiling against higher-dimensional gravitons G, computed for e^+e^- collisions at $\sqrt{s} = 183$ GeV with an angular cut $|\cos\theta| < 0.95$. The dotted curve is the Standard Model expectation. The solid curves show the additional cross section expected in the model of ref. [6] with (a) $n = 2$, $M = 1200$ GeV, (b) $n = 6$, $M = 520$ GeV.

is stronger than the one coming from the macroscopic gravity measurements [17, 96].

Higher-energy studies of e^+e^- annihilation will be done at a linear e^+e^- collider (LC). We have already noted that higher energy alone should lead to much higher sensitivity to KK graviton production. But the LC also offers another advantage, the possibility of electron beam polarization, which can be used to suppress the dominant t-channel W exchange piece of the SM background process. At $\sqrt{s} = 1$ TeV, with electron polarization $P = +0.9$ (right-handed), integrating over the kinematic region 50 GeV $< E_\gamma <$ 400 GeV, $|\cos\theta_\gamma| < 0.95$, we find a SM background cross section of 82 fb and a signal cross section of

$$\sigma = 20/M^4 \ , \ 46/M^6 \ , \ 55/M^8 \text{ pb} \ , \tag{3.3.4}$$

for $n = 2, 4, 6$ and M in TeV. To quantify the effect of this measurement, we assume that this cross section can be measured with 5% accuracy, and that the value to be found agrees with the SM. Then the measurement would give very strong limits on R and M which are listed in the third line of Table 3.1.

3.3.2 Proton-Antiproton Collisions

In a similar way, proton-antiproton collisions can lead to processes in which a single parton is produced at large transverse momentum recoiling against a KK graviton. This leads to a monojet signature of graviton production—a jet plus missing transverse energy ($\not{E}_T$). Theoretical predictions for this process are less robust than in the e^+e^- case considered above because of the hadronic uncertainties. On the other hand, the higher energy available in hadron collisions gives a substantial enhancement of the event rates, so a search for this process complements the searches at e^+e^- colliders.

The large-E_T jets recoiling against KK gravitons can be produced in parton subprocesses $q\bar{q} \to Gg$, $qg \to qG$, $\bar{q}g \to \bar{q}G$, and $gg \to gG$. The polarization- and color-averaged cross section for $q\bar{q} \to gG$ can be obtained directly from Eq. (3.3.2)

$$\frac{d\sigma}{d\cos\theta} = \frac{2}{9}\frac{\pi\alpha_s G_N}{1-m^2/s}\left[(2-\frac{4ut}{(s-m^2)^2})\left(1+(\frac{m^2}{s})^4\right)\right.$$
$$\left.+\left(2\frac{(s-m^2)^2}{4ut}-5+4\frac{4ut}{(s-m^2)^2}\right)\frac{m^2}{s}\left(1+(\frac{m^2}{s})^2\right)+6\left(\frac{u-t}{s-m^2}\right)^2(\frac{m^2}{s})^2\right], \quad (3.3.5)$$

where s, t, u are the Mandelstam variables: $t, u = -\frac{1}{2}s(1-m^2/s)(1 \mp \cos\theta)$. The cross section for $qg \to qG$ can be obtained from this expression by crossing $s \leftrightarrow t$:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha_s G_N(-t/s)(1-m^2/s)}{12\,(1-m^2/t)^2}\left[(2-\frac{4us}{(t-m^2)^2})\left(1+(\frac{m^2}{t})^4\right)\right.$$
$$\left.+\left(2\frac{(t-m^2)^2}{4us}-5+4\frac{4us}{(t-m^2)^2}\right)\frac{m^2}{t}\left(1+(\frac{m^2}{t})^2\right)+6\left(\frac{s-u}{t-m^2}\right)^2(\frac{m^2}{t})^2\right]. \quad (3.3.6)$$

The cross section for $\bar{q}g \to \bar{q}G$ is also given by (3.3.6). For the process $gg \to gG$, we find the polarization- and color-averaged cross section[2] [98, 4]

$$\frac{d\sigma}{d\cos\theta} = \frac{3}{16}\frac{\pi\alpha_s G_N}{(1-m^2/s)(1-\cos^2\theta)}\Big[(3+\cos^2\theta)^2\left(1+(\frac{m^2}{s})^4\right) - 4\,(7+\cos^4\theta)\frac{m^2}{s}\left(1+(\frac{m^2}{s})^2\right) + 6\,(9-2\cos^2\theta+\cos^4\theta)(\frac{m^2}{s})^2\Big]\,. \tag{3.3.7}$$

All of these formulae must be integrated over the KK graviton mass spectrum using (3.3.3). The rate of monojet production can then be found by integrating these cross sections with appropriate parton distributions.

The main irreducible background to this search comes from the Standard Model processes $q\bar{q} \to gZ^0$, $qg \to qZ^0$, followed by an invisible decay of the Z^0. Unlike the case of e^+e^- reactions, the detector does not measure the imbalance in longitudinal momentum, and there is not enough kinematic information from the single observed jet to exclude the kinematic region in which the Z^0 is on-shell. Still, it was observed in [98] that the signal-to-background ratio can be improved by separating the events containing jets with large E_T. To obtain the maximal efficiency, the minimal accepted value $E_{T,min}$ should be pushed up as high as statistics allow. Other important background sources, such as mismeasured jets and W production with forward leptons, also decrease sharply as the lower bound on missing E_T is increased. The SM background rate can be calibrated experimentally by comparing to the corresponding process in which the Z^0 decays to a lepton pair.

A search for events with a single jet and missing transverse energy has been performed by the CDF collaboration at the Fermilab Tevatron collider. The results of this analysis have been presented in [101]. Of the five possible cuts on missing E_T presented in this analysis, the cut $E_T > 200$ GeV gives the best sensitivity. The resulting bounds on the parameters M and R are given in the fourth line Table 3.1.

Hadron-hadron collisions will be studied at higher energy at the CERN LHC. At the LHC, most collisions are between gluons, since the gluon structure functions rise

[2] As $m^2 \to 0$, this expression has a smooth limit in which the only nonzero helicity amplitudes are those related by crossing to $g(+)g(+)g(-) \to G(+2)$ and $g(-)g(-)g(+) \to G(-2)$. Though it is not so obvious in this representation, the squared matrix element contributing to (3.3.7) is symmetric under permutations of s, t, and u.

rapidly at low x. This suppresses the SM contribution, since gluon-gluon collisions cannot lead to Z^0 production at the leading order in α_s. However, we find that the most important contributions to KK graviton production also involve quarks, since the enhancement of the cross section at high energy partially compensates the falloff of the structure functions. Applying the kinematic cuts $E_T > 1000$ GeV, $|y| < 5$, we find that the irreducible background cross section at LHC (14 TeV beam center-of-mass energy) is 4.41 fb, and signal cross sections are in the ratios

$$S/B = 4.4 \times 10^3/M^4 \; , \; 3.2 \times 10^4/M^6 \; , \; 3.5 \times 10^5/M^8 \; , \tag{3.3.8}$$

for $n = 2, 4, 6$ and M in TeV. Assuming that this measurement can be performed with 20% accuracy, and that the value to be found agrees with the SM, we find the potential limits on R and M listed in the last line of Table 3.1.

One may worry that, in the case $n = 6$, the dominant parton-parton center of mass energies are comparable to the quoted limit on M, so the effective coupling (3.2.8) might not be appropriate for this case. In section 3.4, we will discuss the effects of string theory on the KK graviton emission in this case. We will argue that the stringy effects do not significantly change the bounds on M obtained here.

3.4 String Theory At Colliders

In section 3.2, we have developed the framework for studying processes involving higher-dimensional gravitons at energies well below the fundamental gravitational scale M. Such processes can be described by an effective Lagrangian (3.2.8), which at leading order is independent of the details of physics at the scale M. Here, we will discuss the effects which depend upon the nature of the fundamental theory of gravity. We will assume that quantum gravity is described by string theory with a string scale of order TeV. The feature of string theory most important for our analysis is the appearance of String Regge (SR) excitations of the Standard Model particles at the string scale. At low energies, these particles can appear as virtual states in scattering processes. We will show that this can lead to observable corrections to

Standard Model cross sections. We will also discuss stringy corrections to the cross sections of the missing energy processes discussed above.

3.4.1 String Toy Model

Several quasi-realistic realizations of the ADD proposal within string theory have been proposed [19, 20]. These models are rather complicated, so here instead we will use a simple toy model to quantify the effects of SR excitations on physical processes.

Let us first consider a very simple embedding of QED into Type IIB string theory. We will assume the same toroidal background geometry as before, with $n = 6$. (This is the most natural case since the total number of dimensions in string theory is 10.) In Type IIB theory, there exists a stable BPS object, the D3-brane, which is a 4-dimensional hypersurface on which open strings may end. We will assume that "our brane" consists of N coincident D3-branes stretched out in the 4 extended dimensions. The massless states associated with open strings that end on the branes are described by an $N = 4$ supersymmetric Yang-Mills theory with a gauge group $U(N)$. These states include gauge bosons $A^{\mu a}$, gauginos $\tilde{g}^{ai}$, and complex scalars ϕ^a, where a is an index of the adjoint representation of $U(N)$ and i runs from 1 to 4. We will project this theory down to a $U(1)$ gauge theory with two massless Weyl fermions and identify the gauge boson and fermions of that theory with the photon and electron of QED. Explicitly, consider the $SU(2)$ subgroup of $U(N)$ with generators

$$t^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad t^- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.4.1}$$

(In general, we normalize $SU(N)$ generators to $\mathrm{tr}[t^a(t^b)^\dagger] = \frac{1}{2}\delta^{ab}$.) We can identify the left-handed electron e_L^-, the left-handed positron e_L^+, and the photon A_μ as

$$e_L^- = \tilde{g}^{-1}, \qquad e_L^+ = \tilde{g}^{+1}, \qquad A_\mu = A_\mu^3, \tag{3.4.2}$$

where the superscript ($+, -$ or 3) denotes the matrix from (3.4.1) which would be used in computing the Chan-Paton factor.

At tree level, the string amplitudes which have only the states (3.4.2) on external lines also involve only these states on internal lines. Therefore, in the low-energy limit they will coincide with the usual QED amplitudes, provided that we identify the Yang-Mills coupling of the brane $N = 4$ theory g with the electric charge of the electron e. In this sense, our model is an embedding of QED into string theory. The same property holds for amplitudes involving external gravitons in addition to the states (3.4.2).

We take the parameters of this theory to be the string scale $M_S = \alpha'^{-1/2}$ and the (dimensionless) Yang-Mills coupling constant g. (Except for this definition of g, we adopt the conventions of [94].) Note that M_S is directly observable: The SR resonances occur at masses $M_n = \sqrt{n} M_S$, for $n = 1, 2, \ldots$. The gravitational constant and other physical scales in the theory are derivable from M_S and g. The fundamental gravitational scale M defined in (3.2.2) and the D3-brane tension are related to M_S via

$$\frac{M}{M_S} \sim \frac{f}{M_S} \sim \alpha^{-1/4}, \tag{3.4.3}$$

where $\alpha = g^2/4\pi$. The numerical coefficients in these formulae are model-dependent.

3.4.2 Stringy Corrections to $e^+e^- \to \gamma\gamma$ and Bhabha scattering

We can use our toy model to compute the effects of TeV scale strings on Bhabha scattering and $\gamma\gamma$ production in e^+e^- collisions. We will compute the tree-level scattering amplitudes in string theory, using the external states described in the previous section.

Tree amplitudes of open-string theory are given as sums of ordered amplitudes multiplied by group theory Chan-Paton factors [94]. (This is analogous to the color decomposition technique for gauge theory calculations, discussed in section 1.3.1.) We consider amplitudes with all momenta directed inward. Let the ordered amplitude with external states $(1, 2, 3, 4)$ be denoted $g^2 A(1, 2, 3, 4)$. Then the full scattering amplitude $\mathcal{A}(1,2,3,4)$ is given by

$$\begin{aligned} \mathcal{A}(1,2,3,4) = g^2 A(1,2,3,4) \cdot \mathrm{tr}[t^1t^2t^3t^4 + t^4t^3t^2t^1] \\ +\ g^2 A(1,3,2,4) \cdot \mathrm{tr}[t^1t^3t^2t^4 + t^4t^2t^3t^1] \\ +\ g^2 A(1,2,4,3) \cdot \mathrm{tr}[t^1t^2t^4t^3 + t^3t^4t^2t^1] \ . \end{aligned} \tag{3.4.4}$$

To compute string QED amplitudes with fixed external states, we substitute for each t^i the appropriate matrix from (3.4.1) (or, for outgoing states, the Hermitian conjugate matrix).

The four-point ordered tree amplitudes for open string states living on a D brane are given by [102]

$$A(1,2,3,4) = \mathcal{S}(s,t) A_{\mathrm{YM}}(1,2,3,4), \tag{3.4.5}$$

where A_{YM} is the corresponding ordered amplitude in the Yang-Mills field theory describing the low-energy limit of the theory at hand, and the string form factor $\mathcal{S}(s,t)$ is given by

$$\mathcal{S}(s,t) = \frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)} \ . \tag{3.4.6}$$

The amplitudes with other orderings can be obtained by crossing.

For Bhabha scattering, only the first Chan-Paton factor in (3.4.4) is nonzero. Using the Yang-Mills amplitudes in figure 3.2, we find

$$\begin{aligned} \mathcal{A}(e^-_L e^+_R \to e^-_L e^+_R) &= -2e^2 \frac{u^2}{st} \mathcal{S}(s,t) \ , \\ \mathcal{A}(e^-_L e^+_R \to e^-_R e^+_L) &= -2e^2 \frac{t}{s} \mathcal{S}(s,t) \ , \\ \mathcal{A}(e^-_L e^+_L \to e^-_L e^+_L) &= -2e^2 \frac{s}{t} \mathcal{S}(s,t) \ , \end{aligned} \tag{3.4.7}$$

and the same results for the parity-reflected processes. In general, all helicity amplitudes for Bhabha scattering are given by their field theory expressions multiplied by $\mathcal{S}(s,t)$. This form factor has SR poles in the s- and t-channels. A u-channel pole cannot appear, because the open string contains no states with electric charge ± 2.

For $e^+e^- \to \gamma\gamma$, the result is more complex. The string form factor appears in all

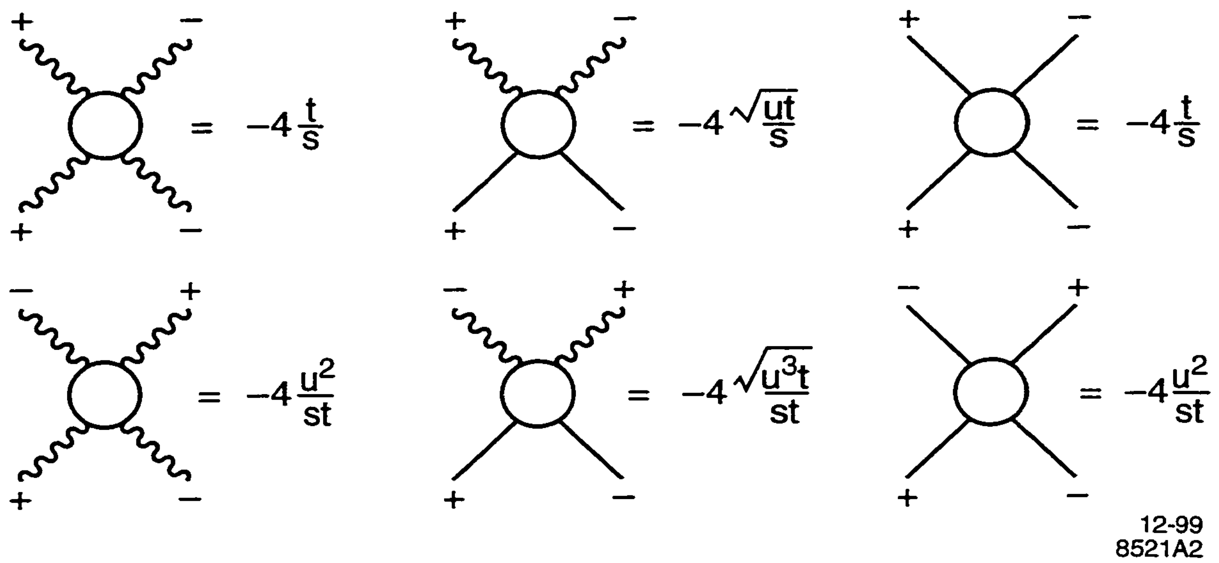


Figure 3.2: Nonzero 4-point ordered tree amplitudes of Yang-Mills theory. Wavy lines represent gauge bosons; straight lines represent fermions. The sign for each line is the helicity, directed inward.

three possible channels, and we find

$$\mathcal{A}(e^-_L e^+_R \to \gamma_L \gamma_R) = e^2 \sqrt{\frac{u}{t}} \left[\frac{u}{s} \mathcal{S}(s,t) + \frac{t}{s} \mathcal{S}(s,u) - \mathcal{S}(t,u) \right] . \tag{3.4.8}$$

The other nonzero helicity amplitudes are derived from this one by parity reflection and crossing. In particular, the amplitude for production of $\gamma_R\gamma_R$ remains zero. The amplitude (3.4.8) contains massive SR poles in all three channels.

Both two-photon production and Bhabha scattering have been studied at LEP 2 at the highest available energies. We consider first the case of two-photon production. Deviations from the Standard Model cross section have been analyzed by the LEP experiments in terms of Drell's parametrization [103]

$$\frac{d\sigma}{d\cos\theta} = \left. \frac{d\sigma}{d\cos\theta} \right|_{SM} \cdot \left(1 \pm \frac{2ut}{\Lambda_\pm^4} \right) . \tag{3.4.9}$$

To compare our string theory results to this expression, we expand the stringy form factors in (3.4.8) according to

$$\mathcal{S}(s,t) = \left(1 - \frac{\pi^2}{6}\frac{st}{M_S^4} + \cdots\right) . \tag{3.4.10}$$

This gives

$$\mathcal{A}(e_L^- e_R^+ \to \gamma_L \gamma_R) = -2e^2 \sqrt{\frac{u}{t}} \left[1 + \frac{\pi^2}{12}\frac{ut}{M_S^4} + \cdots\right] . \tag{3.4.11}$$

Squaring this expression, and noting that the correction is invariant to crossing $t \leftrightarrow u$, we can identify

$$\Lambda_+ = \left(12/\pi^2\right)^{1/4} M_S . \tag{3.4.12}$$

The OPAL collaboration [104] has reported a limit $\Lambda_+ > 304$ GeV from measurements at 183 and 189 GeV in the center of mass. The ALEPH, DELPHI, and L3 collaborations have reported similar constraints [105, 106, 107]. The OPAL result corresponds to a limit

$$M_S > 290 \text{ GeV} \ , \ 95\% \text{ conf.} \tag{3.4.13}$$

The comparison of string predictions to the data on Bhabha scattering brings in a new complication: Bhabha scattering at energies above the Z^0 resonance includes Z^0 exchange as an important contribution, while the Z^0 was not a part of our string QED. To find a prescription for including both γ and Z^0 exchange, we recall that all QED Bhabha scattering amplitudes are multiplied by the common form factor $\mathcal{S}(s,t)$. Thus, we suggest comparing the data on Bhabha scattering to the simple formula

$$\frac{d\sigma}{d\cos\theta}(e^-e^+ \to e^-e^+) = \left.\frac{d\sigma}{d\cos\theta}\right|_{SM} \cdot |\mathcal{S}(s,t)|^2 . \tag{3.4.14}$$

This is essentially the assumption that the SR excitations of the photon and the Z^0 have the same spectrum, up to contributions of size M_Z^2 that we can ignore in computing their masses, and that the SR excitations of the Z^0 have the same polarization asymmetry as the Z^0 in their coupling to electrons.

The comparison of the expression (3.4.14) to data has been performed, for example, by L3 collaboration at LEP 2 [108]. They obtain a lower bound on the string scale $M_S > 490$ GeV. Using (3.4.3), this bound can be converted to a bound on the fundamental gravity scale M of around 1700 GeV. (Here we assumed that the model-dependent numerical coefficient in (3.4.3) is close to one.) This bound is higher than any of the direct bounds in Table 3.1 for the relevant case $n = 6$. Of course, it is also less robust, and could be shifted substantially by model-dependent effects. In particular, the value of the coupling we have used here is $\alpha = 1/137$. A realistic string model should also include strong and weak interactions. It is likely that in such a model the true string coupling which has to be used in (3.4.3) will be substantially larger than e, resulting in a less stringent lower bound on M.

It is interesting that, in our toy model, the leading corrections are proportional to M_S^{-4}, corresponding to an operator of dimension 8. This is a consequence of the fact that the first higher-dimension operator with $N = 4$ supersymmetry appears at dimension 8 [109]. For the case of $e^+e^- \to \gamma\gamma$, there are no gauge-invariant operators of dimension lower than 8, so our results are robust. [3] On the other hand, Bhabha scattering can in principle receive contributions from dimension 6 operators. It is likely that in more general string models with broken supersymmetry, the first stringy corrections in this channel would be proportional to M_S^{-2}. The bounds on M_S in the context of such models are expected to be higher than the ones derived here.

At a higher-energy e^+e^- collider, the SR excitations of the Standard Model particles can be produced directly. At the first level, there are resonances with spins up to 2. The production and decay rates of these resonances were computed in [4]. The observation of these particles would give a very strong reason to believe that the ADD scenario is realized in Nature.

3.4.3 Stringy Corrections to Graviton Emission

Our toy model includes the process of graviton emission in electron-positron annihilation, $e^+e^- \to \gamma G$. This process gives a missing-energy signature which was discussed in section 3.3. Here, we will study the stringy corrections to this process.

In our stringy toy model, the graviton is a part of the closed string massless spectrum, while the electrons and photons are described by massless states of open

[3] This statement is only true in the approximation $m_e = 0$.

strings. Therefore, to study the process $e^+e^- \to \gamma G$ we consider the string scattering amplitude involving three open strings and a closed string. The calculation of this amplitude is very similar to the calculation of the four open-string scattering discussed above. Again, we find that the string theory amplitudes are proportional to their field theory counterparts, multiplied by a stringy form factor. The cross section for emission of a KK graviton with mass m is given by

$$\frac{d\sigma}{d\cos\theta} = \left.\frac{d\sigma}{d\cos\theta}\right|_{\text{ft}} \cdot \left|\mathcal{F}(s,t,u,m^2)\right|^2 , \tag{3.4.15}$$

where $d\sigma/d\cos\theta|_{\text{ft}}$ is the field theory cross section, given in (3.3.2), and the form factor $\mathcal{F}$ is given by

$$\mathcal{F}(s,t,u,m^2) = \frac{1}{\sqrt{\pi}} e^{-(\log 2)\alpha' m^2} \Gamma(\frac{1}{2} - \frac{1}{2}\alpha' m^2) \cdot \frac{\Gamma(1-\frac{1}{2}\alpha' s)\Gamma(1-\frac{1}{2}\alpha' t)\Gamma(1-\frac{1}{2}\alpha' u)}{\Gamma(1+\frac{1}{2}\alpha'(s-m^2))\Gamma(1+\frac{1}{2}\alpha'(t-m^2))\Gamma(1+\frac{1}{2}\alpha'(u-m^2))} . \tag{3.4.16}$$

The expression (3.4.16) has an interesting pole structure [110]. The poles in the s channel occur for $s = 2nM_S^2$, and correspond to producing SR states with an even excitation number. The SR states with an odd excitation number cannot decay into a graviton and an open string massless state. On the other hand, these states can mix with the graviton, leading to the appearance of extra poles at $m^2 = (2n+1)M_S^2$. These poles were also observed by Hashimoto and Klebanov [102] in their calculation of the gluon-gluon-graviton vertex. Their presence is essential for the correct factorization properties of the form factor (3.4.16).

In section 3.3 we have remarked that the cross section for KK graviton emission (3.3.2), summed over all possible KK modes, appears to violate the unitarity bound for sufficiently high s. Of course, the formula (3.3.2) cannot be applied for $s \sim M^2$, since in this regime KK modes with $m \sim M$ can be emitted. The coupling of these modes is no longer described by (3.2.8). Our stringy toy model, however, can be used to describe the graviton emission in this case. The form factor (3.4.16) suppresses the emission of modes with high m. Assume that the kinematic variables are sufficiently far away from any of the poles in (3.4.16). (Near the poles, the effects of finite width of the resonances have to be taken into account. This is beyond the scope of our analysis here.) For the radiation of states of very high mass, we can evaluate $\mathcal{F}$ at the threshold $s = m^2$, $t = u = 0$, and then take m^2 large. Using Stirling's formula, we find

$$\mathcal{F} \sim \exp[-(\log 2)\alpha' m^2] \ . \tag{3.4.17}$$

Thus, unitarity is restored in our stringy model.

The form factor (3.4.16) also cuts off the amplitudes for KK graviton emission in other relevant high-energy limits. In the limit of fixed mass, $s \to \infty$, and fixed angle, we find

$$\mathcal{F} \sim \exp[-\alpha' s f(\cos\theta)] \ , \tag{3.4.18}$$

where $f(c)$ is given by

$$f(c) = -\frac{1+c}{2}\log\frac{1+c}{2} - \frac{1-c}{2}\log\frac{1-c}{2} \ . \tag{3.4.19}$$

In the high-energy limit in which s, t, u, m^2 all become large together, we find the more complicated formula

$$\mathcal{F} \sim \exp[-\frac{1}{2}\alpha' s g(x, \cos\theta)] \ , \tag{3.4.20}$$

where $x = m^2/s$ and $f(x, c)$ is given by

$$\begin{aligned} g(x,c) =& x\log 4x - (1-x)\frac{(1+c)}{2}\log\frac{(1+c)}{2} - (1-x)\frac{(1-c)}{2}\log\frac{(1-c)}{2} \\ & - (\frac{(1+c)}{2} + x\frac{(1-c)}{2})\log(\frac{(1+c)}{2} + x\frac{(1-c)}{2}) \\ & - (\frac{(1-c)}{2} + x\frac{(1+c)}{2})\log(\frac{(1-c)}{2} + x\frac{(1+c)}{2}) \ . \end{aligned} \tag{3.4.21}$$

The function $g(x, c)$ is positive for the allowed values of c and x, even though this property is not manifest in (3.4.21). Thus, the string correction (3.4.16) gives a form factor suppression in all hard-scattering regions.

As we have mentioned in section 3.3, the effects of string theory on the KK graviton production could become important phenomenologically at the LHC. In particular, the LHC bound for the case $n = 6$ in Table 3.1 is of the same order as the dominant parton-parton center of mass energies. While our toy model does not include gluons and quarks, the structure of (3.4.15) suggests that this relation will also hold for simple stringy extensions of QCD. Assuming that (3.4.15) holds for the parton subprocesses which contribute to the jet+KK graviton production in $\bar{p}p$ collisions, we can study the effects of string theory on the LHC analysis in section 3.3. Naively, one would expect these effects to lower the sensitivity of the LHC searches, since the form factor $\mathcal{F}$ suppresses the cross section at high energies. It turns out, however, that for values of the string scale in the few-TeV range, this suppression does not significantly alter the signal rates at the LHC. For typical parton-parton center-of-mass energies there is a relatively small suppression of the cross section. But when partons can combine to form the SR resonances, we find a dramatic enhancement of the signal rate. A typical resonant process is

$$gg \to g^{**} \to gG \, , \tag{3.4.22}$$

where g^{**} is the spin-2 SR excitation of the gluon[4]. In the situation in which these states are present, they would also be seen as resonances in the two-jet invariant mass distribution. We conclude that in either case, whether the resonances are observed or not, the bounds in the last line of Table 3.1 would not be significantly decreased by stringy physics.

3.5 Discussion

Models with large extra dimensions have been suggested as a potential solution of the hierarchy problem of the Standard Model. If this possibility is realized in Nature, the fundamental gravitational scale, M, should not be much higher than a TeV. In the first part of this chapter, we have discussed how this scale can be constrained experimentally. The results of our analysis are summarized in Table 3.1. Except for the case $n = 2$, where the supernova data puts a very strong constraint on M,

[4]These resonances correspond to the poles at $s = 2nM_S^2$ and $m^2 = (2n+1)M_S^2$ in the form factor (3.4.16).

the values of M in the TeV range are compatible with the current experimental situation. At future colliders, such as the LHC, most of the interesting range for M will be probed.

We have specified the assumptions of the analysis which led us to bounds in Table 3.1 in section 3.2. A number of these assumptions could be relaxed. For example, it is trivial to extend our analysis to the case of asymmetric compactifications, as well as curved compactification manifolds. Another assumption that is likely to break down in realistic models is that gravity is the only field living in the extra dimensions. Extra bulk particles would produce additional, model-dependent, missing-energy signatures beyond those we consider here. It is important to point out that the processes we consider here contain a higher-dimensional graviton in the final state. Since there can be no interference between processes with different final states, the presence of additional degrees of freedom in the bulk will *increase* the anomalous missing-energy event rates. This means that our assumption here is a conservative one: In a realistic model, one is likely to obtain higher signal rates, and therefore higher sensitivities to the fundamental scale of gravity M, than the ones calculated here.

The constraints in Table 3.1 are independent of the structure of the theory at the fundamental scale. A number of other, model-dependent, bounds on M can also be obtained. For example, in section 3.4 we have assumed that the physics at the scale M is described by string theory. Using a simple stringy toy model, we have obtained a bound on the string scale, M_S, from the data on Bhabha scattering and two-photon production at LEP 2. This bound can be converted into a bound on M using the relation between the two scales in string theory. This relation is highly model-dependent, so while the constraint on M obtained here is quite strong, it could be shifted significantly in a more realistic stringy model.

The stringy toy model we have used here is very elementary, and does not possess many essential features of a realistic model. For example, it would be interesting to incorporate QCD into the same framework. This would allow one to obtain constraints on M_S from experiments at hadron colliders. The first attempt to do this was presented in [5], but the analysis presented there can clearly be improved on.

Appendix A

KLT Relations for an Arbitrary Number of External Legs

The KLT relations [7] between gravity and gauge amplitudes are, in the field theory limit and for an arbitrary number n of external particles,

$$\begin{aligned} M_n^{\text{tree}}(1,2,\ldots,n) \quad &= i\,(-1)^{n+1}\Big[A_n^{\text{tree}}(1,2,\ldots,n)\sum_{\text{perms}} f(i_1,\ldots,i_j)\bar{f}(l_1,\ldots,l_{j'}) \\ &\qquad \times A_n^{\text{tree}}(i_1,\ldots,i_j,1,n-1,l_1,\ldots,l_{j'},n)\Big] \\ &\qquad + \mathcal{P}(2,\ldots,n-2), \end{aligned} \tag{A.1}$$

where 'perms' are $(i_1,\ldots,i_j) \in \mathcal{P}(2,\ldots,n/2)$, $(l_1,\ldots,l_{j'}) \in \mathcal{P}(n/2+1,\ldots,n-2)$, $j = n/2-1$, $j' = n/2-2$, giving a total of $(n/2-1)! \times (n/2-2)!$ terms in the square brackets. We have assumed that n is even here; the case of odd n is completely analogous. The functions f and $\bar{f}$ are given by

$$\begin{aligned} f(i_1,\ldots,i_j) \quad &= \quad s(1,i_j)\prod_{m=1}^{j-1}\Big(s(1,i_m)+\sum_{k=m+1}^{j} g(i_m,i_k)\Big)\,, \\ \bar{f}(l_1,\ldots,l_{j'}) \quad &= \quad s(l_1,n-1)\prod_{m=2}^{j'}\Big(s(l_m,n-1)+\sum_{k=1}^{m-1} g(l_k,l_m)\Big)\,, \end{aligned} \tag{A.2}$$

with

$$g(i,j) \;=\; \begin{cases} s(i,j) \equiv s_{ij}, & i > j, \\ 0, & \text{otherwise.} \end{cases} \tag{A.3}$$

These definitions are used to compute the first term in the 'big' permutation sum; the rest of the terms ($(n-3)!$ all together) are obtained by simply permuting the arguments of the $s(i,j)$'s and $A_n^{\rm tree}$'s in the square brackets. As a consistency check, we have numerically verified that these relations correctly give the MHV amplitudes of BGK, eq. (1.2.14), up to $n = 8$.

Appendix B

One-Loop Integrals

In this appendix we define our notation, and collect explicit expressions (through $\mathcal{O}(\epsilon^0)$) for the loop momentum integrals used in chapter 1. The one-loop scalar m-point integral in D dimensions is defined by

$$\begin{aligned}
\mathcal{I}_m^{K_1 K_2 \cdots K_m} &\equiv \int \frac{d^D L}{(2\pi)^D} \frac{1}{L^2 (L-K_1)^2 (L-K_1-K_2)^2 \ldots (L - \sum_{i=1}^{m-1} K_i)^2} \\
&\equiv \int \frac{d^4 \ell}{(2\pi)^4} \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \frac{1}{(\ell^2-\mu^2)((\ell-K_1)^2-\mu^2)\ldots((\ell-\sum_{i=1}^{m-1} K_i)^2-\mu^2)},
\end{aligned} \tag{B.1}$$

where $K_1, K_2, \ldots, K_m$ are the (four-dimensional) external momenta for the integral, and L is the D-dimensional loop momentum, decomposed in the second line into 4- and (-2ϵ)-dimensional components, $L = \ell + \mu$. In general, the K_i may be either individual massless external momenta k_j for the amplitude under consideration, or else sums of such external momenta. To simplify the notation, in the former case we will replace K_i in the argument of $\mathcal{I}_m$ simply by the appropriate integer index j; in the latter case we will often replace it by the set of integers entering the momentum sum, enclosed in parentheses. For example, one of the box integrals encountered in six-point amplitudes contains two diagonally-opposite massive external legs, with masses s_{23} and s_{56}, and is given by

$$\mathcal{I}_4^{1(23)4(56)} \equiv \int \frac{d^D L}{(2\pi)^D} \frac{1}{L^2(L-k_1)^2(L-k_1-k_2-k_3)^2(L+k_5+k_6)^2}\,. \tag{B.2}$$

Integrals where an additional factor of $(\mu^2)^r \equiv \mu^{2r}$ has been inserted into the loop integrand are denoted by

$$\mathcal{I}_m^{K_1K_2\cdots K_m}[\mu^{2r}] = \int \frac{d^4\ell}{(2\pi)^4}\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \frac{\mu^{2r}}{(\ell^2-\mu^2)((\ell-K_1)^2-\mu^2)\cdots((\ell-\sum_{i=1}^{m-1}K_i)^2-\mu^2)}, \tag{B.3}$$

These integrals can be expressed, via eq. (1.3.9), in terms of the integrals $\mathcal{I}_m^{D=4+2r-2\epsilon}$, that is, $\mathcal{I}_m$ in eq. (B.1) with D replaced by $D+2r$.

The $N=4$ super-Yang-Mills and $N=8$ supergravity amplitudes contain scalar box integrals with one and two external masses. The two-mass box integral with two diagonally-opposite massive legs (see fig. 1.1), evaluated in $D=4-2\epsilon$, is [52]

$$\begin{aligned}\mathcal{I}_4^{aK_1bK_2} &= i\,\frac{c_\Gamma}{S_{1a}S_{1b}-K_1^2K_2^2}\Big\{\frac{2}{\epsilon^2}\Big[(-S_{1a})^{-\epsilon}+(-S_{1b})^{-\epsilon}-(-K_1^2)^{-\epsilon}-(-K_2^2)^{-\epsilon}\Big]\\ &-2\,\mathrm{Li}_2\left(1-\frac{K_1^2}{S_{1a}}\right)-2\,\mathrm{Li}_2\left(1-\frac{K_1^2}{S_{1b}}\right)-2\,\mathrm{Li}_2\left(1-\frac{K_2^2}{S_{1a}}\right)-2\,\mathrm{Li}_2\left(1-\frac{K_2^2}{S_{1b}}\right)\\ &+2\,\mathrm{Li}_2\left(1-\frac{K_1^2K_2^2}{S_{1a}S_{1b}}\right)-\ln^2\left(\frac{S_{1a}}{S_{1b}}\right)\Big\}+\mathcal{O}(\epsilon)\,,\end{aligned} \tag{B.4}$$

where $S_{1a}=(K_1+k_a)^2$, $S_{1b}=(K_1+k_b)^2$, and

$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{B.5}$$

One-mass box integrals also appear in the amplitudes, corresponding to the case where K_1 reduces to a single external momentum. They are also given by eq. (B.4) — one may set $K_1^2=0$ after dropping the $(-K_1^2)^{-\epsilon}$ term. (One may similarly set $K_1^2=K_2^2=0$ to obtain the box integral with no external masses encountered in the four-point amplitudes.)

For the all-plus gauge amplitudes (1.3.7) and gravity amplitudes (1.4.12), (1.4.20) and (1.4.25), only the ultraviolet-singular parts ($1/\epsilon$ poles) of the higher-dimensional integrals $\mathcal{I}_m^{D=4+2r-2\epsilon}$ contribute in the four-dimensional limit $\epsilon \to 0$, due to the overall prefactor of ϵ in eq. (1.3.9). Such terms are given by elementary integrals of polynomials in the Feynman parameters.

In the gauge theory case, the only divergent integrals that appear are $D = 8 - 2\epsilon$ box integrals and $D = 10 - 2\epsilon$ pentagon integrals. The box integral is

$$\begin{aligned}
\mathcal{I}_4[\mu^4] &= -\epsilon(1-\epsilon)(4\pi)^2 \mathcal{I}_4^{D=8-2\epsilon} \\
&= -i\epsilon(1-\epsilon)\frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \int d^4 a_i \, \delta\Big(1 - \sum_i a_i\Big) \, (-s_{12}a_1 a_3 - s_{23}a_2 a_4 + \cdots)^{-\epsilon} \\
&= -\frac{i}{(4\pi)^2}\frac{1}{6} + \mathcal{O}(\epsilon) \, .
\end{aligned} \tag{B.6}$$

Since $\mathcal{I}_4^{D=8-2\epsilon}$ is dimensionless, the pole in ϵ does not depend on the particular kinematic configuration, so we have suppressed the labels describing the kinematics. (External masses would contribute to the '$+\cdots$' terms in eq. (B.6).) Similarly, the pentagon integral that appears is

$$\begin{aligned}
\mathcal{I}_5[\mu^6] &= -\epsilon(1-\epsilon)(2-\epsilon)(4\pi)^3 \mathcal{I}_5^{D=10-2\epsilon} \\
&= i\epsilon(1-\epsilon)(2-\epsilon)\frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \int d^5 a_i \, \delta\Big(1 - \sum_i a_i\Big) \, (-s_{45}a_4 a_1 + \text{cyclic} + \cdots)^{-\epsilon} \\
&= \frac{i}{(4\pi)^2}\frac{1}{12} + \mathcal{O}(\epsilon) \, ,
\end{aligned} \tag{B.7}$$

where the leading term again does not depend on the kinematic configuration. The hexagon integral in eq. (1.3.7) is

$$\mathcal{I}_6[\mu^6] = -\epsilon(1-\epsilon)(2-\epsilon)(4\pi)^3 \mathcal{I}_6^{D=10-2\epsilon} = \mathcal{O}(\epsilon) \, , \tag{B.8}$$

since $\mathcal{I}_6^{D=10-2\epsilon}$ is finite.

In the all-plus gravity amplitudes, the required higher-dimensional integrals are dimensionful, and therefore depend on the kinematics. We give here the box and pentagon integrals with the maximum required number of external masses; those with fewer masses can be obtained by setting masses to zero. The two-mass box integral that appears in the amplitudes is

$$\begin{aligned}
\mathcal{I}_4^{aK_1bK_2}[\mu^8] &= -\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)(4\pi)^4\,\mathcal{I}_4^{aK_1bK_2,\,D=12-2\epsilon} \\
&= -i\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)\frac{\Gamma(-2+\epsilon)}{(4\pi)^{2-\epsilon}} \\
&\quad \times\int d^4a_i\;\delta\Big(1-\sum_i a_i\Big)\;(-S_{1a}\,a_1a_3 - S_{1b}\,a_2a_4 - K_1^2\,a_2a_3 - K_2^2\,a_4a_1)^{2-\epsilon} \\
&= -\frac{i}{(4\pi)^2}\frac{1}{840}\Big[2S_{1a}^2 + 2S_{1b}^2 + 2(K_1^2)^2 + 2(K_2^2)^2 + S_{1a}S_{1b} + K_1^2K_2^2 \\
&\qquad +2(S_{1a}+S_{1b})(K_1^2+K_2^2)\Big] + \mathcal{O}(\epsilon)\,,
\end{aligned} \tag{B.9}$$

where the kinematic configuration is depicted in fig. 1.1.

The one-mass pentagon integral that appears in the six-point all-plus gravity amplitude is

$$\begin{aligned}
\mathcal{I}_5^{1234(56)}[\mu^{10}] &= -\epsilon(1-\epsilon)\cdots(4-\epsilon)(4\pi)^5\,\mathcal{I}_5^{1234(56),\,D=14-2\epsilon} \\
&= i\epsilon(1-\epsilon)\cdots(4-\epsilon)\,\frac{\Gamma(-2+\epsilon)}{(4\pi)^{2-\epsilon}}\int d^5a_i\;\delta\Big(1-\sum_i a_i\Big) \\
&\quad \times(-s_{12}a_1a_3 - s_{23}a_2a_4 - s_{34}a_3a_5 - t_{456}a_4a_1 - t_{561}a_5a_2 - s_{56}a_5a_1)^{2-\epsilon} \\
&= \frac{i}{(4\pi)^2}\frac{1}{1680}\Big[2s_{12}^2 + 2s_{23}^2 + 2s_{34}^2 + 2s_{56}^2 + 2t_{456}^2 + 2t_{561}^2 + 2s_{12}s_{34} + 2s_{12}s_{56} \\
&\qquad +2s_{34}s_{56} + 2s_{12}t_{456} + 2s_{34}t_{561} + 2(s_{23}+s_{56})(t_{456}+t_{561}) \\
&\qquad +s_{12}s_{23} + s_{23}s_{34} + s_{23}s_{56} + s_{12}t_{561} + s_{34}t_{456} + t_{456}t_{561}\Big] + \mathcal{O}(\epsilon).
\end{aligned} \tag{B.10}$$

To obtain the massless pentagon integral appearing in the five-point amplitude from eq. (B.10), simply replace $k_5 + k_6$ by k_5, i.e., $s_{56} \to 0$, $t_{456} \to s_{45}$, and $t_{561} \to s_{51}$.

Appendix C

Equivalence of Recursive and Non-Recursive Definitions of Half-Soft Functions

In this appendix, we shall show that the non-recursive formula (1.6.10) for the half-soft functions $h(q, M, r)$, where $M = \{1, 2, \ldots, n\}$, is equivalent to the recursive definition (1.6.4) in section 1.6.1. In the recursive solution, $h(q, M, r)$ is represented as a sum of terms of the form,

$$\frac{[X]}{\langle X\rangle} \prod_{j=1}^{n} (\langle q\, j\rangle \langle j\, r\rangle)^{i_j - 1}, \tag{C.1}$$

where $[X]/\langle X\rangle$ is one of the terms in the sum for the factor $\phi(i_1, i_2, \ldots, i_n)$. That is, $[X]$ is a product (obeying certain rules) of $n-1$ square brackets $[j\, k]$, with $j, k \in M$; $\langle X\rangle$ is the product of the corresponding angle brackets; and $i_j =$ (number of appearances of j in $[X]$) -1. Through the recursive definition (1.6.5) of the ϕ's, each term (C.1) can be built recursively, starting from

$$\phi(0,0) \frac{1}{\langle q\, 1\rangle \langle 1\, r\rangle} \frac{1}{\langle q\, 2\rangle \langle 2\, r\rangle} = \frac{[1\, 2]}{\langle 1\, 2\rangle} \frac{1}{\langle q\, 1\rangle \langle 1\, r\rangle} \frac{1}{\langle q\, 2\rangle \langle 2\, r\rangle}\,, \tag{C.2}$$

and multiplying by

$$\frac{[m\,l]}{\langle m\,l\rangle}\frac{\langle q\,m\rangle}{\langle q\,l\rangle}\frac{\langle m\,r\rangle}{\langle l\,r\rangle} \tag{C.3}$$

at each step of the recursion, where m is one of the 'legs' added in a previous step, and l is the new 'leg' we add at this step.

Now, let us look at eq. (1.6.10). In each of the $(n-1)!$ terms in the permutation sum we can expand each factor,

$$\langle q^-|\not{K}_{1,l-1}|l^-\rangle \;=\; \sum_{j=1}^{l-1}\langle q\,j\rangle\,[j\,l] \;=\; \langle q\,1\rangle\,[1\,l]+\langle q\,2\rangle\,[2\,l]+\cdots+\langle q,\,l-1\rangle\,[l-1,\,l]\;, \tag{C.4}$$

and collect all terms having a given product of $(n-1)$ square brackets, which we shall also call $[X]$. We wish to show that

(a) the types of square bracket products $[X]$ that appear here are the same as in the recursive construction, and

(b) the products of angle bracket factors multiplying them are also the same.

To prove part (b) we will have to apply the Schouten identity on angle brackets,

$$\langle i\,j\rangle\,\langle k\,l\rangle \;=\; \langle i\,k\rangle\,\langle j\,l\rangle+\langle i\,l\rangle\,\langle k\,j\rangle\;. \tag{C.5}$$

We will *not* need the corresponding identity for square brackets; i.e. the square brackets are already in the correct form. Our strategy for part (b) will be a recursive one, based on demonstrating that the multiplicative factor (C.3) arises in going from $n-1$ to n.

First we establish part (a). Note from eq. (1.6.10) that each $j \in M$ (for $j \neq 1$) appears in $[X]$ once 'from the back' (i.e. from $\langle q^-|\not{K}_{\ldots}|j^-\rangle$), and the remaining $i_j \geq 0$ times 'from the front' (i.e. from $\langle q^-|\ldots+\not{k}_j+\ldots|l^-\rangle$). Because there are $n-1$ square brackets in $[X]$, each containing two arguments, there are a total of $2(n-1) = \sum_{j=1}^{n}(i_j+1)$ arguments in $[X]$, and we have $\sum_{j=1}^{n} i_j = n-2$, just as in the recursive formula. Since all the i_j are non-negative, at least two of the i_j must vanish. Thus we can always find some l, with $l \neq 1$, such that $i_l = 0$; i.e. l appears exactly once in $[X]$, from the back. It appears in the factor $[m\,l]$, for some $m \neq l$. Consider the product $[X]$ with $[m\,l]$ removed from it, which no longer contains l. That is, define $[\tilde{X}] \equiv [X]/[m\,l]$, and for $j \neq l$ let $\tilde{i}_j =$ (number of appearances of j in $[\tilde{X}]$) -1. We see that $[\tilde{X}]$ obeys the same 'counting rules' as $[X]$, except with $n-1$ arguments, because l is no longer present: $\sum_{j \neq l} \tilde{i}_j = n-3$, with all $\tilde{i}_j \geq 0$, and each $j \neq l$ ($j \neq 1$) still appears at least once from the back. Hence we can again find some l', with $l' \neq 1$, such that $\tilde{i}_{l'} = 0$; i.e. l' appears exactly once in $[\tilde{X}]$, from the back. Clearly, this procedure can be repeated until we have only two arguments left, at which stage we obtain the square bracket structure $[1\,j]$ corresponding to $\phi(0,0)$. We have thus established part (a) by working backwards through the recursive construction of the ϕ factors, eq. (1.6.5).

To prove part (b), first note that the $\langle q\,j\rangle$ factors in the non-recursive formula already agree precisely with the recursive form, in *every* contributing permutation. As noted above, in the numerator of each term in eq. (1.6.10) each j appears once from the back and i_j times from the front. (For $j = 1$, we count its appearance in $[1\,2]$ as 'from the back'.) In the second case, it is always accompanied by $\langle q\,j\rangle$, so we get $\langle q\,j\rangle^{i_j}$. But we also have $\langle q\,j\rangle$ in the denominator for each j, so overall we get $\langle q\,j\rangle^{i_j-1}$, as required by the recursive eq. (1.6.4). Thus we only have to rearrange the spinor products of the form $\langle j\,k\rangle$ and $\langle k\,r\rangle$, where $j,k \in M$.

In general, an individual term in the permutation sum for $h(q,M,r)$ in eq. (1.6.10) either contains the square bracket structure $[X]$ *once*, or does not contain it at all. (The fact that $j < l$ in eq. (C.4) means that, for a given permutation, one can uniquely determine which of the arguments in $[X]$ come from the front, and which from the back.) However, $[X]$ can appear in many *different* terms in the sum over $(n-1)!$ permutations. The coefficient of

$$[X] \prod_{j=1}^{n} \langle q\,j\rangle^{i_j-1} \tag{C.6}$$

is

$$\frac{1}{\langle 1\,r\rangle} \sum_{\sigma \in \mathcal{P}_{[X]}} \frac{1}{\langle 1, \sigma_2 \ldots \sigma_n, r\rangle}, \tag{C.7}$$

where the sum is over permutations σ containing $[X]$, $\{\sigma_2 \ldots \sigma_n\}$ is a permutation of

$\{2\ldots n\}$, and

$$\langle a, bc\ldots d, e\rangle \equiv \langle a\, b\rangle\, \langle b\, c\rangle \cdots \langle d\, e\rangle\ . \tag{C.8}$$

As in the proof of (a), let $l \neq 1$ be the leg appearing exactly once in $[X]$, in the combination $[m\, l]$. Since l appears from the back, only the permutations where l appears *after* m will contribute. In the permutation sum in eq. (C.7), let us hold fixed the positions of all legs except l, and just sum over the insertions of l after m. From the Schouten identity (C.5) it is easy to derive the 'eikonal' identity

$$\sum_{i=m}^{r-1} \frac{\langle i, i+1\rangle}{\langle i\, l\rangle\, \langle l, i+1\rangle} = \frac{\langle m\, r\rangle}{\langle m\, l\rangle\, \langle l\, r\rangle}\ . \tag{C.9}$$

This identity allows us to simplify the sum over l insertions,

$$\sum_{\substack{l \text{ after } m \\ \sigma_2\ldots\hat{l}\ldots\sigma_n \text{ fixed}}} \frac{1}{\langle 1, \sigma_2\ldots l\ldots\sigma_n, r\rangle} = \frac{\langle m\, r\rangle}{\langle m\, l\rangle\, \langle l\, r\rangle} \frac{1}{\langle 1, \sigma_2\ldots\hat{l}\ldots\sigma_n, r\rangle}\,, \tag{C.10}$$

where a hat over l signifies that it is no longer present.

We find that the terms in eq. (1.6.10) containing $[X]$ can be rewritten in terms of $[\tilde{X}] \equiv [X]/[m\, l]$ and $\tilde{i}_j$ as,

$$\frac{[m\, l]}{\langle m\, l\rangle} \frac{\langle q\, m\rangle}{\langle q\, l\rangle} \frac{\langle m\, r\rangle}{\langle l\, r\rangle} \times [\tilde{X}] \prod_{\substack{j=1 \\ j\neq l}}^{n} \langle q\, j\rangle^{\tilde{i}_j - 1} \times \frac{1}{\langle 1\, r\rangle} \sum_{\sigma\in\mathcal{P}_{[\tilde{X}]}} \frac{1}{\langle 1, \sigma_2\ldots\hat{l}\ldots\sigma_n, r\rangle}\ . \tag{C.11}$$

Note that the first factor is exactly the desired factor (C.3) appearing in the recursion relation when l is added, and that the third factor is precisely the same as eq. (C.7) with l removed. As in the proof of part (a), we can repeat this argument for $[\tilde{X}]/[m'\, l']$, etc., until we arrive at the case $n = 2$, which works by inspection. Thus we have proven part (b), by showing that the coefficient of $[X]$ is correct, inductively in n.

Appendix D

Extraction of Ultraviolet Infinities from Two-Loop Integrals

In this appendix, we shall evaluate the ultraviolet divergences of the dimensionally-regulated planar and non-planar double-box integrals, defined in eqs. (2.2.11) and (2.2.12), respectively . These integrals are ultraviolet finite in $D \leq 6$; since there is already a one-loop divergence for $D = 8$ and $D = 10$, the cases of $D = 7$, 9, and 11 are of more interest. (If one uses a different regulator, for example a proper time cutoff, there may also be one-loop divergences in $D = 9$ and 11, but these linear and higher order divergences are absent in dimensional regularization.)

A straightforward Feynman parameterization of the integrals (2.2.11) and (2.2.12) gives

$$\mathcal{I}_4^{\text{2-loop},X}(s,t) = \frac{\Gamma(7-D)}{(4\pi)^D} \int_0^1 d^7a \; \delta\Big(1 - \sum_{i=1}^{7} a_i\Big) \left(-Q_X(s,t,a_i)\right)^{D-7} (\Delta_X(a_i))^{7-3D/2} \,, \tag{D.1}$$

where $X = \mathrm{P}, \mathrm{NP}$, and

$$\begin{aligned} \Delta_{\mathrm{P}}(a_i) &= (a_1 + a_2 + a_3)(a_4 + a_5 + a_6) + a_7(1 - a_7)\,, \\ \Delta_{\mathrm{NP}}(a_i) &= (a_1 + a_2)(a_3 + a_4) + (a_1 + a_2 + a_3 + a_4)(a_5 + a_6 + a_7)\,, \end{aligned} \tag{D.2}$$

and

$$
\begin{aligned}
Q_{\mathrm{P}}(a_i) &= s\,(a_1 a_3 (a_4 + a_5 + a_6) + a_4 a_6 (a_1 + a_2 + a_3) + a_7 (a_1 + a_4)(a_3 + a_6)) \\
&\quad + t\, a_2 a_5 a_7 , \\
Q_{\mathrm{NP}}(a_i) &= s\,(a_1 a_3 a_5 + a_2 a_4 a_7 + a_5 a_7 (a_1 + a_2 + a_3 + a_4)) + t\, a_2 a_3 a_6 + u\, a_1 a_4 a_6 \,.
\end{aligned}
\tag{D.3}
$$

Fig. D.3 depicts the two integrals with the labels on the propagators specifying the Feynman parametrization used in eq. (D.1).

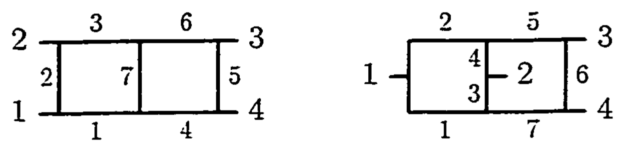


Figure D.3: The planar and non-planar double box integrals, labeled with Feynman parameters.

For $D = 7$, 9, and 11, the absence of one-loop sub-divergences implies that the two-loop divergence should have only a single $1/\epsilon$ pole, which is contributed by the prefactor $\Gamma(7 - D)$. In extracting the coefficient of this pole, one might therefore hope to be able to set $\epsilon = 0$ immediately in the remaining parameter integral. This is possible for $D = 7$, but it is not directly possible for $D = 9$ or 11, because the remaining parameter integrals do not converge for $\epsilon = 0$, and have to be defined through analytic continuation.

Note that the factors $\Delta_X(a_i)$ depend only on special combinations of Feynman parameters, namely the sums corresponding to the three lines of the vacuum diagram obtained by deleting the four external legs. In contrast, the $Q_X(s, t, a_i)$ depend on all the parameters, making it difficult to perform the parameter integrals if Q_X should appear raised to a non-integer power. Fortunately, in extracting the $1/\epsilon$ pole term for $D = n - 2\epsilon$ it is legitimate to replace $\mathcal{I}_4^{\text{2-loop},X}(s,t)$ with

$$
\mathcal{I}_4^{\text{2-loop},X}(s,t)\Big|_{\text{pole}} = \frac{\Gamma(7-D)}{(4\pi)^D} \int_0^1 d^7 a\, \delta\Big(1 - \sum_{i=1}^{7} a_i\Big)\, (-Q_X(s,t,a_i))^{n-7} (\Delta_X(a_i))^{7-n-D/2}.
\tag{D.4}
$$

This can be shown by integrating by parts on the uniform scaling parameter. (Alternatively, one can use the fact that the leading divergence has a known, uniform dimension [111].) Now the factor containing Q_X is a polynomial in the Feynman parameters, and it is straightforward to integrate all but two of the six remaining Feynman parameter integrals.

For the case $D = 7 - 2\epsilon$, where we can set $\epsilon = 0$ from the beginning (except in the Γ-function prefactor), we get:

$$\begin{aligned} \mathcal{I}_4^{\text{2-loop, P}, D=7-2\epsilon}\Big|_{\text{pole}} &= \frac{1}{2\epsilon\,(4\pi)^7}\frac{1}{4}\int_0^1 dy\, y^2(1-y)^2 \int_0^1 dx \frac{x^{3/2}}{[1-x(1-y(1-y))]^{7/2}} \\ &= \frac{1}{2\epsilon\,(4\pi)^7}\frac{\pi}{10}. \end{aligned} \tag{D.5}$$

For $D = 9 - 2\epsilon$, setting $\epsilon = 0$ would lead to

$$\begin{aligned} \mathcal{I}_4^{\text{2-loop, P}, D=9-2\epsilon}\Big|_{\text{pole}} &= \frac{1}{4\epsilon\,(4\pi)^9}\frac{1}{498960}\int_0^1 \frac{dy}{[y(1-y)]^{3/2}}\Big[3\,s^2\,\Big(16y^2(1-y)^2 - \\ &\quad 77y(1-y)+132\Big) + 8\,st\,y(1-y)\Big(2y(1-y)+11\Big) + 80\,t^2\,y^2(1-y)^2\Big]. \end{aligned} \tag{D.6}$$

The terms in eq. (D.6) that are proportional to st and to t^2 have convergent, elementary y-integrals. However, the s^2 term is divergent, so we must retain the full ϵ-dependence from eq. (D.4) in evaluating it. Doing this, we find that the (x, y)-integral for the s^2 term can be conveniently written as the sum of two terms,

$$\mathcal{I}_4^{\text{2-loop, P}, D=9-2\epsilon}\Big|_{\text{pole, } s^2 \text{ term}} = \frac{s^2}{4\epsilon\,(4\pi)^9}\int_0^1 dy \int_0^1 dx\Big[C_1(x,y) + C_2(x,y)\Big], \tag{D.7}$$

where

$$\begin{aligned} C_1(x,y) &= \frac{[y(1-y)]^4}{480}\frac{x^{5/2+\epsilon}\Big(-x^2y(1-y)+(1-x)(2-3x)\Big)}{[1-x(1-y(1-y))]^{13/2-\epsilon}}, \\ C_2(x,y) &= \frac{[y(1-y)]^2}{360}\frac{x^{5/2+\epsilon}}{[1-x(1-y(1-y))]^{9/2-\epsilon}}. \end{aligned} \tag{D.8}$$

The integral of C_1 converges for $\epsilon = 0$, and $\int_0^1 dx\, dy\; C_1(x,y) = -5\pi/11088$.

The integral over C_2 requires analytic continuation in ϵ, which we handle with identities (3.197.3) and (9.131.2) of Gradshteyn and Ryzhik [112]. Somewhat more generally, we need

$$
\begin{aligned}
I(p,q,\alpha) &\equiv \int_0^1 dy\; [y(1-y)]^p \int_0^1 dx \frac{x^{\alpha-q-2+\epsilon}\,(1-x)^q}{[1-x(1-y(1-y))]^{\alpha-\epsilon}} \\
&= \frac{\Gamma(\alpha-q-1+\epsilon)\Gamma(q+1)}{\Gamma(\alpha+\epsilon)} \\
&\quad \times \int_0^1 dy\; [y(1-y)]^p\; {}_2F_1(\alpha-\epsilon, \alpha-q-1+\epsilon; \alpha+\epsilon; 1-y(1-y)) \\
&= \frac{\Gamma(\alpha-q-1+\epsilon)\Gamma(q+1)}{\Gamma(\alpha+\epsilon)} \int_0^1 dy\; [y(1-y)]^p \\
&\times \left\{ \frac{\Gamma(\alpha+\epsilon)\Gamma(-\alpha+q+1+\epsilon)}{\Gamma(2\epsilon)\Gamma(q+1)}\; {}_2F_1(\alpha-\epsilon, \alpha-q-1+\epsilon; \alpha-q-\epsilon; y(1-y)) + \right. \\
&\left. [y(1-y)]^{-\alpha+q+1+\epsilon} \frac{\Gamma(\alpha+\epsilon)\Gamma(\alpha-q-1-\epsilon)}{\Gamma(\alpha-\epsilon)\Gamma(\alpha-q-1+\epsilon)}\, {}_2F_1(2\epsilon, q+1; -\alpha+q+2+\epsilon; y(1-y)) \right\},
\end{aligned}
\tag{D.9}
$$

where p and q are positive integers and α is a positive half-integer. In the limit $\epsilon \to 0$, the factor of $1/\Gamma(2\epsilon)$ in eq. (D.9) causes the term containing it to vanish, and the surviving hypergeometric function can be set to 1. Performing the remaining y-integral gives

$$
\begin{aligned}
I(p,q,\alpha) &= \frac{\Gamma(\alpha-q-1-\epsilon)\Gamma(q+1)}{\Gamma(\alpha-\epsilon)} \int_0^1 dy\; [y(1-y)]^{-\alpha+p+q+1+\epsilon} \\
&= \frac{\Gamma(\alpha-q-1)\Gamma(q+1)\Gamma^2(-\alpha+p+q+2)}{\Gamma(\alpha)\Gamma(2(-\alpha+p+q+2))}\,.
\end{aligned}
\tag{D.10}
$$

Thus $I(p,q,\alpha) = 0$ (after analytic continuation in ϵ) unless $\alpha < p+q+2$. In the present case, $p=2$, $q=0$, and $\alpha = 9/2$, so the integral of C_2 vanishes.

The case of $D = 11-2\epsilon$ is similar, although the leading s^4 term in $D = 11-2\epsilon$ requires a bit more separation, along the lines of eq. (D.7), into convergent terms, and terms to which eq. (D.9) can be applied. The final result for the planar double-box pole at $D = 9 - 2\epsilon$ and $D = 11 - 2\epsilon$ is then

$$\mathcal{I}_4^{\text{2-loop, P}, D=9-2\epsilon}\Big|_{\text{pole}} = \frac{1}{4\epsilon\,(4\pi)^9}\frac{\pi}{99792}(-45s^2 + 18st + 2t^2)\,, \tag{D.11}$$

$$\mathcal{I}_4^{\text{2-loop, P}, D=11-2\epsilon}\Big|_{\text{pole}} = \frac{1}{48\epsilon\,(4\pi)^{11}}\frac{\pi}{196911000}(2100s^4 - 880s^3t + 215s^2t^2 + 30st^3 + 12t^4)\,. \tag{D.12}$$

The non-planar double-box integrals are handled analogously, with the results:

$$\mathcal{I}_4^{\text{2-loop, NP}, D=7-2\epsilon}\Big|_{\text{pole}} = \frac{1}{2\epsilon\,(4\pi)^7}\frac{\pi}{15}\,, \tag{D.13}$$

$$\mathcal{I}_4^{\text{2-loop, NP}, D=9-2\epsilon}\Big|_{\text{pole}} = \frac{1}{4\epsilon\,(4\pi)^9}\frac{-\pi}{83160}(75s^2 + 2tu)\,, \tag{D.14}$$

$$\mathcal{I}_4^{\text{2-loop, NP}, D=11-2\epsilon}\Big|_{\text{pole}} = \frac{1}{48\epsilon\,(4\pi)^{11}}\frac{\pi}{1654052400}(40383s^4 - 1138s^2tu + 144t^2u^2)\,. \tag{D.15}$$

As mentioned in section 2.3.1, for $N = 8$ supergravity we are also interested in the divergences at $D = 10 - 2\epsilon$, which require subtractions of one-loop sub-divergences. Here we followed the approach of ref. [111], differentiating the subtracted momentum integrals with respect to the external momenta until they are only logarithmically divergent. For completeness, we also quote the results for $D = 8 - 2\epsilon$, which were obtained in the same way. In each case a minimal $(1/\epsilon)$ subtraction of the subdivergence was used:

$$\mathcal{I}_4^{\text{2-loop, P}, D=8-2\epsilon}\Big|_{\text{subtracted, pole}} = \frac{1}{2\,(4\pi)^8}\left[-\frac{1}{72}\frac{s}{\epsilon^2} + \frac{1}{864}\frac{5s+2t}{\epsilon}\right], \tag{D.16}$$

$$\begin{aligned}\mathcal{I}_4^{\text{2-loop, P}, D=10-2\epsilon}\Big|_{\text{subtracted, pole}} &= \frac{1}{12\,(4\pi)^{10}}\left[-\frac{1}{25200}\frac{s^2(4s+t)}{\epsilon^2}\right.\\ &\left.+\frac{1}{21168000}\frac{-704s^3 + 55s^2t + 252st^2 + 63t^3}{\epsilon}\right],\end{aligned} \tag{D.17}$$

$$\mathcal{I}_4^{\text{2-loop, NP}, D=8-2\epsilon}\Big|_{\text{subtracted, pole}} = \frac{1}{2\,(4\pi)^8}\left[-\frac{1}{144}\frac{s}{\epsilon^2} - \frac{1}{864}\frac{s}{\epsilon}\right], \tag{D.18}$$

$$\mathcal{I}_4^{\text{2-loop, NP}, D=10-2\epsilon}\Big|_{\text{subtracted, pole}} = \frac{1}{12\,(4\pi)^{10}}\left[\frac{1}{7200}\frac{s^3}{\epsilon^2} - \frac{1}{4536000}\frac{s(301s^2+10tu)}{\epsilon}\right]. \tag{D.19}$$

Appendix E

Two-loop Three-Particle Cuts

We now evaluate the D-dimensional two-loop three-particle cuts for the $N = 8$ four-point amplitude, by recycling the analogous calculation for $N = 4$ Yang-Mills theory.

Consider first the three-particle s-channel cut (2.2.2). Just as for the two-particle sewing, the sum over three-particle states for $N = 8$ supergravity may be expressed as a double sum over $N = 4$ states. Then we may apply the five-point KLT formula (1.2.2), which expresses the $N = 8$ supergravity tree amplitudes appearing in eq. (2.2.2) in terms of the corresponding $N = 4$ Yang-Mills tree amplitudes.

The tree amplitude on the left side of the cut is

$$\begin{aligned} M_5^{\rm tree}(\ell_1, 1, 2, \ell_3, \ell_2) &= i\,(\ell_1 + k_1)^2 (\ell_3 + k_2)^2 A_5^{\rm tree}(\ell_1, 1, 2, \ell_3, \ell_2) \\ & \quad \times A_5^{\rm tree}(1, \ell_1, \ell_3, 2, \ell_2) + \{1 \leftrightarrow 2\}\,, \end{aligned} \tag{E.1}$$

where we have chosen a convenient representation in terms of the gauge theory amplitudes. (The $\{1 \leftrightarrow 2\}$ interchange acts only on k_1 and k_2, and *not* on ℓ_1 and ℓ_2.) Similarly, for the tree amplitude on the right side of the cut,

$$\begin{aligned} M_5^{\rm tree}(-\ell_3, 3, 4, -\ell_1, -\ell_2) &= i\,(\ell_3 - k_3)^2 (\ell_1 - k_4)^2 A_5^{\rm tree}(-\ell_3, 3, 4, -\ell_1, -\ell_2) \\ & \quad \times A_5^{\rm tree}(3, -\ell_3, -\ell_1, 4, -\ell_2) + \{3 \leftrightarrow 4\}\,. \end{aligned} \tag{E.2}$$

Thus we have,

$$
\begin{aligned}
\sum_{N=8 \text{ states}} & M_5^{\text{tree}}(1,2,\ell_3,\ell_2,\ell_1) M_5^{\text{tree}}(3,4,-\ell_1,-\ell_2,-\ell_3) \\
& = -(\ell_1+k_1)^2(\ell_3+k_2)^2(\ell_3-k_3)^2(\ell_1-k_4)^2 \\
& \times \Big[\sum_{N=4 \text{ states}} A_5^{\text{tree}}(\ell_1,1,2,\ell_3,\ell_2) A_5^{\text{tree}}(-\ell_3,3,4,-\ell_1,-\ell_2) \Big] \\
& \times \Big[\sum_{N=4 \text{ states}} A_5^{\text{tree}}(1,\ell_1,\ell_3,2,\ell_2) A_5^{\text{tree}}(3,-\ell_3,-\ell_1,4,-\ell_2) \Big] \\
& \qquad + \{1 \leftrightarrow 2\} + \{3 \leftrightarrow 4\} + \{1 \leftrightarrow 2,\, 3 \leftrightarrow 4\}\,, \qquad \text{(E.3)}
\end{aligned}
$$

where all momenta are to be taken on-shell. The sum over $N = 4$ states in each set of brackets can be simplified using the results for the two-loop $N = 4$ Yang-Mills amplitudes (2.2.13). Taking the three-particle cuts of the planar contributions (see figure 3 of ref. [74]) yields the on-shell phase-space integral of

$$
\begin{aligned}
\sum_{N=4 \text{ states}} & A_5^{\text{tree}}(\ell_1,1,2,\ell_3,\ell_2) A_5^{\text{tree}}(-\ell_3,3,4,-\ell_1,-\ell_2) = -i\, st\, A_4^{\text{tree}}(1,2,3,4) \\
\times \Big[& \frac{s}{(\ell_1-k_4)^2(\ell_3+k_2)^2(\ell_1+\ell_2)^2(\ell_2+\ell_3)^2} + \frac{s}{(\ell_3-k_3)^2(\ell_1+k_1)^2(\ell_1+\ell_2)^2(\ell_2+\ell_3)^2} \\
& + \frac{t}{(\ell_3-k_3)^2(\ell_1-k_4)^2(\ell_3+k_2)^2(\ell_1+k_1)^2} \Big]\,. \qquad \text{(E.4)}
\end{aligned}
$$

This equation actually holds even before carrying out the loop-momentum (or phase-space) integral. In the calculations used to derive the results of ref. [74] eq. (E.4) was obtained at the level of the integrands, with all states carrying four-dimensional momenta and helicities, but then it was argued that the light-cone superspace power-counting of Mandelstam [113] ruled out corrections coming from the (-2ϵ)-dimensional components of the loop momenta. Since this argument is based on superspace cancellations it applies to the integrands before integration over loop momenta, and works for D-dimensional external states as well.

The second sum over $N = 4$ states is similar, but more complicated, involving three different cuts of planar double-boxes and ten of non-planar ones,

$$
\begin{aligned}
\sum_{N=4 \text{ states}} & A_5^{\text{tree}}(1,\ell_1,\ell_3,2,\ell_2) A_5^{\text{tree}}(3,-\ell_3,-\ell_1,4,-\ell_2) = -i\, st A_4^{\text{tree}}(1,2,3,4) \\
\times \Big[& \frac{t}{(\ell_1+k_1)^2(\ell_2+k_2)^2(\ell_2-k_3)^2(\ell_1-k_4)^2} - \frac{s}{(\ell_1+\ell_3)^2(\ell_3+k_2)^2(\ell_1+k_1)^2(\ell_2-k_3)^2} \\
+ & \frac{t}{(\ell_1+k_1)^2(\ell_3+k_2)^2(\ell_1-k_4)^2(\ell_2-k_3)^2} - \frac{s}{(\ell_1+\ell_3)^2(\ell_2+k_2)^2(\ell_3-k_3)^2(\ell_1-k_4)^2} \\
+ & \frac{t}{(\ell_1+k_1)^2(\ell_2+k_2)^2(\ell_1-k_4)^2(\ell_3-k_3)^2} + \frac{t}{(\ell_1+k_1)^2(\ell_3+k_2)^2(\ell_1-k_4)^2(\ell_3-k_3)^2} \\
- & \frac{u}{(\ell_2+k_1)^2(\ell_3+k_2)^2(\ell_1-k_4)^2(\ell_2-k_3)^2} - \frac{s}{(\ell_1+\ell_3)^2(\ell_2+k_1)^2(\ell_3-k_3)^2(\ell_1-k_4)^2} \\
+ & \frac{t}{(\ell_2+k_1)^2(\ell_3+k_2)^2(\ell_3-k_3)^2(\ell_1-k_4)^2} - \frac{s}{(\ell_1+\ell_3)^2(\ell_1+k_1)^2(\ell_3+k_2)^2(\ell_2-k_4)^2} \\
- & \frac{u}{(\ell_1+k_1)^2(\ell_2+k_2)^2(\ell_3-k_3)^2(\ell_2-k_4)^2} + \frac{t}{(\ell_1+k_1)^2(\ell_3+k_2)^2(\ell_3-k_3)^2(\ell_2-k_4)^2} \\
+ & \frac{t}{(\ell_2+k_1)^2(\ell_3+k_2)^2(\ell_3-k_3)^2(\ell_2-k_4)^2} \Big] .
\end{aligned} \tag{E.5}
$$

The complete $N = 8$ result is given by simply inserting the $N = 4$ results (E.4) and (E.5) into eq. (E.3).

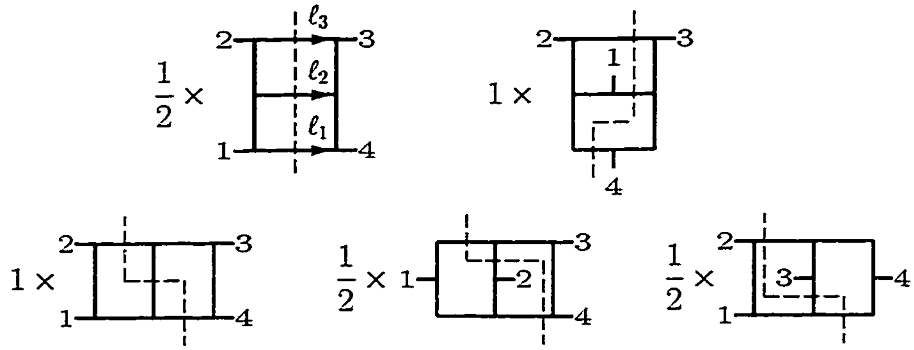


Figure E.4: The three-particle s-channel cuts for gravity. We must also sum over $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ and the 3! permutations of ℓ_1, ℓ_2 and ℓ_3. We have included the appropriate combinatoric factors necessary for eliminating double counts.

This may be compared with the s-channel three-particle cuts of eq. (2.3.12). Taking all s-channel three-particle cuts of the scalar integrals, as shown in fig. E.4, we obtain

$$
\begin{aligned}
& stuM_4^{\rm tree} \int \frac{d^D\ell_1}{(2\pi)^D}\frac{d^D\ell_2}{(2\pi)^D}\,\frac{1}{\ell_1^2\,\ell_2^2\,\ell_3^2}\bigg\{\bigg[\Big(\frac{1}{2}\,\frac{t^2}{(\ell_1+k_1)^2(\ell_3+k_2)^2(\ell_3-k_3)^2(\ell_1-k_4)^2} \\
& +\frac{t^2}{(\ell_2+k_1)^2(\ell_3+k_2)^2(\ell_3-k_3)^2(\ell_1-k_4)^2}+\frac{1}{2}\,\frac{s^2}{(\ell_2+\ell_3)^2(\ell_3+k_1)^2(\ell_2+k_2)^2(\ell_1-k_4)^2} \\
& +\frac{s^2}{(\ell_1+\ell_2)^2(\ell_2+\ell_3)^2(\ell_3+k_2)^2(\ell_1-k_4)^2}+\frac{1}{2}\,\frac{s^2}{(\ell_1+\ell_2)^2(\ell_3+k_2)^2(\ell_2-k_3)^2(\ell_1-k_4)^2}\Big) \\
& +\text{perms}(\ell_1,\ell_2,\ell_3)\bigg] \qquad +\{1\leftrightarrow 2\}+\{3\leftrightarrow 4\}+\{1\leftrightarrow 2,\,3\leftrightarrow 4\}\bigg\}\bigg|_{\ell_i^2=0}, \qquad \text{(E.6)}
\end{aligned}
$$

where we have included all kinematic factors associated with each scalar diagram in fig. E.4. The $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ permutations act on all terms in the square brackets. Note that the cut result (E.3) also contains a sum over $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ interchanges.

We have verified that the expressions in eqs. (E.6) and (E.3) are indeed equal. The t- and u-channel cuts are identical, up to relabelings. Since all two- and three-particle cuts are correctly given by eq. (2.3.12) in arbitrary dimensions, it is the complete expression for the two-loop four-point $N = 8$ supergravity amplitude, expressed in terms of scalar loop integrals.

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