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TOPOLOGICAL STRING THEORY AND ENUMERATIVE GEOMETRY

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TOPOLOGICAL STRING THEORY AND ENUMERATIVE GEOMETRY

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Abstract

In this thesis we investigate several problems which have their roots in both topological string theory and enumerative geometry. In the former case, underlying theories are topological field theories, whereas the latter case is concerned with intersection theories on moduli spaces. A permeating theme in this thesis is to examine the close interplay between these two complementary fields of study.

The main problems addressed are as follows: In considering the Hurwitz enumeration problem of branched covers of compact connected Riemann surfaces, we completely solve the problem in the case of simple Hurwitz numbers. In addition, utilizing the connection between Hurwitz numbers and Hodge integrals, we derive a generating function for the latter on the moduli space $\overline{M}_{g,2}$ of 2-pointed, genus-g Deligne-Mumford stable curves. We also investigate Givental's recent conjecture regarding semisimple Frobenius structures and Gromov-Witten invariants, both of which are closely related to topological field theories; we consider the case of a complex projective line $\mathbf{P}^1$ as a specific example and verify his conjecture at low genera. In the last chapter, we demonstrate that certain topological open string amplitudes can be computed via relative stable morphisms in the algebraic category.

Acknowledgments

Merely mentioning the names of those who have made the past four years of my life altogether gratifying and meaningful, seems infinitely inadequate for expressing my gratitude towards them. What I barely sketch in the present short section, I will again and again come back to in the future and continue to paint in my mind its worthy countenance. As such, the main text of this thesis should be considered a supplement to what has merely begun in this section, but not vice versa.

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With warm respect I thank my wife Young, whom I met as a confused graduate student. She has given a new life to my once desiccated soul and has filled my time with richness.

To Emily Dickinson, whose natural verses
have many times consoled my soul.

It was not death, for I stood up,
And all the dead lie down.
It was not night, for all the bells
Put out their tongues for noon.

It was not frost, for on my flesh
I felt siroccos crawl,
Nor fire, for just my marble feet
Could keep a chancel cool.

And yet it tasted like them all,
The figures I have seen
Set orderly for burial
Reminded me of mine,

As if my life were shaven
And fitted to a frame
And could not breathe without a key,
And 'twas like midnight, some,

When everything that ticked has stopped
And space stares all around,
Or grisly frosts, first autumn morns,
Repeal the beating ground;

But most like chaos, stopless, cool,
Without a chance, or spar,
Or even a report of land
To justify despair.

– Emily Dickinson

Contents

Chapter 1

Introduction

Over the past two decades, the close interplay between mathematics and physics has been very fruitful, often leading to many far-reaching consequences in each discipline. A subject which encompasses such an intricate mixture of physics and mathematics is string theory. Aside from being the best candidate for the theory of quantum gravity, string theory has branched off many new and exciting ideas in mathematics. String theory would not, of course, have seen its success without all the important concurrent developments in mathematics. A general theme that runs through this thesis is to elucidate the aforementioned interplay between string theory and mathematics by investigating a few intriguing connections between the two fields of study. In this introductory chapter, we shall lay out some elementary concepts in topological string theory and discuss the relevant physical motivations that underlie the succeeding chapters.

Mathematical implications of string theory become most clear in the context of topological string theory, which consists of a two-dimensional "matter" topological field theory coupled to two-dimensional topological gravity. A very important and interesting fact is that two-dimensional topological gravity, which is associated to the algebraic topology of the suitably compactified moduli space $\overline{M}_{g,n}$ of n-pointed, genus-g Riemann surfaces, is but one of several equivalent formulations of two-dimensional quantum gravity [W3, W4]. In the matrix model approach, two-dimensional quantum gravity is studied by counting inequivalent triangulations of Riemann surfaces, and this formulation sometimes leads to complete solvability. The so-called "one matrix model," for example, is a solvable theory which is related to integrable hierarchies of the KdV-type.[1] Since correlation functions of topological gravity encode intersection numbers of cohomology classes of the moduli space $\overline{M}_{g,n}$, that topological gravity is proposed to be equivalent to the one matrix model has far-reaching mathematical consequences. In particular, it suggests that the intersection theory on the moduli space $\overline{M}_{g,n}$ of stable curves[2] is governed by the KdV hierarchy, a conjecture originally put forth by E.Witten [W4].

As the stable intersection theory on $\overline{M}_{g,n}$ had not been known to have any algebraic structure, Witten's conjecture came as a surprise. The conjecture was later proved by mathematician M.Kontsevich, who constructed a combinatorial model to study the intersection theory as sums of tri-valent graphs on Riemann surfaces. The tri-valent graphs were interpreted as Feynman diagrams that arose in a new matrix model, and the KdV equations were then derived by analyzing the matrix integral [Kont1]. The combinatorial formula that appears in Kontsevich's proof is, however, quite difficult to digest, and many mathematicians have tried to understand the proof using different approaches. Recently A.Okounkov and R.Pandharipande have used the enumeration problem of Hurwitz numbers to provide a new path between matrix models and the intersection theory on $\overline{M}_{g,n}$ [OP].

1.1 Appearance of Hurwitz Numbers in Physics

The Hurwitz enumeration problem, which concerns counting the number of inequivalent branched covers of a compact connected Riemann surface by compact connected Riemann surfaces with specified ramifications, was first posed by Hurwitz more than 100 years ago [Hur], and has been receiving renewed interest recently. In fact, Hurwitz numbers arise naturally in topological string theories obtained by coupling topologi-

[1] The N matrix model is related to the N^{th} generalized KdV hierarchy[W7].

[2] The moduli space of stable curves is defined in §1.2.

cal sigma models [W1] to topological gravity. In Chapter 2 of this thesis, we compute all simple Hurwitz numbers $\mu^{g,n}_{h,n}$, which count the number of inequivalent degree-n branched covers of a genus-h Riemann surface by genus-g Riemann surfaces with no non-simple branch points, for arbitrary g, h, and n. We also derive generating functions—which, in the case of $h = 1$, reproduce the partition function, the exponential of the free energy, of topological string theory with an elliptic curve target space—for the simple Hurwitz numbers $\mu^{g,n}_{h,n}$.

There is another physical motivation for studying the Hurwitz enumeration problem. Before the discovery of AdS/CFT correspondence [Mal, W8, GuKP] and holography [Su, tH], physicists were interested in finding a possible connection between a gauge theory in d dimensions and a string theory with a target space also of d dimensions. In particular, in two dimensions, there exists a rigorous connection between a two-dimensional gauge theory on a Riemann surface and a string theory with the same manifold as its target space. Recall that in two dimensions pure Yang-Mills theory has no propagating degrees of freedom and is locally trivial. But, if the theory is defined on a compact manifold, global geometry plays an important role and gives the theory nontrivial characters.

One of the remarkable things about the two-dimensional Yang-Mills theory is that it is exactly solvable. This fact was established by Migdal [Mig], Rusakov [Rus] and Witten [W5]. The first two authors used a renormalization group invariant lattice formulation, and Witten explained it in terms of localization of path integrals in topological Yang-Mills theory to a moduli space of classical configuration. Migdal and Rusakov showed that the partition function of the Euclidean $SU(N)$ or $U(N)$ Yang-Mills theory on a genus-h Riemann surface of area A can be expressed in terms of the group theory of the gauge group as follows:

$$Z_{\mathrm{YM}_2}(h, \lambda A, N) = \int [\mathcal{D}A^\mu] e^{-S_{\mathrm{YM}}} = \sum_{R \in \mathscr{R}} (\dim R)^{2-2h} \exp\left(\frac{-\lambda A}{2N} C_2(R) \right).$$

Here, λ is the 't Hooft coupling related to the gauge coupling g by $\lambda = g^2 N$, $\mathscr{R}$ denotes the set of all irreducible representations of the gauge group, and $C_2(R)$ is the quadratic Casimir. Another breakthrough came in 1993 when D. Gross and W. Taylor explained how the $1/N$ expansion of this expression can be given an interpretation as a string theory partition function to all orders in $1/N$ [GroT]. More precisely, they expanded the free energy $\mathscr{F} = \ln Z$ in powers of $1/N$ and obtained

$$\mathscr{F} = \sum_{g=0}^{\infty} N^{2-2g} \sum_{n=0}^{\infty} \sum_{i=0}^{2(g-1)-2n(h-1)} \omega_{g,h}^{n,i} e^{-\frac{n\lambda A}{2}} (\lambda A)^i.$$

Furthermore, they showed that the coefficients $\omega_{g,h}^{n,i}$, up to normalization, in the expansion have a geometric interpretation of counting the number of topologically inequivalent maps from the source Riemann surface to the target Riemann surface, i.e. the Hurwitz numbers. The index n is interpreted as the winding number and the index i is supposed to specify the number of branch points. For i saturating the upper limit—which would give the Riemann-Hurwitz formula $2(g-1) - 2n(h-1) = i$, where i is the number of simple branch points—$\omega_{g,h}^{n,i}$ counts the number of topologically inequivalent simple branched covers. For smaller values of i, the corresponding covering maps have certain degenerations. It turns out that when the gauge group is $U(N)$, the non-vanishing coefficients are precisely the ones with the index i saturating the upper limit. And, therefore, the free energy of two-dimensional $U(N)$ YM-theory encodes the number of branched covers of a Riemann surface by another Riemann surface.

1.2 Hodge Integrals and Topological String Theory

We have discussed in the previous section why physicists have recently become interested in the Hurwitz numbers. Before we elaborate further on this point, let us consider in parallel some of the mathematical motivations for examining anew the Hurwitz enumeration problem. One of the most fascinating results in this respect is that the Hurwitz numbers are closely related to the intersection theory on the moduli space of stable curves. As we mentioned before, the underlying physical theory in the latter case is topological gravity. In a topological string theory with a target space X, the physical operators are the cohomology classes of X and their gravitational descendants. In particular, if one considers the pure topological gravity without coupling any matter topological sigma model, then the only observables are the gravitational descendants of the identity. We now briefly describe what they are in terms of mathematics.

A point in the moduli space $\overline{M}_{g,n}$ corresponds to a projective, connected, nodal curve C_g of arithmetic genus g, with n distinct marked points $\{p_1, \ldots, p_n\}$ which are nonsingular. In addition, it satisfies the stability condition $2g - 2 + n > 0$, which guarantees that there are no infinitesimal automorphisms fixing the marked points. For each marking i, there is an associated cotangent line bundle $\mathscr{L}_i \to \overline{M}_{g,n}$ whose fiber over the point $[C_g; p_1, \ldots, p_n] \in \overline{M}_{g,n}$ is $T^*C_g(p_i)$. Then, the observables in the pure topological gravity are the first Chern classes

$$\psi_i = c_1(\mathscr{L}_i) \in H^2(\overline{M}_{g,n}, \mathbb{Q}).$$

Therefore, for a Riemann surface with n marked points, there are n distinct ψ-classes. We can now restate Witten's conjecture as follows: Starting from the intersection

$$\langle \tau_0 \tau_0 \tau_0 \rangle_0 := \int_{\overline{M}_{0,3}} \psi_1^0 \psi_2^0 \psi_3^0 = 1\,,$$

all other intersections of the ψ-classes

$$\langle \tau_{k_1} \tau_{k_2} \cdots \tau_{k_n} \rangle_g := \int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}\ , \tag{1.2.1}$$

where $\sum_{i=1}^n k_i = 3g - 3 + n = \dim(\overline{M}_{g,n})$, can be completely determined using the KdV equations and the so-called string equation.

To relate the Hurwitz numbers to the intersection theory on $\overline{M}_{g,n}$, we actually need to generalize the integral in (1.2.1) by considering a new class of objects; that is, in addition to the cotangent line bundles described above, we need to consider the Hodge bundle $\mathbb{E} \to \overline{M}_{g,n}$. The Hodge bundle is the rank-g vector bundle whose fiber over $[C_g; p_1, \ldots, p_n]$ is $H^0(C_g, \omega_{C_g})$, spanned by g independent holomorphic 1-forms on $C_g, g > 1$. The λ-classes are defined as $\lambda_j = c_j(\mathbb{E})$, the jth Chern class of $\mathbb{E}$, and a Hodge integral is an integral over $\overline{M}_{g,n}$ of products of the ψ and λ classes[3].

So far we have not said anything to motivate studying Hodge integrals. Let us now try to fill in this gap. The importance of Hodge integrals is most apparent in Gromov-Witten theory, which is a study of intersection theory on the moduli space $\overline{M}_{g,n}(X)$ of *stable maps*. Physically, Gromov-Witten theory corresponds to coupling a topological sigma model with target space X to topological gravity. If $X = \{\mathrm{pt}\}$, then $\overline{M}_{g,n}(X)$ is naturally isomorphic to $\overline{M}_{g,n}$ and we end up with pure topological gravity. In Gromov-Witten theory the virtual localization formula of T.Graber and R.Pandharipande in [GraP] reduces the computation of intersection numbers on $\overline{M}_{g,n}(X)$ to explicit graphical sums involving only Hodge integrals over $\overline{M}_{g,n}$. Moreover, if the stable map to the target X has degree zero, the so-called Gromov-Witten invariants of X are given by the classical cohomology ring of X and Hodge integrals over $\overline{M}_{g,n}$.

Let us now consider a specific example in topological string theory that demonstrates the importance of Hodge integrals. In [FabP1] C.Faber and R.Pandharipande have derived the following generating function for Hodge integrals:

$$F(t,k) := 1 + \sum_{g\geq 1} t^{2g} \sum_{i=0}^{g} k^i \int_{\overline{M}_{g,1}} \psi^{2g-2+i}\,\lambda_{g-i} = \left(\frac{t/2}{\sin(t/2)}\right)^{k+1}, \qquad (1.2.2)$$

where t and k are some formal parameters. This generating function $F(t,k)$ has a direct application in string theory. For instance, consider the topological A-model on the local Calabi-Yau manifold $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. Its free energy has an expansion of the form

$$\mathscr{F}_g = \int_{\overline{M}_{g,0}} \lambda_{g-1}^3 + \sum_{n=1}^{\infty} C(g,n) e^{-n\hat{t}},$$

where $\hat{t}$ is the Kähler parameter of the fixed rational curve, i.e. the base $\mathbf{P}^1$, and the quantities $C(g,n)$ are given by degree-n multiple covers of that fixed $\mathbf{P}^1$. For $n \geq 1$, the genus-0 answer was obtained in [AsM, CdGP] to be

$$C(0,n) = \frac{1}{n^3},$$

[3]The total degree of these cohomology elements should equal the dimension $3g-3+n$ of the moduli space $\overline{M}_{g,n}$.

whereas the genus-1 answer

$$C(1,n) = \frac{1}{12n}$$

was independently obtained by mathematicians [GraP] and physicists [BerCOV]. A significant progress in this line of research was made when C.Faber and R.Pandharipande [FabP1] used the method of virtual localization [Kont2, GraP] to show that, for all $g > 1$, $C(g,n)$ are given by the expression

$$C(g,n) = n^{2g-3} \sum_{\substack{g_1+g_2=g \\ g_1,g_2 \geq 0}} \left(\int_{\overline{M}_{g_1,1}} \psi_1^{2g_1-2} \lambda_{g_1} \int_{\overline{M}_{g_2,1}} \psi_1^{2g_2-2} \lambda_{g_2} \right),$$

which can be evaluated using the Hodge integral generating function $F(t,0)$ in (1.2.2). After expressing the right hand side of (1.2.2) in terms of Bernoulli numbers B_n, Faber and Pandharipande showed that

$$C(g,n) = \frac{|B_{2g}| n^{2g-3}}{2g(2g-2)!}.$$

Inspired by the impressive usefulness of Faber and Pandharipande's generating function $F(t,k)$ for Hodge integrals, we have extended their result and obtained a generating function for Hodge integrals over the moduli space $\overline{M}_{g,2}$. This work is described in detail in Chapter 2 of this thesis. We only mention here that we use the idea of T.Ekedahl, S.Lando, M.Shapiro and A.Vainshtein [ELSV] which relates the Hurwitz numbers to Hodge integrals. After first obtaining a generating function for appropriate Hurwitz numbers, we then find a closed-form generating function for the Hodge integrals over $\overline{M}_{g,2}$.

1.3 Frobenius Structures of 2-Dimensional Topological Field Theories

Topological field theories have well-defined properties which can be formulated as axioms. In particular, in two-dimensional topological field theories, a beautiful underlying structure emerges from the axiomatic approach of M.Atiyah, in which one associates a Hilbert space to a connected boundary component and certain operations to cobordisms [At]. One very important result of such constructions is that in two-dimensional topological field theories—whose necessary data are encoded in the genus-0 surface with one, two or three holes—the Hilbert space $\mathscr{H}$ associated to a circle carries an algebra structure. More precisely, the algebra structure is that of a commutative Frobenius algebra, which is defined as follows:

Definition 1.1 (Frobenius Algebra) *Let $(\mathscr{A}, *)$ be a commutative, associative algebra with a unit $\mathbf{1}$. If there exists a non-degenerate inner product $\langle \cdot, \cdot \rangle$ on $\mathscr{A}$, such that*

$$\langle a * b, c \rangle = \langle a, b * c \rangle \tag{1.3.3}$$

for all $a, b, c \in \mathscr{A}$, then $(\mathscr{A}, *)$ is called a commutative Frobenius algebra.

The disk gives a unit $\mathbf{1}$ of the algebra, and the cylinder gives the non-degenerate inner product $\langle \cdot, \cdot \rangle$ on $\mathscr{H}$. The sphere with three holes gives rise to a bilinear map

$$\mathscr{H} \otimes \mathscr{H} \to \mathscr{H},$$

which gives the product $*$ on $\mathscr{H}$. Moreover, it follows from the axioms of topological field theory that the product $*$ is commutative and associative, and that (1.3.3) is satisfied.

Let $\{\mathcal{O}^{(0)}\}$ denote the set of local physical observables, which can be taken as operator-valued zero-forms on the domain manifold $\mathfrak{M}$. Topological invariance of the theory then implies that $\{\mathcal{O}^{(0)}\}$ are in fact closed up to BRST commutators, that is

$$\mathrm{d}\mathcal{O}^{(0)} = \{Q, \mathcal{O}^{(1)}\},$$

where Q is a BRST symmetry generator. The above equation, in turn, implies that $\mathcal{O}^{(1)}$ is an operator-valued one-form on $\mathfrak{M}$ that is BRST-invariant up to an exact form. Hence one can now construct a new class of non-local physical observables as

$$\int_{C_1} \mathcal{O}^{(1)}(x),$$

where C_1 is a one-dimensional closed submanifold of $\mathfrak{M}$. Topological invariance further implies the descent equations

$$\mathrm{d}\mathcal{O}^{(k)} = \{Q, \mathcal{O}^{(k+1)}\}$$

for $1 \leq k \leq \dim(\mathfrak{M}) - 1$, and $\mathcal{O}^{(k)}$ can be given a similar interpretation as above. In particular, one can construct a non-local physical observable

$$\int_{C_k} \mathcal{O}^{(k)}(x)$$

for $C_k \in H_k(\mathfrak{M})$.

Let us now restrict our attention to the case where the domain manifold $\mathfrak{M}$ is a Riemann surface Σ_g. A very important class of non-local observables in this case takes the form

$$\int_{\Sigma_g} \mathcal{O}^{(2)}(x),$$

and they can be used to perturb the two-dimensional topological field theory by modifying the action as

$$S \mapsto S - \sum_{\alpha} t^{\alpha} \int_{\Sigma_g} \mathcal{O}^{(2)}_{\alpha} .$$

Such a perturbation preserves the topological invariance [DiVV], and therefore the coupling constants t^{α} parametrize a family of two-dimensional topological field theories; that is, t^{α} are coordinates on the parameter space of two-dimensional topological field theories. Another important fact is that the perturbed algebra $\mathscr{H}_t$ still has a Frobenius algebra structure. The associativity of the product on $\mathscr{H}_t$ is equivalent to the fact that the free energy $\mathscr{F}_g(t)$ of the two-dimensional topological field theory satisfies a very complicated system of differential equations, often called the WDVV equations after their discovery in [W3, DiVV].

In a topological sigma model with target space X, the Hilbert space $\mathscr{H}$ is isomorphic to $H^*(X, \mathbb{C})$ as a vector space and the product $*$ is identified with the quantum cup product $\circ$. Quantum cohomology is an important subject that arises in the mathematical formulation of mirror symmetry, and is defined using the genus-zero Gromov-Witten invariants, which, as we have mentioned before, are closely related to topological sigma models. The associativity of the quantum cup product $\circ$ is equivalent to the condition that the genus-zero Gromov-Witten potential, a generating function for the genus-zero Gromov-Witten invariants, satisfies the WDVV equations. In the context of topological field theory, the genus-g Gromov-Witten potential is identified with the genus-g free energy $\mathscr{F}_g(t)$, which is defined as follows: Let $f_*[\Sigma_g] = \beta \in H_2(X)$, where f is a holomorphic map $f : \Sigma_g \to X$. Then,

$$\mathscr{F}_g(t) := \sum_{\beta} \langle \exp \sum_{\alpha} t^{\alpha} \int_{\Sigma_g} \mathcal{O}_{\alpha}^{(2)} \rangle_{g,\beta} \,,$$

where $\mathcal{O}_{\alpha}^{(2)}$ is a $H^*(X, \mathbb{C})$-valued two-form on Σ_g.

Recently a remarkable conjecture was put forth by A.Givental regarding the Gromov-Witten potentials of semisimple Frobenius manifolds[4] [Giv2]. Givental's conjecture expresses higher genus Gromov-Witten invariants in terms of the genus-0 data and the intersection theory on the moduli space $\overline{M}_{g,n}$ of stable curves. In Chapter 3 of this thesis, we investigate the conjecture in the case of a complex projective line, whose cohomology $H^*(\mathbf{P}^1, \mathbb{C})$ is, when viewed as a super-manifold, a semisimple Frobenius manifold. Although we are not able to prove Givental's conjecture, we do make some simple checks supporting his conjecture.

1.4 String Instanton Amplitudes from Algebraic Geometry

A supersymmetric non-linear σ-model whose target space is a Kähler manifold has an extended $\mathcal{N} = 2$ supersymmetry, of which the extra supersymmetry generator is constructed from the complex structure of the Kähler manifold. Of particular interest to string theorists is the case where the source manifold is a Riemann surface Σ and the target manifold X is a Calabi-Yau manifold, which of course is Kähler. It is well-known that an $\mathcal{N} = 2$ supersymmetric non-linear σ-model with maps $\phi : \Sigma \to X$

[4] The definition of Frobenius manifold is given in §3.1.1.

can be twisted[5] in two inequivalent ways to yield a pair of topological field theories, often called the A-model and the B-model. Furthermore, under mirror symmetry, the A-model and the B-model of X are related to the B-model and the A-model, respectively, of the mirror manifold (See [W6] for a very clear exposition on this point).

The twisted theories have many intriguing properties, one of which is that physical answers can be obtained purely from geometrical considerations. In the parent untwisted theories, it follows from the non-renormalization theorems of $\mathcal{N} = 2$ supersymmetry that certain correlation functions do not get corrected from their classical values to all orders in perturbation theory. One of the important results of twisting is that the twisted theories focus on these special correlation functions. For example, the correlation functions in the A-model are topological invariants of the target manifold X, and they receive only non-perturbative corrections from world sheet instantons, which are characterized by the holomorphic maps

$$\partial_{\bar{z}}\phi^i = 0 = \partial_z\phi^{\bar{i}}$$

minimizing the bosonic part of the σ-model action. Instanton amplitudes in the A-model also admit a geometrical interpretation, this time as intersection numbers on the space of maps, and it is in this context that mathematical implications become most transparent. As briefly mentioned before, the study of the intersection theory on the moduli space of stable maps is the subject of Gromov-Witten theory. In §3.1.2 we will define the so-called Gromov-Witten invariants that arise in the theory.

Various closed string instanton amplitudes have been computed from the algebro-geometric point of view. As already discussed in §1.2, a concrete example, in which the usefulness of algebraic geometry is manifest, is the problem of multiple covers of a fixed rational curve $\mathbf{P}^1$ with the normal bundle $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. In that case, using the method of virtual localization and the generating function for Hodge integrals obtained by Faber and Pandharipande, one can completely solve the multiple

[5]Physically, twisting involves changing the spins of the quantum fields and performing a projection on the Hilbert space.

cover problem. There are many equally impressive results in the link between topological *closed* string theories and intersection theories on moduli spaces. One of the factors responsible for the emergence of such a beautiful link is the recent advances in mathematics regarding Gromov-Witten theory. When one tries, however, to extend the picture to topological *open* string theories, one immediately faces the lack of complementary development in mathematics for studying maps between manifolds with boundaries.

In [LS] J.Li and the present author try to narrow the gap between topological *open* string theories and intersection theories on moduli spaces[6]. In the paper we show how topological open string theory amplitudes can be computed by using relative stable morphisms in the algebraic category. We achieve our goal by explicitly working through an example which has been previously considered by Ooguri and Vafa from the point of view of physics. In fact, the example is a variant of the multiple cover problem mentioned above. By using the method of virtual localization, we successfully reproduce Ooguri and Vafa's results for multiple covers of a holomorphic disc, whose boundary lies in a Lagrangian submanifold of a Calabi-Yau 3-fold, by Riemann surfaces with arbitrary genus and number of boundary components. In particular we show that, in the case we consider, there are no open string instantons with more than one boundary component ending on the Lagrangian submanifold. In [LS] we define the moduli space of relative stable morphisms, investigate the obstruction theory of the moduli space, and describe how multiple covers of a holomorphic disc can be viewed as a problem regarding relative stable morphisms. In Chapter 4 of this thesis, we will discuss in detail the computational part of that paper.

(**Remark**: Chapters 2, 3 and 4 of this thesis are based on the preprints [MSS], [SS] and [LS], respectively. The present author was intimately involved with each joint research.)

[6]From a different perspective, Katz and Liu also address the same problem in [KL].

Chapter 2

The Hurwitz Enumeration Problem and Hodge Integrals

In this chapter, we use algebraic methods to compute the simple Hurwitz numbers for arbitrary source and target Riemann surfaces. In the case of an elliptic curve target space, we reproduce the results previously obtained by string theorists. Motivated by the Gromov-Witten potentials, we find a general generating function for the simple Hurwitz numbers in terms of the representation theory of the symmetric group S_n. We also find a generating function for Hodge integrals on the moduli space $\overline{M}_{g,2}$ of genus-g stable curves with two marked points, similar to that found by Faber and Pandharipande [FabP1] for the case of one marked point.

2.1 Introduction

Many classical problems in enumerative geometry have been receiving renewed interests in recent years, the main reason being that they can be translated into the modern language of Gromov-Witten theory and, moreover, that they can be consequently solved. One such classical problem which has been under recent active investigation is the Hurwitz enumeration problem of branched covers. Let Σ_g be a compact connected Riemann surface of genus g, and Σ_h a compact connected Riemann surface of genus h. Here, $g \geq h \geq 0$. A degree-n *branched covering* of Σ_h by Σ_g is a non-constant holomorphic map $f : \Sigma_g \to \Sigma_h$ such that $|f^{-1}(q)| = n$ for all $q \in \Sigma_h$ except for a finite number of points called *branch points*. A branched covering f is called *almost simple* if $|f^{-1}(p)| = n - 1$ for each branch point, with the possible exception of one degenerate point, often denoted by $\infty \in \Sigma_h$. The *ramification type* of that special degenerate point's pre-images are specified by an ordered partition α of the degree n of the covering. Let $\alpha = (\alpha_1, \ldots, \alpha_w)$ be such an ordered partition of n, denoted by $\alpha \vdash n$, of length $\ell(\alpha) = w$. Then, the number r of simple branch points is determined by the Riemann-Hurwitz formula to be:

$$r = (1 - 2h)n + w + 2g - 2 \ , \tag{2.1.1}$$

where g and h are the genera of the source and the target Riemann surfaces, respectively. Two branched coverings f_1 and f_2 are said to be equivalent if there exists a homeomorphism $\phi : \Sigma_g \to \Sigma_g$ such that $f_2 = f_1 \circ \phi$. We define $\mu^{g,n}_{h,w}(\alpha)$ to be the number of inequivalent, almost simple, degree-n branched covering of Σ_h by Σ_g with ramification type $\alpha = (\alpha_1, \ldots, \alpha_w)$. The problem of determining $\mu^{g,n}_{h,w}(\alpha)$ is the Hurwitz enumeration problem. Many mathematicians and physicists have contributed to determining $\mu^{g,n}_{h,w}(\alpha)$ explicitly, and we here list some of their works:

Mathematicians	Cases Considered
J. Dénes (1959) [De]	$g = h = 0;\ \ell(\alpha) = 1$
V.I. Arnol'd (1996) [Ar]	$g = h = 0;\ \ell(\alpha) = 2$
I. Goulden & D. Jackson (1997-1999) [GouJ]	$g = 0, 1, 2;\ h = 0$
R. Vakil (1998) [V]	$g = 0, 1;\ h = 0$

Physicists	Cases Considered
R. Rudd (1994) [Rud]	$g = 1, \ldots, 8;\ h = 1;\ \alpha = (1^n)$
M. Crescimanno & W. Taylor (1995) [CT]	$g = h = 0;\ \alpha = (1^n)$

In the first part of this chapter, we mostly restrict ourselves to simple Hurwitz numbers $\mu^{g,n}_{h,n}(1^n)$, for which there is no ramification over ∞. Hurwitz numbers appear in many branches of mathematics and physics. In particular, they arise naturally in combinatorics, as they count factorizations of permutations into transpositions, and the original idea of Hurwitz expresses them in terms of the representation theory of the symmetric group. Indeed in this respect, the most general problem of counting covers of Riemann surfaces by Riemann surfaces, both reducible and irreducible, with arbitrary branch types, has been completely solved by Mednykh [Med1, Med2]. His formulas however generally do not allow explicit computations of the numbers, except in a few cases.

It turns out that one can successfully obtain the simple Hurwitz numbers using Mednykh's works, and in the first part of chapter, we shall compute them at low degrees for arbitrary target and source Riemann surfaces. Hurwitz numbers also appear in physics: when the target is an elliptic curve, they are[1] the coefficients in the expansion of the free energies of the large N two-dimensional quantum Yang-Mills theory on the elliptic curve, which has in fact a string theory interpretation [Gro, GroT]. The total free energy and the partition function, which is its exponential, can be thought of as generating functions for simple Hurwitz numbers $\mu^{g,n}_{1,n}$. Generalizing this analogy, we have determined the generating functions for target Riemann surfaces of arbitrary genus in terms of the representation theory of the symmetric group S_n.

In the framework of Gromov-Witten theory, simple Hurwitz numbers can be considered as certain cohomological classes evaluated over the virtual fundamental class of the moduli space of stable maps to $\mathbf{P}^1$ [FanP]. By exploiting this reformulation, many new results such as new recursion relations [FanP, So] have been obtained. Furthermore, a beautiful link with Hodge integrals has been discovered, both by virtual localization [FanP, GraV] and by other methods [ELSV]. It is therefore natural to expect that the knowledge of Hurwitz numbers might be used to gain new insights into Hodge integrals. This line of investigations has previously led to a closed-form formula for a generating function for Hodge integrals over the moduli space $\overline{M}_{g,1}$ of curves with one marked point [ELSV, FabP1]. Similarly, in this thesis, we consider

[1] Up to over-all normalization constants.

the following generating function for Hodge integrals over $\overline{M}_{g,2}$:

$$G(t,k) := \frac{1}{2} + \sum_{g\geq 1} t^{2g} \sum_{i=0}^{g} k^i \int_{\overline{M}_{g,2}} \frac{\lambda_{g-i}}{(1-\psi_1)(1-\psi_2)} .$$

For negative integral values of k, we have managed to compute $G(t,k)$ in a closed form by relating the integrals to the almost simple Hurwitz numbers $\mu_{0,2}^{g,2k}(k,k)$. We then conjecture a simplified version of our rigorously obtained result, and this conjectural counterpart can then be analytically continued to all values of k. We have checked that the conjectural form of our formula holds true for $-60 \leq k \leq 1$, but unfortunately, we have not been able to prove it for arbitrary k. The success of the computation makes us speculate that in more general cases, similar results might be within reach, and the simplicity of the results suggests that new yet undiscovered structures might be present.

This chapter is organized as follows: in §2.2, we briefly explain the work of Mednykh and apply it to compute the simple Hurwitz numbers; in §2.3, we find the generating functions for all simple Hurwitz numbers; §2.4 discusses our closed-form formula for the generating function for Hodge integrals over $\overline{M}_{g,2}$; and, we conclude in §2.5 by drawing the reader's attention to some important open questions.

NOTATIONS: We here summarize our notations to be used throughout this chapter:

$\mu^{g,n}_{h,n}$	The usual degree-n simple Hurwitz numbers for covers of a genus-h Riemann surface by genus-g Riemann surfaces.
$\tilde{\mu}^{g,n}_{h,n}$, $N_{n,h,r}$	Mednykh's definition of simple Hurwitz numbers, including the fixed point contributions of the S_n action. (See §2.2.2 and §2.2.6 for details.)
$\mathscr{R}_n$	The set of all ordinary irreducible representations of the symmetric group S_n.
$\chi_\gamma(1^{\alpha_1}\cdots n^{\alpha_n})$	The character of the irreducible representation $\gamma \in \mathscr{R}_n$ evaluated at the conjugacy class $[(1^{\alpha_1}\cdots n^{\alpha_n})]$. For those α_i which are zero, we omit the associated cycle in our notation.
f^γ	The dimension of the irreducible representation $\gamma \in \mathscr{R}_n$.
$\mathscr{B}_{n,h,\sigma}$	See (2.2.3).
$\mathscr{T}_{n,h,\sigma}$	Subset of $\mathscr{B}_{n,h,\sigma}$, generating a transitive subgroup of S_n.
$\mathcal{H}^g_h$	Generating functions for $\mu^{g,n}_{h,n}$, for fixed g and h. See (2.4.19).
$\widetilde{\mathcal{H}}^g_h$	Generating functions for $\tilde{\mu}^{g,n}_{h,n}$, for fixed g and h.
$H_{g,n}$	Simplified notation for $\mu^{g,n}_{0,n}$. Not to be confused with $\mathcal{H}^g_h$.
t^p_k	Entries of the branching type matrix σ.
$\hat{t}^i_j$	Coordinates on the large phase space in the Gromov-Witten theory.

In this chapter, all simple Hurwitz numbers count irreducible covers, unless specified otherwise.

2.2 Computations of Simple Hurwitz Numbers

This section describes our computations of the simple Hurwitz numbers $\tilde{\mu}^{g,n}_{h,n}$. The simple covers of an elliptic curve by elliptic curves are actually unramified, and we obtain the numbers $\tilde{\mu}^{1,n}_{1,n}$ by using the standard theory of two-dimensional lattices[2]. For other values of g and h, we simplify the general formulas of Mednykh [Med1] and explicitly compute the numbers for low degrees.

2.2.1 Unramified Covers of a Torus by Tori

For covers of an elliptic curve by elliptic curves, the Riemann-Hurwitz formula (2.1.1) becomes

$$r = w - n\,;$$

but since $n \geq w$, there cannot be any simple branch points and the special point ∞ also has no branching. As a result, the computation for this case reduces to determining the number of degree n unbranched covers of an elliptic curve by elliptic curves. Equivalently, for a given lattice L associated with the target elliptic curve, we need to find the number of inequivalent sublattices $L' \subset L$ of index $[L : L'] = n$.

Lemma 2.1 *Let $L = \langle e_1, e_2\rangle := \mathbb{Z}e_1 + \mathbb{Z}e_2$ be a two-dimensional lattice generated by e_1 and e_2. Then, the number of inequivalent sublattices $L' \subset L$ of index $[L : L'] = n$ is given by $\sigma_1(n) := \sum_{d|n} d$.*

Proof: Let $f_1 = de_1 \in L'$ be the smallest multiple of e_1. Then, there exists $f_2 = ae_1 + be_2 \in L', a < d$, such that L' is generated by f_1 and f_2 over $\mathbb{Z}$. It is clear that the index of this lattice is db. Thus, for each d dividing the index n, we have the following d inequivalent sublattices: $\langle de_1, (n/d)e_2\rangle, \langle de_1, e_1 + (n/d)e_2\rangle, \ldots, \langle de_1, (d-1)e_1 + (n/d)e_2\rangle$. The lemma now follows. ■

[2] We thank R. Vakil for explaining this approach to us.

It hence follows from the above lemma that

$$\tilde{\mu}^{1,n}_{1,n} = \sigma_1(n)\,,$$

where, as usual, $\sigma_k(n) = \sum_{d|n} d^k$. Note that we are doing the actual counting of distinct covers, and our answer $\tilde{\mu}^{1,n}_{1,n}$ is not equal to $\mu^{1,n}_{1,n}$ which is defined by incorporating the automorphism group of the cover differently. This point will become clear in our ensuing discussions.

The generating function for the number of inequivalent simple covers of an elliptic curve by elliptic curves is thus given by

$$\widetilde{\mathcal{H}}^1_1 = \sigma_1(n) q^n = -\left(\frac{d\log\eta(q)}{dt} - \frac{1}{24}\right)\,, \tag{2.2.2}$$

where $q = e^{\hat{t}}$ is the exponential of the Kähler parameter $\hat{t}$ of the target elliptic curve. Up to the constant 1/24, our answer (2.2.2) is a derivative of the genus-1 free energy $\mathscr{F}_1$ of string theory on an elliptic curve target space. The expression (2.2.2) can also be obtained by counting distinct orbits of the action of S_n on a set $\mathscr{T}_{n,1,0}$, which will be discussed subsequently. The string theory computation of $\mathscr{F}_1$, however, counts the number $\mu^{1,n}_{1,n} := |\mathscr{T}_{n,1,0}|/n!$ without taking the fixed points of the S_n action into account, and it is somewhat surprising that our counting is related to the string theory answer by simple multiplication by the degree. It turns out that this phenomenon occurs for $g = 1$ because the function $\sigma_1(n)$ can be expressed as a sum of products of $\pi(k)$, where $\pi(k)$ is the number of distinct partitions of the integer k into positive integers, and because this sum precisely appears in the definition of $T_{n,1,0} = |\mathscr{T}_{n,1,0}|$. We will elaborate upon this point in §2.2.6. In other cases, the two numbers $\mu^{g,n}_{h,n}$ and $\tilde{\mu}^{g,n}_{h,n}$ are related by an additive term which generally depends on g, h, and n.

2.2.2 Application of Mednykh's Master Formula

The most general Hurwitz enumeration problem for an arbitrary branch type has been formally solved by Mednykh in [Med1]. His answers are based on the original idea of Hurwitz of reformulating the ramified covers in terms of the representation theory of S_n [Hur]. Let $f : \Sigma_g \to \Sigma_h$ be a degree-n branched cover of a compact connected Riemann surface of genus-h by a compact connected Riemann surface of genus-g, with r branch points, the orders of whose pre-images being specified by the partitions $\alpha^{(p)} = (1^{t_1^p}, \ldots, n^{t_n^p}) \vdash n$, $p = 1, \ldots, r$. The ramification type of the covering f is then denoted by the matrix $\sigma = (t_s^p)$. Two such branched covers f_1 and f_2 are equivalent if there exists a homeomorphism $\phi : \Sigma_g \to \Sigma_g$ such that $f_2 = f_1 \circ \phi$.

Let $L = \{z_1, \ldots, z_r\} \subset \Sigma_h$ be the *branch locus*, consisting of all branch points of $f : \Sigma_g \to \Sigma_h$. Then, there exists a homomorphism from the fundamental group $\pi_1(\Sigma_h \backslash L)$ to the symmetric group S_n. A presentation of the fundamental group $\pi_1(\Sigma_h \backslash L)$ is given by

$$(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_h, \beta_h, \gamma_1, \ldots, \gamma_r : \prod_{i=1}^{h} [\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j = \mathbb{I}),$$

and we can define the set $\mathscr{B}_{n,h,\sigma}$ of homomorphisms of the above generators as

$$\begin{aligned} \mathscr{B}_{n,h,\sigma} \;=\; & \Big\{ \Big(a_1, b_1, \ldots, a_h, b_h, (1^{t_1^1}, \ldots, n^{t_n^1}), \ldots, (1^{t_1^r}, \ldots, n^{t_n^r}) \Big) \in (S_n)^{2h+r} \;\Big| \\ & \prod_{i=1}^{h} [a_i, b_i] \prod_{p=1}^{r} (1^{t_1^p}, \ldots, n^{t_n^p}) = \mathbb{I} \Big\} . \end{aligned} \tag{2.2.3}$$

Furthermore, we can define $\mathscr{T}_{n,h,\sigma} \subset \mathscr{B}_{n,h,\sigma}$ as the subset whose elements are free sets of generators that generate transitive subgroups of S_n. Then, according to Hurwitz, there is a one-to-one correspondence between irreducible branched covers and elements of $\mathscr{T}_{n,h,\sigma}$. Furthermore, the equivalence relation of branched covers gets translated into conjugation by a permutation in S_n; that is, two elements of $\mathscr{T}_{n,h,\sigma}$ are considered equivalent if and only if they are conjugate to each other. Thus, the Hurwitz enumeration problem reduces to counting the number of orbits in $\mathscr{T}_{n,h,\sigma}$ under the action of S_n by conjugation.

Let us denote the orders of the sets by $B_{n,h,\sigma} = |\mathscr{B}_{n,h,\sigma}|$ and $T_{n,h,\sigma} = |\mathscr{T}_{n,h,\sigma}|$. Then, using the classical Burnside's formula, Mednykh obtains the following theorem for the number $N_{n,h,\sigma}$ of orbits:

Theorem 2.2 (Mednykh) *The number of degree-n inequivalent branched covers of the ramification type $\sigma = (t_s^p)$, for $p = 1, \ldots, r$, and $s = 1, \ldots, n$ is given by*

$$N_{n,h,\sigma} = \frac{1}{n} \sum_{\substack{\ell | v \\ m\ell = n}} \sum_{\frac{1}{(t,\ell)}|d|\ell} \frac{\mathfrak{m}(\frac{\ell}{d})\, d^{(2h-2+r)m+1}}{(m-1)!} \left[\sum_{j^s_{k,p}} T_{n,h,(s_k^p)} \times \right.$$
$$\left. \times \sum_{x=1}^{d} \prod_{s,k,p} \left[\frac{\Psi(x, s/k)}{d} \right]^{j^s_{k,p}} \prod_{k,p} \binom{s_k^p}{j^1_{k,p}, \ldots, j^{md}_{k,p}} \right] \qquad (2.2.4)$$

where $t := GCD\{t_s^p\}$, $v := GCD\{s\, t_s^p\}$, $(t,\ell) = GCD(t,\ell)$, $s_k^p = \sum_{s=1}^{md} j^s_{k,p}$, and the sum over $j^s_{k,p}$ ranges over all collections $\{j^s_{k,p}\}$ satisfying the condition

$$\sum_{\substack{1 \le k \le s t_s^p/\ell \\ (s/(s,d))|k|s}} k\, j^s_{k,p} = \frac{s\, t_s^p}{\ell}$$

where $j^s_{k,p}$ is non-zero only for $1 \le k \le s t_s^p/\ell$ and $(s/(s,d))|k|s$. The functions $\mathfrak{m}(n)$ and $\Psi(x,n)$ are the Möbius and von Sterneck functions defined below.

In the following definitions, let n be a positive integer.

Definition 2.3 (Möbius Function) *The Möbius function $\mathfrak{m}(n)$ is defined to be $(-1)^k$ if n is a product of k distinct primes, and 0 if n is divisible by a square greater than 1.*

Definition 2.4 (Euler's Totient Function) *The Euler's totient function $\varphi(n)$ gives the number of positive integers $m < n$ such that GCD(m,n)=1.*

Definition 2.5 (Von Sterneck Function) *The von Sterneck function $\Psi(x,n)$ is defined in terms of the Möbius function $\mathfrak{m}(n)$ and Euler's totient function $\varphi(n)$ as*

$$\Psi(x,n) = \frac{\varphi(n)}{\varphi(n/(x,n))}\, \mathfrak{m}(n/(x,n)),$$

where $(x,n) = GCD(x,n)$.

As is apparent from its daunting form, the expression in (2.2.4) involves many conditional sums and does not immediately yield the desired numerical answers. Mednykh's works, even though quite remarkable, are thus of dormant nature for obtaining the closed-form numerical answers[3] of the Hurwitz enumeration problem.

Interestingly, the general formula (2.2.4) still has some applicability. For example, in [Med2], Mednykh considers the special case of branch points whose orders are all equal to the degree of the cover and obtains a simplified formula which is suitable for practical applications. In a similar vein, we discover that for simple branched covers, Mednykh's formula simplifies dramatically and that for some low degrees, we are able to obtain closed-form answers for simple Hurwitz numbers of ramified coverings of genus-h Riemann surfaces by genus-g Riemann surfaces.

The Simplifications for Simple Hurwitz Numbers

We consider degree-n simple branch covers of a genus-h Riemann surface by genus-g Riemann surfaces. A simple branch point has order $(1^{n-2}, 2)$, and thus the branch type is characterized by the matrix $\sigma = (t^p_s)$, for $p = 1, \ldots, r$, and $s = 1, \ldots, n$, where

$$t^p_s = (n-2)\delta_{s,1} + \delta_{s,2}.$$

To apply Mednykh's master formula (2.2.4), we need to determine $t = GCD\{t^p_s\}$ and $v = GCD\{s\, t^p_s\}$, which are easily seen to be

$$t = 1 \qquad \text{and} \qquad v = \begin{cases} 2 & \text{for } n \text{ even} \ , \\ 1 & \text{for } n \text{ odd} \ . \end{cases}$$

Because v determines the range of the first sum in the master formula, we need to distinguish when the degree n is odd or even.

[3] Recently, closed-form answers for coverings of a Riemann sphere by genus-0,1,2 Riemann surfaces with one non-simple branching have been obtained in [GouJ, GouJV].

Odd Degree Covers

For degree-n odd, we have $\ell = d = (t,\ell) = 1$ and $m = n$. The constraints $(s/(s,d))|k|s$ and

$$\sum_{1\le k\le s\,t_s^p/\ell} k\, j_{k,p}^s = \frac{s\,t_s^p}{\ell}$$

then determine the collection $\{j_{k,p}^s\}$ to be

$$j_{k,p}^s = t_s^p\,\delta_{k,s}\,.$$

Noting that $\Psi(1,1) = 1$, we see that the master formula now reduces to

$$N_{n,h,\sigma} = \frac{T_{n,h,(s_k^p)}}{n!} \qquad (n \text{ odd})\,, \tag{2.2.5}$$

where

$$s_k^p = \sum_{s=1}^{n} j_{k,p}^s = t_k^p = (n-2)\,\delta_{k,1} + \delta_{k,2}\,. \tag{2.2.6}$$

Even Degree Covers

For degree-n even, $v = 2$ and thus $\ell = 1$ or 2.

$\underline{\ell = 1}$: The variables take the same values as in the case of n odd, and the $\ell = 1$ contribution to $N_{n,h,\sigma}$ is thus precisely given by (2.2.5).

$\underline{\ell = 2}$: In this case, the summed variables are fixed to be

$$m = \frac{n}{2} \qquad \text{and} \qquad d = \ell = 2\,.$$

Then, one determines that

$$j_{k,p}^s = \frac{t_1^p}{2}\delta_{s,1}\delta_{k,1} + t_2^p\delta_{s,2}\delta_{k,1}\ ,$$

from which it follows that

$$\tilde{s}_k^p = \frac{n}{2}\,\delta_{k,1}\ ,$$

where we have put a tilde over s_k^p to distinguish them from (2.2.6). Using the fact that the number r of simple branch points is even, and the values $\Psi(2,1) = \Psi(2,2) = -\Psi(1,2) = 1$, one can now show that the $\ell = 2$ contribution to $N_{n,h,\sigma}$ is

$$\frac{2^{(h-1)n+1}}{(\frac{n}{2}-1)!}\left(\frac{n}{2}\right)^{r-1} T_{\frac{n}{2},h,(\tilde{s}_k^p)} \ .$$

The sum of both contributions is finally given by

$$N_{n,h,\sigma} = \frac{1}{n!} T_{n,h,(s_k^p)} + \frac{2^{(h-1)n+1}}{(\frac{n}{2}-1)!}\left(\frac{n}{2}\right)^{r-1} T_{\frac{n}{2},h,(\tilde{s}_k^p)} \qquad (n \text{ even})\,. \tag{2.2.7}$$

NOTATIONS: For simple branch types, i.e. for $\sigma = (t_k^p)$ where $t_k^p = (n-2)\delta_{k,1} + \delta_{k,2}$, for $p = 1, \ldots, r$ and $k = 1, \ldots, n$, we will use the notation $T_{n,h,\sigma} =: T_{n,h,r}$.

The computations of fixed-degree-n simple Hurwitz numbers are thus reduced to computing the two numbers $T_{n,h,(s_k^p)}$ and $T_{\frac{n}{2},h,(\tilde{s}_k^p)}$, only the former being relevant when n is odd. We now compute these numbers for some low degrees and arbitrary genera h and g. The nature of the computations is such that we only need to know the characters of the identity and the transposition elements in S_n.

The term $T_{\frac{n}{2},h,(\tilde{s}_k^p)}$ can be easily computed:

Lemma 2.6 *Let $\tilde{s}_k^p = n\delta_{k,1}$. Then,*

$$T_{n,h,(\tilde{s}_k^p)} = n! \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{n_1+\cdots+n_k=n} \prod_{i=1}^{k} \left[\sum_{\gamma \in \mathscr{R}_{n_i}} \left(\frac{n_i!}{f^\gamma} \right)^{2h-2} \right] \ .$$

where n_i are positive integers, $\mathscr{R}_{n_i}$ the set of all irreducible representations of S_{n_i}, and f^γ the dimension of the representation γ.

For $h = 0$, we can explicitly evaluate this contribution:

Lemma 2.7 *Let $\tilde{s}_k^p = n\delta_{k,1}$. Then,*

$$T_{n,0,(\tilde{s}_k^p)} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_1+\cdots+n_k=n \\ n_i>0}} \binom{n}{n_1,\ldots,n_k} = \begin{cases} 1, \text{ for } n=1 \\ 0, \text{ for } n>1 \ . \end{cases}$$

PROOF: The first equality follows from the fact that the order of a finite group is equal to the sum of squares of the dimension of its irreducible representations. The second equality follows by noticing that the expression for $T_{n,0,(\tilde{s}_k^p)}/n!$ is the n-th coefficient of the formal q-expansion of $\log(\sum_{n=0}^{\infty} q^n/n!)$, which is a fancy way of writing q. ■

Using (2.2.5) and (2.2.7) we have computed closed-form formulas for the simple Hurwitz numbers for arbitrary source and target Riemann surfaces for degrees less than 8. For explicit computations of $\tilde{\mu}_{h,n}^{g,n} = N_{n,h,r}$, we will need the following relation among the numbers of irreducible and reducible covers [Med1]:

$$T_{n,h,\sigma} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_1+\cdots+n_k=n \\ \sigma_1+\cdots\sigma_k=\sigma}} \binom{n}{n_1,\ldots,n_k} B_{n_1,h,\sigma_1} \cdots B_{n_k,h,\sigma_k} \tag{2.2.8}$$

where

$$B_{n,h,r} = (n!)^{2h-1} \binom{n}{2}^r \left[\sum_{\gamma \in \mathscr{R}_n} \frac{1}{(f^\gamma)^{2h-2}} \left(\frac{\chi_\gamma(2)}{f^\gamma} \right)^r \right].$$

Furthermore, in these computations, we assume that r is positive unless indicated otherwise.

2.2.3 Degree One and Two

It is clear that the degree-one simple Hurwitz numbers are given by

$$\tilde{\mu}_{h,1}^{g,1}(1) = \delta_{g,h} .$$

The number of simple double covers of a genus-h Riemann surface by by genus-g Riemann surfaces can be obtained by using the work of Mednykh on Hurwitz numbers for the case where all branchings have the order equal to the degree of the covering [Med2].

Proposition 2.8 *The simple Hurwitz numbers $\tilde{\mu}_{h,2}^{g,2}(1,1)$ are equal to 2^{2h} for $g > 2(h-1)+1$.*

PROOF: For $g > 2(h-1)+1$, the number r of simple branch points is positive, and we can use the results of Mednykh [Med2]. Let p be a prime number and D_p the set of all irreducible representations of the symmetric group S_p. Then, Mednykh shows that the number $N_{p,h,r}$ of inequivalent degree-p covers of a genus-h Riemann surface by genus-g Riemann with r branch points[4] of order-p is given by

$$N_{p,h,r} = \frac{1}{p!} T_{p,h,r} + p^{2h-2}[(p-1)^r + (p-1)(-1)^r] \,,$$

where

$$T_{p,h,r} = p! \sum_{\gamma \in D_p} \left(\frac{\chi_\gamma(p)}{p} \right)^r \left(\frac{p!}{f^\gamma} \right)^{2h-2+r} \,,$$

where $\chi_\gamma(p)$ is the character of a p-cycle in the irreducible representation γ of S_p and f^γ is the dimension of γ. For $p = 2$, S_2 is isomorphic to $\mathbb{Z}_2$, and the characters of the transposition for two one-dimensional irreducible representations are 1 and -1, respectively. It follows that

$$N_{2,h,r} = T_{2,h,r} = \begin{cases} 2^{2h} & \text{for } r \text{ even} \,, \\ 0 & \text{for } r \text{ odd} \,, \end{cases}$$

and therefore that

$$\tilde{\mu}^{g,2}_{h,2}(1,1) \equiv N_{2,h,r} = 2^{2h} \,,$$

which is the desired result. ■

Remark: The answer for the case $g = 1$ and $h = 1$ is 3, which follows from Lemma 2.1. For $h = 1$ and $g > 1$, we have $\tilde{\mu}^{g,2}_{1,2}(1,1) = 4$.

2.2.4 Degree Three

The following lemma will be useful in the ensuing computations:

Lemma 2.9 *Let* $t^p_k = 2\,\delta_{k,1} \sum_{i=1}^{j} \delta_{p,i} + \delta_{k,2} \sum_{i=j+1}^{r} \delta_{p,i}$. *Then,*

$$B_{2,h,(t^p_k)} = \begin{cases} 2^{2h} & \textit{for } j \textit{ even} \,, \\ 0 & \textit{for } j \textit{ odd} \,. \end{cases}$$

[4] The Riemann-Hurwitz formula determines r to be $r = [2(1-h)p + 2(g-1)]$.

PROOF: The result follows trivially from the general formula for $B_{n,h,\sigma}$ by noting that the character values of the transposition for the two irreducible representations of S_2 are 1 and -1. ■

We now show

Proposition 2.10 *The degree-3 simple Hurwitz numbers are given by*

$$\tilde{\mu}^{g,3}_{h,3} \equiv N_{3,h,r} = 2^{2h-1}(3^{2h-2+r} - 1) = 2^{2h-1}(3^{2g-4h+2} - 1),$$

where $r = 6(1-h) + 2(g-1)$ is the number of simple branch points.

PROOF: $T_{3,h,r}$ receives non-zero contributions from the following partitions of 3: (3) and $(1,2)$. There are three irreducible representations of S_3 of dimensions 1,1, and 2, whose respective values of their characters on a transposition are $1, -1$, and 0. Taking care to account for the correct combinatorial factors easily yields the desired result. ■

2.2.5 Degree Four

The degree-4 answer is slightly more complicated:

Proposition 2.11 *The degree-4 simple Hurwitz numbers are given by*

$$\begin{aligned} N_{4,h,r} &= 2^{2h-1}\left[(3^{2h-2+r}+1)2^{4h-4+r} - 3^{2h-2+r} - 2^{2h-3+r} + 1\right] + 2^{4h-4+r}(2^{2h}-1) \\ &= 2^{2h-1}\left[(3^{2g-6h+4}+1)2^{2g-4h+2} - 3^{2g-6h+4} - 2^{2g-6h+3} + 1\right] + \\ &\quad + 2^{2g-4h+2}(2^{2h}-1). \end{aligned} \tag{2.2.9}$$

PROOF: The last term in (2.2.9) comes from the second term in (2.2.7) by applying Lemma 2.6. To compute $T_{4,h,r}$, we note that the only consistent partitions of 4 and σ are when 4 has the following partitions: $(4), (1,3), (2,2)$, and $(1,1,2)$. The only non-immediate sum involves

$$\sum_{\sigma_1+\sigma_2=\sigma} B_{2,h,\sigma_1}\, B_{2,h,\sigma_2}\,,$$

which, upon applying Lemma 2.9, becomes 2^{4h+r-1}. ■

Higher degree computations are similarly executed. But, one must keep track of some combinatorial factors arising from inequivalent distributions of σ in (2.2.8), and we thus omit their proofs in our presentation and summarize our results in Appendix B.1.

2.2.6 Cautionary Remarks

Hurwitz numbers are sometimes *defined* to be $T_{n,h,\sigma}/n!$, counting orbits as if there were no fixed points of the action S_n on $\mathscr{T}_{n,h,\sigma}$. The master formula obtained by Mednykh uses the Burnside's formula to account for the fixed points. In the case of simple Hurwitz numbers, this will lead to an apparent discrepancy between our results and those obtained by others for even degree covers, the precise reason being that for even degree covers, say of degree-$2n$, the action of $(2^n) \in S_{2n}$ on $\mathscr{T}_{2n,h,\sigma}$ has fixed points which are counted by the second term in (2.2.7). Consequently, to obtain the usual even degree Hurwitz numbers, we just need to consider the contribution of the first term in (2.2.7). For odd degree cases, there is no non-trivial fixed points, and our formula needs no adjustment. The following examples of the discussion would be instructive:

<u>Example One</u>

Let us explicitly count the double covers of an elliptic curve by genus-g Riemann surfaces. The set $\mathscr{T}_{2,1,2g-2}$ is given by

$$\mathscr{T}_{2,1,2g-2} = \left\{(a, b, (2)^{2g-2}) \in S_2^{2g} \,\|\, aba^{-1}b^{-1}(2)^{2g-2} = 1\right\} \, .$$

Since S_2 is commutative and $(2)^2 = 1$, any pair $(a, b) \in S_2 \times S_2$ satisfies the required condition. Hence, the order of $\mathscr{T}_{2,1,2g-2}$ is four. Now, to count non-equivalent coverings, we need to consider the action of S_2 on the set $\mathscr{T}_{2,1,2g-2}$ by conjugation. Again, since S_2 is Abelian, it is clear that it acts trivially on the set and thus that there are 4 inequivalent double covers of an elliptic curve by genus-g Riemann surfaces. The commonly adopted definition of Hurwitz number, however, specifies that we should take the order of the set $\mathscr{T}_{2,1,2g-2}$ and divide it by the dimension of S_2, yielding 2 as its answer. This number 2 is precisely the first contribution in the Burnside's formula:

$$N_{2,1,2g-2} = \frac{1}{|S_2|} \sum_{\sigma \in S_2} |F_\sigma| = \frac{F_{(1^2)}}{2!} + \frac{F_{(2)}}{2!} = 2 + 2 = 4$$

where $|F_\sigma|$ is the order of the fixed-point set under the action of $\sigma \in S_2$. For odd n, S_n acts freely on the set $\mathscr{T}_{n,h,r}$, but for even n, it has fixed points and our formula (2.2.7) accounts for the phenomenon, truly *counting* the number of inequivalent covers.

To avoid possible confusions, we thus use the following notations to distinguish the two numbers:

$$\mu^{g,n}_{h,n} := \frac{T_{n,h,r}}{n!}, \qquad \text{for all } n,$$

and

$$\tilde{\mu}^{g,n}_{h,n} := N_{n,h,r} \, .$$

It turns out that current researchers are mostly interested in $\mu^{g,n}_{h,n}$; for example, it is this definition of simple Hurwitz numbers that appears in the string theory literature and in relation to Gromov-Witten invariants. In this thesis, we will compute the numbers $\tilde{\mu}^{g,n}_{h,n}$ and indicate the $\ell = 2$ contributions which can be subtracted to yield $\mu^{g,n}_{h,n}$. We will however find generating functions only for the case $\mu^{g,n}_{h,n}$.

<u>Example Two</u>

The above discussion shows that the two numbers $\tilde{\mu}^{g,n}_{h,n}$ and $\mu^{g,n}_{h,n}$ differ by the second term in (2.2.7) and thus are not related by simple multiplicative factors. For $h = 1$ and $g = 1$, however, we have previously observed that $\widetilde{\mathcal{H}}^1_1$ given in (2.2.2) is equal to $\partial_t \mathscr{F}_1$, up to an additive constant, implying that

$$\tilde{\mu}^{1,n}_{1,n} = n \, \mu^{1,n}_{1,n} \, . \tag{2.2.10}$$

Since we know that $\tilde{\mu}^{1,n}_{1,n} = \sigma_1(n)$ and since one can show that

$$\mu^{1,n}_{1,n} := \frac{T_{n,1,0}}{n!} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{m_1 + \cdots + m_k = n} \left(\prod_{i=1}^{k} \pi(m_i) \right) ,$$

the following lemma establishes the special equality in (2.2.10):

Lemma 2.12 *Let $\pi(m)$ be the number of distinct ordered partitions of a positive integer m into positive integers. Then, the function $\sigma_1(n)$ has the following expression:*

$$\sigma_1(n) = n \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{m_1+\cdots m_k=n} \left(\prod_{i=1}^{k} \pi(m_i) \right) .$$

PROOF: As is well-known, the functions $\pi(m)$ arise as coefficients of the expansion of $q^{1/24}\,\eta(q)^{-1}$, i.e.

$$\frac{q^{1/24}}{\eta(q)} = 1 + \sum_{m=1}^{\infty} \pi(m)\, q^m \ . \tag{2.2.11}$$

We take log of both sides of (2.2.11) and q-expand the resulting expression on the right hand side. Now, using the fact that

$$\log \left(q^{1/24} \eta(q)^{-1} \right) = \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} q^n \ ,$$

we match the coefficients of q^n to get the desired result. ∎

2.2.7 Recursive Solutions to $T_{n,h,r}$ for an Elliptic Curve ($h = 1$)

Elliptic curve is the simplest Calabi-Yau manifold and is of particular interest to string theorists. The free energies $\mathscr{F}_g$ count the numbers $\mu^{g,n}_{h,n}$, and string theorists have computed $\mathscr{F}_g$ for $g \leq 8$ [Rud]. Using the approach described in the previous subsection, we have obtained the closed-form formulas for $N_{n,h,r}$ for $n < 8$. For $h = 1$, its $\ell = 1$ parts agree with the known free energies $\mathscr{F}_g$. Although our results are rewarding in that they give explicit answers for all g and h, further computations become somewhat cumbersome beyond degree 8. For higher degrees, we therefore adopt a recursive method to solve $T_{n,1,r}$ on a case-by-case basis.

The number of reducible covers $B_{n,h,\sigma}/n!$ and that of irreducible covers $T_{n,h,\sigma}/n!$ are related by exponentiation [Med2]:

$$\sum_{\sigma \geq 0} \frac{B_{n,h,\sigma}}{n!} w^{\sigma} = \exp \left(\sum_{\sigma \geq 0} \frac{T_{n,h,\sigma}}{n!} w^{\sigma} \right) \ , \tag{2.2.12}$$

where w^{σ} denotes the multi-product

$$w^{\sigma} := \prod_{p=1}^{r} \prod_{k=1}^{n} w_{pk}^{t_k^p}$$

in the indeterminates w_{pk} and $\sigma \geq 0$ means $t_k^p \geq 0, \forall p, k$. From (2.2.12), one can derive

$$B_{n,h,\sigma} = \sum_{k=1}^{n} \frac{1}{k!} \binom{n}{n_1, \ldots, n_k} \sum_{\substack{n_1 + \cdots + n_k = n \\ \sigma_1 + \cdots + \sigma_k = \sigma}} T_{n_1,h,\sigma_1} \cdots T_{n_k,h,\sigma_k} .$$

In particular, for simple covers of an elliptic curve, partitioning σ appropriately yields

$$\frac{T_{n,1,r}}{n!} = \frac{B_{n,1,r}}{n!} - \sum_{k=2}^{n} \frac{1}{k!} \sum_{\substack{n_1 + \cdots + n_k = n \\ 2\ell_1 + \cdots + 2\ell_k = r}} \binom{r}{2\ell_1, \ldots, 2\ell_k} \prod_{i=1}^{k} \frac{T_{n_i,1,2\ell_i}}{n_i!} , \tag{2.2.13}$$

where n_i and ℓ_i are positive and non-negative integers, respectively. For fixed degree n, (2.2.13) expresses $T_{n,1,r}$ in terms of lower degree and lower genus Hurwitz numbers, and $B_{n,1,r}$. The number $B_{n,1,r}$ in this case reduces to

$$B_{n,1,r} = \frac{n!}{2^r} \sum_{k=1}^{n} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_1 \geq n_2 \geq \cdots \geq n_k}} \left[\sum_{i \in \mathcal{I}} p_i(p_i - 1) \prod_{j \neq i} \left(\frac{p_i - 2 - p_j}{p_i - p_j} \right) \right]^r ,$$

where $p_i = n_i + k - i$ and $\mathcal{I} = \{i \in \{1, \ldots, k\} \,\|\, (p_i - 2) \geq 0\}$. In Appendix B.2, we provide the explicit values of $B_{n,h,r}$ for $n \leq 10$.

We have implemented the recursion into a *Mathematica* program which, using our results from the previous subsection as inputs, computes $T_{n,1,r}$ for $n \geq 8$. For the sake of demonstration, we present some numerical values of $T_{n,1,r}/n!$ for $n \leq 10$ in Appendix B.3.

2.3 Generating Functions for Simple Hurwitz Numbers

Recently, Göttsche has conjectured an expression for the generating function for the number of nodal curves on a surface S, with a very ample line bundle L, in terms of certain universal power series and basic invariants [Gott]. More precisely, he conjectures that the generating function $T(S, L)$ for the number of nodal curves may have the form

$$T(S,L) = \exp\left(c_2(S)\, A + K_S^2\, B + K_S \cdot L\, C + L^2\, D\right),$$

where A, B, C, D are universal power series in some formal variables and K_S the canonical line bundle of S.

In a kindred spirit, it would be interesting to see whether such universal structures exist for Hurwitz numbers. For a curve, the analogues of K_S and $c_2(S)$ would be the genus of the target and L the degree of the branched cover. It turns out that for simple Hurwitz numbers, we are able to find their generating functions in closed-forms, but the resulting structure is seen to be more complicated than that for the case of surfaces.

2.3.1 Summing up the String Coupling Expansions

The free energies $\mathscr{F}_g$ on an elliptic curve have been computed in [Rud] up to $g = 8$, and their q-expansions[5] agree precisely with our results shown in Appendix B.3.

For a fixed degree $n < 8$, we know $\mathscr{F}_g$ for all g, so we can sum up the expansion

$$\mathscr{F} = \sum_g \lambda^{2g-2}\, \mathscr{F}_g \ , \qquad (2.3.14)$$

up to the given degree n in the world-sheet instanton expansion. That is, we are summing up the string coupling expansions, and this computation is a counterpart of "summing up the world-sheet instantons" which string theorists are accustomed to studying.

Consider the following generating function for simple Hurwitz numbers:

$$\Phi(h) = \sum_{g,n} \frac{T_{n,h,r}}{n!}\, \frac{\lambda^r}{r!}\, q^n = \sum_{g,n} \mu_{h,n}^{g,n}\, \frac{\lambda^r}{r!}\, q^n \ , \qquad (2.3.15)$$

which coincides with the total free energy (2.3.14) for $h = 1$. For low degree simple covers of an elliptic curve, we can use our results (B.3.1) to perform the summation

[5] Here, $q = \exp(\hat{t})$, where $\hat{t}$ is a formal variable dual to the Kähler class of the elliptic curve.

over the number r of simple branch points and get

$$\begin{aligned}
\Phi(1) &= \sum_g \lambda^{2g-2}\,\mathscr{F}_g \\
&= -\log(q^{-1/24}\,\eta(q)) + 2\,[\cosh(\lambda)-1]\,q^2 + 2\,[\cosh(3\lambda)-\cosh(\lambda)]\,q^3 \\
&\quad + 2\left[\cosh(6\lambda) + \frac{1}{2}\cosh(2\lambda) - \cosh(3\lambda) + \cosh(\lambda) - \frac{3}{2}\right] q^4 \\
&\quad + 2\,[1 + \cosh(10\lambda) - \cosh(6\lambda) + \cosh(5\lambda) - \cosh(4\lambda) + \cosh(3\lambda) - 2\,\cosh(\lambda)]\,q^5 \\
&\quad + 2\Big[\cosh(15\lambda) - \cosh(10\lambda) + \cosh(9\lambda) - \cosh(7\lambda) + \frac{1}{2}\cosh(6\lambda) - \cosh(5\lambda) \\
&\quad + 2\cosh(4\lambda) - \frac{2}{3}\cosh(3\lambda) + \frac{1}{2}\cosh(2\lambda) + 2\cosh(\lambda) - \frac{11}{3}\Big]\,q^6 + \mathcal{O}(q^7)\ .
\end{aligned}$$

The partition function $\mathcal{Z} = e^{\Phi(1)}$ is then given by

$$\begin{aligned}
\mathcal{Z} &= 1 + q + 2\,\cosh(\lambda)\,q^2 + [1 + 2\,\cosh(3\lambda)]\;q^3 + [1 + 2\,\cosh(2\lambda) + 2\,\cosh(6\lambda)]\;q^4 \\
&\quad + [1 + 2\,\cosh(2\lambda) + 2\,\cosh(5\lambda) + 2\,\cosh(10\lambda)]\;q^5 \\
&\quad + [1 + 4\,\cosh(3\lambda) + 2\cosh(5\lambda) + 2\,\cosh(9\lambda) + 2\,\cosh(15\lambda)]\,q^6 \\
&\quad + [1 + 2\cosh(\lambda) + 2\cosh(3\lambda) + 2\cosh(6\lambda) + 2\cosh(7\lambda) + 2\cosh(9\lambda) + 2\cosh(14\lambda) \\
&\quad + 2\cosh(21\lambda)]\,q^7 + 2\,[1 + \cosh(2\lambda) + 2\cosh(4\lambda) + \cosh(7\lambda) + \cosh(8\lambda) + \cosh(10\lambda) \\
&\quad + \cosh(12\lambda) + \cosh(14\lambda) + \cosh(20\lambda) + \cosh(28\lambda)]\,q^8 + \mathcal{O}(q^9).
\end{aligned}$$

At this point, we can observe a pattern emerging, and indeed, the partition function can be obtained to all degrees from the following statement which, we subsequently discovered, was also given in [Dijk]:

Proposition 2.13 *The partition function $\mathcal{Z}$, or the exponential of the generating function for simple Hurwitz numbers, for an elliptic curve target is given by*

$$\mathcal{Z} = 1 + q + \sum_{n\geq 2}\left(\sum_{\gamma\in\mathscr{R}_n} \cosh\left[\binom{n}{2}\frac{\chi_\gamma(2)}{f^\gamma}\,\lambda\right]\right) q^n\,. \tag{2.3.16}$$

Proof: From (2.2.13), we see that

$$\frac{B_{n,1,r}}{n!\,r!} = \sum_{k=1}^{n}\frac{1}{k!}\sum_{\substack{n_1+\cdots+n_k=n\\ 2\ell_1+\cdots+2\ell_k=r}}\prod_{i=1}^{k}(\mathscr{F}_{\ell_i+1})_{q^{n_i}}\ , \tag{2.3.17}$$

where, as before, n_i and ℓ_i are positive and non-negative integers, respectively, and $(\mathscr{F}_g)_{q^m}$ is the coefficient of q^m in the genus-g free energy. The numbers $B_{n,1,r}$ are determined to be

$$B_{n,1,r} = \begin{cases} n! \begin{pmatrix} n \\ 2 \end{pmatrix}^r \left[\sum_{\gamma\in\mathscr{R}_n} \left(\frac{\chi_\gamma(2)}{f^\gamma}\right)^r\right] & \text{, for } n \geq 2 \\ \delta_{r,0} & \text{, for } n \leq 1 \ . \end{cases}$$

Now, multiplying both sides of (2.3.17) by $\lambda^r q^n$ and summing over all even $r \geq 0$ and all $n \geq 0$ proves the claim. ■

The argument of hyperbolic-cosine is known as the central character of the irreducible representation γ and can be evaluated as in (A.1.4).

Further Recursions for Closed-Form Answers

The above explicit form of the partition function gives rise to a powerful way of recursively solving for the simple Hurwitz numbers $\mu_{1,n}^{g,n}$ for a given degree n, similar to those given in (B.3.1). Let us consider this more closely. Suppose that, knowing closed-form formulas for $\mu_{1,n_i}^{g_i,n_i}$ for all $n_i < n$ and arbitrary g_i, we are interested in deriving a closed-form formula for $\mu_{1,n}^{g,n}$, where g is again arbitrary. The key idea is to match the coefficient of $\lambda^{2g-2}q^n$ in the expansion of the partition function $\mathcal{Z}$ with the coefficient of the same term in the expansion

$$\exp[\Phi(1)] \;=\; 1 + \Phi(1) + \frac{1}{2}[\Phi(1)]^2 + \cdots + \frac{1}{k!}[\Phi(1)]^k + \cdots .$$

The coefficient of $\lambda^{2g-2}q^n$ in $\Phi(1)$ contains precisely what we are looking for, namely $\mu_{1,n}^{g,n}$. On the other hand, the coefficients of $\lambda^{2g-2}q^n$ in $[\Phi(1)]^k$, for $k > 1$, are given in terms of $\mu_{1,n_i}^{g_i,n_i}$, where $n_i < n$ and $g_i \leq g$. But, by hypothesis, we know $\mu_{1,n_i}^{g_i,n_i}$ for all $n_i < n$, and therefore we can solve for $\mu_{1,n}^{g,n}$ in a closed-form. Using this method, we have obtained the degree-8 Hurwitz numbers, and the answer agrees with the known results as well as the computation done by our earlier recursive method.

This recursive method also works for determining the general simple Hurwitz numbers $\mu_{h,n}^{g,n}$, upon using the general "partition function" (2.3.18) in place of $\mathcal{Z}$.

2.3.2 The Generating Functions for Target Curves of Arbitrary Genus

For arbitrary genus targets, there is a natural generalization of the above discussion on the generating functions. We have previously defined the generating function $\Phi(h)$ to be

$$\Phi(h) = \sum_{r,n\geq 0} \mu^{g,n}_{h,n} \frac{\lambda^r}{r!} q^n \ ,$$

and seen that for $h = 1$, it coincides with the total free energy of string theory on an elliptic curve target, where λ is identified with the string coupling constant. For $h \neq 1$, however, the formal parameter λ should be actually viewed as the parameter[6] $\hat{t}^1_1$ dual to the first gravitational descendant of the Kähler class. We do not need an extra genus-keeping parameter, because for simple covers of a fixed target space with a given number of marked points r, choosing the degree of the map fixes the genus of the source Riemann surface uniquely. For the purpose of finding a nice generating function, it is thus convenient to treat r and n as independent indices, with the requirement that they be both non-negative.

For $r = 0$, our previous computations of the simple Hurwitz numbers need to be modified as

$$\frac{T_{n,h,0}}{n!} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_i > 0}} \prod_{i=1}^{k} (n_i)!^{2h-2} \left[\sum_{\gamma \in D_{n_i}} (f^\gamma)^{2-2h} \right] .$$

Also, note that $N_{1,h,r} = \delta_{r,0}$. Then, we have

Proposition 2.14 *The generalized "partition function" $\mathcal{Z}(h) = \exp(\Phi(h))$ for all h is given by*

$$\mathcal{Z}(h) = 1 + q + \sum_{n\geq 2} \sum_{\gamma \in \mathscr{R}_n} \left(\frac{n!}{f^\gamma}\right)^{2h-2} \cosh\left[\binom{n}{2} \frac{\chi_\gamma(2)}{f^\gamma} \lambda \right] q^n. \tag{2.3.18}$$

[6]Unfortunately, we have previously used the notation t^p_k to denote the branching matrix. Here, to avoid confusions, we use $\hat{t}$ for the coordinates that appear in the Gromov-Witten theory.

PROOF: The proof is exactly the same as that of Proposition 2.13. One just needs to keep track of extra factors in the general form of $B_{n,h,r}$. For genus $h = 0$, when applying the Riemann-Hurwitz formula, we must remember to use the correctly defined arithmetic genus of reducible curves and, as a result, sum over all even $r \geq 0$ in $B_{n,0,r}$; doing so takes into account the degree-1 covers in the exponential. ■

2.4 Hodge Integrals on $\overline{M}_{g,2}$ and Hurwitz Numbers

In the modern language of Gromov-Witten theory, the simple Hurwitz numbers are equal to

$$\mu_{h,n}^{g,n} := \frac{T_{n,h,r}}{n!} = \langle \tau_{1,1}^r \rangle_{g,n} \ ,$$

where $r = 2(1-h)n + 2(g-1)$ and $\tau_{k,1}$ is the k-th gravitational descendant of the Kähler class of the target genus-h Riemann surface. We can organize these numbers into a generating function as follows:

$$\mathcal{H}_h^g := \sum_n \frac{1}{r!} \langle \tau_{1,1}^r \rangle_{g,n} \, (\hat{t}_1^1)^r \, e^{n\hat{t}} = \sum_n \frac{1}{r!} \frac{T_{n,h,r}}{n!} \, (\hat{t}_1^1)^r \, e^{n\hat{t}} \ , \tag{2.4.19}$$

where $\hat{t}_1^1$ and $\hat{t}$ are coordinates dual to $\tau_{1,1}$ and $\tau_{0,1}$, respectively. In this thesis, we have determined (2.4.19) for all g and h up to degree $n = 7$.

For $h = 0$ and $h = 1$, these generating functions arise as genus-g free energies of string theory on $\mathbf{P}^1$ and an elliptic curve as target spaces, respectively, evaluated by setting all coordinates to zero except for $\hat{t}_1^1$ and $\hat{t}$. For definitions of Hodge integrals, see [FabP1, FabP2].

2.4.1 Generating Functions for Hodge Integrals

The Hurwitz enumeration problem has been so far investigated intensely mainly for branched covers of the Riemann sphere. In this case, the almost simple Hurwitz numbers for covers with one general branch point can be expressed explicitly in terms of certain Hodge integrals. An interesting application of this development is to use the generating function for Hurwitz numbers $\mu^{g;d}_{0,1}(d)$ to derive a generating function for Hodge integrals over the moduli space $\overline{M}_{g,1}$. More precisely, consider the formula

$$F(t,k) := 1 + \sum_{g\geq 1} t^{2g} \sum_{i=0}^{g} k^i \int_{\overline{M}_{g,1}} \psi_1^{2g-2+i}\, \lambda_{g-i} = \left(\frac{t/2}{\sin(t/2)}\right)^{k+1}, \qquad (2.4.20)$$

where the ψ and λ classes are defined as in §1.2. That is, $\psi_1 = c_1(\mathscr{L}_1) \in H^2(\overline{M}_{g,1}, \mathbb{Q})$, where $\mathscr{L}_1$ is the cotangent line bundle $\mathscr{L}_1 \to \overline{M}_{g,1}$ associated to the single marked point, and $\lambda_j = c_j(\mathbb{E})$, where $\mathbb{E} \to \overline{M}_{g,1}$ is the Hodge bundle. This formula was first obtained by Faber and Pandharipande in [FabP1] by using virtual localization techniques and has been re-derived by Ekedahl *et al.* in [ELSV] by using the generating function for Hurwitz numbers for branched covers whose only non-simple branch point has order equal to the degree of the cover.

In this thesis, we speculate a possible connection between the Hurwitz numbers for $\mathbf{P}^1$ and generating functions for Hodge integrals on $\overline{M}_{g,n}, n \geq 1$. For this purpose, let us rewrite $F(t,k)$ as

$$F(t,k) = 1 + \sum_{g\geq 1} t^{2g} \sum_{i=0}^{g} k^i \int_{\overline{M}_{g,1}} \frac{\lambda_{g-i}}{1-\psi_1}\,. \qquad (2.4.21)$$

Now, recall that the simple Hurwitz numbers $\mu^{g,n}_{0,n}(1^n)$, henceforth abbreviated $H_{g,n}$, have the following Hodge integral expression [FanP]:

$$H_{g,n} := \mu^{g,n}_{0,n} = \frac{(2g-2+2n)!}{n!} \int_{\overline{M}_{g,n}} \frac{1-\lambda_1+\cdots+(-1)^g\lambda_g}{\prod_{i=1}^{n}(1-\psi_i)} \qquad (2.4.22)$$

for $(g,n) \neq (0,1),(0,2)$. The degree-1 simple Hurwitz numbers are $H_{g,1} = \delta_{g,0}$, thus (2.4.22) yields the relation

$$\int_{\overline{M}_{g,1}} \frac{1-\lambda_1+\cdots+(-1)^g\lambda_g}{1-\psi_1} = 0, \quad \text{for } g \geq 1.$$

which implies from (2.4.21) that $F(t,-1) = 1$, in accord with the known answer (2.4.20). Naively, we thus see that the simple Hurwitz numbers are coefficients of $F(t,k)$ evaluated at special k.

In a similar spirit, we can speculate a crude generating function for Hodge integrals with two marked points:

$$G(t,k) = \frac{1}{2} + \sum_{g\geq 1} t^{2g} \sum_{i=0}^{g} k^i \int_{\overline{M}_{g,2}} \frac{\lambda_{g-i}}{(1-\psi_1)(1-\psi_2)} \ .$$

Our goal is to find a closed-form expression for this generating function $G(t,k)$. Without much work, we can immediately evaluate $G(t,k)$ at certain special values of k:

Proposition 2.15 *The generating function $G(t,k)$ can be evaluated at $k=-1$ to be*

$$G(t,-1) = \frac{1}{2} - \frac{1}{t^2}\left(\cos t + \frac{t^2}{2} - 1\right) = \frac{1}{2}\left(\frac{\sin(t/2)}{t/2}\right)^2 \ ,$$

and similarly at $k=0$ to be

$$G(t,0) = \frac{1}{2}\left(\frac{t}{\sin t}\right) = \frac{1}{2}\,\frac{t/2}{\sin(t/2)}\,\frac{1}{\cos(t/2)} \ .$$

PROOF: At $k=-1$, we can use (2.4.22) to get

$$G(t,-1) = \sum_{g\geq 0} (-1)^g \frac{2\,t^{2g}}{(2g+2)!} H_{g,2} \ . \tag{2.4.23}$$

We have previously computed $H_{g,2} = N_{2,0,2g+2}/2 = 1/2$, and we can then perform the summation in (2.4.23) and get the desired result. To evaluate $G(t,0)$, we use the following λ_g-conjecture, which has been recently proven by Faber and Pandharipande [FabP2]:

$$\int_{\overline{M}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}\,\lambda_g = \binom{2g+n-3}{\alpha_1,\ldots,\alpha_n} \int_{\overline{M}_{g,1}} \psi_1^{2g-2}\,\lambda_g \,.$$

One can now compute

$$\int_{\overline{M}_{g,2}} \frac{\lambda_g}{(1-\psi_1)(1-\psi_2)} \quad = \quad \frac{(2^{2g-1}-1)}{(2g)!}\,|B_{2g}|$$

and obtain the result. ∎

To extract the terms without λ_k insertions, consider the scaling limit

$$\begin{aligned} G(t\,k^{\frac{1}{2}}, k^{-1}) \quad &= \quad \frac{1}{2} + \sum_{g\geq 1} t^{2g} \sum_{i=0}^{g} k^{g-i} \int_{\overline{M}_{g,2}} \frac{\lambda_{g-i}}{(1-\psi_1)(1-\psi_2)} \\ &\xrightarrow{k\to 0} \quad \frac{1}{2} + \sum_{g\geq 1} t^{2g} \sum_{i=0}^{g} \int_{\overline{M}_{g,2}} \frac{1}{(1-\psi_1)(1-\psi_2)}. \end{aligned} \tag{2.4.24}$$

The asymptotic behavior (2.4.24) can be explicitly evaluated as follows:

Proposition 2.16 *The asymptotic limit of* $G(t,k)$ *is*

$$G(t\,k^{\frac{1}{2}},k^{-1}) \overset{k\to 0}{\longrightarrow} \frac{\exp{(t^2/3)}}{2\,t}\sqrt{\pi}\,\mathrm{Erf}\left[\frac{t}{2}\right]\ ,$$

and thus, the integrals can be evaluated to be

$$\int_{\overline{M}_{g,2}} \frac{1}{(1-\psi_1)(1-\psi_2)} \ = \ \frac{1}{2}\sum_{m=0}^{g}\frac{1}{m!\,12^m}\,\frac{(g-m)!}{(2g-2m+1)!}\ .$$

Proof: This is an easy consequence of the following Dijkgraaf's formula which appeared in the work of Faber [Fab1]:

$$\langle\tau_0\tau(w)\tau(z)\rangle = \exp\left(\frac{(w^3+z^3)\hbar}{24}\right)\sum_{n\geq 0}\frac{n!}{(2n+1)!}\left[\frac{1}{2}wz(w+z)\hbar\right]^n . \qquad (2.4.25)$$

Here, $\tau(w) := \sum_{n\geq 0}\tau_n w^n$, where τ_n are defined as in §1.2, and $\hbar$ is a formal genus-expansion parameter defined by

$$\langle\quad\rangle = \sum_{g\geq 0}\langle\quad\rangle_g\,\hbar^g\ .$$

Setting $w=z=\hbar^{-1}=t$ in (2.4.25) and noting that

$$\sum_{n\geq 0}\frac{1}{(2n+1)!!}\,t^{2n+1} = e^{t^2/2}\sqrt{\frac{\pi}{2}}\,\mathrm{Erf}\left[\frac{t}{\sqrt{2}}\right]$$

gives the result, upon using the string equation on the left-hand side. ■

For future reference, it would be desirable to find an explicit series expansion of $G(t,k)$. Using Faber's Maple program for computing the intersection numbers on $\overline{M}_{g,n}$ [Fab2], the generating function can be seen to have an expansion of the form

$$\begin{aligned} G(t,k) \ =\ & \frac{1}{2}+\left(\frac{1}{12}+\frac{1}{8}k\right)t^2+\left(\frac{7}{720}+\frac{73}{2880}k+\frac{49}{2880}k^2\right)t^4+\\ & +\left(\frac{31}{30240}+\frac{253}{72576}k+\frac{983}{241920}k^2+\frac{1181}{725760}k^3\right)t^6+\\ & +\left(\frac{127}{1209600}+\frac{36413}{87091200}k+\frac{37103}{58060800}k^2+\frac{38869}{87091200}k^3+\frac{467}{3870720}k^4\right)t^8+\\ & +\left(\frac{73}{6842880}+\frac{38809}{821145600}k+\frac{122461}{1437004800}k^2+\frac{86069}{1094860800}k^3+\right.\\ & \left.\quad+\frac{53597}{1437004800}k^4+\frac{33631}{4598415360}k^5\right)t^{10}+\mathcal{O}(t^{12})\,. \end{aligned} \qquad (2.4.26)$$

2.4.2 Relation to the Hurwitz Numbers $\mu_{h,2}^{g,2k}(k,k)$

We now relate the generating function $G(t,k)$ to the Hurwitz numbers $\mu_{h,2}^{g,2k}(k,k)$, which we are able to compute explicitly. This connection allows us to evaluate $G(t,k)$ for all $k \in \mathbb{Z}_{<0}$. In [ELSV], T.Ekedahl, S.Lando, M.Shapiro and A.Vainshtein have proved the following useful relation between Hurwitz numbers and Hodge integrals:

$$\mu_{0,w}^{g,n}(\alpha) = \frac{(2g-2+n+w)!}{|\mathrm{Aut}(\alpha)|}\left(\prod_{i=1}^{w}\frac{\alpha_i^{\alpha_i}}{\alpha_i!}\right)\int_{\overline{M}_{g,w}}\frac{\sum_{j=0}^{g}(-1)^j\lambda_j}{\prod_{i=1}^{w}(1-\alpha_i\psi_i)}\,.$$

For $\alpha=(k,k)$, which means $n=2k$ and $w:=\ell(\alpha)=2$, this relation reduces to

$$\mu_{0,2}^{g,2k}(k,k) = \frac{(2k+2g)!}{2}\frac{k^{2k}}{(k!)^2}\int_{\overline{M}_{g,2}}\frac{\sum_{j=0}^{g}(-1)^j\lambda_j}{(1-k\psi_1)(1-k\psi_2)},$$

which we can rewrite as

$$\mu_{0,2}^{g,2k}(k,k) = \frac{(2k+2g)!}{2}\frac{k^{2k+2g-1}}{(k!)^2}\sum_{i=0}^{g}k^i\int_{\overline{M}_{g,2}}\frac{(-1)^{g-i}\lambda_{g-i}}{(1-\psi_1)(1-\psi_2)}\,.$$

This implies that for integers $k>0$,

$$G(it,-k) = \frac{1}{2}+\sum_{g\geq 1}\frac{2\,t^{2g}}{(2k+2g)!}\frac{(k!)^2}{k^{2k+2g-1}}\,\mu_{0,2}^{g,2k}(k,k)\,. \tag{2.4.27}$$

By using the expansion (2.4.26) and matching coefficients with (2.4.27), one can thus obtain the Hurwitz numbers $\mu_{0,2}^{g,2k}(k,k)$. We have listed the numbers for $g\leq 6$ in Appendix B.4.

It is in fact possible to determine the Hurwitz numbers $\mu_{0,2}^{g,2k}(k,k)$ from the work of Shapiro *et al.* on enumeration of edge-ordered graphs [ShShV]. According to theorem 9 of their paper[7], the Hurwitz numbers $\mu_{0,2}^{g,2k}(k,k)$ are given by

$$\begin{aligned}\mu_{0,2}^{g,2k}(k,k) &= N(2k,2k+2g,(k,k)) - \binom{2k}{k}\frac{(2k+2g)!}{(2k)!}k^{2k-2+2g}\times\\ &\times\frac{1}{2}\left[\sum_{s=0}^{g+1}\delta_{2s}^{k}\,\delta_{2g+2-2s}^{k}\right],\end{aligned} \tag{2.4.28}$$

[7] Actually, their formula has a minor mistake for the case when $n=2k$ is partitioned into (k,k) for odd genus. More precisely, when the summation variable s in their formula equals $(g+1)/2$, for an odd genus g, there is a symmetry factor of $1/2$ in labeling the edges because the two disconnected graphs are identical except for the labels.

where the numbers δ^k_{2g} are defined by

$$\sum_{g=0}^{\infty} \delta^k_{2g}\, t^{2g} = \left(\frac{\sinh(t/2)}{t/2}\right)^{k-1}$$

and can be written explicitly as

$$\delta^k_{2g} = \frac{1}{(k+2g-1)!} \sum_{m=0}^{k-1} \binom{k-1}{m} (-1)^m \left(\frac{k-1}{2} - m\right)^{k+2g-1} .$$

The number $N(2k, 2k+2g, (k,k))$, which counts the number of certain edge-ordered graphs, is given by

$$N(2k, 2k+2g, (k,k)) \;=\; \frac{|C(k,k)|}{[(2k)!]^2} \sum_{\rho \vdash 2k} f^\rho (h(\rho') - h(\rho))^{2k+2g} \chi_\rho(k,k), \qquad (2.4.29)$$

where $|C(k,k)|$ is the order of the conjugacy class $C(k,k)$, ρ' is the partition conjugate to ρ, and $h(\rho) = \sum_i^m (i-1)\rho_i$ for $\rho = (\rho_1, \ldots, \rho_m) \vdash 2k$. Hence, the problem of finding $\mu^{g,2k}_{0,2}(k,k)$ reduces down to evaluating (2.4.29).

Proposition 2.17 *For $k \geq 2$,*

$$\begin{aligned} N(2k, 2k+2g, (k,k)) \;=\; & \frac{(k-1)!}{2\,k \cdot k!\,(2\,k)!} \left\{ \frac{2\,[k\,(k-2)]^{2\,g+2\,k}\,(2\,k)!}{k!\,(1+k)!} + \right. \\ & + \sum_{m=0}^{k-1} \binom{2\,k-1}{m} (-1)^m \, [k\,(2k-2m-1)]^{2\,k+2\,g} + \\ & + \sum_{m=k}^{2k-1} \binom{2\,k-1}{m} (-1)^{m-1} \, [k\,(2k-2m-1)]^{2\,k+2\,g} + \\ & + 2 \sum_{m=0}^{k-3} \sum_{p=1}^{k-m-1} \binom{k-1}{m} \binom{k-1}{m+p} \times \\ & \left. \times \frac{p^2}{k^2-p^2} \frac{(2k)!}{(k!)^2} (-1)^{p+1} [k\;(k-2\,m-p-1)]^{2\,k+2\,g} \right\} . \end{aligned}$$

Proof: To each irreducible representation labeled by $\rho = (\rho_1, \ldots, \rho_j) \vdash 2k$, we can associate a Young diagram with j rows, the i^{th} row having length ρ_i. According to the *Murnaghan-Nakayama* rule, which we review in Appendix A.2, the diagram corresponding to an irreducible representation ρ for which $\chi_\rho(k,k) \neq 0$, must be either (a) a hook or (b) a union of two hooks. After long and tedious computations, we arrive at the following results:

(a) There are $2k$ "one-hook" diagrams.

(i)

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
m{+}1 & \cdots & k{-}1 & k & 1 & 2 & \cdots & k{-}1 & k \\
\hline
m \\
\cline{1-1}
\vdots \\
\cline{1-1}
2 \\
\cline{1-1}
1 \\
\cline{1-1}
\end{array}$$

A diagram of this kind with leg-length m for $0 \le m \le k-1$ gives

$$f^\rho = \binom{2k-1}{m}, \ \chi_\rho(k,k) = (-1)^m ,$$

$$h(\rho') - h(\rho) = k(2k-2m-1) .$$

(ii)

$$\begin{array}{|c|c|c|c|}
\hline
m'{+}1 & \cdots & k{-}1 & k \\
\hline
m' \\
\cline{1-1}
\vdots \\
\cline{1-1}
2 \\
\cline{1-1}
1 \\
\cline{1-1}
k \\
\cline{1-1}
k{-}1 \\
\cline{1-1}
\vdots \\
\cline{1-1}
2 \\
\cline{1-1}
1 \\
\cline{1-1}
\end{array}$$

A diagram of this kind with leg-length $m = m' + k$ for $0 \le m' \le k-1$ gives

$$f^\rho = \binom{2k-1}{m}, \ \chi_\rho(k,k) = (-1)^{m-1} ,$$

$$h(\rho') - h(\rho) = k(2k-2m-1) .$$

(b) There are $k(k-1)/2$ "two-hook" diagrams.

(i)

k	k
$k-1$	$k-1$
$\vdots$	$\vdots$
2	2
1	1

One diagram has 2 columns and k rows. It corresponds to the irreducible representation with

$$f^\rho = \frac{(2k)!}{k!(k+1)!}\ ,\ \chi_\rho(k,k) = 2\ ,$$

$$h(\rho') - h(\rho) = k(k-2)\,.$$

(ii)

$m{+}1$	$m{+}2$	$m{+}3$	$\cdots$	$k{-}p{+}1$	$\cdots$	$k{-}1$	k
m	$p{+}m{+}1$	$p{+}m{+}2$	$\cdots$	k			
$\vdots$	$\vdots$						
2	$p{+}3$						
1	$p{+}2$						
p	$p{+}1$						
$p{-}1$							
$\vdots$							
2							
1							

For each value of m and p satisfying $0 \le m \le k-3$ and $1 \le p \le k-m-1$, respectively, there is a diagram with $k-m$ columns and $p+m+1$ rows. Such diagram has

$$f^\rho = \binom{k-1}{m}\binom{k-1}{m+p}\frac{p^2}{k^2-p^2}\frac{(2k)!}{(k!)^2}\,,$$

$$\chi_\rho(k,k) = 2(-1)^{p+1}\ ,\ h(\rho') - h(\rho) = k(k-2m-p-1)\,.$$

Furthermore, after some simple combinatorial consideration, we find that $|C(k,k)| = (2k)!(k-1)!/(2k\cdot k!)$. Finally, substituting in (2.4.29) the values of $f^\rho, \chi_\rho(k,k)$ and

$h(\rho') - h(\rho)$ for the above $k(k+3)/2$ irreducible representations gives the desired result. ■

By using (2.4.27) and (2.4.28), we can now rewrite $G(it, -k)$ as

Proposition 2.18 *For integral $k \geq 2$,*

$$
\begin{aligned}
G(it,-k) &= \frac{2\,(k-1)!}{(k+1)!\,t^{2k}} \cosh[(k-2)t] \\
&+ \frac{2\,(k!)^2}{k\,(2k)!\,t^{2k}} \sum_{m=0}^{k-1} \binom{2k-1}{m} (-1)^m \cosh[(2k-2m-1)t] \\
&+ \frac{2}{k\,t^{2k}} \sum_{m=0}^{k-3} \sum_{p=1}^{k-m-1} \binom{k-1}{m} \binom{k-1}{m+p} \frac{p^2}{k^2-p^2} (-1)^{p+1} \cosh[(k-2m-p-1)t] \\
&- \frac{1}{k\,t^2} \left(\frac{\sinh[t/2]}{(t/2)} \right)^{2k-2} .
\end{aligned}
$$

Proof: By substituting the expression (2.4.28) into (2.4.27) and summing over the δ terms, we get

$$
\begin{aligned}
G(it,-k) &= \frac{1}{2} + \sum_{g \geq 1} \frac{2\,(k!)^2\, t^{2g}}{(2k+2g)!\, k^{2k+2g-1}} N(2k, 2k+2g, (k,k)) \\
&\quad - \frac{1}{kt^2} \left(\frac{\sinh(t/2)}{t/2} \right)^{2k-2} + \frac{\delta_0^k \delta_0^k}{kt^2} + \frac{2}{k} \delta_0^k \delta_2^k \\
&= \frac{1}{2} + \sum_{\tilde{\ell} \geq 0} \frac{2\,(k!)^2\, t^{2\tilde{\ell}-2k}}{(2\tilde{\ell})!\, k^{2\tilde{\ell}-1}} N(2k, 2\tilde{\ell}, (k,k)) - \sum_{\ell \geq 0}^{k} \frac{2\,(k!)^2\, t^{2\ell-2k}}{(2\ell)!\, k^{2\ell-1}} N(2k, 2\ell, (k,k)) \\
&\quad - \frac{1}{kt^2} \left(\frac{\sinh(t/2)}{t/2} \right)^{2k-2} + \frac{\delta_0^k \delta_0^k}{kt^2} + \frac{2}{k} \delta_0^k \delta_2^k .
\end{aligned}
\tag{2.4.30}
$$

Before we proceed with our proof, we need to establish two minor lemmas. As in [ShShV], let $N(n, m, \nu)$ be the number of edge-ordered graphs with n vertices, m edges, and ν cycle partition, and $N_c(n, m, \nu)$ the number of connected such graphs. Then,

Lemma 2.19 $N(2k, 2\ell, (k,k)) = 0$ *for $\ell \leq k-2$.*

PROOF: These constraints follow from Theorem 4 of [ShShV] which states that the length l of the cycle partition must satisfy the conditions $c \le l \le \min(n, m-n+2c)$ and $l = m-n (\mathrm{mod}\ 2)$, where c is the number of connected components. In our case, $l = 2$ and the second condition is always satisfied. The first condition, however, is violated for all $\ell \le k-2$ because $c \le 2$ and thus $\min(2\ell - 2k + 2c) \le 0$. ■

Similarly, one has

Lemma 2.20 $N_c(2k, 2k-2, (k,k)) = 0.$

PROOF: This fact again follows from Theorem 4 of [ShShV]. Here, $c = 1$ and $\min(n, m-n+2c) = 0$, whereas $\ell = 2$, thus violating the first condition of the theorem. ■

By Lemma 2.19, the third term in (2.4.30) is non-vanishing only for $\ell = k-1$ and $\ell = k$. But the $\ell = k-1$ piece and the fifth term in (2.4.30) combine to give

$$-\frac{2(k!)^2}{(2k-2)!k^{2k-3}t^2} N(2k, 2k-2, (k,k)) + \frac{\delta_0^k \delta_0^k}{kt^2} \propto N_c(2k, 2k-2, (k,k)) = 0,$$

which follows from Lemma 2.20. Furthermore, the $\ell = k$ piece and the last term in (2.4.30) give

$$-\frac{2(k!)^2}{(2k)!\,k^{2k-1}} N(2k, 2k, (k,k)) + \frac{2}{k}\,\delta_0^k \delta_2^k = -\frac{2(k!)^2}{(2k)!\,k^{2k-1}} \mu_{0,2}^{0,2k}(k,k) = -\frac{1}{2}\,,$$

where we have used the known fact [ShShV] that

$$\mu_{0,2}^{0,2k}(k,k) = \binom{2k}{k} \frac{k^{2k-1}}{4}\ .$$

Thus, we have

$$G(it, -k) = \sum_{\tilde{\ell} \ge 0} \frac{2\,(k!)^2\, t^{2\tilde{\ell}-2k}}{(2\tilde{\ell})!\,k^{2\tilde{\ell}-1}}\, N(2k, 2\tilde{\ell}, (k,k)) - \frac{1}{kt^2}\left(\frac{\sinh(t/2)}{t/2}\right)^{2k-2},$$

and the first term can now be easily summed to yield our claim. ■

It turns out that there are some magical simplifications, and we find for a few low values of k that

$$
\begin{aligned}
G(t,-1) &= \frac{1}{2}\left(\frac{\sin(t/2)}{t/2}\right)^2,\\
G(t,-2) &= \frac{1}{6}\,[2+\cos(t)]\left(\frac{\sin(t/2)}{t/2}\right)^4,\\
G(t,-3) &= \frac{1}{30}\,[8+6\,\cos(t)+\cos(2t)]\left(\frac{\sin(t/2)}{t/2}\right)^6,\\
G(t,-4) &= \frac{1}{140}\,[32+29\,\cos(t)+8\,\cos(2t)+\cos(3t)]\left(\frac{\sin(t/2)}{t/2}\right)^8,\\
G(t,-5) &= \frac{1}{630}\,[128+130\,\cos(t)+46\,\cos(2t)+10\,\cos(3t)+\cos(4t)]\left(\frac{\sin(t/2)}{t/2}\right)^{10},\\
G(t,-6) &= \frac{1}{2772}\,[512+562\,\cos(t)+232\,\cos(2t)+67\,\cos(3t)+12\,\cos(4t)\\
&\qquad +\cos(5t)]\left(\frac{\sin(t/2)}{t/2}\right)^{12},\\
G(t,-7) &= \frac{1}{4(3003)}\,[2048+2380\,\cos(t)+1093\,\cos(2t)+378\,\cos(3t)+92\,\cos(4t)\\
&\qquad +14\,\cos(5t)+\cos(6t)]\left(\frac{\sin(t/2)}{t/2}\right)^{14},
\end{aligned}
$$

and so forth. We have explicitly computed $G(t,-k)$ for $k \leq 60$, and based on these computations, we conjecture the following general form:

Conjecture 2.21 *For integers $k \geq 1$, the generating function is given by*

$$
G(t,-k) = \frac{2(k-1)!\,k!}{(2k)!}\left(\frac{\sin(t/2)}{t/2}\right)^{2k}\left[2^{2(k-2)+1}+\sum_{n=1}^{k-1}\left[\sum_{i=0}^{k-n-1}\binom{2k-1}{i}\right]\cos(nt)\right].
$$

Let us rewrite the summation as follows:

$$
\begin{aligned}
\sum_{n=1}^{k-1}\left[\sum_{i=0}^{k-n-1}\binom{2k-1}{i}\right]\cos(nt) &= \sum_{\ell=0}^{k-2}\binom{2k-1}{\ell}\left(\sum_{n=1}^{k-1-\ell}\cos(nt)\right)\\
&= \frac{1}{2}\sum_{\ell=0}^{k-2}\binom{2k-1}{\ell}\left[\frac{\sin\left[(2k-1-2\ell)t/2\right]}{\sin(t/2)}-1\right].
\end{aligned}
\tag{2.4.31}
$$

The last expression in (2.4.31) can now be explicitly summed, leading to an expression which can be analytically continued to all values of k. After some algebraic manipulations, we obtain the following corollary to Conjecture 2.21:

Conjecture 2.22 *For all[8] k, the generating function as a formal power series in $\mathbb{Q}[k][[t]]$ is given by*

$$\begin{aligned} G(t,-k) \;&=\; \frac{2^{2k-1}}{\sqrt{\pi}}\,\frac{\Gamma(k)\,\Gamma(\frac{1}{2}+k)}{\Gamma(2k+1)}\left(\frac{\sin(t/2)}{t/2}\right)^{2k}\frac{1}{\sin(t/2)} \\ &\quad\times\left[\sin(t/2)+\Re\left(i\,e^{i\,t/2}\,{}_2F_1(1,-k;k;-e^{-i\,t})\right)\right], \qquad (2.4.32)\end{aligned}$$

where $\Re$ denotes the real part and ${}_2F_1(a,b;c;z)$ is the generalized hypergeometric function defined as

$${}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},$$

where $(a)_k := \Gamma(a+k)/\Gamma(a)$.

We have checked that our conjectural formula (2.4.32) indeed reproduces all the terms in (2.4.26).

2.4.3 Possible Extensions

Motivated by our results, let us consider a similar generating function for the case of more marked points:

$$G_n(t,k) := \frac{n!}{(2n-2)!}H_{0,n} + \sum_{g\geq 1} t^{2g}\sum_{i=0}^{g} k^i\int_{\overline{M}_{g,n}}\frac{\lambda_{g-i}}{(1-\psi_1)\cdots(1-\psi_n)}.$$

At $k=-1$, it can be evaluated in terms of simple Hurwitz numbers as

$$G_n(t,-1) = n!\sum_{g=0}^{\infty}\frac{(-1)^g\,H_{g,n}}{(2g+2n-2)!}\,t^{2g}.$$

[8] For k non-positive integers and half-integers, the below expression of $G(t,-k)$ appears to be divergent. For these cases, one might try first expanding $G(t,-k)$ in t and setting k equal to the desired values.

Interestingly, our previous generating function for simple Hurwitz numbers (2.3.15), with $\lambda = it$, is related to $G_n(t,-1)$:

$$\Phi(0)\Big|_{\lambda=it} = \log \mathcal{Z}(0) = \sum_{n\geq 1} \frac{(it)^{2n-2}}{n!}\, G_n(t,-1)\, q^n\,.$$

Hence, we have

$$G_n(t,-1) = \frac{n!}{t^{2n-2}} \sum_{k=1}^{n} \frac{(-1)^{k-n}}{n} \sum_{\substack{m_1+\cdots+m_k=n \\ m_i>0}} W_{m_i}\cdots W_{m_k}$$

where $W_1 = 1$ and

$$W_{m_i} = \sum_{\gamma\in\mathscr{R}_{m_i}} \left(\frac{f^\gamma}{m_i!}\right)^2 \cos\left[\binom{m_i}{2} \frac{\chi_\gamma(2)}{f^\gamma}\, t\right]\,.$$

This relation might suggest a possible connection between the symmetric group S_n and the geometry of the moduli space of marked Riemann surfaces.

Of course, $G_n(t,-1)$ can be also explicitly computed from our previous computations of the simple Hurwitz numbers $H_{g,n}$. For example, we find that

$$\begin{aligned}
G_3(t,-1) &= \frac{(2+\cos(t))}{3}\left(\frac{\sin(t/2)}{t/2}\right)^4,\\
G_4(t,-1) &= \frac{(20+21\,\cos(t)+6\,\cos(2\,t)+\cos(3\,t))}{12}\left(\frac{\sin(t/2)}{t/2}\right)^6,\\
G_5(t,-1) &= \left(\frac{\sin(t/2)}{t/2}\right)^8 \frac{1}{60}\,[422+608\,\cos(t)+305\,\cos(2\,t)+\\
&\qquad +120\,\cos(3\,t)+36\,\cos(4\,t)+8\,\cos(5\,t)+\cos(6\,t)]\,,\\
G_6(t,-1) &= \left(\frac{\sin(t/2)}{t/2}\right)^{10} \frac{1}{360}\times[16043+26830\,\cos(t)+17540\,\cos(2\,t)+\\
&\qquad +9710\,\cos(3\,t)+4670\,\cos(4\,t)+1966\,\cos(5\,t)+715\,\cos(6\,t)+\\
&\qquad +220\,\cos(7\,t)+55\,\cos(8\,t)+10\,\cos(9\,t)+\cos(10\,t)]\,.
\end{aligned}$$

Similarly, $G_n(t,0)$ can be computed by using the λ_g-conjecture. For example, one can easily show that

$$G_3(t,0) = \frac{(3t/2)}{\sin(3t/2)}\,,$$

et cetera. Although we are able to compute the generating function $G_n(t,k)$ at these particular values, it seems quite difficult–nevertheless possible–to determine its closed-form expression for all k. It would be a very intriguing project to search for the answer.

2.5 Conclusion

To recapitulate, the first part of this chapter studies the simple branched covers of compact connected Riemann surfaces by compact connected Riemann surfaces of arbitrary genera. Upon fixing the degree of the irreducible covers, we have obtained closed form answers for simple Hurwitz numbers for arbitrary source and target Riemann surfaces, up to degree 7. For higher degrees, we have given a general prescription for extending our results. Our computations are novel in the sense that the previously known formulas fix the genus of the source and target curves and vary the degree as a free parameter. Furthermore, by relating the simple Hurwitz numbers to descendant Gromov-Witten invariants, we have obtained the explicit generating functions (2.3.18) for the number of inequivalent reducible covers for arbitrary source and target Riemann surfaces. For an elliptic curve target, the generating function (2.3.16) is known to be a sum of quasi-modular forms. More precisely, in the expansion

$$\mathcal{Z} = \sum_{n=0}^{\infty} A_n(q)\, \lambda^{2n} \ ,$$

the series $A_n(q)$ are known to be quasi-modular of weight $6n$ under the full modular group $PSL(2,\mathbb{Z})$. Our general answer (2.3.18) for an arbitrary target genus differs from the elliptic curve case only by the pre-factor $(n!/f^{\gamma})^{2h-2}$. Naively, it is thus tempting to hope that the modular property persists, so that in the expansion

$$\mathcal{Z}(h) = \sum_{n=0}^{\infty} A_n^h(q)\, \lambda^{2n},$$

the series $A_n^h(q)$ are quasi-automorphic forms, perhaps under a genus-h subgroup of $PSL(2,\mathbb{Z})$.

Throughout the chapter, we have taken caution to distinguish two different conventions of accounting for the automorphism groups of the branched covers and have clarified their relations when possible. The recent developments in the study of Hurwitz numbers involve connections to the relative Gromov-Witten theory and Hodge integrals on the moduli space of stable curves. In particular, Li *et al.* have obtained a set of recursion relations for the numbers $\mu_{h,w}^{g,n}(\alpha)$ by applying the gluing formula to the relevant relative Gromov-Witten invariants [LiZZ]. Incidentally, these recursion relations require as initial data the knowledge of simple Hurwitz numbers, and our work would be useful for applying the relations as well.

Although we cannot make any precise statements at this stage, our work may also be relevant to understanding the conjectured Toda hierarchy and the Virasoro constrains for Gromov-Witten invariants on $\mathbf{P}^1$ and elliptic curve. It has been shown in [So] that Virasoro constraints lead to certain recursion relations among simple Hurwitz numbers for a $\mathbf{P}^1$ target. It might be interesting to see whether there exist further connections parallel to these examples. The case of an elliptic curve target seems, however, more elusive at the moment. The computations of the Gromov-Witten invariants for an elliptic curve are much akin to those occurring for Calabi-Yau three-folds. For instance, a given n-point function receives contributions from the stable maps of all degrees, in contrast to the Fano cases in which only a finite number of degrees yields the correct dimension of the moduli space. Consequently, the recursion relations and the Virasoro constraints seem to lose their efficacy when one considers the Gromov-Witten invariants of an elliptic curve. It is similar to the ineffectiveness of the WDVV equations for determining the number of rational curves on a Calabi-Yau three-fold.

Chapter 3

Semisimple Frobenius Structures and Gromov-Witten Invariants

This chapter is devoted to an investigation of Givental's recent conjecture regarding semisimple Frobenius manifolds. The conjecture expresses higher genus Gromov-Witten invariants in terms of the data obtained from genus-0 Gromov-Witten invariants and the intersection theory of tautological classes on the Deligne-Mumford moduli space $\overline{M}_{g,n}$ of stable curves. We limit our investigation to the case of a complex projective line $\mathbf{P}^1$, whose Gromov-Witten invariants are well-known and easy to compute. We make some simple checks supporting Givental's conjecture.

3.1 Introduction

In the first two subsections of this rather long introduction, we define semisimple Frobenius structures and Gromov-Witten invariants. Givental's conjecture and our investigation of it are summarized in the last subsection.

3.1.1 Semisimple Frobenius Manifold

In §1.3 we defined Frobenius algebra. In this subsection, we define what it means for a manifold to have a semisimple Frobenius structure.

Definition 3.1 (Frobenius Manifold) [Du, Giv2]

$\mathfrak{H}$ is a Frobenius manifold if, at any $t \in \mathfrak{H}$, a Frobenius algebra structure, which smoothly depends on t, is defined on the tangent space $T_t\mathfrak{H}$ such that the following conditions hold:

(i) The non-generate inner product $\langle \cdot, \cdot \rangle$ is a flat pseudo-Riemannian metric on $\mathfrak{H}$.

(ii) There exists a function $\mathscr{F}$ whose third covariant derivatives are structure constants $\langle a * b, c \rangle$ of a Frobenius algebra structure on $T_t\mathfrak{H}$.

(iii) The vector field of unities $\mathbf{1}$, which preserve the algebra multiplication $*$, is covariantly constant with respect to the Levi-Civita connection of the flat metric $\langle \cdot, \cdot \rangle$.

If the algebras $(T_t\mathfrak{H}, *)$ are semisimple at generic $t \in \mathfrak{H}$, then $\mathfrak{H}$ is called semisimple. For example, if X is a complex projective space, then $\mathfrak{H} = H^*(X, \mathbb{Q})$ carries a semisimple Frobenius structure defined by the genus-0 Gromov-Witten potential $\mathscr{F}_0$ [Giv2].

3.1.2 Gromov-Witten Invariants

Let X be a smooth projective variety. In [Kont2] Kontsevich introduced the compactified moduli space $\overline{M}_{g,n}(X, \beta)$ of *stable maps* $(f : C \to X; p_1, p_2, \ldots, p_n)$, where C is a connected, projective curve of arithmetic genus $g = h^1(C, \mathcal{O}_C)$, possibly with ordinary double points, which are the only allowed singularities; $p_1, p_2, \ldots, p_n$ are pairwise distinct non-singular points of C; and f is a morphism from C to X such that $f_*([C]) = \beta \in H_2(X, \mathbb{Z})$. The stability of $(f : C \to X; p_1, p_2, \ldots, p_n)$ means that it has only finite automorphisms. Equivalently, $(f : C \to X; p_1, p_2, \ldots, p_n)$ is stable if every irreducible component $\mathscr{C} \in C$ satisfies the following two conditions:

(i) If $\mathscr{C} \simeq \mathbf{P}^1$ and f is constant on $\mathscr{C}$, then $\mathscr{C}$ must contain at least three special points, which can be either marked points or nodal points where $\mathscr{C}$ meets the other irreducible components of C.

(ii) If $\mathscr{C}$ has arithmetic genus 1 and f is constant on $\mathscr{C}$, then $\mathscr{C}$ must contain at least one special point.

The expected complex-dimension of the moduli space $\overline{M}_{g,n}(X,\beta)$ is

$$\delta := (3-\dim(X))(g-1) + \int_\beta c_1(X) + n, \tag{3.1.1}$$

but in general $\overline{M}_{g,n}(X,\beta)$ may contain components whose dimensions exceed the above expected dimension. A crucial fact in Gromov-Witten theory is that $\overline{M}_{g,n}(X,\beta)$ carries a canonical perfect obstruction theory which allows one to construct a well-defined algebraic cycle [BehF, LT2]

$$[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}} \in A_{2\delta}(\overline{M}_{g,n}(X,\beta),\mathbb{Q})$$

in the rational Chow group of the expected dimension. $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}$ is called the *virtual fundamental class*, and all intersection invariants of cohomology classes in Gromov-Witten theory are evaluated on $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}$.

Cohomology classes of $\overline{M}_{g,n}(X,\beta)$ can be constructed from that of X as follows. For $1 \le i \le n$, let

$$\mathrm{ev}_i : \overline{M}_{g,n}(X,\beta) \longrightarrow X$$

be the evaluation map at the marked point p_i such that

$$\mathrm{ev}_i : (f : C \to X; p_1, p_2, \ldots, p_n) \longmapsto f(p_i).$$

Then a cohomology class $\gamma \in H^*(X,\mathbb{Q})$ can be pulled back by the *evaluation map* to yield $\mathrm{ev}^*(\gamma) \in H^*(\overline{M}_{g,n}(X,\beta),\mathbb{Q})$. For $\gamma_1,\ldots,\gamma_n \in H^*(X,\mathbb{Q})$, *Gromov-Witten invariants* are defined as

$$\int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \cup \mathrm{ev}_2^*(\gamma_2) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n),$$

which are defined to vanish unless the total dimension of the integrand is equal to the expected dimension in (3.1.1). Let $\tilde{\gamma}_1,\ldots,\tilde{\gamma}_m$ be a homogeneous basis of $H^*(X,\mathbb{Q})$, and define $\tilde{\gamma} := \sum_{i=0}^{m} t^\alpha \tilde{\gamma}_\alpha$, where t^α are formal variables. Then, the genus-g *Gromov-Witten potential*, which is a generating function for genus-g Gromov-Witten invariants, is defined as

$$\mathscr{F}_g(\tilde{\gamma}) := \sum_{n=0}^{\infty} \sum_{\beta \in H_2(X,\mathbb{Z})} \frac{\mathfrak{q}^\beta}{n!} \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\tilde{\gamma}) \cup \mathrm{ev}_2^*(\tilde{\gamma}) \cup \cdots \cup \mathrm{ev}_n^*(\tilde{\gamma}), \tag{3.1.2}$$

where $\mathfrak{q}^\beta := e^{2\pi i \int_\beta \omega}$ for a complexified Kähler class ω of X.

3.1.3 Brief Summary

Let X be a compact symplectic manifold whose cohomology space $H^*(X,\mathbb{Q})$ carries a semisimple Frobenius structure, and let $\mathscr{F}_g(t)$ be its genus-g Gromov-Witten potential. Then, Givental's conjecture, whose equivariant counter-part he has proved [Giv2], is

$$e^{\sum_{g\geq 2} \lambda^{g-1} \mathscr{F}_g(t)} = \left[e^{\frac{\lambda}{2} \sum_{k,l\geq 0} \sum_{i,j} V_{kl}^{ij} \sqrt{\Delta_i} \sqrt{\Delta_j} \partial_{q_k^i} \partial_{q_l^j}} \prod_j \tau(\lambda \Delta_j; \{q_m^j\}) \right] \Bigg|_{q_m^j = T_m^j}, \tag{3.1.3}$$

where $i,j = 1,\ldots,\dim H^*(X,\mathbb{Q})$; V_{kl}^{ij}, Δ_j, and T_n^j are functions of $t \in H^*(X,\mathbb{Q})$ and are defined by solutions to the flat-section equations associated with the genus-0 Frobenius structure of X [Giv2]; and τ is the KdV tau-function governing the intersection theory on the Deligne-Mumford space $\overline{M}_{g,n}$ and is defined as follows:

$$\tau(\lambda, \{q_k\}) = \exp\left[\sum_{g=0}^{\infty} \lambda^{g-1} \mathscr{F}_g^{\mathrm{pt}}(\{q_k\}) \right],$$

where

$$\mathscr{F}_g^{\mathrm{pt}}(\{q_k\}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\overline{M}_{g,n}} q(\psi_1) \cup \cdots \cup q(\psi_n).$$

We have used the notation $q(\psi_i) := \sum_{k=0}^{\infty} q_k \psi_i^k$, where q_k are formal variables. The ψ classes are the gravitational descendants defined in §1.2, i.e. the first Chern classes of the universal cotangent line bundles over $\overline{M}_{g,n}$.

Givental's remarkable conjecture organizes the higher genus Gromov-Witten invariants in terms of the genus-0 data and the τ-function for a point. The motivation for our work lies in verifying the conjecture for $X = \mathbf{P}^1$, which is the simplest example whose cohomology space $H^*(X, \mathbb{Q})$ carries a semisimple Frobenius structure and whose Gromov-Witten invariants can be easily computed.

We have obtained two particular solutions to the flat-section equations (3.3.7), an analytic one encoding the two-point descendant Gromov-Witten invariants of $\mathbf{P}^1$ and a recursive one corresponding to Givental's fundamental solution. According to Givental, both of these two solutions are supposed to yield the same data V_{kl}^{ij}, Δ_j, and T_n^j. Unfortunately, we were not able to produce the desired information using our analytic solutions, but the recursive solutions do lead to sensible quantities which we need. Combined with an expansion scheme which allows us to verify the conjecture at each order in λ, we thus use our recursive solutions to check the conjecture (3.1.3) for $\mathbf{P}^1$ up to order λ^2. Already at this order, we need to expand the differential operators in (3.1.3) up to λ^6 and need to consider up to genus-3 free energy in the τ-functions, and the computations quickly become cumbersome with increasing order. We have managed to re-express the conjecture for this case into a form which resembles the Hirota-bilinear relations, but at this point, we have no insights into a general proof. It is nevertheless curious how the numbers work out, and we hope that our results would provide a humble support for Givental's master equation.

We have organized this chapter as follows: in §3.2, we review the canonical coordinates for $\mathbf{P}^1$, to be followed by our solutions to the flat-section equations in §3.3. Our computations are presented in §3.4, and we conclude with some remarks in §3.5.

3.2 Canonical Coordinates for $\mathbf{P}^1$.

We here review the canonical coordinates $\{u_\pm\}$ for $\mathbf{P}^1$ [Du, DZ, Giv1]. Recall that a Frobenius structure on $H^*(\mathbf{P}^1, \mathbb{Q})$ carries a flat pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$ defined by the Poincaré intersection pairing. The canonical coordinates are defined by the property that they form the basis of idempotents of the quantum cup-product, denoted in the present thesis by $\circ$. The flat metric $\langle \cdot, \cdot \rangle$ is diagonal in the canonical coordinates, and following Givental's notation, we define $\Delta_\pm := 1/\langle \partial_{u_\pm}, \partial_{u_\pm} \rangle$.

Let $\{t^\alpha\}, \alpha \in \{0, 1\}$ be the flat coordinates of the metric and let $\partial_\alpha := \partial/\partial t^\alpha$. The quantum cohomology of $\mathbf{P}^1$ is

$$\partial_0 \circ \partial_\alpha = \partial_\alpha \qquad \text{and} \qquad \partial_1 \circ \partial_1 = e^{t^1}\, \partial_0.$$

The eigenvalues and eigenvectors of $\partial_1 \circ$ are

$$\pm\, e^{t^1/2} \qquad \text{and} \qquad (\pm e^{t^1/4}\, \partial_0 + e^{-t^1/4}\, \partial_1),$$

respectively. So, we have

$$(\pm e^{t^1/4}\, \partial_0 + e^{-t^1/4}\, \partial_1) \circ (\pm e^{t^1/4}\, \partial_0 + e^{-t^1/4}\, \partial_1) = \pm 2\, e^{t^1/4}\, (\pm e^{t^1/4}\, \partial_0 + e^{-t^1/4}\, \partial_1),$$

which implies that

$$\frac{\partial}{\partial u_\pm} = \frac{\partial_0 \pm e^{-t^1/2}\, \partial_1}{2},$$

such that

$$\partial_{u_\pm} \circ \partial_{u_\pm} = \partial_{u_\pm} \qquad \text{and} \qquad \partial_{u_\pm} \circ \partial_{u_\mp} = 0\,.$$

We can solve for $u_\pm$ up to constants as

$$u_\pm = t^0 \pm 2\, e^{t^1/2}\,. \tag{3.2.4}$$

To compute $\Delta_\pm$, note that

$$\frac{1}{\Delta_\pm} := \langle \partial_{u_\pm}, \partial_{u_\pm} \rangle = \pm \frac{1}{2 e^{t^1/2}}\,.$$

The two bases are related by

$$\partial_0 = \partial_{u_+} + \partial_{u_-} \qquad \text{and} \qquad \partial_1 = e^{t^1/2}\,(\partial_{u_+} - \partial_{u_-})\,.$$

Define an orthonormal basis by $f_i = \Delta_i^{1/2}\frac{\partial}{\partial u_i}$. Then the transition matrix Ψ from $\{\frac{\partial}{\partial t_\alpha}\}$ to $\{f_i\}$ is given by

$$\Psi_\alpha^{\ i} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-t^1/4} & -i\,e^{-t^1/4} \\ e^{t^1/4} & i\,e^{t^1/4} \end{pmatrix} = \begin{pmatrix} \Delta_+^{-1/2} & \Delta_-^{-1/2} \\ \frac{1}{2}\Delta_+^{1/2} & \frac{1}{2}\Delta_-^{1/2} \end{pmatrix}, \tag{3.2.5}$$

such that

$$\frac{\partial}{\partial t_\alpha} = \sum_i \Psi_\alpha^{\ i}\, f_i.$$

We will also need the inverse of (3.2.5):

$$(\Psi^{-1})_i^{\ \alpha} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{t^1/4} & e^{-t^1/4} \\ i\,e^{t^1/4} & -i\,e^{-t^1/4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\Delta_+^{1/2} & \Delta_+^{-1/2} \\ \frac{1}{2}\Delta_-^{1/2} & \Delta_-^{-1/2} \end{pmatrix}. \tag{3.2.6}$$

3.3 Solutions to the Flat-Section Equations

The relevant data V_{kl}^{ij}, Δ_j and T_n^j are extracted from the solutions to the flat-section equations of the genus-0 Frobenius structure for $\mathbf{P}^1$. We here find two particular solutions. The analytic solution correctly encodes the two-point descendant Gromov-Witten invariants, while the recursive solution is used in the next section to verify Givental's conjecture.

3.3.1 Analytic Solution

The genus-0 free energy for $\mathbf{P}^1$ is

$$\mathscr{F}_0 = \frac{1}{2}(t^0)^2 t^1 + e^{t^1}.$$

Flat sections S_α of $TH^*(\mathbf{P}^1, \mathbb{Q})$ satisfy the equations

$$z\,\partial_\alpha\, S_\beta = \mathscr{F}_{\alpha\beta\mu}\, g^{\mu\nu}\, S_\nu\ , \tag{3.3.7}$$

where $z \neq 0$ is an arbitrary parameter and $\mathscr{F}_{\alpha\beta\mu} := \partial^3 \mathscr{F}_0 / \partial t^\alpha \partial t^\beta \partial t^\mu$. Since the only non-vanishing components of $\mathscr{F}_{\alpha\beta\mu}$ for $\mathbf{P}^1$ are

$$\mathscr{F}_{\underline{0}\underline{0}\underline{1}} = 1 \qquad \text{and} \qquad \mathscr{F}_{\underline{1}\underline{1}\underline{1}} = e^{t^1} ,$$

(3.3.7) gives the following set of equations:

$$\begin{aligned} z\,\partial_0 S_0 &= S_0 \,, \\ z\,\partial_0 S_1 &= S_1 \,, \\ z\,\partial_1 S_0 &= S_1 \,, \\ z\,\partial_1 S_1 &= e^{t^1}\, S_0 . \end{aligned}$$

The first two equations imply

$$S_0 = \mathcal{A}(t^1)\, e^{t^0/z} \qquad \text{and} \qquad S_1 = \mathcal{B}(t^1)\, e^{t^0/z} ,$$

while the last two imply

$$z\,\mathcal{A}'(t^1) = \mathcal{B}(t^1) \qquad \text{and} \qquad z\,\mathcal{B}'(t^1) = e^{t^1}\,\mathcal{A}(t^1) \,.$$

These coupled differential equations together imply

$$z^2 \mathcal{A}''(t^1) = e^{t^1}\,\mathcal{A}(t^1) \qquad \text{and} \qquad z^2 \mathcal{B}''(t^1) = z^2 \mathcal{B}'(t^1) + e^{t^1}\,\mathcal{B}(t^1) \,,$$

and we can now solve for $\mathcal{A}(t)$ and $\mathcal{B}(t)$ as follows:

$$\begin{aligned} \mathcal{B}(t) &= e^{t^1/2} \left[c_1\, I_1(2 e^{t^1/2}/z) + c_2\, K_1(2 e^{t^1/2}/z) \right] , \\ \mathcal{A}(t) &= c_1\, I_0(2\, e^{t/2}/z) - c_2\, K_0(2\, e^{t/2}/z) \,, \end{aligned}$$

where $I_n(x)$ and $K_n(x)$ are modified Bessel functions, and c_i are integration constants which may depend on z. Hence, we find that the general solutions to the flat-section equations (3.3.7) are

$$S_0 = e^{t^0/z} \left[c_1\, I_0(2\, e^{t^1/2}/z) - c_2\, K_0(2\, e^{t^1/2}/z) \right] \tag{3.3.8}$$

and

$$S_1 = e^{t^0/z}\, e^{t^1/2} \left[c_1\, I_1(2e^{t^1/2}/z) + c_2\, K_1(2e^{t^1/2}/z)\right] .$$

We would now like to find two particular solutions corresponding to the following Givental's expression:

$$S_{\alpha\beta}(z) = g_{\alpha\beta} + \sum_{n\geq 0,(n,d)\neq(0,0)} \frac{1}{n!} \langle \phi_\alpha \cdot \frac{\phi_\beta}{z-\psi} \cdot (t^0\phi_0 + t^1\phi_1)^n \rangle_d \,, \tag{3.3.9}$$

where $S_{\alpha\beta}$ denotes the α-th component of the β-th solution. Here, $\{\phi_\alpha\}$ is a homogeneous basis of $H^*(\mathbf{P}^1, \mathbb{Q})$, $g_{\alpha\beta}$ is the intersection paring $\int_{\mathbf{P}^1} \phi_\alpha \cup \phi_\beta$ and $\psi \in H^2(\overline{M}_{0,n+2}(\mathbf{P}^1, d), \mathbb{Q})$ is the first Chern class of the universal cotangent line bundle over the moduli space $\overline{M}_{0,n+2}(\mathbf{P}^1, d)$. In order to find the particular solutions, we compare our general solution (3.3.8) with the 0-th components of $S_{0\beta}$ in (3.3.9) *at the origin of the phase space.* The two-point functions appearing in (3.3.9) have been computed at the origin in [So] and have the following forms:

$$S_{00}|_{t^\alpha=0} = -\sum_{m=1}^{\infty} \frac{1}{z^{2m+1}} \frac{2\, d_m}{(m!)^2} \;, \quad \text{where } d_m = \sum_{k=1}^{m} 1/k \,, \tag{3.3.10}$$

and

$$S_{01}|_{t^\alpha=0} = 1 + \sum_{m=1}^{\infty} \frac{1}{z^{2m}} \frac{1}{(m!)^2}. \tag{3.3.11}$$

Using the standard expansion of the modified Bessel function K_0, we can evaluate (3.3.8) at the origin of the phase space to be

$$c_1\, I_0\left(\frac{2}{z}\right) - c_2\, K_0\left(\frac{2}{z}\right) \;=\; c_1\, I_0\left(\frac{2}{z}\right) - c_2 \left[-\left(-\log(z) + \gamma_{\scriptscriptstyle E}\right) I_0\left(\frac{2}{z}\right) + \sum_{m=1} \frac{c_m}{z^{2m}(m!)^2}\right] , \tag{3.3.12}$$

where $\gamma_{\scriptscriptstyle E}$ is Euler's constant. Now matching (3.3.12) with (3.3.10) gives

$$c_1 = -c_2 \log(1/z) - c_2\gamma_{\scriptscriptstyle E} \qquad \text{and} \qquad c_2 = \frac{2}{z} \,,$$

while noticing that (3.3.11) is precisely the expansion of $I_0(2/z)$ and demanding that our general solution coincides with (3.3.11) at the origin yields

$$c_1 = 1 \qquad \text{and} \qquad c_2 = 0 \ .$$

To recapitulate, we have found

$$
\begin{aligned}
S_{00} &= -\frac{2e^{t^0/z}}{z}\left[(\gamma_E - \log(z))\; I_0\left(\frac{2\,e^{t^1/2}}{z}\right) + K_0\left(\frac{2\,e^{t^1/2}}{z}\right)\right], \\
S_{10} &= \frac{2e^{t^0/z}\,e^{t^1/2}}{z}\left[K_1\left(\frac{2e^{t^1/2}}{z}\right) - (\gamma_E - \log(z))\, I_1\left(\frac{2e^{t^1/2}}{z}\right)\right], \\
S_{01} &= e^{t^0/z}\, I_0\left(\frac{2\,e^{t^1/2}}{z}\right), \\
S_{11} &= e^{t^0/z}\,e^{t^1/2}\, I_1\left(\frac{2e^{t^1/2}}{z}\right).
\end{aligned}
$$

We have checked that these solutions correctly reproduce the corresponding descendant Gromov-Witten invariants obtained in [So].

If the inverse transition matrix in (3.2.6) is used to relate the matrix elements $S_\alpha^{\ i}$ to $S_{\alpha\beta}$ as $S_\alpha^{\ i} = S_{\alpha\beta}((\psi^{-1})^t)^\beta_{\ j}\,\delta^{ji}$, then we should have

$$
S_\alpha^{\ \pm} = \sqrt{\pm 2}\,e^{t^1/4}\left(\frac{1}{2}S_{\alpha 0} \pm \frac{e^{-t^1/2}}{2}S_{\alpha 1}\right). \tag{3.3.13}
$$

3.3.2 Recursive Solution

In [Giv1, Giv2], Givental has shown that near a semisimple point, the flat-section equations (3.3.7) have a fundamental solution given by

$$
S_\alpha^{\ i} = \Psi_\alpha^{\ j}(R_0 + zR_1 + z^2R_2 + \cdots + z^nR_n + \cdots)_{jk}\,[\exp(U/z)]^{ki},
$$

where $R_n = (R_n)_{jk}$, $R_0 = \delta_{jk}$ and U is the diagonal matrix of canonical coordinates. The matrix R_1 satisfies the relations

$$
\Psi^{-1}\frac{\partial\Psi}{\partial t^1} = [\frac{\partial U}{\partial t^1}, R_1] \tag{3.3.14}
$$

and

$$
\left[\frac{\partial R_1}{\partial t^1} + \Psi^{-1}\left(\frac{\partial\Psi}{\partial t^1}\right)R_1\right]_{\pm\pm} = 0\,, \tag{3.3.15}
$$

which we use to find its expression. From the transition matrix given in (3.2.5) we see that

$$
\Psi^{-1}\frac{\partial\Psi}{\partial t^1} = \frac{1}{4}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},
$$

while taking the $(+-)$ component of the relation (3.3.14) gives

$$\frac{i}{4} = \frac{\partial U_{++}}{\partial t^1}(R_1)_{+-} - (R_1)_{+-}\frac{\partial U_{--}}{\partial t^1} = 2e^{t^1/2}(R_1)_{+-} \,,$$

where in the last step we have used the definition (3.2.4) of canonical coordinates. We therefore have

$$(R_1)_{+-} = \frac{i}{8}e^{-t^1/2} \,,$$

and similarly considering the $(-+)$ component of (3.3.14) gives

$$(R_1)_{-+} = \frac{i}{8}e^{-t^1/2}.$$

The diagonal components of R_1 can be obtained from (3.3.15), which implies that

$$\frac{\partial (R_1)_{++}}{\partial t^1} = (R_1)_{+-}\frac{\partial U_{--}}{\partial t^1}(R_1)_{-+} - \frac{\partial U_{++}}{\partial t^1}(R_1)_{+-}(R_1)_{-+} = \frac{\exp(-t^1/2)}{32} = -\frac{\partial (R_1)_{--}}{\partial t^1}.$$

Hence, $(R_1)_{++} = -\exp(-t^1/2)/16 = -(R_1)_{--}$ and the matrix R_1 can be written as

$$(R_1)_{jk} = \frac{1}{16}e^{-t^1/2}\begin{pmatrix} -1 & 2i \\ 2i & 1 \end{pmatrix}. \tag{3.3.16}$$

In general, the matrices R_n satisfy the recursion relations [Giv1]

$$\left(d + \Psi^{-1}d\Psi\right) R_n = [dU, R_{n+1}] \,, \tag{3.3.17}$$

which, for our case, imply the following set of equations:

$$\frac{\partial R_n}{\partial t^0} = 0 \,, \tag{3.3.18}$$

$$\frac{\partial (R_n)_{++}}{\partial t^1} = -\frac{i}{4}(R_n)_{-+} \,, \tag{3.3.19}$$

$$(R_{n+1})_{-+} = -\frac{1}{2}e^{-t^1/2}\left[\frac{\partial (R_n)_{-+}}{\partial t^1} - \frac{i}{4}(R_n)_{++}\right] \,, \tag{3.3.20}$$

$$\frac{\partial (R_n)_{--}}{\partial t^1} = \frac{i}{4}(R_n)_{+-} \,, \tag{3.3.21}$$

$$(R_{n+1})_{+-} = \frac{1}{2}e^{-t^1/2}\left[\frac{\partial (R_n)_{+-}}{\partial t^1} + \frac{i}{4}(R_n)_{--}\right] \,. \tag{3.3.22}$$

Proposition 3.2 *For $n \geq 1$, the matrices R_n in the fundamental solution are given by*

$$(R_n)_{ij} = \frac{(-1)^n}{(2n-1)} \frac{\alpha_n}{2^n} e^{-nt^1/2} \begin{pmatrix} -1 & (-1)^{n+1} 2n\, i \\ 2n\, i & (-1)^{n+1} \end{pmatrix}, \tag{3.3.23}$$

where

$$\alpha_n = (-1)^n \frac{1}{8^n n!} \prod_{\ell=1}^{n} (2\ell-1)^2 \ , \ \alpha_0 = 1.$$

These solutions satisfy the unitarity condition

$$\boldsymbol{R}(z)\boldsymbol{R}^t(-z) := (1+zR_1+z^2R_2+\cdots+z^nR_n+\cdots)(1-zR_1^t+z^2R_2^t+\cdots+(-1)^n z^n R_n^t+\cdots) = 1$$

and the homogeneity condition and, thus, are unique.

Proof: For $n = 1$, $\alpha_1 = -1/8$ and (3.3.23) is equal to the correct solution (3.3.16). The proof now follows by an induction on n. Assume that (3.3.23) holds true up to and including $n = m$. Using the fact that

$$\alpha_{m+1} = -\frac{(2m+1)^2}{8(m+1)} \alpha_m,$$

we can show that R_{m+1} in (3.3.23) satisfies the relations (3.3.19)–(3.3.22) as well as (3.3.18).

To check unitarity, consider the z^k-term $P_k := \sum_{\ell=0}^{k} (-1)^\ell R_{k-\ell} R_\ell^t$ in $\boldsymbol{R}(z)\boldsymbol{R}^t(-z) = \sum_{k=0} P_k z^k$. As shown by Givental, the equations satisfied by the matrices R_n imply that the off-diagonal entries of P_k vanish. As a result, combined with the anti-symmetry of P_k for odd k, we see that P_k vanishes for k odd. Hence, we only need to show that for our solution, P_k vanishes for all positive even k as well. To this end, we note that Givental has also deduced from the equation $dP_k + [\Psi^{-1}d\Psi, P_k] = [dU, P_{k+1}]$ that the diagonal entries of P_k are constant. The expansion of P_{2k} is

$$P_{2k} = R_{2k} + R_{2k}^t + \cdots ,$$

where the remaining terms are products of R_ℓ, for $\ell < 2k$. Now, we proceed inductively. We first note that R_1 and R_2 given in (3.3.23) satisfy the condition $P_2 = 0$, and assume that R_ℓ's in (3.3.23) for $\ell < 2k$ satisfy $P_\ell = 0$. Then, since the off-diagonal entries of P_n vanish for all n, the expansion of P_{2k} is of the form

$$P_{2k} = A\, e^{-2k\,t^1/2} + B,$$

where A is a constant diagonal matrix resulting from substituting our solution (3.3.23) and B is a possible diagonal matrix of integration constants for R_{2k}. But, since the diagonal entries of P_n are constant for all n, we know that $A = 0$. We finally choose the integration constants to be zero so that $B = 0$, yielding $P_{2k} = 0$. Hence, the matrices in our solution (3.3.23) satisfy the unitarity condition and are manifestly homogeneous. It then follows by the proposition in [Giv2] that our solutions R_n are unique. ■

Let $\boldsymbol{R} := (R_0 + zR_1 + z^2 R_2 + \cdots + z^n R_n + \cdots)$. Then, we can use the matrices R_n from Proposition 3.2 to find

$$\begin{aligned} S_0{}^{+} &= (\boldsymbol{R}_{++} - i\,\boldsymbol{R}_{-+})\,\frac{\exp(u_+/z)}{\sqrt{\Delta_+}} \\ &= \left[1 + \sum_{n=1}^{\infty} \frac{\alpha_n}{2^n} \exp\left(\frac{-nt^1}{2}\right)(-z)^n\right] \frac{\exp(u_+/z)}{\sqrt{\Delta_+}}\ , \qquad (3.3.24) \\ S_0{}^{-} &= (\boldsymbol{R}_{--} + i\,\boldsymbol{R}_{+-})\,\frac{\exp(u_-/z)}{\sqrt{\Delta_-}} \\ &= \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{\alpha_n}{2^n} \exp\left(\frac{-nt^1}{2}\right)(-z)^n\right] \frac{\exp(u_-/z)}{\sqrt{\Delta_-}}\ , \qquad (3.3.25) \\ S_1{}^{+} &= (\boldsymbol{R}_{++} + i\,\boldsymbol{R}_{-+})\,\frac{\sqrt{\Delta_+}}{2}\exp(u_+/z) \\ &= \left[1 - \sum_{n=1}^{\infty} \frac{(2n+1)}{(2n-1)} \frac{\alpha_n}{2^n} \exp\left(\frac{-nt^1}{2}\right)(-z)^n\right] \frac{\sqrt{\Delta_+}}{2}\exp(u_+/z)\ , \qquad (3.3.26) \\ S_1{}^{-} &= (\boldsymbol{R}_{--} - i\,\boldsymbol{R}_{+-})\,\frac{\sqrt{\Delta_-}}{2}\exp(u_-/z) \\ &= \left[1 - \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)}{(2n-1)} \frac{\alpha_n}{2^n} \exp\left(\frac{-nt^1}{2}\right)(-z)^n\right] \frac{\sqrt{\Delta_-}}{2}\exp(u_-/z)\ . \qquad (3.3.27) \end{aligned}$$

Using the above expressions for ${S_\alpha}^i(z)$, we can also find $V^{ij}(z,w)$, which is given by the expression

$$V^{ij}(z,w) := \frac{1}{z+w}\,[{S_\mu}^i(w)]^t\,[g^{\mu\nu}]\,[{S_\nu}^j(z)].$$

If we define

$$A_{p,q} := \frac{(4p\,q-1)}{(2p-1)(2q-1)}\,\frac{\alpha_p\,\alpha_q}{2^{p+q}}e^{\frac{-(p+q)t^1}{2}}$$

and

$$B_{p,q} := \frac{2(p-q)}{(2p-1)(2q-1)}\,\frac{\alpha_p\,\alpha_q}{2^{p+q}}e^{\frac{-(p+q)t^1}{2}},$$

then after some algebraic manipulations we obtain

$$V^{++}(z,w) = e^{u_+/w+u_+/z}\left\{\frac{1}{z+w}+\sum_{k,l=0}^{\infty}\left[\sum_{n=0}^{k}(-1)^n A_{l+n+1,k-n}\right](-1)^{k+l}\,w^k z^l\right\}, \quad (3.3.28)$$

$$V^{--}(z,w) = e^{u_-/w+u_-/z}\left\{\frac{1}{z+w}-\sum_{k,l=0}^{\infty}\left[(-1)^{k+l}\sum_{n=0}^{k}(-1)^n A_{l+n+1,k-n}\right](-1)^{k+l}\,w^k z^l\right\},$$

$$V^{+-}(z,w) = e^{u_+/w+u_-/z}\left\{\sum_{k,l=0}^{\infty}\left[i\,(-1)^l\sum_{n=0}^{k}B_{l+n+1,k-n}\right]\;(-1)^{k+l}\,w^k z^l\right\}, \quad (3.3.29)$$

$$V^{-+}(z,w) = e^{u_-/w+u_+/z}\left\{\sum_{k,l=0}^{\infty}\left[i\,(-1)^k\sum_{n=0}^{k}B_{l+n+1,k-n}\right]\;(-1)^{k+l}\,w^k z^l\right\}.$$

3.3.3 A Puzzle

Incidentally, we note that in the asymptotic limit $z \to 0$,

$${S_0}^+ = \Re\left[\sqrt{\frac{2\pi}{z}}e^{t^0/z}I_0\left(\frac{2e^{t^1/2}}{z}\right)\right]$$

and

$${S_0}^- = -i\sqrt{\frac{2}{\pi z}}e^{t^0/z}K_0\left(\frac{2e^{t^1/2}}{z}\right)$$

reproduce the expansions in (3.3.24) and (3.3.25). This is in contrast to what was expected from the discussion leading to (3.3.13). Despaired of matching the two expressions, it seems to us that the analytic correlation functions obtained in §3.3.1

do not encode the right information that appear in Givental's conjecture. In the following section, we will use the recursive solutions from §3.3.2 to check Givental's conjectural formula at low genera.

3.4 Checks of the Conjecture at Low Genera

The T_n^i that appear in Givental's formula (3.1.3) are defined by the equations [Giv2]

$$S_0{}^{\pm} := \left[1 - \sum_{n=0}^{\infty} T_n^{\pm}(-z)^{n-1}\right] \frac{\exp(u_\pm/z)}{\sqrt{\Delta_\pm}}\,.$$

From the computations of $S_0{}^{+}$ and $S_0{}^{-}$ in (3.3.24) and (3.3.25), respectively, one can extract T_n^i to be

$$T_n^+ \;=\; \begin{cases} 0\,, & n = 0, 1\,, \\ -\dfrac{\alpha_{n-1}}{2^{n-1}} \exp\left[\dfrac{-(n-1)t^1}{2}\right], & n \geq 2\,, \end{cases} \tag{3.4.30}$$

$$T_n^- \;=\; \begin{cases} 0\,, & n = 0, 1\,, \\ -(-1)^{n-1}\dfrac{\alpha_{n-1}}{2^{n-1}} \exp\left[\dfrac{-(n-1)t^1}{2}\right], & n \geq 2\,. \end{cases} \tag{3.4.31}$$

Notice that

$$T_n^- = (-1)^{n-1}\, T_n^+\,. \tag{3.4.32}$$

The functions V_{kl}^{ij} are defined[1] by the expansion [Giv2]

$$V^{ij}(z,w) = e^{u^i/w + u^j/z}\left[\frac{\delta^{ij}}{z+w} + \sum_{k,l=0}^{\infty}(-1)^{k+l}\, V_{kl}^{ij}\, w^k z^l\right],$$

and from (3.3.28) and (3.3.29) we see that

$$V_{kl}^{++} = \sum_{n=0}^{k}(-1)^n A_{l+n+1,k-n} = \sum_{n=0}^{k}\frac{(-1)^n(4(l+n+1)(k-n)-1)}{(2l+2n+1)(2k-2n-1)}\, T_{l+n+2}^+ T_{k-n+1}^+\,,$$

$$V_{kl}^{+-} = i(-1)^l\sum_{n=0}^{k} B_{l+n+1,k-n} = i(-1)^l\sum_{n=0}^{k}\frac{2(l+2n+1-k)}{(2l+2n+1)(2k-2n-1)}\, T_{l+n+2}^+ T_{k-n+1}^+.$$

[1] There seems to be a misprint in the original formula for V_{kl}^{ij} in [Giv2], i.e. we believe that w and z should be exchanged, as in our expression here.

Now, the τ-function for the intersection theory on the Deligne-Mumford moduli space $\overline{M}_{g,n}$ of stable curves is defined by

$$\tau(\lambda; \{q_k\}) = \exp\left(\sum_{g=0}^{\infty} \lambda^{g-1}\, \mathscr{F}_g^{\mathrm{pt}}(\{q_k\})\right)$$

and has the following nice scaling invariance: consider the scaling of the phase-space variables q_k given by

$$q_k \mapsto s^{k-1}\, q_k \tag{3.4.33}$$

for some constant s. Then, since a non-vanishing intersection number $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle$ must satisfy

$$\sum_{i=1}^{n} (k_i - 1) = \dim(\overline{M}_{g,n}) - n = 3g - 3\,,$$

we see that under the transformation (3.4.33), the genus-g generating function $\mathscr{F}_g^{\mathrm{pt}}$ must behave as

$$\mathscr{F}_g^{\mathrm{pt}}(\{s^{k-1}\, q_k\}) = (s^3)^{g-1}\, \mathscr{F}_g^{\mathrm{pt}}(\{q_k\}).$$

Hence, upon scaling the "string coupling constant" λ to $s^{-3}\,\lambda$, we see that

$$\tau(s^{-3}\lambda;\, \{s^{k-1} q_k\}) = \tau(\lambda; \{q_k\})\,. \tag{3.4.34}$$

Now, consider the function

$$F(\{q_n^+\}, \{q_n^-\}) := \left[\mathrm{e}^{\frac{\lambda}{2} \sum_{k,l\geq 0} \sum_{i,j\in\{\pm\}} V_{kl}^{ij} \sqrt{\Delta_i}\sqrt{\Delta_j}\, \partial_{q_k^i} \partial_{q_l^j}}\; \tau(\lambda\Delta_+; \{q_n^+\})\tau(\lambda\Delta_-; \{q_n^-\}) \right]. \tag{3.4.35}$$

Then, since the Gromov-Witten potentials of $\mathbf{P}^1$ for $g \geq 2$ all vanish, Givental's conjectural formula for $\mathbf{P}^1$ is

$$F(\{T_n^+\}, \{T_n^-\}) = 1\,,$$

where it is understood that one sets $q_k^i = T_k^i$ after taking the derivatives with respect to q_k^i. Since T_n^+ and T_n^- are related by (3.4.32), let us rescale $q_k^- \mapsto (-1)^{k-1} q_k^-$ in (3.4.35). Then, since $\Delta_+ = -\Delta_-$, we observe from (3.4.34) that

$$F(\{T_n^+\},\{T_n^-\}) = \Bigg\{\exp\left[\frac{\lambda}{2}\Delta_+\sum_{k,l\geq 0}\Big(V_{kl}^{++}\partial_{q_k^+}\partial_{q_l^+} + i(-1)^{l-1}\,V_{kl}^{+-}\,\partial_{q_k^+}\partial_{q_l^-} + \right.$$
$$\left. + i(-1)^{k-1}\,V_{kl}^{-+}\partial_{q_k^-}\partial_{q_l^+} \; - (-1)^{k+l}V_{kl}^{--}\,\partial_{q_k^-}\partial_{q_l^-}\Big)\right]\,\tau(\lambda\Delta_+;\{q_n^+\})\tau(\lambda\Delta_+;\{q_n^-\})\Bigg\}\Bigg|_{q_n^+,q_n^-=T_n^+}.$$

But, the V_{kl}^{ij} satisfy the relations $V_{kl}^{--} = -(-1)^{k+l}V_{kl}^{++}$ and $V_{kl}^{+-} = V_{lk}^{-+}$, so

$$\begin{aligned} F(\{T_n^+\},\{T_n^-\}) \;&=\; \Bigg\{\exp\left[\frac{\lambda}{2}\Delta_+\sum_{k,l\geq 0}\Big(\,V_{kl}^{++}(\partial_{q_k^+}\partial_{q_l^+} + \partial_{q_k^-}\partial_{q_l^-}) + \right. \\ &\left. + 2i\,(-1)^{l-1}\,V_{kl}^{+-}\,\partial_{q_k^+}\partial_{q_l^-}\Big)\right]\;\tau(\lambda\Delta_+;\{q_n^+\})\tau(\lambda\Delta_+;\{q_n^-\})\Bigg\}\Bigg|_{q_n^+,q_n^-=T_n^+}. \end{aligned} \tag{3.4.36}$$

Now, consider the following transformations of the variables:

$$q_k^+ = x_k + y_k \qquad \text{and} \qquad q_k^- = x_k - y_k$$

so that

$$\partial_{q_k^+} = \frac{1}{2}\left(\partial_{x_k} + \partial_{y_k}\right) \qquad \text{and} \qquad \partial_{q_k^-} = \frac{1}{2}\left(\partial_{x_k} - \partial_{y_k}\right).$$

Then, in these new coordinates, (3.4.36) becomes

$$F(\{T_n^+\},\{T_n^-\}) = G(\{T_n^+\},\{0\}),$$

where the new function $G(\{x_k\},\{y_k\})$ is defined[2] by

$$\begin{aligned} G(\{x_n\},\{y_n\}) \;&=\; \\ &\exp\left[\frac{\lambda}{4}\Delta_+\sum_{k,l\geq 0}\left(V_{kl}\partial_{x_k}\partial_{x_l} + W_{kl}\partial_{y_k}\partial_{y_l}\right)\right]\tau(\lambda\Delta_+;\{x_n+y_n\})\tau(\lambda\Delta_+;\{x_n-y_n\}), \end{aligned} \tag{3.4.37}$$

where

$$\begin{aligned} V_{kl} \;&:=\; V_{kl}^{++} + i(-1)^{l-1}\,V_{kl}^{+-}\;, \\ W_{kl} \;&:=\; V_{kl}^{++} - i(-1)^{l-1}\,V_{kl}^{+-}\;. \end{aligned}$$

[2]We have simplified the expression by noting that the mixed derivative terms cancel because of the identity $V_{kl}^{+-} = (-1)^{k-l}\,V_{lk}^{+-}$.

Remark: The conjecture expressed in terms of (3.4.37), i.e. that $G(\{T_k^+\},\{0\})=1$, is now in a form which resembles the Hirota bilinear relations, which might be indicating some kind of an integrable hierarchy, perhaps of Toda-type.

Because the tau-functions are exponential functions, upon acting on them by the differential operators, we can factor them out in the expression of $G(\{x_k\},\{y_k\})$. We thus define

Definition 3.3 *$P(\lambda\Delta_+,\{x_k\},\{y_k\})$ is a formal power series in the variables $\lambda\Delta_+$, $\{x_k\}$ and $\{y_k\}$ such that*

$$G(\{x_k\},\{y_k\}) = P(\lambda\Delta_+,\{x_k\},\{y_k\})\,\tau(\lambda\Delta_+,\{x_k+y_k\})\,\tau(\lambda\Delta_+,\{x_k-y_k\}).$$

Hence, Givental's conjecture for $\mathbf{P}^1$ can be restated as

Conjecture 3.4 (Givental) *The generating function $G(\{T_k^+\},\{0\})$ is equal to one, or equivalently*

$$P(\lambda\Delta_+,\{T_k^+\},\{0\}) = \frac{1}{\tau(\lambda\Delta_+,\{T_k^+\})^2}. \tag{3.4.38}$$

This conjecture can be verified order by order[3] in λ.

Let us check (3.4.38) up to order λ^2, for which we need to consider up to λ^6 expansions in the differential operators acting on the τ-functions. Let $h=\lambda\Delta_+$. The low-genus free energies for a point target space can be easily computed using the KdV hierarchy and topological axioms; they can also be verified using Faber's program [Fab1]. The terms relevant to our computation are given by the following expression:

[3]This procedure is possible because when $q_0=q_1=0$, only a finite number of terms in the free-energies and their derivatives are non-vanishing. In particular, the genus-0 and genus-1 free energies vanish when $q_0=q_1=0$.

$$\begin{aligned}
\frac{\mathscr{F}_0^{\rm pt}}{h} + \mathscr{F}_1^{\rm pt} + h\mathscr{F}_2^{\rm pt} = \frac{1}{h}\Bigg[& \frac{(q_0)^3}{3!} + \frac{(q_0)^3 q_1}{3!} + 2!\frac{(q_0)^3(q_1)^2}{3!\,2!} + 3!\frac{(q_0)^3(q_1)^3}{3!\,3!} + \frac{(q_0)^4 q_2}{4!} + \\
& + 3\frac{(q_0)^4 q_1 q_2}{4!} + 12\frac{(q_0)^4(q_1)^2 q_2}{4!\,2!} + \frac{(q_0)^5 q_3}{5!} + 4\frac{(q_0)^5 q_1 q_3}{5!} + \\
& + 6\frac{(q_0)^5(q_2)^2}{5!2!} + 30\frac{(q_0)^5 q_1 (q_2)^2}{5!\,2!} + \frac{(q_0)^6 q_4}{6!} + 10\frac{(q_0)^6 q_2 q_3}{6!} + \\
& + 90\frac{(q_0)^6(q_2)^3}{6!\,3!} + \cdots \Bigg] + \\
& + \Bigg[\frac{1}{24}q_1 + \frac{1}{24}\frac{(q_1)^2}{2!} + \frac{1}{12}\frac{(q_1)^3}{3!} + \frac{1}{4}\frac{(q_1)^4}{4!} + \frac{1}{24}q_0q_2 + \frac{1}{12}q_0q_1q_2 + \\
& + \frac{1}{4}\frac{q_0(q_1)^2q_2}{2!} + \frac{q_0(q_1)^3q_2}{3!} + \frac{1}{6}\frac{(q_0)^2(q_2)^2}{2!\,2!} + \frac{10}{3}\frac{(q_0)^2(q_1)^2(q_2)^2}{2!\,2!\,2!} + \\
& + \frac{2}{3}\frac{(q_0)^2q_1(q_2)^2}{2!\,2!} + \frac{1}{24}\frac{(q_0)^2q_3}{2!} + \frac{1}{8}\frac{(q_0)^2q_1q_3}{2!} + \frac{1}{2}\frac{(q_0)^2(q_1)^2q_3}{2!\,2!} + \\
& + \frac{7}{24}\frac{(q_0)^3q_2q_3}{3!} + \frac{35}{24}\frac{(q_0)^3q_1q_2q_3}{3!} + 2\frac{(q_0)^3(q_2)^3}{3!\,3!} + 12\frac{(q_0)^3q_1(q_2)^3}{3!\,3!} + \\
& + \frac{1}{24}\frac{(q_0)^3q_4}{3!} + \frac{1}{6}\frac{(q_0)^3q_1q_4}{3!} + 48\frac{(q_0)^4(q_2)^4}{4!\,4!} + \frac{59}{12}\frac{(q_0)^4(q_2)^2q_3}{4!\,2!} + \\
& + \frac{7}{12}\frac{(q_0)^4(q_3)^2}{4!\,2!} + \frac{11}{24}\frac{(q_0)^4q_2q_4}{4!} + \frac{1}{24}\frac{(q_0)^4q_5}{4!} + \cdots\Bigg] + \\
& + h\Bigg[\frac{7}{240}\frac{(q_2)^3}{3!} + \frac{29}{5760}q_2q_3 + \frac{1}{1152}q_4 + \frac{7}{48}\frac{q_1(q_2)^3}{3!} + \frac{7}{8}\frac{(q_1)^2(q_2)^3}{2!\,3!} + \\
& + \frac{29}{1440}q_1q_2q_3 + \frac{29}{288}\frac{(q_1)^2q_2q_3}{2!} + \frac{1}{384}q_1q_4 + \frac{1}{96}\frac{(q_1)^2q_4}{2!} + \\
& + \frac{7}{12}\frac{q_0(q_2)^4}{4!} + \frac{49}{12}\frac{q_0q_1(q_2)^4}{4!} + \frac{5}{72}\frac{q_0(q_2)^2q_3}{2!} + \frac{5}{12}\frac{q_0q_1(q_2)^2q_3}{2!} + \\
& + \frac{29}{2880}\frac{q_0(q_3)^2}{2!} + \frac{29}{576}\frac{q_0q_1(q_3)^2}{2!} + \frac{11}{1440}q_0q_2q_4 + \frac{11}{288}q_0q_1q_2q_4 + \\
& + \frac{1}{1152}q_0q_5 + \frac{1}{288}q_0q_1q_5 + \frac{245}{12}\frac{(q_0)^2(q_2)^5}{2!\,5!} + \frac{11}{6}\frac{(q_0)^2(q_2)^3q_3}{2!\,3!} + \\
& + \frac{109}{576}\frac{(q_0)^2q_2(q_3)^2}{2!\,2!} + \frac{17}{960}\frac{(q_0)^2q_3q_4}{2!} + \frac{7}{48}\frac{(q_0)^2(q_2)^2q_4}{2!\,2!} + \\
& + \frac{1}{90}\frac{(q_0)^2q_2q_5}{2!} + \frac{1}{1152}\frac{(q_0)^2q_6}{2!} + \cdots\Bigg].
\end{aligned}$$

This expression gives the necessary expansion of $\tau(\lambda\Delta_+;\{x_k \pm y_k\})$ for our consideration, and upon evaluating $G(\{T_k^+\},\{0\})$, we find

$$P(h,\{T_k^+\},\{0\}) = 1 - \frac{17}{2359296}e^{-3t^1/2}h + \frac{41045}{695784701952}e^{-3t^1}h^2 + \mathcal{O}(h^3). \quad (3.4.39)$$

At this order, the expansion of the right-hand-side of (3.4.38) is

$$\tau(h,\{T_k^+\})^{-2} = 1 - 2\,\mathscr{F}_2^{\rm pt}\,h + 2\left[(\mathscr{F}_2^{\rm pt})^2 - \mathscr{F}_3^{\rm pt}\right]h^2 + \mathcal{O}(h^3).$$

At $q_n = T_n^+$, $\forall\, n$, the genus-2 free energy is precisely given by

$$\mathscr{F}_2^{\rm pt}\Big|_{q_n^i=T_n^+} = \frac{1}{1152}T_4 + \frac{29}{5760}T_3\,T_2 + \frac{7}{240}\frac{T_2^3}{3!} = \frac{17}{4718592}e^{-3t^1/2},$$

and the genus-3 free energy is

$$\begin{aligned}\mathscr{F}_3^{\rm pt}\Big|_{q_n^i=T_n^+} &= \frac{1}{82944}T_7 + \frac{77}{414720}T_2T_6 + \frac{503}{1451520}T_3T_5 + \frac{17}{11520}(T_2)^2T_5 + \\ &\quad + \frac{607}{2903040}(T_4)^2 + \frac{1121}{241920}T_2T_3T_4 + \frac{53}{6912}(T_2)^3T_4 + \frac{583}{580608}(T_3)^3 + \\ &\quad + \frac{205}{13824}(T_2)^2(T_3)^2 + \frac{193}{6912}(T_2)^4T_3 + \frac{245}{20736}(T_2)^6 \\ &= -\frac{656431}{22265110462464}\,e^{-3t^1}.\end{aligned}$$

Thus, we have

$$\tau(h,\{T_k^+\})^{-2} = 1 - \frac{17}{2359296}\,e^{-3t^1/2}\,h + \frac{41045}{695784701952}\,e^{-3t^1}\,h^2 + \mathcal{O}(h^3),$$

which agrees with our computation of $P(\lambda,\{T_k^+\},\{0\})$ in (3.4.39).

3.5 Conclusion

It would be very interesting if one could actually prove Givental's conjecture, but even our particular example remains elusive and verifying its validity to all orders seems intractable using our method.

Many confusions still remain – for instance, the discrepancy between our analytic and recursive solutions. As mentioned above, Givental's conjecture for $\mathbf{P}^1$ can be rewritten in a form which resembles the Hirota-bilinear relations for the KdV hierarchies (see (3.4.37)). It would thus be interesting to speculate a possible relation between his conjecture and the conjectural Toda hierarchy for $\mathbf{P}^1$.

Chapter 4

Open String Instantons and Relative Stable Morphisms

In this chapter, we describe how certain topological open string amplitudes may be computed via algebraic geometry. We consider an explicit example which has been also considered by Ooguri and Vafa using Chern-Simons theory and M-theory. Utilizing the method of virtual localization, we successfully reproduce the predicted results for multiple covers of a holomorphic disc whose boundary lies in a Lagrangian submanifold of a Calabi-Yau three-fold.

4.1 Introduction

The astonishing link between intersection theories on moduli spaces and topological *closed* string theories has by now taken a well-established form, a progress for which E.Witten first plowed the ground in his seminal papers [W1, W3, W4]. As a consequence, there now exist rigorous mathematical theories of Gromov-Witten invariants, which naturally arise in the aforementioned link. In the symplectic category, Gromov-Witten invariants were first constructed for semi-positive symplectic manifolds by Y.Ruan and G.Tian [RT]. To define the invariants in the algebraic category, J.Li and G.Tian constructed the virtual fundamental class of the moduli space of stable maps by endowing the moduli space with an extra structure called a perfect tangent-obstruction complex [LT2].[1] Furthermore, Gromov-Witten theory was later extended to general symplectic manifolds by Fukaya and Ono [FO], and by J.Li and G.Tian [LT1]. In contrast to such an impressive list of advances just described, no clear link currently exists between topological *open* string theories and intersection theories on moduli spaces. One of the most formidable obstacles that stand in the way to progress is that it is not yet known how to construct well-defined moduli spaces of maps between manifolds with boundaries. The main goal of the work described in this chapter is to contribute to narrowing the existing gap between topological open string theory and Gromov-Witten theory. In so doing we hope that our work will serve as a stepping-stone that will take us a bit closer to answering how relative stable morphisms can be used to study topological open string theory.

In order to demonstrate the proposed link between topological open string theory and Gromov-Witten theory, we will focus on an explicit example throughout this chapter. The same example was also considered by string theorists H.Ooguri and C.Vafa in [OV], where they used results from Chern-Simons theory and M-theory to give two independent derivations of open string instanton amplitudes. A more detailed description of the problem will be presented later in the chapter. We just mention here that, by using our mathematical approach, we have successfully reproduced their answers for multiple covers of a holomorphic disc by Riemann surfaces of arbitrary genera and number of holes. In fact we show that there are no open string instantons with more than one hole, a result which was anticipated in [OV] from their physical arguments.

The invariants we compute are a generalization of *absolute* Gromov-Witten invariants that should be more familiar to string theorists. Our case involves relative stable maps which intersect a specified complex-codimension-two submanifold of the target space in a finite set of points with multiplicity. It will become clear later in the chapter that the theory of relative stable maps is tailor-made for studying topological open string theory. The construction of relative stable maps was first developed

[1] Alternative constructions were also made by Y.Ruan [Rua] and by B.Siebert [Si].

in the symplectic category [LiR, IP1, IP2]. Recently in [Li1, Li2] J.Li has given an algebro-geometric definition of the moduli space of relative stable morphisms and has constructed relative Gromov-Witten invariants in the algebraic category. The foundation of our work will be based on those papers.

The organization of this chapter is as follows: In §4.2 we give a brief description of the multiple cover problem that arose in [OV] and state what we wish to reproduce using relative stable morphisms. The basic idea of localization is reviewed in §4.3. In §4.4 we describe the moduli space of ordinary relative stable morphisms and its localization, compute the equivariant Euler class of the virtual normal complex to the fixed loci, and obtain the contribution from the obstruction bundle that arises in studying multiple covers. In §4.5 we evaluate the relevant invariants which agree with the expected open string instanton amplitudes. We conclude in §4.6 with some comments.

4.2 A Brief Description of the Problem

The notion of duality has been one of the most important common threads that run through modern physics. A duality draws intricate connections between two seemingly unrelated theories and often allows one to learn about one theory from studying the other. A very intriguing duality correspondence has been proposed in [GopV], where the authors provide several supporting arguments for a duality between the large-N expansion of $SU(N)$ Chern-Simons theory on S^3 and a topological *closed* string theory on the total space of the vector bundle $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ over $\mathbf{P}^1$.[2] The equivalence was established in [GopV] at the level of partition functions. We know from Witten's work in [W2], however, that there are Wilson loop observables in Chern-Simons theory which correspond to knot invariants. The question then is, "What do those invariants that arise in Chern-Simons theory correspond to on the topological string theory side?"

The first explicit answer to the above question was given by Ooguri and Vafa in

[2]See [GopV] and references therein for a more precise account of the proposal.

[OV]. In the case of a simple knot on S^3, by following through the proposed duality in close detail, they showed that the corresponding quantities on the topological string theory side are open string instanton amplitudes. More precisely, in the particular example they consider, the open string instantons map to either the upper or the lower hemisphere of the base $\mathbf{P}^1$.[3]

According to [OV], the generating function for topological open string amplitudes is

$$F(t,V) = \sum_{g=0}^{\infty}\sum_{h=0}^{\infty}\sum_{d_1,\ldots,d_h}^{\infty} \lambda^{2g-2+h} F_{g;d_1,\ldots,d_h}(t) \prod_{i=1}^{h} \mathrm{tr} V^{d_i}, \tag{4.2.1}$$

where t is the Kähler modulus of $\mathbf{P}^1$; V is a path-ordered exponential of the gauge connection along the equator and $\mathrm{tr}V^{d_i}$ arises from the i^{th} boundary component which winds around the equator $|d_i|$-times with orientation, which determines the sign of d_i; λ is the string coupling constant; and $F_{g;d_1,\ldots,d_h}$ is the topological open string amplitude on a genus-g Riemann surface with h boundary components. Furthermore, by utilizing the aforementioned duality with Chern-Simons theory, Ooguri and Vafa concluded that

$$F(t,V) = i\sum_{d=1}^{\infty} \frac{\mathrm{tr}V^d + \mathrm{tr}V^{-d}}{2d\,\sin(d\lambda/2)} e^{-dt/2}, \tag{4.2.2}$$

which they confirmed by using an alternative approach in the M-theory limit of type IIA string theory.[4] By comparing (4.2.1) and (4.2.2), one immediately sees that there are no open string instantons with more than one boundary component ending on the equator; that is, $F_{g;d_1,\ldots,d_h} = 0$ for $h > 1$. To extract the topological open string amplitude on a genus-g Riemann surface with one boundary component ($h = 1$), we need to expand (4.2.2) in powers of λ. After some algebraic manipulation, we see

[3] We clarify that the geometric set up in the present case is no longer that described above. There is a unique Lagrangian 3-cycle $\mathcal{C}_K$ in T^*S^3 which intersects S^3 along a given knot K in S^3. Associated to such a 3-cycle $\mathcal{C}_K$ in T^*S^3 there is a Lagrangian 3-cycle $\tilde{\mathcal{C}}_K$ in the local Calabi-Yau three-fold X of the topological string theory side. For the simple knot S considered by Ooguri and Vafa, the latter 3-cycle $\tilde{\mathcal{C}}_S$ intersects the base $\mathbf{P}^1$ of X along its equator. It is the presence of this 3-cycle that allows for the existence of holomorphic maps from Riemann surfaces with boundaries to either the upper or the lower hemisphere. See [OV] for a more detailed discussion.

[4] We refer the reader to the original reference [OV] for further description of this approach.

that

$$F(t,V) \;=\; i\sum_{d=1}^{\infty}\left(\frac{1}{d^2}\lambda^{-1}+\sum_{g=1}^{\infty}d^{2g-2}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}\lambda^{2g-1}\right)e^{-dt/2}\left(\mathrm{tr}V^d+\mathrm{tr}V^{-d}\right),$$

where B_{2g} are the Bernoulli numbers defined by

$$\sum_{n=0}^{\infty}B_n\frac{x^n}{n!} \;=\; \frac{x}{e^x-1}.$$

Hence, topological open string amplitudes, which correspond to multiple covers of either the upper or the lower hemisphere inside the local Calabi-Yau three-fold described above, are

$$-iF_{g;d_1,\ldots,d_h}(0)=\begin{cases} d^{-2}, & g=0, h=1, |d_1|=d>0,\\ d^{2g-2}\left(\dfrac{2^{2g-1}-1}{2^{2g-1}}\dfrac{|B_{2g}|}{(2g)!}\right), & g>0, h=1, |d_1|=d>0,\\ 0, & \text{otherwise.}\end{cases} \tag{4.2.3}$$

In the remainder of this chapter, we will work towards reproducing these results using relative stable morphisms.

4.3 Mathematical Preliminaries

In this section, we describe the method of localization, which is an indispensable tool in Gromov-Witten theory. We will closely follow [OP, CK] in our presentation.

4.3.1 The Localization Theorem of Atiyah and Bott

Let X be a smooth algebraic variety with an algebraic $\mathbb{C}^*$-action. Then, the $\mathbb{C}^*$-fixed locus X^{f} is a union of connected components $\{\mathscr{X}_i\}$, which are also smooth [Iv]. The gist of the Atiyah-Bott localization theorem is that integrals of equivariant cohomology classes over X can be expressed in terms of equivariant integrals over $\{\mathscr{X}_i\}$ [AtB]. Let us now try to make this statement a bit more precise.

Let $B\mathbb{C}^*$ be the classifying space of the algebraic torus $\mathbb{C}^*$, and $M(\mathbb{C}^*)$ its character group. Then, there exists an isomorphism $\omega : M(\mathbb{C}^*) \xrightarrow{\simeq} H^2(B\mathbb{C}^*,\mathbb{Q})$, from which results a ring isomorphism $H^*(B\mathbb{C}^*, \mathbb{Q}) \simeq \mathbb{Q}[t]$, where $t := \omega(\chi)$ is called the *weight* of $\chi \in M(\mathbb{C}^*)$. Note that the equivariant cohomology ring $H^*_{\mathbb{C}^*}(X, \mathbb{Q})$ of X is a $H^2(B\mathbb{C}^*, \mathbb{Q})$-module.

Each $\mathbb{C}^*$-fixed component $\mathscr{X}_i$ is mapped to X by the inclusion $\iota_i : \mathscr{X}_i \hookrightarrow X$. Let $\mathscr{N}_i$ denote the equivariant normal bundle to $\mathscr{X}_i$ in X, and let $e(\mathscr{N}_i) \in H^*_{\mathbb{C}^*}(\mathscr{X}_i, \mathbb{Q})$ denote its equivariant Euler class. Then, the Atiyah-Bott localization theorem [AtB] says that

$$[X] = \sum_i \frac{\iota_{i*}[\mathscr{X}_i]}{e(\mathscr{N}_i)} \ \in \ H^*_{\mathbb{C}^*}(X, \mathbb{Q}) \otimes \mathbb{Q}[t, \frac{1}{t}] \, .$$

A direct consequence of the localization theorem is that, if $\alpha \in H^*_{\mathbb{C}^*}(X, \mathbb{Q})$, then

$$\int_X \alpha = \sum_i \int_{\mathscr{X}_i} \frac{\iota_i^*(\alpha)}{e(\mathscr{N}_i)} \, . \tag{4.3.4}$$

4.3.2 Localization of the Virtual Fundamental Class

We now describe how the localization theorem of Atiyah and Bott extends to virtual classes. Let V be an algebraic variety which may be singular. Furthermore, let V admit a $\mathbb{C}^*$-action and carry a $\mathbb{C}^*$-equivariant perfect obstruction theory. We denote by $\{\mathscr{V}_i\}$ the connected components—which may also be singular—of the scheme theoretic $\mathbb{C}^*$-fixed locus. In [GraP] it was shown that each $\mathbb{C}^*$-fixed component $\mathscr{V}_i$ carries a canonical perfect obstruction theory, which allows one to construct its virtual fundamental class $[\mathscr{V}_i]^{\mathrm{vir}}$.

Analogous to the equivariant normal bundle of $\mathscr{X}_i$ in the previous subsection, in the present case there is a normal complex $\mathscr{N}_i$ associated to each connected component $\mathscr{V}_i$. The normal complex $\mathscr{N}_i$ is defined in terms of the dual complex $E_{i\bullet}$ of a two-term complex $E_i^\bullet$ of vector bundles that arises in the perfect obstruction theory of $\mathscr{V}_i$. More precisely, $\mathscr{N}_i$ is defined by the "moving" part $E^m_{i\bullet}$, which have non-zero $\mathbb{C}^*$-characters. As before, we denote the Euler class of $\mathscr{N}_i$ by $e(\mathscr{N}_i)$.

Let $\iota_i : [\mathscr{V}_i]^{\mathrm{vir}} \hookrightarrow [V]^{\mathrm{vir}}$ be the inclusion. Then, the virtual localization formula of T.Graber and R.Pandharipande is [GraP]

$$[V]^{\mathrm{vir}} = \sum_i \frac{\iota_{i*}[\mathscr{V}_i]^{\mathrm{vir}}}{e(\mathscr{N}_i^{\mathrm{vir}})} \in A^*_{\mathbb{C}^*}(X, \mathbb{Q}) \otimes \mathbb{Q}[t, \frac{1}{t}] \, , \tag{4.3.5}$$

where $A^*_{\mathbb{C}^*}(X, \mathbb{Q})$ is the equivariant Chow ring of X with rational coefficients. In proving the above formula, the authors of [GraP] assume the existence of a $\mathbb{C}^*$-equivariant embedding of V into a nonsingular variety. In the usual context of Gromov-Witten theory, the algebraic variety V of interest is the moduli space $\overline{M}_{g,n}(X, \beta)$ of stable morphisms defined in §3.1.2. If the smooth target manifold X is equipped with a $\mathbb{C}^*$-action, then there is an induced $\mathbb{C}^*$-action on $\overline{M}_{g,n}(X, \beta)$, and the authors of [GraP] have proved the existence of a $\mathbb{C}^*$-equivariant embedding of $\overline{M}_{g,n}(X, \beta)$ into a smooth variety. Hence, as in (4.3.4), the virtual localization formula (4.3.5) allows one to evaluate equivariant integrals in Gromov-Witten theory by summing up contributions from the $\mathbb{C}^*$-fixed components of the virtual fundamental class.

4.4 Solutions via Algebraic Geometry

Before we address the problem of our interest involving open strings, let us recall how multiple cover contributions are computed in close string theory. Consider multiple covers of a fixed rational curve $\mathscr{C} \simeq \mathbf{P}^1 \subset X$, where X is a Calabi-Yau three-fold. The rigidity condition implies that the normal bundle of $\mathscr{C}$ is $N = \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. The contribution of degree d multiple covers of $\mathscr{C}$ to the genus-g Gromov-Witten invariant of X is given by restricting the virtual fundamental class $[\overline{M}_{g,0}(X, d[\mathscr{C}])]^{\mathrm{vir}}$ to $[\overline{M}_{g,0}(\mathbf{P}^1, d)]^{\mathrm{vir}}$. This restriction of the virtual fundamental class is represented by the rank $2g + 2d - 2$ obstruction bundle $R^1\pi_* e_1^* N$, where $\pi : \overline{M}_{g,1}(\mathbf{P}^1, d) \to \overline{M}_{g,0}(\mathbf{P}^1, d)$ and $e_1 : \overline{M}_{g,1}(\mathbf{P}^1, d) \to \mathbf{P}^1$ are canonical forgetful and evaluation maps, respectively, from the universal curve over the moduli stack $\overline{M}_{g,0}(\mathbf{P}^1, d)$. In summary, the contribution from degree d multiple covers is given by the integral

$$\int_{[\overline{M}_{g,0}(\mathbf{P}^1,d)]^{\mathrm{vir}}} c_{\mathrm{top}}(R^1\pi_* e_1^* N)\,.$$

In the case of open string instantons, it is proposed in [LS] that multiple covers of a holomorphic disc embedded in X can be studied using relative stable morphisms. In that paper, the problem is reduced to looking at the space of maps to $\mathbf{P}^1$ with specified contact conditions. More precisely, the topological open string amplitudes we want to reproduce are computed by the expression

$$\int_{[\mathbf{M}^{\rm rel}_{g,\mu}(\mathbf{P}^1)_{\rm o}]^{\rm vir}} c_{\rm top}(V), \tag{4.4.6}$$

where, as we will define presently, $\mathbf{M}^{\rm rel}_{g,\mu}(\mathbf{P}^1)_{\rm o}$ is the moduli space of ordinary relative stable morphisms and V is an appropriate obstruction bundle.

4.4.1 The Moduli Space of Ordinary Relative Stable Morphisms

Let $\mu = (d_1, \ldots, d_h)$ be an ordered h-tuple of positive integers. μ is said to have length $\ell(\mu) = h$ and degree $\deg(\mu) = d_1 + d_2 + \cdots + d_h = d$. Throughout this chapter, we fix two points $q_0 := [0,1] \in \mathbf{P}^1$ and $q_\infty := [1,0] \in \mathbf{P}^1$. A genus-$g$ *ordinary* relative stable morphism of ramification order μ consists of a connected h-pointed nodal curve $(C; x_1, x_2, \ldots, x_h)$ and a stable morphism $f : C \to \mathbf{P}^1$ such that

$$f^{-1}(q_\infty) = d_1 x_1 + \cdots + d_h x_h$$

as a divisor [LS]. The moduli space of such ordinary relative stable morphisms is denoted by $\mathbf{M}^{\rm rel}_{g,\mu}(\mathbf{P}^1)_{\rm o}$, where the subscript "o" is used to indicate "ordinary."

As discussed in [LS], there exists an S^1-action that leaves invariant the boundary condition associated with the Lagrangian submanifold where the disc ends. This action, in turn, induces a natural S^1-action on $\mathbf{M}^{\rm rel}_{g,\mu}(\mathbf{P}^1)_{\rm o}$, and the idea is to use this S^1-action to carry out localization.

We now describe this group action. Given a homogeneous coordinate $[w_1, w_2]$ of $\mathbf{P}^1$, define $w := w_1/w_2$ such that $[w,1] \simeq [w_1, w_2]$ for $w_2 \neq 0$. If we denote by g_t the S^1-action, then

$$g_t \cdot [w,1] = [tw, 1],$$

where $t = e^{2\pi i\theta}$, $\theta \in \mathbb{R}$. When w is viewed as a section, we will use $g_{t*}(w) = g^*_{t^{-1}} w = t^{-1} w$ to define the weight of the S^1-action on w. If we use $\tilde{t}$ to denote the weight of the S^1-action, then the function w has weight $-\tilde{t}$. The two fixed points of the S^1-action on $\mathbf{P}^1$ are $q_0 = [0,1] \in \mathbf{P}^1$ and $q_\infty = [1,0] \in \mathbf{P}^1$.

In the above notation, considering source Riemann surfaces with one hole corresponds to setting $h = 1$, and in the remainder of this section that is what we will do. In this case $\mu = (d)$ and $f^{-1}(q_\infty) = d \cdot x$. For genus $g = 0$, there is only one S^1-fixed point in $\mathbf{M}^{\mathrm{rel}}_{0,\mu}(\mathbf{P}^1)_{\mathrm{o}}$. It is given by the map

$$f : \mathbf{P}^1 \longrightarrow \mathbf{P}^1 , \quad f : [z, 1] \longmapsto [z^d, 1].$$

For genus $g > 0$, the fixed locus of the S^1-action is given by the image of the embedding

$$\overline{M}_{g,1} \longrightarrow \mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)_{\mathrm{o}},$$

where $\overline{M}_{g,1}$ is the smooth moduli stack of genus-g, 1-pointed Deligne-Mumford stable curves. Under the embedding, any $(C_2; p) \in \overline{M}_{g,1}$ is mapped to the ordinary relative stable morphism $f : (C; x) \longrightarrow \mathbf{P}^1$, where the curve C is given by gluing a rational curve $C_1 \equiv \mathbf{P}^1$ with the genus-g curve C_2 along $[0, 1] \in C_1$ and $p \in C_2$. Furthermore, if we use f_i to denote the restriction of the map f to the component C_i, then f_1 sends $[z, 1] \in C_1$ to $[w, 1] = [z^d, 1] \in \mathbf{P}^1$ and f_2 is a constant map such that $f_2(y) = q_0 \in \mathbf{P}^1$, $\forall y \in C_2$. As before, $f^{-1}(q_\infty) = d \cdot x$. Since $w = z^d$ and the weight of the S^1-action on w is $-\tilde{t}$, the weight on the function z is given by $-\tilde{t}/d$. In what follows, we will use p to denote the node in C.

In [LS] the full moduli space $\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)$ of relative stable morphisms is defined and is shown to contain $\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)_{\mathrm{o}}$ as its open substack. A relative stable morphism in $\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)$ consists of a connected h-pointed algebraic curve $(C; x_1, x_2, \ldots, x_h)$ and a morphism $f : C \to \mathbf{P}^1[m]$ such that

$$f^{-1}(q_\infty) = d_1 x_1 \, + \, \cdots \, + \, d_h x_h \, .$$

Here, $\mathbf{P}^1[m]$ is defined to have m ordered irreducible components, each of which being isomorphic to $\mathbf{P}^1$, and q_∞ is contained in the first component of $\mathbf{P}^1[m]$—the reader should refer to [LS] for a more precise definition. If we consider the full moduli space $\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)$, there are S^1-fixed loci other than the ones described above. The S^1-action extends to $\mathbf{P}^1[m]$ with two fixed points on each of its components, and there exists an induced S^1-action on $\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)$. The moduli space of relative stable morphisms $\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)$ thus has two classes of fixed locus. Henceforth, the S^1-fixed loci in $\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)_{\mathrm{o}}$ will be denoted by Θ_I, and we will be only interested in those loci.

4.4.2 The Equivariant Euler Class of $N^{\mathrm{vir}}_{\Theta_I}$

As discussed in [LS], $\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)$ admits a perfect obstruction theory and hence it is possible to define the virtual fundamental class $[\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)]^{\mathrm{vir}}$. Furthermore, the S^1-equivariant version of $[\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)]^{\mathrm{vir}}$ can be constructed and one can apply the localization formula of [GraP]. The connected component Θ_I of the fixed point loci carries an S^1-fixed perfect obstruction theory, which determines the virtual fundamental class $[\Theta_I]^{\mathrm{vir}}$. In this section, we will compute the equivariant Euler class $e(N^{\mathrm{vir}}_{\Theta_I})$ of the virtual normal complex $N^{\mathrm{vir}}_{\Theta_I}$ to the fixed loci Θ_I.

The tangent space of the moduli stack $\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)$ at $(f, C; x)$ is

$$\mathrm{Ext}^1_C([f^*\Omega_{\mathbf{P}^1}(\log q_\infty) \to \Omega_C(x)], \mathcal{O}_C),$$

whereas the obstructions lie in

$$\mathrm{Ext}^2_C([f^*\Omega_{\mathbf{P}^1}(\log q_\infty) \to \Omega_C(x)], \mathcal{O}_C).$$

These two terms fit into the perfect tangent-obstruction complex [LT2]

$$\mathrm{Ext}^\bullet_C([f^*\Omega_{\mathbf{P}^1}(\log q_\infty) \to \Omega_C(x)], \mathcal{O}_C).$$

As in [GraP], we let A_{Θ_I} be the automorphism group of Θ_I and define sheaves $\mathcal{T}^1$ and $\mathcal{T}^2$ on Θ_I/A_{Θ_I} by taking the sheaf cohomology of the perfect obstruction theory on $\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)$ restricted to Θ_I/A_{Θ_I}. Then, we have the following tangent-obstruction exact sequence of sheaves on the substack Θ_I/A_{Θ_I}:

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^0_C(\Omega_C(x), \mathcal{O}_C) \longrightarrow \mathrm{Ext}^0_C(f^*\Omega_{\mathbf{P}^1}(\log q_\infty), \mathcal{O}_C) \longrightarrow \mathcal{T}^1 \longrightarrow \\ \longrightarrow \mathrm{Ext}^1_C(\Omega_C(x), \mathcal{O}_C) \longrightarrow \mathrm{Ext}^1_C(f^*\Omega_{\mathbf{P}^1}(\log q_\infty), \mathcal{O}_C) \longrightarrow \mathcal{T}^2 \longrightarrow 0. \end{aligned} \quad (4.4.7)$$

In the notation of Graber and Pandharipande [GraP], the equivariant Euler class $e(N^{\mathrm{vir}}_{\Theta_I})$ is given by

$$e(N^{\mathrm{vir}}_{\Theta_I}) = \frac{e(B^m_{II})e(B^m_{IV})}{e(B^m_I)e(B^m_V)}, \quad (4.4.8)$$

where B_i^m denotes the moving part of the i^{th} term in the sequence (4.4.7).

We now proceed to study the individual terms that appear in the above definition of $e(N_{\Theta_I}^{\text{vir}})$. First, we let $i_1 : C_1 \to C$ and $i_2 : C_2 \to C$ be inclusion maps. Then

$$\Omega_C = i_{1*}\Omega_{C_1} \oplus i_{2*}\Omega_{C_2} \oplus \mathbb{C}_p \ .$$

$e(B_I^m)$: The first term in the sequence (4.4.7)

We will first carry out our analysis for genus $g > 0$. The genus-zero case will be discussed subsequently. For $g > 0$, we have

$$\begin{aligned}\text{Ext}_C^0(\Omega_C(x), \mathcal{O}_C) &= \text{Hom}_C(i_{1*}\Omega_{C_1}(x), \mathcal{O}_C) \oplus \text{Hom}_C(i_{2*}\Omega_{C_2}(x), \mathcal{O}_C) \oplus \text{Hom}_C(\mathbb{C}_p, \mathcal{O}_C) \\ &= \text{Hom}_{C_1}(\Omega_{C_1}(x), \mathcal{O}_{C_1}(-p)) = H_{C_1}^0(T_{C_1}(-p-x)).\end{aligned}$$

A basis of $H_{C_1}^0(T_{C_1})$ is given by

$$\left\{\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}\right\},$$

and recalling that the weight of the S^1-action on z is $-\tilde{t}/d$, we see that the weights on the above basis elements are $\{\tilde{t}/d, 0, -\tilde{t}/d\}$, respectively. $H_{C_1}^0(T_{C_1}(-p-x))$ is 1-dimensional and its basis is $z\frac{\partial}{\partial z}$, whose weight under the S^1-action is 0. The second arrow in (4.4.7) is injective, and when we compute the equivariant Euler class $e(N_{\Theta_I}^{\text{vir}})$ using (4.4.8), the above-mentioned zero weight contribution will cancel the zero weight term that appears below in $e(B_{II}^m)$.

In the genus $g = 0$ case, $C = C_1$ and there is no node p. Hence, $\text{Ext}_C^0(\Omega_C(x), \mathcal{O}_C) = H_{C_1}^0(T_{C_1}(-x))$. Its basis is $\{\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}\}$, on which S^1 acts with weights $\{\tilde{t}/d, 0\}$. Again, the zero weight term will cancel out in the computation of $e(N_{\Theta_I}^{\text{vir}})$.

$e(B_{II}^m)$: The second term in the sequence (4.4.7)

Note that

$$\text{Ext}_C^0(f^*\Omega_{\mathbf{P}^1}(\log q_\infty), \mathcal{O}_C) = H_{C_1}^0(f_1^*T_{\mathbf{P}^1}(-d\cdot x)).$$

$H_{C_1}^0(f_1^*T_{\mathbf{P}^1}(-dx))$ has dimension $d+1$ and its basis is given by

$$\left\{\frac{\partial}{\partial w}, z\frac{\partial}{\partial w}, z^2\frac{\partial}{\partial w}, \ldots, z^{d-1}\frac{\partial}{\partial w}, z^d\frac{\partial}{\partial w}\right\},$$

whose S^1-action weights are

$$\left\{\tilde{t}\,, \frac{d-1}{d}\,\tilde{t}\,, \frac{d-2}{d}\,\tilde{t}\,, \ldots, \frac{1}{d}\,\tilde{t}\,, 0\right\}.$$

Thus, modulo the zero weight piece, the Euler class $e(B^m_{II})$ is given by

$$e(B^m_{II}) = \prod_{j=0}^{d-1} \frac{d-j}{d}\,\tilde{t} = \frac{d!}{d^d}\tilde{t}^d. \tag{4.4.9}$$

$e(B^m_{IV})$: <u>The fourth term in the sequence (4.4.7)</u>
In this case we have

$$\begin{aligned} \mathrm{Ext}^1_C(\Omega_C(x), \mathcal{O}_C) &= \mathrm{Ext}^0_C(\mathcal{O}_C, \Omega_C(x)\otimes\omega_C)^\vee \\ &= \mathrm{Ext}^0_{C_2}(\mathcal{O}_{C_2}, \omega^{\otimes 2}_{C_2}(p))^\vee \oplus \mathrm{Ext}^0_C(\mathcal{O}_C, \mathbb{C}_p\otimes\omega_C)^\vee \\ &= \mathrm{Ext}^1_{C_2}(\Omega_{C_2}(p), \mathcal{O}_{C_2}) \oplus T^\vee_{C_1,p}\otimes T^\vee_{C_2,p}. \end{aligned}$$

$\mathrm{Ext}^1_{C_2}(\Omega_{C_2}(p), \mathcal{O}_{C_2})$ gives deformations of the contracted component $(C_2; p)$ and lies in the fixed part of $\mathrm{Ext}^1_C(\Omega_C(p), \mathcal{O}_C)$. Therefore, it does not contribute to $e(B^m_{IV})$. The moving part is $T^\vee_{C_1,p}\otimes T^\vee_{C_2,p}$ and it corresponds to the deformations of C which smooth the node at p for $g > 0$ (There is no node in the genus-zero case). The total contribution to $e(B^m_{IV})$ is

$$e(B^m_{IV}) = \frac{1}{d}\tilde{t} - \psi = \frac{\tilde{t} - d\cdot\psi}{d}, \quad g > 0, \tag{4.4.10}$$

where $\tilde{t}/d$ comes from the tangent space of the non-contracted component C_1. ψ is defined to be the first Chern class $c_1(\mathscr{L}_p)$ of the line bundle $\mathscr{L}_p \to \overline{M}_{g,1}$ whose fiber at $(C_2; p)$ is $T^\vee_{C_2,p}$.

In genus zero there is no node and $e(B^m_{IV})$ is simply 1.

$e(B^m_V)$: <u>The fifth term in the sequence (4.4.7)</u>
It is straightforward to compute that

$$\mathrm{Ext}^1_C(f^*\Omega_{\mathbf{P}^1}(\log q_\infty), \mathcal{O}_C) = H^1_{C_2}(\mathcal{O}_{C_2})\otimes T_{q_0}\mathbf{P}^1\,.$$

$H^1_{C_2}(\mathcal{O}_{C_2})$ gives the dual of the Hodge bundle $\mathbb{E}$ on $\overline{M}_{g,1}$. After twisting by $T_{q_0}\mathbf{P}^1$, we can compute the equivariant top Chern class of the bundle $H^1_{C_2}(\mathcal{O}_{C_2}) \otimes T_{q_0}\mathbf{P}^1$ as follows. Let $\mathcal{E}$ be a rank k vector bundle which admits a decomposition into a sum of k line bundles, i.e. $\mathcal{E} = L_1 \oplus L_2 \oplus \cdots \oplus L_k$. If we use η_i to denote $c_1(L_i)$ and let ρ be the first Chern class of a line bundle $\mathcal{L}$, then the splitting principle implies that the equivariant top Chern class of $\mathcal{E} \otimes \mathcal{L}$ is

$$\begin{aligned} c_{\text{top}}(\mathcal{E} \otimes \mathcal{L}) &= (\rho + \eta_1)(\rho + \eta_2) \cdots (\rho + \eta_k) \\ &= \rho^k + s_1(\eta)\rho^{k-1} + s_2(\eta)\rho^{k-2} + \cdots + s_k(\eta)\rho^0 \\ &= \rho^k \left(1 + c_1(\mathcal{E})\rho^{-1} + c_2(\mathcal{E})\rho^{-2} + \cdots + c_k(\mathcal{E})\rho^{-k}\right), \qquad (4.4.11) \end{aligned}$$

where $s_i(\eta) := s_i(\eta_1, \eta_2, \ldots, \eta_k)$ is the i^{th} elementary symmetric function. In the present case, $\mathcal{E} = \mathbb{E}^\vee$ and $\mathcal{L} = T_{q_0}\mathbf{P}^1$. Since the induced S^1-action on $\mathcal{L} = T_{q_0}\mathbf{P}^1$ at q_0 has weight $+\tilde{t}$, the equivariant top Chern class of $H^1_{C_2}(\mathcal{O}_{C_2}) \otimes T_{q_0}\mathbf{P}^1$ is

$$\begin{aligned} e(B^m_V) &= c_{\text{top}}(\mathbb{E}^\vee \otimes T_{q_0}\mathbf{P}^1) \qquad (4.4.12) \\ &= \left(\tilde{t}^g + c_1(\mathbb{E}^\vee)\,\tilde{t}^{g-1} + c_2(\mathbb{E}^\vee)\,\tilde{t}^{g-2} + \cdots + c_g(\mathbb{E}^\vee)\right). \qquad (4.4.13) \end{aligned}$$

We now have all the necessary ingredients to obtain $e(N^{\text{vir}}_{\Theta_I})$ in (4.4.8). For later convenience, we summarize our final results in the following form:

$$\frac{1}{e(N^{\text{vir}}_{\Theta_I})} = \begin{cases} \dfrac{1}{d} \cdot \dfrac{d^d}{d!}\,\tilde{t}^{1-d}, & g = 0, \\[2ex] \dfrac{d^d}{d!}\left(\dfrac{\tilde{t}^{-d}\,d}{\tilde{t} - d \cdot \psi}\right)\left(\tilde{t}^g + c_1(\mathbb{E}^\vee)\,\tilde{t}^{g-1} + c_2(\mathbb{E}^\vee)\,\tilde{t}^{g-2} + \cdots + c_g(\mathbb{E}^\vee)\right), & g > 0. \end{cases} \qquad (4.4.14)$$

4.4.3 Contribution from the Obstruction Bundle

Now that we have analyzed the moduli space $\mathbf{M}^{\text{rel}}_{g,\mu}(\mathbf{P}^1)_0$, we need to investigate the obstruction bundle that arises in (4.4.6). In [LS], the obstruction bundle V is found to be a vector bundle over $\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)_{\mathrm{o}}$ whose fibers over $(f, C; x_1, \ldots, x_h)$ are

$$H^1(C, \mathcal{O}_C(-\sum_{i=1}^{h} d_i x_i) \oplus \mathcal{O}_C(-\sum_{i=1}^{h} x_i)).$$

We can evaluate the integral in (4.4.6) using the localization theorem [Kont2, GraP], which implies that

$$\left[\int_{[\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)_{\mathrm{o}}]^{\mathrm{vir}}} c_{\mathrm{top}}(V)\right] = \frac{1}{|\mathrm{A}_{\Theta_I}|}\int_{[\Theta_I]^{\mathrm{vir}}} \frac{\iota^*(c_{\mathrm{top}}(V))}{e(N^{\mathrm{vir}}_{\Theta_I})}, \tag{4.4.15}$$

where ι is the inclusion map $\iota : \Theta_I \to \mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)_{\mathrm{o}}$. Incidentally, we note that by the Riemann-Roch theorem,

$$\begin{aligned}
\dim_{\mathbb{C}} \mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)_{\mathrm{o}} &= 2d + (1-g)(\dim_{\mathbb{C}} \mathbf{P}^1 - 3) - (\deg(\mu) - \ell(\mu)) \\
&= 2g - 2 + h + d \\
&= \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C(-\sum_{i=1}^{h} d_i x_i) \oplus \mathcal{O}_C(-\sum_{i=1}^{h} x_i)).
\end{aligned}$$

In this subsection, we focus on source Riemann surfaces with one hole ($h = 1$), in which case we need to find the weights of the S^1-action on $H^1(C, \mathcal{O}_C(-dx)) \oplus H^1(C, \mathcal{O}_C(-x))$. As described in [LS], the sheaf $\mathcal{O}_C$ in the first cohomology group has weight 0, while the sheaf $\mathcal{O}_C$ in the second cohomology group has weight $-\tilde{t}$. We will work out the genus-zero and higher genus cases separately.

<u>Genus $g = 0$</u> ($C \simeq \mathbf{P}^1$)

The dimension of $H^1(C, \mathcal{O}_C(-x))$ is zero and we only need to consider $H^1(C, \mathcal{O}_C(-dx))$. To obtain the contribution from the latter, we use the exact sequence

$$0 \longrightarrow \mathcal{O}_C(-dx) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{dx} \longrightarrow 0$$

and the induced cohomology exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_{dx}) \longrightarrow H^1(\mathcal{O}_C(-dx)) \longrightarrow 0.$$

A basis of $H^0(\mathcal{O}_C)$ is just $\{1\}$ and that of $H^0(\mathcal{O}_{dx})$ is $\{1, z^{-1}, z^{-2}, \ldots, z^{-(d-1)}\}$. This information lets us construct the following explicit basis of $H^1(\mathcal{O}_C(-dx))$:

$$\left\{\frac{1}{z}, \frac{1}{z^2}, \ldots, \frac{1}{z^{d-1}}\right\}.$$

S^1 acts on the above basis with weights

$$\left\{\frac{1}{d}\,\tilde{t}, \frac{2}{d}\,\tilde{t}, \ldots, \frac{d-1}{d}\,\tilde{t}\right\}. \tag{4.4.16}$$

Thus the equivariant top Chern class of the obstruction bundle V is

$$c_{\text{top}}(V) = \prod_{j=1}^{d-1} \frac{j}{d}\,\tilde{t} = \frac{(d-1)!}{d^{d-1}}\tilde{t}^{d-1}. \tag{4.4.17}$$

Genus $g \geq 1$

For genus $g > 0$, C is a union of the two irreducible components $C_1 = \mathbf{P}^1$ and $C_2 = \Sigma_g$ which intersect at a node, denoted by p. Then there is the normalization exact sequence

$$0 \longrightarrow \mathcal{O}_C(-dx) \longrightarrow \mathcal{O}_{C_1}(-dx) \oplus \mathcal{O}_{C_2} \longrightarrow \mathcal{O}_C(-dx)\big|_p \longrightarrow 0, \tag{4.4.18}$$

which gives the following long exact sequence of cohomology:

$$0 \longrightarrow H^0(C_2, \mathcal{O}_{C_2}) \xrightarrow{\simeq} H^0(C, \mathcal{O}_C(-dx)\big|_p) \longrightarrow H^1(C, \mathcal{O}_C(-dx)) \longrightarrow$$
$$\longrightarrow H^1(C_1, \mathcal{O}_{C_1}(-dx)) \oplus H^1(C_2, \mathcal{O}_{C_2}) \longrightarrow 0.$$

Therefore, $H^1(C, \mathcal{O}_C(-dx))$ is given by

$$H^1(C, \mathcal{O}_C(-dx)) = H^1(C_1, \mathcal{O}_{C_1}(-dx)) \oplus H^1(C_2, \mathcal{O}_{C_2}). \tag{4.4.19}$$

We have already computed in (4.4.16) the contribution of $H^1(C_1, \mathcal{O}_{C_1}(-dx))$ to the equivariant top Chern class $c_{\text{top}}(V)$. Since the linearization of $\mathcal{O}_{C_2}$ in (4.4.19) has weight zero, the contribution of $H^1(C_2, \mathcal{O}_{C_2})$ to $c_{\text{top}}(V)$ can be obtained from (4.4.11) by letting $\rho = 0$ and $\mathcal{E} = \mathbb{E}^\vee$. This gives $c_g(\mathbb{E}^\vee) = (-1)^g c_g(\mathbb{E})$ as the contribution of

$H^1(C_2, \mathcal{O}_{C_2})$. Combining this result with (4.4.16), we see that the total contribution of $H^1(C, \mathcal{O}_C(-dx))$ to $c_{\rm top}(V)$ is

$$(-1)^g \frac{(d-1)!}{d^{d-1}} c_g(\mathbb{E})\, \tilde{t}^{d-1}.$$

We can use a similar line of reasoning to determine the weights on $H^1(C, \mathcal{O}_C(-x))$. Examining the long exact sequence of cohomology that follows from the exact normalization sequence

$$0 \longrightarrow \mathcal{O}_C(-x) \longrightarrow \mathcal{O}_{C_1}(-x) \oplus \mathcal{O}_{C_2} \longrightarrow \mathcal{O}_C(-x)|_p \longrightarrow 0,$$

we obtain

$$H^1(C, \mathcal{O}_C(-x)) = H^1(C_1, \mathcal{O}_{C_1}(-x)) \oplus\ H^1(C_2, \mathcal{O}_{C_2}).$$

As discussed in the genus-zero case, $H^1(C_1, \mathcal{O}_{C_1}(-x))$ is of zero dimension and does not contribute. Moreover, as we have mentioned in the beginning of this subsection, since the S^1-action lifts to the present $\mathcal{O}_C$ with weight $-\tilde{t}$, the contribution of $H^1(C_2, \mathcal{O}_{C_2})$ to the equivariant top Chern class $c_{\rm top}(V)$ is

$$(-1)^g \left(\tilde{t}^g - c_1(\mathbb{E}^\vee)\, \tilde{t}^{g-1} + c_2(\mathbb{E}^\vee)\, \tilde{t}^{g-2} + \cdots + (-1)^g c_g(\mathbb{E}^\vee)\right) .$$

Thus the final expression for the equivariant top Chern of the obstruction bundle is

$$\begin{aligned} c_{\rm top}(V) &= \frac{(d-1)!}{d^{d-1}}\, c_g(\mathbb{E})\, \tilde{t}^{d-1} \left(\tilde{t}^g - c_1(\mathbb{E}^\vee)\, \tilde{t}^{g-1} + c_2(\mathbb{E}^\vee)\, \tilde{t}^{g-2} + \cdots + (-1)^g c_g(\mathbb{E}^\vee)\right) \\ &= \frac{(d-1)!}{d^{d-1}}\, c_g(\mathbb{E})\, \tilde{t}^{d-1} \left(\tilde{t}^g + c_1(\mathbb{E})\, \tilde{t}^{g-1} + c_2(\mathbb{E})\, \tilde{t}^{g-2} + \cdots + c_g(\mathbb{E})\right) . \end{aligned} \tag{4.4.20}$$

4.5 Evaluation of Invariants

We are now ready to evaluate the integral (4.4.15) in the case of $h = 1$. For $h > 1$, an inductive argument can be used to show that all open string instanton amplitudes vanish in the particular problem we are considering.

4.5.1 Invariants for $h = 1$

Genus $g = 0$

After taking into account the automorphism group A_{Θ_I}, which has order d, the genus-zero answer is obtained by multiplying the expressions in (4.4.14) and (4.4.17). This gives the genus-zero invariant

$$\frac{1}{d}\cdot\frac{1}{d}\frac{d^d}{d!}\,\tilde{t}^{1-d}\cdot\frac{(d-1)!}{d^{d-1}}\,\tilde{t}^{d-1} = \frac{1}{d^2}, \tag{4.5.21}$$

which agrees with the expected answer (4.2.3).

Genus $g \geq 1$

We need to use the results in (4.4.14) and (4.4.20) to compute higher genus invariants. It follows from Mumford's formula

$$(1 + c_1(\mathbb{E}^\vee) + c_2(\mathbb{E}^\vee) + \cdots + c_g(\mathbb{E}^\vee))\cdot(1 + c_1(\mathbb{E}) + c_2(\mathbb{E}) + \cdots + c_g(\mathbb{E})) = 1$$

that

$$(\tilde{t}^g + c_1(\mathbb{E}^\vee)\,\tilde{t}^{g-1} + c_2(\mathbb{E}^\vee)\,\tilde{t}^{g-2} + \cdots + c_g(\mathbb{E}^\vee))\cdot(\tilde{t}^g + c_1(\mathbb{E})\,\tilde{t}^{g-1} + c_2(\mathbb{E})\,\tilde{t}^{g-2} + \cdots + c_g(\mathbb{E})) = \tilde{t}^{2g}.$$

Hence we have

$$\begin{aligned}\left[\int_{[\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)_{\mathrm{o}}]^{\mathrm{vir}}} c_{\mathrm{top}}(V)\right] &= \left[\frac{1}{|\mathrm{A}_{\Theta_I}|}\int_{\overline{M}_{g,1}} \frac{d\cdot\tilde{t}^{2g-1}}{\tilde{t}-d\cdot\psi}c_g(\mathbb{E})\right]\\ &= d^{2g-2}\int_{\overline{M}_{g,1}}\psi^{2g-2}c_g(\mathbb{E}),\end{aligned} \tag{4.5.22}$$

where in the last equality we have used the fact that the moduli space $\overline{M}_{g,1}$ of Deligne-Mumford stable curves has dimension $\dim_{\mathbb{C}}\overline{M}_{g,1} = 3g - 2$. The above Hodge integral can easily be evaluated by using C.Faber and R.Pandharipande's generating function for Hodge integrals over the moduli space $\overline{M}_{g,1}$ [FabP1]. Taking the result from [FabP1], we conclude that

$$\left[\int_{[\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)_o]^{\mathrm{vir}}} c_{\mathrm{top}}(V)\right] = d^{2g-2}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}. \tag{4.5.23}$$

As promised, (4.5.23) is precisely equal to the expected result in (4.2.3).

4.5.2 Invariants for $h > 1$

As mentioned before, all invariants for $h > 1$ vanish. The main idea that underlies our argument is that at least one of the weights of the S^1-action on the obstruction bundle is zero. We will present our argument for genus-zero and higher genus cases separately.

Genus $g = 0$

We will first consider the genus-zero case. Assume that $h = 2$, in which case $\mu = (d_1, d_2)$, where $d_1 + d_2 = d$. In genus zero $C = C_1 \sqcup_p C_2$, where C_1 and C_2 both are rational curves and p is a node that gets mapped to q_0. For $i = 1$ or 2, $d_i > 0$ is the degree of the map f_i that maps C_i to $\mathbf{P}^1$. If we denote the pre-images of q_∞ by $x_1 \in C_1$ and $x_2 \in C_2$, then we have the normalization exact sequence

$$0 \longrightarrow \mathcal{O}_C(-d_1x_1 - d_2x_2) \longrightarrow \mathcal{O}_{C_1}(-d_1x_1) \oplus \mathcal{O}_{C_2}(-d_2x_2) \longrightarrow \mathcal{O}_C(-d_1x_1 - d_2x_2)|_p \longrightarrow 0,$$

which gives the long exact sequence of cohomology

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^0(C, \mathcal{O}_C(-d_1x_1 - d_2x_2)|_p) \longrightarrow \mathrm{H}^1(C, \mathcal{O}_C(-d_1x_1 - d_2x_2)) \longrightarrow \\ &\longrightarrow \mathrm{H}^1(C_1, \mathcal{O}_{C_1}(-d_1x_1)) \oplus \mathrm{H}^1(C_2, \mathcal{O}_{C_2}(-d_2x_2)) \longrightarrow 0. \end{aligned}$$

From this we immediately see that one of the weights of the S^1-action on $\mathrm{H}^1(C, \mathcal{O}_C(-d_1x_1 - d_2x_2))$ is zero, since the weight on $\mathrm{H}^0(C, \mathcal{O}_C(-d_1x_1 - d_2x_2)|_p)$ is zero. This means that the contribution of $\mathrm{H}^1(C, \mathcal{O}_C(-d_1x_1 - d_2x_2))$ to the equivariant top Chern class of the obstruction bundle vanishes, thus rendering the invariant to vanish as well.

For $h = 3$, C contains a contracted genus-zero component $\tilde{C}_0$ which is connected to 3 rational curves, say C_1, C_2, C_3, at 3 nodes, say p_1, p_2, p_3. Note that since $\tilde{C}_0$ contains 3 special points, it is stable and can be contracted to q_0. Each C_i maps to $\mathbf{P}^1$ with degree $d_i > 0$ and contains a special point x_i that gets mapped to q_∞. As usual there is the exact normalization sequence

$$0 \longrightarrow \mathcal{O}_C(-\sum_i^3 d_i x_i) \longrightarrow \bigoplus_{i=1}^3 \mathcal{O}_{C_i}(-d_i x_i) \oplus \mathcal{O}_{\tilde{C}_0} \longrightarrow \bigoplus_{i=1}^3 \mathcal{O}_C(-\sum_i^3 d_i x_i)|_{p_i} \longrightarrow 0,$$

and the associated cohomology long exact sequence

$$0 \longrightarrow \mathrm{H}^0(\tilde{C}_0, \mathcal{O}_{\tilde{C}_0}) \stackrel{\psi}{\longrightarrow} \bigoplus_i^3 \mathrm{H}^0(C, \mathcal{O}_C(-\sum_i^3 d_i x_i)|_{p_i}) \longrightarrow \mathrm{H}^1(C, \mathcal{O}_C(-\sum_i^3 d_i x_i) \longrightarrow$$
$$\longrightarrow \bigoplus_{i=1}^3 \mathrm{H}^1(C_i, \mathcal{O}_{C_i}(-d_i x_i)) \oplus \mathrm{H}^1(\tilde{C}_0, \mathcal{O}_{\tilde{C}_0}) \longrightarrow 0.$$

Here, ψ is not surjective and it follows that not all of the zero weights on $\bigoplus_i^3 \mathrm{H}^0(C, \mathcal{O}_C(-\sum_i^3 d_i x_i)|_{p_i})$ get cancelled in

$$\begin{aligned} \mathrm{H}^1(C, \mathcal{O}_C(-\sum_i^3 d_i x_i)) &= \bigoplus_{i=1}^3 \mathrm{H}^1(C_i, \mathcal{O}_{C_i}(-d_i x_i)) \oplus \mathrm{H}^1(\tilde{C}_0, \mathcal{O}_{\tilde{C}_0}) - \mathrm{H}^0(\tilde{C}_0, \mathcal{O}_{\tilde{C}_0}) + \\ &\quad + \bigoplus_i^3 \mathrm{H}^0(C, \mathcal{O}_C(-\sum_i^3 d_i x_i)|_{p_i}). \end{aligned}$$

Therefore, the equivariant top Chern class of the obstruction bundle in the localization theorem vanishes. We can perform induction on h and conclude that genus-zero invariants vanish for all $h > 1$. We will now sketch how that works. For $h \leq n$, assume that the S^1-action on $\mathrm{H}^1(C, \mathcal{O}_C(-\sum_i^h d_i x_i))$ has at least one zero weight and that therefore the invariants vanish. At $h = n + 1$, an S^1-fixed stable map can be constructed from that at $h = n$ by attaching a rational curve C_{n+1} to the contracted component, such that $\deg(f|_{C_{n+1}}) = d_{n+1} > 0$. C_{n+1} contains the point x_{n+1} that gets mapped to q_∞ and is joined to the contracted component at a new node. Such an operation increases the number of nodes by 1, and analyzing the exact normalization sequence and its associated cohomology long exact sequence shows that the number of zero weights on $\mathrm{H}^1(C, \mathcal{O}_C(-\sum_i^h d_i x_i))$ has increased by one. Therefore, the total number of zero weights on $\mathrm{H}^1(C, \mathcal{O}_C(-\sum_i^h d_i x_i))$ is again non-zero. This shows that the equivariant top Chern class of the obstruction bundle vanishes at $h = n + 1$.

Genus $g \geq 1$

Now assume that $g \geq 1$ and $h = 2$. In addition to the two rational curves C_1 and C_2, we introduce a stable genus-g curve $\tilde{C}_g$, which gets contracted to q_0. There are two nodes p_1 and p_2 where C_1 and C_2, respectively, intersect $\tilde{C}_g$. In our usual notation, the normalization exact sequence in the present case is

$$0 \longrightarrow \mathcal{O}_C(-d_1x_1 - d_2x_2) \longrightarrow \bigoplus_{i=1}^{2} \mathcal{O}_{C_i}(-d_i x_i) \oplus \mathcal{O}_{\tilde{C}_g} \longrightarrow \bigoplus_{i=1}^{2} \mathcal{O}_C(-d_1x_1 - d_2x_2)\big|_{p_i} \longrightarrow 0.$$

This implies the following long exact sequence of cohomology:

$$\begin{aligned} 0 \to \mathrm{H}^0(\tilde{C}_g, \mathcal{O}_{\tilde{C}_g}) &\xrightarrow{\varphi} \mathrm{H}^0(C, \mathcal{O}_C(-d_1x_1 - d_2x_2)\big|_{p_1}) \oplus \mathrm{H}^0(C, \mathcal{O}_C(-d_2x_2 - d_2x_2)\big|_{p_2}) \to \\ &\longrightarrow \mathrm{H}^1(C, \mathcal{O}_C(-d_1x_1 - d_2x_2)) \longrightarrow \bigoplus_{i=1}^{2} \mathrm{H}^1(C_i, \mathcal{O}_{C_i}(-d_i x_i)) \oplus \mathrm{H}^1(\tilde{C}_g, \mathcal{O}_{\tilde{C}_g}) \longrightarrow 0. \end{aligned}$$

Unlike in the $h = 1$ case, φ is not surjective and we need to compute

$$\begin{aligned} \mathrm{H}^1(C, \mathcal{O}_C(-d_1x_1 - d_2x_2)) \; = \; & \bigoplus_{i=1}^{2} \mathrm{H}^1(C_i, \mathcal{O}_{C_i}(-d_i x_i)) \oplus \mathrm{H}^1(\tilde{C}_g, \mathcal{O}_{\tilde{C}_g}) - \mathrm{H}^0(\tilde{C}_g, \mathcal{O}_{\tilde{C}_g}) + \\ & + \mathrm{H}^0(C, \mathcal{O}_C(-d_1x_1 - d_2x_2)\big|_{p_1}) \oplus \mathrm{H}^0(C, \mathcal{O}_C(-d_2x_2 - d_2x_2)\big|_{p_2}). \end{aligned}$$

The zero weight term from $\mathrm{H}^0(\tilde{C}_g, \mathcal{O}_{\tilde{C}_g})$ will cancel only one of the two zero weight terms from the second line, thus leaving a zero weight term in $\mathrm{H}^1(C, \mathcal{O}_C(-d_1x_1 - d_2x_2))$. Hence the equivariant top Chern class of the obstruction bundle again vanishes, and so does the invariant.

The vanishing of the invariants for $g \geq 1$ again follows from induction on h. As in the genus-zero case, a S^1-fixed stable map at $h = n+1$ can be constructed from that at $h = n$ by attaching a non-contracted rational curve, say C_{n+1}, to the contracted component $\tilde{C}_g$ at a new node. This addition of a node increases the number of zero weights of the S^1-action on $\mathrm{H}^1(C, \mathcal{O}_C(-\sum_i^h d_i x_i))$, and therefore the equivariant top

Chern class of the obstruction bundle vanishes at $h = n+1$ as it does at $h = n$. Hence, all higher genus invariants vanish for $h > 1$.

To recapitulate, we have just established that

$$\left[\int_{[\mathbf{M}^{\mathrm{rel}}_{g,\mu}(\mathbf{P}^1)_{\mathrm{o}}]^{\mathrm{vir}}} c_{\mathrm{top}}(V)\right] = 0 \ , \quad \forall g \geq 0,\ d \geq h > 1,$$

in perfect agreement with what was expected from §4.2.

4.6 Conclusion

In this chapter we have made an explicit connection between topological open string theory and relative stable morphisms. In the particular example we consider, we have successfully reproduced open string instanton multiple cover answers as invariants of relative stable maps. So far several interesting proposals for studying open string instanton effects have been made [OV, KKLM, AgV], but direct computational methods involving integrals over moduli spaces of stable morphisms have been hitherto lacking. This is in marked contrast to the closed string case, where there exist well-developed techniques in the context of Gromov-Witten theory [GraP, FabP1].

Open string instantons play an important role in string theory. For example, in the Strominger-Yau-Zaslow conjecture of mirror symmetry, open string instanton effects are crucial for modifying the geometry of D-brane moduli space [StYZ]. Also, genus-zero topological open string amplitudes are important for computing superpotentials in $\mathcal{N} = 1$ supersymmetric theories in 4-dimension—see [KKLM, OV] and references therein. It is clear that many illuminating implications can stem from understanding better how one can directly compute open string instanton amplitudes. We hope we have made it clear in our work that relative stable morphisms could be the right framework for studying open string instantons in general, and that the proposed link between topological open string theory and relative stable maps well deserve further investigations.

Applying the theory of relative stable morphisms to topological open string theory is in the incipient stage. In a sense we have studied here what could be considered the simplest example. As mentioned in §4.2, the quantities we have reproduced correspond to the invariants of a simple knot in S^3. The authors of [LaMV] have extended the results in [OV] to more non-trivial knots and links, and have described how to construct Lagrangian submanifolds, for torus links at least, on the topological string theory side of the duality. It will be interesting to apply our method to those cases as well. Also, M.Aganagic and C.Vafa have recently announced some interesting results on counting holomorphic discs in Calabi-Yau 3-folds [AgV], and we would like to understand their results by means of relative stable morphisms.

Appendix A

Rudiments of the Symmetric Group S_n

We here review some useful facts regarding the representation theory of the symmetric group S_n. We refer the reader to [FuH] for an in-depth introduction to the subject.

The symmetric group S_n is the group of all permutations on a set $\mathscr{S}$ of n letters. The set $\mathscr{S}$ may be partitioned into disjoint subsets such that a permutation $\sigma \in S_n$ is decomposed into disjoint cycles, each acting on a particular subset. Furthermore, two permutations are conjugate if and only if they can be decomposed into the same number of cycles of each length.

Denote by $[\sigma] = (1^{\alpha_1} \dots n^{\alpha_n})$ the conjugacy class of a permutation that can be decomposed into α_k disjoint k-cycles, where $1 \leq k \leq n$ and $0 \leq \alpha_k \leq n$. Then, since we must have

$$\sum_{k=1}^{n} k \cdot \alpha_k = n,$$

we see that a partition of the integer n determines a particular conjugacy class in S_n.

A.1 Irreducible Characters

The character $\chi(g) = \mathrm{tr}\ \rho(g)$ of an irreducible representation ρ is called *simple* or *irreducible*. We now describe how one can compute the irreducible characters $\chi([\sigma])$

for $[\sigma_1] = (1^n)$ and $[\sigma_2] = (1^{n-2}2)$. Proofs and details can be found in group theory textbooks which discuss the Frobenius' formula for the characters of S_n. The number of inequivalent irreducible representations of any finite group is equal to the number of conjugacy classes in the group. For the symmetric group S_n, there is a one-to-one correspondence between its conjugacy classes and inequivalent partitions of n, so we may label the irreducible representations of S_n by the latter.

Let us consider an irreducible representation of S_n labeled by the ordered partition $\gamma = (n_1, \ldots, n_m) \vdash n$, where $n_1 \geq n_2 \geq \ldots \geq n_m > 0$. Let $p_i = n_i + m - i$ and define the Van der Monde determinant

$$D(p_1, \ldots, p_m) \equiv \begin{vmatrix} p_1^{m-1} & p_1^{m-2} & \cdots & p_1 & 1 \\ p_2^{m-1} & p_2^{m-2} & \cdots & p_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_m^{m-1} & p_m^{m-2} & \cdots & p_m & 1 \end{vmatrix}. \tag{A.1.1}$$

Then, the irreducible characters evaluated at the conjugacy classes (1^n) and $(1^{n-2}2)$ can be written respectively as

$$\chi_\gamma\left(1^n\right) = \frac{n!}{p_1!\ p_2!\ \cdots p_m!} D(p_1, \ldots, p_m) \tag{A.1.2}$$

and

$$\chi_\gamma\left(1^{n-2}2\right) = (n-2)! \sum_{i \in \mathcal{I}} \frac{D(p_1, \ldots, p_{i-1}, p_i - 2, p_{i+1}, \ldots, p_m)}{p_1! \cdots p_{i-1}!\ (p_i - 2)!\ p_{i+1}!\ \cdots p_m!}, \tag{A.1.3}$$

where the index set $\mathcal{I}$ is defined as $\{i \in \{1, \ldots, m\} || (p_i - 2) \geq 0\}$. Furthermore, these irreducible characters satisfy the simple relation

$$\binom{n}{2} \frac{\chi_\gamma\left(1^{n-2}2\right)}{\chi_\gamma\left(1^n\right)} = \frac{1}{2} \sum_{k=1}^{m} n_k(n_k + 1) - \sum_{k=1}^{m} k \cdot n_k\ , \tag{A.1.4}$$

which we utilize in Chapter 2 of this thesis. Incidentally, note that the character $\chi_\gamma(1^n)$ is equal to the dimension f^γ of the representation associated to γ.

A.2 The Murnaghan-Nakayama Rule

To each ordered partition $\alpha = (n_1, \ldots, n_m) \vdash n$, where $n_1 \geq n_2 \geq \ldots \geq n_m > 0$, there is an associated Young diagram of m rows, i^{th} of which has n_i boxes. A *skew* *hook* is a connected region of boxes on the boundary of a Young diagram such that deleting it yields a new Young diagram of smaller size. The *length* of a skew hook is defined to be the total number of its constituent boxes, whereas its *leg-length* is defined to be the number of rows minus 1.

Let $a \in S_n$ be a permutation which can be decomposed as a product of two disjoint elements, namely an n'-cycle and a permutation $b \in S_{n-n'}$. If α is a partition of n and β is a partition of $(n - n')$ that is obtained by deleting a skew hook of length n' and of leg-length $r(\beta)$ from the Young diagram associated to α, then the Murnaghan-Nakayama rule states that

$$\chi_\alpha(a) = \sum_\beta (-1)^{r(\beta)} \chi_\beta(b).$$

Appendix B

Computation of Hurwitz Numbers

B.1 Higher Degree Computation of Simple Hurwitz Numbers

• **Degree Five:**

For the degree 5 computation, we need

Lemma B.1 *Let* $t_k^p = 3\,\delta_{k,1}\,\sum_{i=1}^{j}\delta_{i,p} + (\delta_{k,1}+\delta_{k,2})\,\sum_{i=j+1}^{r}$. *Then,*

$$B_{3,h,(t_k^p)} = \begin{cases} 2^{2h}\,3^{2h-1+r-j} & \textit{for } j < r \textit{ even} \;, \\ 0 & \textit{for } j \textit{ odd} \;, \\ 2\cdot 3^{2h-1}\,(2^{2h-1}+1) & \textit{for } j = r \;, \end{cases}$$

from which follows

Claim B.2 *The degree 5 simple Hurwitz numbers are given by*

$$\begin{aligned} N_{5,h,r} \;=\;& 2^{2h-1}\,(2^{2h+r-2} - 2^{4h+r-4} - 1) - \;3^{2h-2}\,2^{2h-1}\,(1 + 2^{2h+r-2} + 2^{2h+2r-2}) + \\ &+\;3^{2h+r-2}\,2^{2h-1}\,(1 - 2^{4h+r-4}) + 2^{6h+r-5}\,3^{2h-2} + \\ &+\;(1 + 2^{4h+r-4})\,2^{2h-1}\,3^{2h-2}\,5^{2h+r-2}. \end{aligned}$$

• **Degree Six:**

Similarly, by using

Lemma B.3 *Let* $t_k^p = 4\,\delta_{k,1}\,\sum_{i=1}^{j}\delta_{i,p} + (\delta_{k,1}+\delta_{k,2})\,\sum_{i=j+1}^{r}$. *Then,*

$$B_{4,h,(t_k^p)} = \begin{cases} 3\cdot 2^{r-j+6h-2}\,(3^{2h-2+r-j}+1) & \text{for } j<r \text{ even },\\ 0 & \text{for } j \text{ odd },\\ 3\cdot 2^{4h-1}\,(2^{2h-1}3^{2h-2}+2^{2h-1}+3^{2h-2}) & \text{for } j=r \ , \end{cases}$$

and we obtain

Claim B.4 *The degree 6 simple Hurwitz numbers are given by*

$$\begin{aligned} N_{6,h,r} \;=\; & \frac{1}{720}\Bigg[360\cdot 2^{2h} - 135\cdot 2^{4h+r} - 40\cdot 2^{2h}\cdot 3^{2h+r} - \frac{5\cdot 2^{2h}\cdot 3^{4h+r}\left(8+2^{2h+r}\right)}{9} \\ & +20\cdot 2^{2h}\cdot 3^{2h}\left(4+2^{2(h+r)}+2^{2h+r}\right) + \frac{15\cdot 2^{6h}\,(3+3^{r})}{2} \\ & +\frac{5\cdot 2^{6h+r}\left(9+3^{2h+r}\right)}{2} - \frac{2^{2h}\cdot 3^{2h}\left(25\cdot 2^{4h+r}+16\cdot 5^{2h+r}+2^{4h+r}\cdot 5^{2h+r}\right)}{10} \\ & +\frac{2^{6h}\left(100\cdot 3^{4h+r}+25\cdot 2^{2h}3^{4h+r}+25\cdot 2^{2h}3^{4h+2r}+81\cdot 2^{2h}5^{2h+r}+2^{2h}3^{4h+r}5^{2h+r}\right)}{360} \\ & -\frac{5\cdot 2^{6h}\left(9\cdot 2^{2h}+4\cdot 3^{2h}+9\cdot 2^{2h}3^{r}+2^{2h}3^{2h}5^{r}+2^{2h}3^{2h}7^{r}\right)}{8}\Bigg] \\ & +2^{6h-5}3^{r-1}\left[3^{2h-1}(2^{2h-1}+1)-3(2^{2h-1})+1\right]. \end{aligned}$$

• **Degree Seven:**

Claim B.5 *The degree 7 simple Hurwitz numbers are given by*

$$\begin{aligned} N_{7,h,r} \;=\; & \frac{-2^{2h}}{2} - \frac{3\,2^{6h}}{32} + \frac{2^{8h}}{64} + \frac{2^{4h+r}}{4} - \frac{2^{6h+r}}{32} - \frac{2^{2h}\,3^{2h}}{6} + \frac{2^{6h}\,3^{2h}}{96} \\ & -\frac{2^{8h}\,3^{2h}}{576} - \frac{2^{4h+r}\,3^{2h}}{36} - \frac{2^{4h+2r}\,3^{2h}}{24} + \frac{2^{6h+r}\,\frac{5}{2}^{r}\,3^{2h}}{288} - \frac{2^{8h+r}\,\frac{5}{2}^{r}\,3^{2h}}{1152} \\ & -\frac{2^{6h}\,3^{r}}{32} + \frac{2^{8h}\,3^{r}}{64} + \frac{2^{2h}\,3^{2h+r}}{18} + \frac{2^{6h}\,3^{2h+r}}{144} - \frac{2^{8h}\,3^{2h+r}}{1152} - \frac{2^{6h+r}\,3^{2h+r}}{288} \\ & +\frac{2^{2h}\,3^{4h+r}}{81} - \frac{2^{6h}\,3^{4h+r}}{1296} - \frac{2^{8h}\,3^{4h+r}}{10368} + \frac{2^{4h+r}\,3^{4h+r}}{648} - \frac{2^{6h+r}\,3^{4h+r}}{2592} \\ & -\frac{2^{8h}\,3^{4h+2r}}{5184} - \frac{2^{6h}\,5^{2h}}{800} + \frac{2^{8h}\,3^{2h}\,5^{2h}}{28800} - \frac{2^{4h+2r}\,3^{2h}\,5^{2h}}{1800} - \frac{2^{8h}\,3^{2(h+r)}\,5^{2h}}{28800} \\ & -\frac{2^{4h+r}\,3^{2h+r}\,5^{2h}}{1800} + \frac{2^{6h+r}\,3^{4h+r}\,5^{2h}}{64800} + \frac{2^{6h}\,3^{4h+2r}\,5^{2h}}{64800} + \frac{2^{8h}\,3^{2h}\,5^{r}}{576} \end{aligned}$$

$$-\frac{2^{8\,h}\,5^{2\,h+r}}{3200}+\frac{2^{2\,h}\,3^{2\,h}\,5^{2\,h+r}}{450}+\frac{2^{6\,h+r}\,3^{2\,h}\,5^{2\,h+r}}{7200}-\frac{2^{8\,h}\,3^{4\,h+r}\,5^{2\,h+r}}{259200}$$
$$+\frac{2^{8\,h}\,3^{2\,h}\,7^{r}}{576}+\frac{2^{8\,h}\,3^{2\,h}\,7^{2\,h+r}}{56448}+\frac{2^{6\,h+r}\,3^{2\,h}\,5^{2\,h}\,7^{2\,h+r}}{352800}+\frac{2^{8\,h}\,3^{4\,h+r}\,5^{2\,h}\,7^{2\,h+r}}{12700800}$$
$$-\frac{2^{8\,h}\,3^{2\,h}\,5^{2\,h}\,11^{r}}{28800}.$$

B.2 Reducible Covers

$$B_{n,h,r} = (n!)^{2h-1}\binom{n}{2}^{r}\left[\sum_{\gamma\in\mathscr{R}_n}\frac{1}{(f^{\gamma})^{2h-2}}\left(\frac{\chi_{\gamma}(2)}{f^{\gamma}}\right)^{r}\right],$$

$$B_{2,h,r} = 2\cdot 2^{2h-1},$$

$$B_{3,h,r} = 2\cdot 3!^{2h-1}\binom{3}{2}^{r},$$

$$B_{4,h,r} = 2\cdot 4!^{2h-1}\binom{4}{2}^{r}\left[1+\frac{1}{3^{2h-2+r}}\right],$$

$$B_{5,h,r} = 2\cdot 5!^{2h-1}\binom{5}{2}^{r}\left[1+\frac{2^r}{4^{2h-2+r}}+\frac{1}{5^{2h-2+r}}\right],$$

$$B_{6,h,r} = 2\cdot 6!^{2h-1}\binom{6}{2}^{r}\left[1+\frac{3^r}{5^{2h-2+r}}+\frac{3^r}{9^{2h-2+r}}+\frac{2^r}{10^{2h-2+r}}+\frac{1}{5^{2h-2+r}}\right],$$

$$B_{7,h,r} = 2\cdot 7!^{2h-1}\binom{7}{2}^{r}\left[1+\frac{4^r}{6^{2h-2+r}}+\frac{6^r}{14^{2h-2+r}}+\frac{5^r}{15^{2h-2+r}}+\frac{4^r}{14^{2h-2+r}}\right.$$
$$\left.+\frac{5^r}{35^{2h-2+r}}+\frac{1}{21^{2h-2+r}}\right],$$

$$B_{8,h,r} = 2\cdot 8!^{2h-1}\binom{8}{2}^{r}\left[1+\frac{5^r}{7^{2h-2+r}}+\frac{10^r}{20^{2h-2+r}}+\frac{9^r}{21^{2h-2+r}}+\frac{10^r}{28^{2h-2+r}}\right.$$
$$\left.+\frac{16^r}{64^{2h-2+r}}+\frac{5^r}{35^{2h-2+r}}+\frac{4^r}{14^{2h-2+r}}+\frac{10^r}{70^{2h-2+r}}+\frac{4^r}{56^{2h-2+r}}\right],$$

$$B_{9,h,r} = 2\cdot 9!^{2h-1}\binom{9}{2}^{r}\left[1+\frac{6^r}{8^{2h-2+r}}+\frac{15^r}{27^{2h-2+r}}+\frac{14^r}{28^{2h-2+r}}+\frac{20^r}{48^{2h-2+r}}\right.$$

$$+\frac{35^r}{105^{2h-2+r}}+\frac{14^r}{56^{2h-2+r}}+\frac{14^r}{42^{2h-2+r}}+\frac{36^r}{162^{2h-2+r}}+\frac{20^r}{120^{2h-2+r}}$$
$$+\frac{21^r}{189^{2h-2+r}}+\frac{14^r}{84^{2h-2+r}}+\frac{14^r}{168^{2h-2+r}}+\frac{6^r}{216^{2h-2+r}}\Big],$$

$$B_{10,h,r} = 2\cdot 10!^{2h-1}\binom{10}{2}^r\Big[1+\frac{7^r}{9^{2h-2+r}}+\frac{21^r}{35^{2h-2+r}}+\frac{20^r}{36^{2h-2+r}}+\frac{35^r}{75^{2h-2+r}}$$
$$+\frac{64^r}{160^{2h-2+r}}+\frac{28^r}{84^{2h-2+r}}+\frac{34^r}{90^{2h-2+r}}+\frac{91^r}{315^{2h-2+r}}+\frac{55^r}{225^{2h-2+r}}$$
$$+\frac{70^r}{350^{2h-2+r}}+\frac{14^r}{126^{2h-2+r}}+\frac{14^r}{42^{2h-2+r}}+\frac{64^r}{288^{2h-2+r}}+\frac{70^r}{450^{2h-2+r}}$$
$$+\frac{63^r}{567^{2h-2+r}}+\frac{35^r}{525^{2h-2+r}}+\frac{28^r}{252^{2h-2+r}}+\frac{20^r}{300^{2h-2+r}}+\frac{14^r}{210^{2h-2+r}}\Big].$$

B.3 Simple Hurwitz Numbers for an Elliptic Curve Target

We can compare our answers in the case of an elliptic curve target with those obtained from string theory. To do so, we organize $T_{n,1,2g-2}/n!$ into a generating function $\mathcal{H}_1^g(q)$, which is defined as

$$(2g-2)!\mathcal{H}_1^g = \sum_{n=1}^{\infty} \mu_{1,n}^{g,n}\, q^n = \sum_{n=1}^{\infty} \frac{T_{n,1,2g-2}}{n!} q^n.$$

This is equivalent to $\mathcal{H}_1^g$ in (2.4.19) with $\hat{t}_1^1 = 1$. Our explicit formulas for $T_{n,1,2g-2}/n!$, $n \le 7$, from §B.1 and the recursive method discussed in §2.2.7 give rise to the following q-expansions of $\mathcal{H}_1^g(q)$:

$$2!\,\mathcal{H}_1^2 = 2q^2+16q^3+60q^4+160q^5+360q^6+672q^7+1240q^8+1920q^9+3180q^{10}+ +\mathcal{O}(q^{11}),$$

$$4!\,\mathcal{H}_1^3 = 2q^2+160q^3+2448q^4+18304q^5+90552q^6+ +341568q^7+1068928q^8+2877696q^9+7014204q^{10}+\mathcal{O}(q^{11}),$$

$$6!\,\mathcal{H}_1^4 = 2q^2+1456q^3+91920q^4+1931200q^5+21639720q^6+$$

$$
\begin{aligned}
&+160786272q^7+893985280q^8+4001984640q^9+15166797900q^{10}+\mathcal{O}(q^{11})\,,\\
8!\,\mathcal{H}_1^5 \;=\;& 2q^2+13120q^3+3346368q^4+197304064q^5+5001497112q^6+\\
&+73102904448q^7+724280109568q^8+5371101006336q^9+\\
&+31830391591644q^{10}+\mathcal{O}(q^{11})\,,\\
10!\,\mathcal{H}_1^6 \;=\;& 2q^2+118096q^3+120815280q^4+19896619840q^5+1139754451080q^6+\\
&+32740753325472q^7+577763760958720q^8+7092667383039360q^9+\\
&+65742150901548780q^{10}+\mathcal{O}(q^{11})\,,\\
12!\,\mathcal{H}_1^7 \;=\;& 2q^2+1062880q^3+4352505888q^4+1996102225024q^5+258031607185272q^6+\\
&+14560223135464128q^7+457472951327051008q^8+9293626316677061376q^9+\\
&+134707212077147740284q^{10}+\mathcal{O}(q^{11})\,,\\
14!\,\mathcal{H}_1^8 \;=\;& 2q^2+9565936q^3+156718778640q^4+199854951398080q^5+\\
&+58230316414059240q^6+6451030954702152672q^7+\\
&+360793945093731688960q^8+12127449147074861834880q^9+\\
&+274847057797905019237260q^{10}+\mathcal{O}(q^{11})\,,\\
16!\,\mathcal{H}_1^9 \;=\;& 2q^2+86093440q^3+5642133787008q^4+19994654452125184q^5+\\
&+13120458818999011032q^6+2852277353239208548608q^7+\\
&+283889181859169785013248q^8+15786934495235533394850816q^9+\\
&+559374323532926110389380124q^{10}+\mathcal{O}(q^{11})\,,\\
18!\,\mathcal{H}_1^{10} \;=\;& 2q^2+774840976q^3+203119138758000q^4+1999804372817081920q^5+\\
&+2954080786719122704200q^6+1259649848110685616355872q^7+\\
&+223062465532295875789024000q^8+20519169517386068841434851200q^9+\\
&+1136630591006374329359969015340q^{10}+\mathcal{O}(q^{11})\,,\\
20!\mathcal{H}_1^{11} \;=\;& 2q^2+6973568800q^3+7312309907605728q^4+199992876225933468544q^5+\\
&+664875505232132669710392q^6+555937950399900003838125888q^7+
\end{aligned}
$$

$$
\begin{aligned}
& +1751163756152753976748219962 88q^8 + 266432436638127790666087841026 56q^9 + \\
& +2307123097757961461530407199135164q^{10} + \mathcal{O}(q^{11})\,, \\
22!\mathcal{H}_1^{12} = {} & 2q^2 + 62762119216q^3 + 263243344926609360q^4 + 1999974148984228752736 0q^5 + \\
& +149618514702670218774465960q^6 + 24527166945410708985170598 3072q^7 + \\
& +137402588289598470102013264291840q^8 + \\
& +345722665928684748181524713350 48320q^9 + \\
& +46795340459927675680521808276131 55020q^{10} + \mathcal{O}(q^{11})\,.
\end{aligned}
$$

The free energies $\mathscr{F}_g$ of string theory on the target space of an elliptic curve are known to be quasi-modular forms of weight $6g-6$. They have been computed up to genus 8 in [Rud] and have the same expansions in $q = \exp(\hat{t})$, where $\hat{t}$ is the Kähler parameter of the elliptic curve, as what we have above for $\mathcal{H}_1^g$.

For convenience, we also summarize the simple Hurwitz numbers for an elliptic curve target and arbitrary source Riemann surfaces up to degree 7:

$$
\begin{aligned}
\mu_{1,1}^{g,1}(1) &= \delta_{g,1}\,, \\
\mu_{1,2}^{g,2}(1^2) &= 2\,, \\
\mu_{1,3}^{g,3}(1^3) &= 2\left[3^r - 1\right]\,, \\
\mu_{1,4}^{g,4}(1^4) &= 2\left[6^r + 2^{r-1} - 3^r + 1\right]\,, \\
\mu_{1,5}^{g,5}(1^5) &= 2\left[10^r - 6^r + 5^r - 4^r + 3^r - 2\right]\,, \\
\mu_{1,6}^{g,6}(1^6) &= 2\cdot 15^r - 2\cdot 10^r + 2\cdot 9^r - 2\cdot 7^r + 6^r - 2\cdot 5^r + 4\cdot 4^r - 4\cdot 3^{r-1} + 2^r + 4\,, \\
\mu_{1,7}^{g,7}(1^7) &= 2\left[21^r - 15^r + 14^r - 11^r + 10^r - 2\cdot 9^r + 3\cdot 7^r - 6^r + 2\cdot 5^r - 4\cdot 4^r + \right. \\
&\quad \left. + 2\cdot 3^r - 2^r - 4\right]\,,
\end{aligned}
\tag{B.3.1}
$$

where $r = 2g-2$.

B.4 The Hurwitz Numbers $\mu_{0,2}^{g,2k}(k,k)$

$$\mu_{0,2}^{1,2k}(k,k) = \frac{(2k+2)!}{48\,(k!)^2}\,(3k-2)\,k^{2k+1}\,,$$

$$\mu_{0,2}^{2,2k}(k,k) = \frac{(2k+4)!}{2\,(k!)^2}\left(\frac{28-73\,k+49k^2}{2880}\right)\,k^{2k+3}\,,$$

$$\mu_{0,2}^{3,2k}(k,k) = \frac{(2k+6)!}{2\,(k!)^2}\left(\frac{-744+2530\,k-2949\,k^2+1181\,k^3}{725760}\right)\,k^{2k+5}\,,$$

$$\mu_{0,2}^{4,2k}(k,k) = \frac{(2k+8)!}{2\,(k!)^2}\left(\frac{18288-72826\,k+111309\,k^2-77738\,k^3+21015\,k^4}{174182400}\right)\,k^{2k+7}\,,$$

$$\mu_{0,2}^{5,2k}(k,k) = \frac{(2k+10)!}{2\,(k!)^2}\,k^{2k+9}\,\frac{1}{22992076800}\Big(-245280+1086652\,k-1959376\,k^2$$
$$+1807449\,k^3-857552\,k^4+168155\,k^5\Big)\,,$$

$$\mu_{0,2}^{6,2k}(k,k) = \frac{(2k+12)!}{2\,(k!)^2}\,k^{2k+11}\,\frac{1}{753220435968000}\,(814738752-3904894152\,k$$
$$+7889383898\,k^2-8650981635\,k^3+5462073347\,k^4$$
$$-1892825445\,k^5+282513875\,k^6\Big)\,.$$

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