SLAC-83
UC-34, Physics
TID-4500

THE CHARGE DISTRIBUTION FUNCTION
IN
PERTURBATION THEORY OF CLASSICAL ELECTRODYNAMICS

by

E. L. Chu

January 1968

Technical Report
Prepared Under
Contract AT(04-3)-515
for the USAEC
San Francisco Operations Office

Printed in USA. Available from CFSTI, National Bureau of Standards,
U.S. Department of Commerce, Springfield, Virginia 22151
Price: Printed Copy $3.00; Microfiche $0.65.

TABLE OF CONTENTS

SYNOPSIS

In various problems of classical electrodynamics the solution is often found by solving the Boltzmann-Vlasov equation

$$\frac{d}{dt} \psi (x^i, t) = 0 , \qquad i = 1, 2, \ldots , 6,$$

under proper constraints in six-dimensional phase space. A perturbation representation of the charge distribution function ψ is described in this paper. This representation is derived from the unperturbed distribution function ψ_0 by using a six-dimensional displacement vector $\tilde{\zeta}$. It is shown that if $\tilde{\zeta}$ satisfies certain constraints, then $\psi = [\exp(-\tilde{\nabla} \cdot \tilde{\zeta})]\psi_0$ satisfies the Boltzmann-Vlasov equation. The constraints which $\tilde{\zeta}$ must satisfy are dictated by none other than the Lorentz equation of motion of a charged particle. This constitutes one proof of the equivalence between the Boltzmann-Vlasov equation and the Lorentz equation. This proof may have advantageous aspects, especially in connection with perturbation calculations. As an example, the work of Lee, Mills and Morton on multipole oscillations of a throbbing beam is discussed in detail. The salient features of different kinds of phase spaces are compared by considering simple relativistic particle systems.

I. INTRODUCTION

The theory of classical electrodynamics is formally complete. There is available an abundance of mathematical methods in this field. Yet, it is by no means an easy task to solve a specific problem, because new problems are ever increasing in complexity. In attempting to obtain useful results, efficiently and accurately, one is often bewildered by his freedom of choice between several usable approaches. It seems advisable to study the salient features of different methods and the relations that exist between them.

For a given charge-current distribution and known boundary conditions, Maxwell's equations determine the electromagnetic field uniquely. Having determined the field intensities, one may obtain the motion of the charged particles from the Lorentz equation and certain initial conditions. This knowledge of the particle motion, in turn, determines the charge-current distribution. If the latter 4-current $(\underset{\sim}{i}, \rho)$ were correct, the Maxwell field $(\underset{\sim}{E}, \underset{\sim}{B})$ obtained from it would be the same field as was used in the Lorentz equation which yielded the same 4-current.

Instead of solving directly the Lorentz equation in 3-dimensional space, one may solve the so-called Boltzmann-Vlasov equation[1,2] for the charge distribution function in 6-dimensional phase space. The latter equation is obtained by substituting the Lorentz equation into the Liouville equation (also known as the collisionless Boltzmann equation). From the charge distribution function, the 4-current vector is determined readily. The use of Maxwell's equations for determining the electromagnetic field from the 4-current is the same in both methods, and, in this paper, these equations will be assumed to have been solved whenever they need be.

The Lorentz equation, which we are discussing, is the microscopic Lorentz equation. The other Lorentz equation, macroscopic, is a consequence of the Boltzmann-Vlasov equation.[3] To compare the microscopic Lorentz equation with the Boltzmann-Vlasov equation is, in effect, to compare the two forms of the Lorentz equation. Hereafter, unless explicitly stated to be otherwise, the Lorentz equation is meant to be the microscopic equation.

The electromagnetic fields $(\underset{\sim}{E}, \underset{\sim}{B})$ which appear in the Lorentz equation consist of the fields applied externally and the fields induced by the electron beam itself. The induced fields are, supposedly, the microscopic fields; the charge- and the current-density of the beam are the sum of Dirac delta functions representing the contribution of individual point charges,

$$\rho = \sum_q e\,\delta\left(\underset{\sim}{r} - \underset{\sim}{r}_q(t)\right)$$

and

$$\underset{\sim}{i} = \rho\,\dot{\underset{\sim}{r}} = \sum_q e\,\dot{\underset{\sim}{r}}\,\delta\left(\underset{\sim}{r} - \underset{\sim}{r}_q(t)\right) .$$

In the Boltzmann-Vlasov equation one also uses the microscopic particle-density and the microscopic fields, so that this equation and the set of Maxwell equations constitute a closed system of equations for the beam-field problem.[4] This system of microscopic equations may then be averaged statistically. The resulting system is not a closed one, unless the correlation effects arising from the random parts of the fields are taken to be vanishingly small. This assumption is implied in Ref. 3 and will also be used in this paper. In other words, the microscopic fields will be assumed to be no different from the corresponding macroscopic fields; only the kinetic quantities (velocity, momentum etc.) may have random fluctuations.

The equivalence between the Lorentz equation and the Boltzmann-Vlasov equation has been discussed by Watson[5] and Chandrasekhar.[6] Each spatial or momentum variable x^i of a particle may be expressed as a function of time t and the six integration constants α^k required for specifying the initial conditions. If these solutions $x^i = x^i(\alpha^k, t)$ of the Lorentz equation are substituted into the expression of the charge distribution function $\psi(x^i, t)$ and if ψ is such a function that, after the substitution, ψ becomes a function of α^k only, then $d\psi/dt = 0$, which is the Boltzmann-Vlasov equation. The fact that ψ should be totally independent of t in the absence of collisions is well-known, and is usually proved by using the Liouville theorem.[7]

This simple discussion seems to convey the thought that, if there are numerous particles in a system, it would be very hard, if not impossible, to solve the Lorentz equation because of the great number of integration constants. This thought, however, is often questionable. The well-known work of Pierce[8] on electron-beam tubes contains many examples of the judicious solution of the Lorentz equation. In this paper a new proof of the equivalence between the two methods will be given, and their relative merits will be discussed.

Let $\underset{\sim}{K}(\underset{\sim}{r}, t)$ be the Lorentz force acting on a charged particle of charge $\underline{e}$, rest mass $\underline{m}$, and relativistic mass $\underline{m\gamma}$,

$$\underset{\sim}{K} = e(\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B}) . \qquad (1.1a)$$

Let $\underset{\sim}{p}_K(\underset{\sim}{r}, t)$ be the kinetic momentum of this particle,

$$\underset{\sim}{p}_K = m\gamma\underset{\sim}{u} . \qquad (1.1b)$$

The Lorentz equation is simply

$$\frac{d\underset{\sim}{p}_K}{dt} = \left(\frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla}\right) \underset{\sim}{p}_K = \underset{\sim}{K} . \qquad (1.2)$$

The first-order part of this equation may be obtained by inspection,

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_0 \cdot \nabla\right) \mathbf{p}_{K1} + \mathbf{u}_1 \cdot \nabla \, \mathbf{p}_{K0} = \mathbf{K}_1$$

$$= e\left(\mathbf{E}_1 + \mathbf{u}_0 \times \mathbf{B}_1 + \mathbf{u}_1 \times \mathbf{B}_0\right) \; ; \qquad (1.3)$$

so may the second and higher orders.

In these equations of successive orders, the zeroth excepted, the particle aspect of the Lorentz equation is destroyed, because u_0 and $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2 + \dots$ are velocities at the same point $(\mathbf{r}, t)$ in 4-space, not the corresponding velocities of the same charged particle, which in the unperturbed case is located at the point $(\mathbf{r}, t)$ and in the perturbed case at $(\mathbf{r}', t)$. To solve these equations is said to follow the Eulerian approach.

The alternative procedure for solving the problem is the Lagrangian approach. One method of the latter approach makes use of the displacement vector $\boldsymbol{\zeta}$, defined by

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} + \boldsymbol{\zeta}(\mathbf{r}, t) , \\ \boldsymbol{\zeta} &= \boldsymbol{\zeta}_1 + \boldsymbol{\zeta}_2 + \dots . \end{aligned} \qquad (1.4)$$

Any perturbed quantity may be expressed as a function of $\boldsymbol{\zeta}$ and the supposedly known unperturbed quantities. When the Lorentz equation is so transformed, it may be separated into different orders without destroying the particle aspect.

The first-order Lorentz equation in terms of the displacement vector was originally derived by Sturrock[9] and Chu[10] by applying the variational principle to the action function of a general charge-field system. General expressions of the 4-current were derived by Dedrick and Wilson[11], using an integral theorem of the Taylor operator defined by $\boldsymbol{\zeta}$. All these derivations involve no small amount of mathematical analyses.

Subsequent to the works just mentioned, Dedrick and Chu[12] formulated a general theorem in N-dimensional space, concerning the Jacobian of coordinate transformation, the Taylor operator, and the Lagrange operator. Here we consider $N = 3$. Let $\bar{J}$ be the Jocobian of transformation from the unperturbed to the perturbed coordinates.

$$\bar{J} \equiv \det \left| \nabla \mathbf{r}' \right| \quad ; \tag{1.5a}$$

$$\rho_0(\mathbf{r}, t) = \bar{J}\rho(\mathbf{r}', t) \quad . \tag{1.5b}$$

Let $\bar{\Sigma}$ be the Taylor operator,

$$\bar{\Sigma} \equiv \exp(\boldsymbol{\zeta} \cdot \nabla) \tag{1.6a}$$

$$\rho(\mathbf{r}', t) = \bar{\Sigma}\rho(\mathbf{r}, t) \tag{1.6b}$$

$$\mathbf{u}(\mathbf{r}', t) = \bar{\Sigma}\mathbf{u}(\mathbf{r}, t) = \mathbf{u}_0 + \dot{\boldsymbol{\zeta}} \quad . \tag{1.6c}$$

Let $\bar{\Omega}$ be the Lagrange operator.

$$\bar{\Omega} \equiv \exp(- \nabla \cdot \boldsymbol{\zeta}) \quad . \tag{1.7}$$

The operator theorem proved in Ref. 12 is as follows:

$$\bar{J}\,\bar{\Sigma}\,\bar{\Omega} = \bar{\Sigma}\,\bar{\Omega}\,\bar{J} = \bar{\Omega}\,\bar{J}\,\bar{\Sigma} = 1, \tag{1.8}$$

where the operand is any infinitely differentiable function.

According to this theorem, we obtain at once from Eqs. (1.5b) and (1.6b)

$$\rho(\mathbf{r}, t) = \bar{\Omega}\,\rho_0(\mathbf{r}, t) \quad . \tag{1.9}$$

The current density may also be obtained easily by using an adjoint property pertaining to the operators $\bar{\Sigma}^{-1}$ and $\bar{\Omega}^{-1}$ [see Eq. (6.8a) in Ref. 12] .

$$\mathbf{i}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\, \mathbf{u}(\mathbf{r}, t) = \left(\bar{\Omega}\rho_0\right) \bar{\Sigma}^{-1} \left(\mathbf{u}_0 + \dot{\boldsymbol{\zeta}}\right) ,$$

i. e.,

$$\mathbf{i}(\mathbf{r}, t) = \bar{\Omega}\left(\mathbf{i}_0 + \rho_0 \dot{\boldsymbol{\zeta}}\right) . \tag{1.10}$$

Expressed in these forms, the 4-current may be separated into successive orders by simple inspection.

To obtain the corresponding form of the Lorentz equation we apply the Taylor operator on both sides of Eq. (1.2). Since

$$\bar{\Sigma} \underset{\sim}{K} = \bar{\Sigma}\, e\, \underset{\sim}{E} + \left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right) \times \left(\bar{\Sigma}\, e\, B\right) \tag{1.11a}$$

and

$$\bar{\Sigma}\, \frac{d\underset{\sim}{p}_K}{dt} = \bar{\Sigma}\left[\left(\frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla}\right) \underset{\sim}{p}_K\right]$$

$$= \left[\frac{\partial}{\partial t} + \underset{\sim}{u}(r', t) \cdot \underset{\sim}{\nabla}'\right]\left(\bar{\Sigma}\underset{\sim}{p}_K\right) = \frac{d}{dt}\left(\bar{\Sigma}\underset{\sim}{p}_K\right),$$

i.e., $$\bar{\Sigma}\, \frac{d\underset{\sim}{p}_K}{dt} = \left[\frac{\partial}{\partial t} + \underset{\sim}{u}_0(\underset{\sim}{r}, t) \cdot \underset{\sim}{\nabla}\right]\left[m\gamma'\left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right)\right], \tag{1.11b}$$

the resulting equation is the Lorentz equation in the Lagrangian approach:

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{u}_0 \cdot \underset{\sim}{\nabla}\right)\left\{m\left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right)\left[1 - \left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right)\cdot\left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right)c^{-2}\right]^{-1/2}\right\}$$

$$= \bar{\Sigma}\, e\, \underset{\sim}{E} + \left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right) \times \left(\bar{\Sigma}\, e\, B\right). \tag{1.12}$$

This equation may then be separated into different orders quite easily. The first-order equation is as given by Eq. (7.10) in Section VII.

This brief review of the perturbation theory serves to show the effective use of the Taylor and Lagrange operators. In succeeding sections we will use these operators in 6-dimensional phase space to discuss the perturbation solution of the Boltzmann-Vlasov equation. We choose 6-space over 8-space,[13,14,15] because 6-space is used in the majority of references, including Hamilton's early work.[16,17] The use of 6-space does not hamper the relativistic treatment. No new mathematical procedure is involved in adapting two more dimensions, time and energy.

Speaking in general, either the covariant- or the contravariant-momentum components may be used as the non-spatial coordinates in a phase space. We will use the covariant-momentum components. This choice conforms to the usage of the Hamiltonian treatment of dynamics, in which the spatial- and the covariant-momentum components are the so-called conjugate variables. It can readily be shown that, when these variables are the coordinates, the determinant of the 6×6 metric tensor of the phase space is unity, regardless of the spatial coordinate system, whether it be cartesian or curvilinear. When a certain canonical transformation is applied to the phase-space coordinates, the two groups of new coordinates remain conjugate to each other; but no new coordinate may retain its purely spatial or purely non-spatial character.[16,18] The determinant of the new metric tensor remains unchanged, because the Jacobian of any canonical transformation of coordinates is unity.[19] In the following discussion, this property of the metric tensor is not needed.

II. THE DISPLACEMENT VECTOR AND COORDINATE TRANSFORMATIONS

Consider at the same time two simple systems of charged particles having one-to-one correspondence. One system is in the unperturbed state and the other in the perturbed state. An unperturbed particle is located at time $\underline{t}$ at the point $\tilde{\mathbf{r}}$ in the phase space, having coordinates x^i referred to a given 6-dimensional coordinate system. The corresponding perturbed particle is located at the same time, $t' = t$, at the point $\tilde{\mathbf{r}}'$, having coordinates x'^i referred to the same coordinate system.

These two phase points are related by the displacement vector as follows:

$$\tilde{\mathbf{r}}' = \tilde{\mathbf{r}} + \tilde{\boldsymbol{\zeta}} , \qquad \tilde{\boldsymbol{\zeta}} \equiv \tilde{\boldsymbol{\zeta}}(\tilde{\mathbf{r}}, t) . \tag{2.1a}$$

This may, alternatively, be given by

$$x'^i = x^i + \xi^i(x^k, t) . \tag{2.1b}$$

Here, ξ^i is the difference between the two i-th coordinates. The displacement vector $\tilde{\boldsymbol{\zeta}}(\tilde{\mathbf{r}}, t)$ is defined by the set of components $\xi^i(x^k, t)$, and vice versa. The components ξ^i, however, do not in general form a vector themselves. In other words, $\xi^i \neq \zeta^i$ except when the reference coordinate system is cartesian.

The first three coordinates (x^1, x^2, x^3) may be designated to be the spatial coordinates, and the other three (x^4, x^5, x^6) the momentum coordinates, either canonical- or kinetic-momentum as the case may be. Thus, the first three components of any vector are the spatial components, and the other three the non-spatial components. Each component of $\tilde{\boldsymbol{\zeta}}$ is some function of the unperturbed coordinate- and momentum-variables and the time. The three components of $\tilde{\boldsymbol{\zeta}}$ pertaining to the momentum subspace are not independent of the three spatial components. They are connected by kinetic relations. These relations will be discussed later.

Equations (2.1a) and (2.1b) have their counterparts, namely,

$$\tilde{r} = \tilde{r}' - \tilde{\zeta}'(\tilde{r}', t) \tag{2.2a}$$

and

$$x^i = x'^i - \xi'^i(x'^k, t) \ , \tag{2.2b}$$

where

$$\tilde{\zeta}'(\tilde{r}', t) = \tilde{\zeta}(\tilde{r}, t) \tag{2.3a}$$

and

$$\xi'^i(x'^k, t) = \xi^i(x^k, t) \ . \tag{2.3b}$$

Equations (2.1b) and (2.2b) may be interpreted from an alternative but equally valid point of view. The coordinates x^i and x'^i, so far understood to pertain to two different points in the same coordinate system, may also be considered to be the coordinates of one and the same phase point with reference to two different coordinate systems. Equations (2.1b) are the set of equations of coordinate transformation from the unperturbed to the perturbed system, while Eqs. (2.2b) are those of the inverse transformation. If the x (or x') coordinate system is cartesian, then the x' (or x) system is curvilinear.

The Jacobian $J(x'^i/x^k)$ of the transformation (2.1b) is the determinant of the 6×6 matrix $(\partial x'^i/\partial x^k)$, i.e.,

$$J(x'^i/x^k) = \left|\partial x'^i/\partial x^k\right| \equiv J \ . \tag{2.4}$$

Similarly,

$$J(x^i/x'^k) = \left|\partial x^i/\partial x'^k\right| \equiv J^{-1} \ . \tag{2.5}$$

Any scalar, vector, or tensor function may be expanded into a series by operating upon the function with the Taylor operator

$$\Sigma \equiv \exp\left(\xi^i \frac{\partial}{\partial x^i}\right) \ ,$$

i.e.,

$$\Sigma \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \xi^a \xi^b \; \ldots \; \xi^p \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} \ldots \frac{\partial}{\partial x^p} \quad , \tag{2.6}$$

where $\xi^a \xi^b \; \ldots \; \xi^p$ is a product of n factors, and every pair of repeated Latin indices implies summation over six components. For example, Taylor's theorem is

$$\psi(x'^i, t) = \Sigma \psi(x^i, t) \; . \tag{2.7}$$

Conversely,

$$\psi(x^i, t) = \Sigma^{-1} \psi(x'^i, t) . \tag{2.8}$$

The inverse Taylor operator Σ^{-1} is closely related to the Lagrange operator Ω, which is defined as follows:

$$\Omega \equiv \exp\left(- \frac{\partial}{\partial x^i} \xi^i\right) \quad ,$$

i.e.,

$$\Omega \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} \; \ldots \; \frac{\partial}{\partial x^p} \xi^a \xi^b \ldots \xi^p \; . \tag{2.9}$$

Many properties of these operators have been discussed in Ref. 12. A general theorem is proved therein. This theorem is the extension of Eq. (1.8) to N-dimensions.

III. CHARGE DISTRIBUTION FUNCTIONS

Let ψ_0 and ψ denote, respectively, the unperturbed and the perturbed charge distribution function. A certain number of identical charged particles in a 6-space phase element $dx^1 dx^2 \ldots dx^6$ bear an electric charge dq. The charge distribution function is defined to be the charge density per unit phase element. Thus, in the unperturbed state, we have

$$dq = \psi_0(x^i, t)(dx)^6 . \tag{3.1}$$

Here, $(dx)^6 \equiv dx^1 dx^2 \ldots dx^6$. In the perturbed state, these same particles in the corresponding phase element must have the same amount of charge. It is assumed that there occurs no collision between particles. Thus, we also have

$$dq = \psi(x'^i, t)(dx')^6 . \tag{3.2}$$

Since

$$(dx')^6 = J(x'^i/x^k)(dx)^6 ,$$

i.e.,

$$J\psi(x'^i, t) = \psi_0(x^i, t) , \tag{3.3}$$

we obtain from Eqs. (2.7) and (3.3)

$$J\Sigma\psi(x^i, t) = \psi_0(x^i, t) , \tag{3.4}$$

or simply,

$$\psi(x^i, t) = \Omega\psi_0(x^i, t) , \tag{3.5}$$

because $\Omega J \Sigma = 1$. In other words, the perturbed charge distribution function may be represented in general as the result of operating on the unperturbed distribution function with the Lagrange operator. Once ψ is represented in this form, it becomes a routine matter to separate ψ into different orders of magnitude. For example, using Eq. (2.9) and separating ξ^i into different orders, $\xi^i = \xi^i_1 + \xi^i_2 + \ldots$, we obtain

$$\psi_1 = - \frac{\partial}{\partial x^i} \left(\psi_0 \xi_1^i \right) , \tag{3.6a}$$

$$\psi_2 = - \frac{\partial}{\partial x^i} \left(\psi_0 \xi_2^i \right) + \frac{1}{2!} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \left(\psi_0 \xi_1^i \xi_1^j \right) , \tag{3.6b}$$

etc.

Equation (3.5) is the consequence of Eq. (3.3). In tensor language, the charge distribution function is a scalar density, not an absolute but a relative scalar of weight one.[20] Scalar densities of all kinds behave in the same manner.

IV. THE EQUATION OF CONTINUITY

Let us consider the time development of the charge distribution function. As the time increases from t to t', a certain phase point moves from x^i to x'^i and a small phase element around this point changes from δV to $\delta V'$. In the absence of collisions, the number of charged particles under consideration remains the same. Thus,

$$\int_{\delta V} \psi(x, t)(dx)^6 = \int_{\delta V'} \psi(x, t')(dx)^6$$

$$= \int_{\delta V'} \psi(x', t')(dx')^6 \quad . \qquad (4.1)$$

The last integral over $\delta V'$ may be changed by a transformation of variables to an integral over δV. Then Eq. (4.1) becomes

$$\int_{\delta V} \psi(x',t')\left|\frac{\partial x'}{\partial x}\right| (dx)^6 = \int_{\delta V} \psi(x, t)(dx)^6 \quad . \qquad (4.2)$$

Here, t is considered fixed and $t' = t + \delta t$ an auxiliary parameter. The equations of transformation of variables may be written as

$$x'^i = x^i + \xi^i(x^k, t') \quad , \qquad \left(\xi^i \xrightarrow[t' \to t]{} 0\right) , \qquad (4.3)$$

which must exist because the solution to the initial-value problem of the physical system exists. The Jacobian $|\partial x'/\partial x|$ also contains t' as a parameter. Since δV is arbitrary, we must have

$$\psi(x'^i, t')\left|\partial x'/\partial x\right| = \psi(x^i, t) \quad . \qquad (4.4)$$

From this equation, it then follows that

$$\left|\partial x'/\partial x\right| \Sigma\, \psi(x^i, t') = \psi(x^i, t) \quad . \qquad (4.4a)$$

Here, the Taylor operator Σ is as defined by Eq. (2.6) with $\xi^i = \xi^i(x^k, t')$ instead of $\xi^i(x^k, t)$, which, in this case, vanishes. Operating on Eq. (4.4a) from the left with the Lagrange operator Ω, which is also defined by $\xi^i(x^k, t')$, we obtain, because of $\Omega \left| \partial x' / \partial x \right| \Sigma = 1$,

$$\psi(x^i, t') = \Omega \psi(x^i, t) \ . \tag{4.5}$$

The left-hand side of this equation may then be expanded into a series about the point (x^i, t) by using Taylor's theorem.

$$\psi(x^i, t') = \mathfrak{T} \psi(x^i, t) \ ,$$

where

$$\mathfrak{T} \equiv \sum_{n=0}^{\infty} \frac{(\delta t)^n}{n!} \frac{\partial^n}{\partial t^n} \ . \tag{4.6}$$

Therefore, Eq. (4.5) becomes

$$\mathfrak{T} \psi(x^i, t) = \Omega \psi(x^i, t) \ . \tag{4.7}$$

The phase-space coordinates x^i are considered fixed when ψ is operated on with $\mathfrak{T}$. The time variables t and t' are considered fixed when ψ is operated on with Ω. Thus, Eq. (4.7) states that the Taylor development of ψ with respect to time at a fixed phase point is equal to the Lagrange development of ψ with respect to phase variables at fixed times. This is, perhaps, the most general form of the equation of continuity satisfied by ψ or any scalar density. No dynamic principle is used in deriving Eq. (4.7) save for the existence of the solution of the initial-value problem. Hence, Eq. (4.7) is valid, independent of dynamic principles including the theorem of Liouville.

When $\delta t \longrightarrow 0$, we have $\xi^i = x'^i - x^i = \delta x^i \longrightarrow 0$. We may then consider only the first-order terms in Eq. (4.7) and obtain

$$\delta t \frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x^i} (\psi \delta x^i) = 0 .$$

Also to the first order, $\delta x^i = \dot{x}^i \delta t$. Using this relation we obtain the equation of continuity in its usual form, namely,

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x^i}(\psi \dot{x}^i) = 0 . \tag{4.8}$$

Since $\dot{x}^i$ is an absolute contravariant vector, the function $\psi \dot{x}^i$ is a contravariant vector density. The second term in Eq. (4.8) is the so-called density divergence[20] in a 6-space. This term is an invariant with respect to coordinate transformations in the same sense as a scalar density is.

When Hamilton's equations are satisfied, Eq. (4.8) becomes the Liouville equation,

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \dot{x}^i \frac{\partial \psi}{\partial x^i} = 0 \quad , \tag{4.9}$$

because $\partial \dot{x}^i / \partial x^i = 0$ as will be discussed in the next section. This equation may be obtained more readily from Eq. (4.4), because $\left| \partial x' / \partial x \right| = 1$ according to the Liouville theorem, which follows from Hamilton's equations.

V. THE BOLTZMANN-VLASOV EQUATION

We differentiate Eq. (3.4) with respect to time.

$$\frac{d\psi_0}{dt} = \frac{dJ}{dt}\,\Sigma\psi + J\,\frac{d}{dt}\,\Sigma\psi \quad . \tag{5.1}$$

Since

$$\frac{d}{dt}\,\Sigma\psi = \left(\frac{\partial}{\partial t} + \dot{x}'^{k}\,\frac{\partial}{\partial x'^{k}}\right)\psi(x'^{i}, t) = \Sigma\,\frac{d}{dt}\,\psi(x^{i}, t) \quad ,$$

Eq. (5.1) is the same as

$$\frac{d\psi_0}{dt} = \frac{1}{J}\,\frac{dJ}{dt}\,\psi_0 + J\Sigma\,\frac{d\psi}{dt} \quad . \tag{5.2}$$

Operating on this equation with the Lagrange operator Ω used in Section III and using $\Omega\, J\, \Sigma = 1$, we obtain

$$\frac{d\psi}{dt} = \Omega\left(\frac{d\psi_0}{dt} - \frac{\psi_0}{J}\,\frac{dJ}{dt}\right) . \tag{5.3}$$

The unperturbed state is supposedly a physically realizable state. Hence, in the absence of collisions,

$$\frac{d\psi_0}{dt} = 0 \tag{5.4}$$

according to the Liouville theorem. Thus, the first term on the right of Eq. (5.3) vanishes. If the perturbed state is also physically realizable, the last term of Eq. (5.3) must also vanish. In other words,

$$\frac{dJ}{dt} = 0 \tag{5.5}$$

should also follow from the principles of dynamics. While this is obvious in view of the Liouville theorem, we will, nevertheless, give an explicit proof of Eq. (5.5) to elucidate its inner contents, which are most essential for the discussion of our subject.

In this connection, it is more convenient to consider the transformation of coordinates in 7- rather than 6-space. We re-write Eq. (2.1b) as follows:

$$x'^i = x^i + \xi^i(x^k) . \tag{5.6}$$

Here, i and k range from 1 to 7; $x'^7 = t'$; $x^7 = t$; $\xi^7 = 0$. This set of equations (5.6) consists of the same six equations in the set (2.1b) and the 7th equation $t' = t$. The Jacobian of the transformation in 7-space is the same as the one in 6-space, because

$$\frac{\partial t'}{\partial x^i} = \begin{cases} 0\,, & i = 1, 2, \ldots, 6. \\ \dfrac{\partial t'}{\partial t} = 1, & i = 7\,. \end{cases}$$

Let g_{ij} denote the covariant element of the metric tensor of the x-coordinate system and $\hat{g}_{ij}$ the corresponding metric element of the x'-system. Then,

$$\hat{g}_{k\ell} = \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^\ell} g_{ij} .$$

Taking the determinants of the matrices on both sides of this equation, we obtain

$$\hat{g} = J^{-2} \tilde{g} ,$$

where

$$\tilde{g} = |g_{ij}| \quad \text{and} \quad \hat{g} = |\hat{g}_{ij}| .$$

Hence,

$$J = \sqrt{\tilde{g}}/\sqrt{\hat{g}} \tag{5.7}$$

and

$$\frac{1}{J}\frac{dJ}{dt} = \frac{1}{\sqrt{\tilde{g}}}\frac{d\sqrt{\tilde{g}}}{dt} - \frac{1}{\sqrt{\hat{g}}}\frac{d\sqrt{\hat{g}}}{dt} . \tag{5.8}$$

In 7-space, $d/dt = \dot{x}^i(\partial/\partial x^i)$ in the x-system and $d/dt = \dot{x}'^i(\partial/\partial x'^i)$ in the x'-system. Thus, Eq. (5.8) may be written as

$$\frac{1}{J}\frac{dJ}{dt} = \frac{1}{\sqrt{\tilde{g}}}\frac{\partial}{\partial x^i}(\dot{x}^i\sqrt{\tilde{g}}) - \frac{1}{\sqrt{\hat{g}}}\frac{\partial}{\partial x'^i}(\dot{x}'^i\sqrt{\hat{g}}) - \frac{\partial \dot{x}^i}{\partial x^i} + \frac{\partial \dot{x}'^i}{\partial x'^i} . \tag{5.9}$$

The first two terms on the right of this equation cancel each other, because both terms are equal to the absolute divergence of the 7-vector $d\tilde{s}/dt$. The vector $d\tilde{s}$ is an infinitesimal, directed, line element in the 7-space.

$$d\tilde{s} = \tilde{e}_i dx^i = \hat{e}_i dx'^i \quad .$$

The time derivative of this vector is

$$d\tilde{s}/dt = \tilde{e}_i \dot{x}^i = \hat{e}_i \dot{x}'^i \quad .$$

Here, $\tilde{e}_i$ and $\hat{e}_i$ are the covariant <u>base</u> vectors of the two coordinate systems. The absolute divergences of the same vector $d\tilde{s}/dt$ in two different coordinate systems are the same. Because of this, Eq. (5.9) becomes :

$$\frac{1}{J}\ \frac{dJ}{dt} = \frac{\partial \dot{x}'^i}{\partial x'^i} - \frac{\partial \dot{x}^i}{\partial x^i} \quad . \tag{5.10}$$

Since $\partial \dot{t}'/\partial t' = \partial \dot{t}/\partial t = 0$, Eq. (5.10) is valid in 7-space as well as in 6-space. Hereafter, the Latin indices will again be considered to range from 1 to 6.

To continue our proof, it is helpful to recognize that the six coordinate variables are three pairs of conjugate variables. Let x^α, α ranging from 1 to 3, denote the three spatial (contravariant) coordinates. Let $x^{3+\alpha}$ denote the three momentum coordinates, canonical or kinetic as the case may be, $x^{3+\alpha} = p_\alpha$. Thus,

$$\frac{\partial \dot{x}^i}{\partial x^i} = \frac{\partial \dot{x}^\alpha}{\partial x^\alpha} + \frac{\partial \dot{p}_\alpha}{\partial p_\alpha} \quad . \tag{5.11}$$

Each pair of coordinates (x^α, p_α) or $(x^\alpha, x^{3+\alpha})$ are conjugate variables.

In Eq. (5.11), the $\dot{x}^\alpha$ are the three velocity components and the $\dot{p}_\alpha$ the three force components (in the generalized sense) pertaining to a certain particle in the unperturbed state. According to the canonical equations of Hamilton, we have

$$\dot{x}^\alpha = \partial \mathscr{H}_0/\partial p_\alpha \tag{5.12a}$$

and

$$\dot{p}_\alpha = - \partial \mathcal{H}_0 / \partial x^\alpha \ . \tag{5.12b}$$

Here, $\mathcal{H}_0$ is the unperturbed Hamiltonian function. To be more specific,

$$\mathcal{H}_0(x, p, t) = eV_0 + mc^2 \left[1 + \frac{1}{m^2 c^2} \left(p^\alpha - eA_0^\alpha \right) \left(p_\alpha - eA_{0\alpha} \right) \right]^{1/2} . \tag{5.13}$$

In this equation, $V_0 = V_0(x, t)$ is the scalar potential, $A_{0\alpha} = A_{0\alpha}(x, t)$ the vector potential, and p_α the canonical momentum; all entities pertain to the unperturbed state. From Eqs. (5.12a) and (5.12b) it is clear that

$$\frac{\partial \dot{x}^\alpha}{\partial x^\alpha} = - \frac{\partial \dot{p}_\alpha}{\partial p_\alpha} \ . \tag{5.14}$$

Hence, Eq. (5.11) becomes

$$\frac{\partial \dot{x}^i}{\partial x^i} = 0 \ . \tag{5.15}$$

Similarly, the perturbed system is characterized by the Hamiltonian $\mathcal{H}(x', p', t)$,

$$\mathcal{H}(x', p', t) = eV' + mc^2 \left[1 + \frac{1}{m^2 c^2} \left(p'^\alpha - eA'^\alpha \right) \left(p'_\alpha - eA'_\alpha \right) \right]^{1/2} . \tag{5.16}$$

Here, $V' \equiv V(x', t)$ and $A'_\alpha \equiv A_\alpha(x', t)$. Following the same procedure as used in deriving Eq. (5.15), we find

$$\frac{\partial \dot{x}'^i}{\partial x'^i} = 0 \ . \tag{5.17}$$

Thus, both terms on the right of Eq. (5.10) should vanish because of Hamilton's equations. This proves Eq. (5.5) in any general case, because J is finite.

The general representation of the charge distribution function as given by Eq. (3.5) has thus been proved to satisfy the Liouville equation, provided that the two conditions as stated by Eqs. (5.4) and (5.5) are satisfied. The Liouville equation becomes the Boltzmann-Vlasov equation, when the electromagnetic field quantities in the Hamiltonian $\mathcal{H}$ are consistent with the 4-current derived from the charge distribution function ψ.

The Jacobian J is determined solely by the 6-dimensional displacement vector $\tilde{\zeta}(x, p, t)$. To satisfy Eq. (5.5), $\tilde{\zeta}$ must be such a vector function as to satisfy Eq. (5.17). The latter equation is solved by the pair of Hamilton's equations:

$$\dot{x}'^{\alpha} = \frac{\partial}{\partial p'_{\alpha}} \mathcal{H}(x', p', t) \; ; \tag{5.18a}$$

$$\dot{p}'_{\alpha} = - \frac{\partial}{\partial x'^{\alpha}} \mathcal{H}(x', p', t) \; . \tag{5.18b}$$

The first equation of this pair is simply the definition of the 3-velocity vector,

$$u'^{\alpha} \equiv \dot{x}'^{\alpha} = \left(p'^{\alpha} - eA'^{\alpha}\right)/ m\gamma' \; , \tag{5.19}$$

where γ' is the ratio of the relativistic mass to the rest mass,

$$\gamma' = \left[1 + \frac{1}{m^2c^2} \left(p'^{\alpha} - eA'^{\alpha}\right)\left(p'_{\alpha} - eA'_{\alpha}\right)\right]^{1/2} \tag{5.20a}$$

$$= \left[1 - \left(u'^2/c^2\right)\right]^{-1/2} \; . \tag{5.20b}$$

The second canonical equation (5.18b) is usually known in classical electrodynamics as the Lorentz equation of motion of a charged particle. The Hamiltonian function $\mathcal{H}$ may contain other interaction terms not included in Eq. (5.16). To solve for ψ from the Boltzmann-Vlasov equation is, in all cases, the same as to solve for $\tilde{\zeta}$ from the Lorentz equation. Each method of solution has its advantages. It should always be helpful to know the detailed connection between them.

In Appendix A, it is shown that if ψ_0 satisfies the equation of continuity, then $\psi = \Omega\psi_0$ will also satisfy this equation without requiring any condition on dJ/dt. In Section X we will discuss the velocity phase-space, for which $dJ/dt \neq 0$.

VI. FURTHER DISCUSSION OF THE DISPLACEMENT VECTOR

To evaluate (1/J)(dJ/dt) in terms of the displacement vector, we first differentiate Eq. (2.1b) with d/dt, then with $\partial/\partial x'^{\ell}$, and then apply contraction by putting $\ell = i$. Thus,

$$\frac{\partial \dot{x}'^{i}}{\partial x'^{i}} = \frac{\partial x^{k}}{\partial x'^{i}} \frac{\partial}{\partial x^{k}} (\dot{x}^{i} + \dot{\xi}^{i}) . \tag{6.1}$$

According to Eq. (5.10),

$$\frac{1}{J}\frac{dJ}{dt} = \left(\frac{\partial x^{k}}{\partial x'^{i}} - \delta_{i}^{k}\right)\frac{\partial \dot{x}^{i}}{\partial x^{k}} + \frac{\partial x^{k}}{\partial x'^{i}}\frac{\partial \dot{\xi}^{i}}{\partial x^{k}} , \tag{6.2}$$

where $\delta_{i}^{k} = \partial x^{k}/\partial x^{i}$ is the Kronecker delta.

Since,

$$\frac{\partial \dot{\xi}^{i}}{\partial x^{k}} = \frac{d}{dt}\frac{\partial \xi^{i}}{\partial x^{k}} + \frac{\partial \dot{x}^{\ell}}{\partial x^{k}}\frac{\partial \xi^{i}}{\partial x^{\ell}} ,$$

Eq. (6.2) becomes:

$$\frac{1}{J}\frac{dJ}{dt} = \left(\frac{\partial x^{k}}{\partial x'^{i}} - \delta_{i}^{k} + \frac{\partial x^{k}}{\partial x'^{\ell}}\frac{\partial \xi^{\ell}}{\partial x^{i}}\right)\frac{\partial \dot{x}^{i}}{\partial x^{k}} + \frac{\partial x^{k}}{\partial x'^{i}}\frac{d}{dt}\frac{\partial \xi^{i}}{\partial x^{k}} .$$

The first term on the right of this equation vanishes, because

$$\frac{\partial x^{k}}{\partial x'^{i}} - \delta_{i}^{k} + \frac{\partial x^{k}}{\partial x'^{\ell}}\frac{\partial \xi^{\ell}}{\partial x^{i}} = \frac{\partial x^{k}}{\partial x'^{i}} - \frac{\partial x^{k}}{\partial x'^{\ell}}\frac{\partial x^{\ell}}{\partial x^{i}} = 0 .$$

Hence,

$$\frac{1}{J}\frac{dJ}{dt} = \frac{\partial x^{k}}{\partial x'^{i}}\frac{d}{dt}\frac{\partial \xi^{i}}{\partial x^{k}} . \tag{6.3}$$

Considering the x-coordinate system to be Cartesian, we may easily write Eq. (6.3) in its vector form, namely,

$$\begin{aligned}\frac{1}{J}\frac{dJ}{dt} &= \left(\tilde{\mathbf{r}}'\tilde{\nabla}\right)^{-1} : \frac{d}{dt}\left(\tilde{\nabla}\tilde{\xi}\right) \\ &= \left(\mathcal{I} + \tilde{\xi}\tilde{\nabla}\right)^{-1} : \frac{d}{dt}\left(\tilde{\nabla}\tilde{\xi}\right),\end{aligned} \tag{6.4a}$$

i.e.,

$$\frac{1}{J}\frac{dJ}{dt} = \left[\mathcal{I} - \tilde{\xi}\tilde{\nabla} + \left(\tilde{\xi}\tilde{\nabla}\right)\cdot\left(\tilde{\xi}\tilde{\nabla}\right) - \left(\tilde{\xi}\tilde{\nabla}\right)^3 + \ldots\right] : \frac{d}{dt}\left(\tilde{\nabla}\tilde{\xi}\right) \quad . \tag{6.4b}$$

In these equations, the double dot product is performed as in the equation $\tilde{a}\tilde{b} : \tilde{c}\tilde{d} = (\tilde{a}\cdot\tilde{c})(\tilde{b}\cdot\tilde{d})$; $\mathcal{I} = \tilde{e}^i\tilde{e}_i = \tilde{e}_i\cdot\tilde{e}^i$ is the unity dyadic. Both Eq. (6.3) and Eqs.(6.4a) or (6.4b) are valid in any coordinate system. Equation (6.4b) can easily be separated into different orders of magnitude. The first two terms are as follows:

$$\left(\frac{1}{J}\frac{dJ}{dt}\right)_1 = \frac{d}{dt}\left(\tilde{\nabla}\cdot\tilde{\xi}_1\right) \tag{6.5a}$$

$$\left(\frac{1}{J}\frac{dJ}{dt}\right)_2 = \frac{d}{dt}\left[\tilde{\nabla}\cdot\tilde{\xi}_2 - \frac{1}{2}\left(\tilde{\xi}_1\tilde{\nabla}\right) : \left(\tilde{\nabla}\tilde{\xi}_1\right)\right] \quad . \tag{6.5b}$$

To satisfy the condition $\left[(1/J)\,dJ/dt\right]_1 = 0$ so that $(d\psi/dt)_1 = 0$, we must require that

$$\frac{d}{dt}\left(\tilde{\nabla}\cdot\tilde{\xi}_1\right) = \frac{d}{dt}\frac{\partial \xi_1^i}{\partial x^i} = 0 \tag{6.6}$$

identically. Thus $\tilde{\nabla}\cdot\tilde{\xi}_1$ must be a constant or zero. If $\tilde{\nabla}\cdot\tilde{\xi}_1$ were a non-vanishing constant, then according to Eq. (3.6a) ψ_1 would contain a small constant fraction of ψ_0 equal to $\left(\partial\xi_1^i/\partial x^i\right)\psi_0$. This part is certainly of no interest in the perturbation problem and may be excluded from consideration. Therefore, we may simply require, instead of Eq. (6.6),

$$\tilde{\nabla}\cdot\tilde{\xi}_1 = \frac{1}{\sqrt{\tilde{g}}}\frac{\partial}{\partial x^i}\left(\xi_1^i\sqrt{\tilde{g}}\right) = \frac{\partial\xi_1^i}{\partial x^i} = 0 \, . \tag{6.7}$$

If this is satisfied, then the Jacobian J is equal to unity, to the first order.

Now we proceed to find the connection between the two parts of the displacement vector, the spatial and the momentum part. Let us denote $\xi^{3+\alpha} \equiv \chi_\alpha$, and separate the set of Eqs. (2.1b) into its two parts:

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha} \quad ; \tag{6.8a}$$

$$p'_{\alpha} = p_{\alpha} + \chi_{\alpha} \quad . \tag{6.8b}$$

These yield, on differentiation with d/dt ,

$$u'^{\alpha} = u_0^{\alpha} + \dot{\xi}^{\alpha} \quad ; \tag{6.9a}$$

$$\dot{p}'_{\alpha} = \dot{p}_{\alpha} + \dot{\chi}_{\alpha} \quad . \tag{6.9b}$$

In Eq. (6.9a), $u_0^{\alpha} = \dot{x}^{\alpha}$ and $u'^{\alpha} = \dot{x}'^{\alpha}$ are, respectively, the unperturbed and the corresponding perturbed particle velocity.

Let $g_{\alpha\beta}$ be the metric tensor elements of the 3-space evaluated at the point $(\underset{\sim}{r})$ and $g'_{\alpha\beta}$ the metric elements evaluated at the point $(\underset{\sim}{r}')$, $g'_{\alpha\beta} = g_{\alpha\beta}(\underset{\sim}{r}')$. Then $u_{0\alpha} = g_{\alpha\beta} u_0^{\beta}$ and $u'_{\alpha} = g'_{\alpha\beta} u'^{\beta}$. By definition,

$$p_{\alpha} = m\gamma_0 u_{0\alpha} + eA_{0\alpha} \quad ; \tag{6.10a}$$

$$p'_{\alpha} = m\gamma' u'_{\alpha} + eA'_{\alpha} \quad . \tag{6.10b}$$

Substituting these relations in Eq. (6.8b) we obtain

$$\chi_{\alpha} = m\gamma_0 g_{\alpha\beta} \dot{\xi}^{\beta} + m\left(\gamma' g'_{\alpha\beta} - \gamma_0 g_{\alpha\beta}\right)\left(u_0^{\beta} + \dot{\xi}^{\beta}\right) + e\left(A'_{\alpha} - A_{0\alpha}\right) \tag{6.11}$$

Equations (6.8a) and (6.8b) have their corresponding 3-dimensional vector forms, namely,

$$\underset{\sim}{r}' = \underset{\sim}{r} + \underset{\sim}{\zeta} \quad , \tag{6.12a}$$

$$\underset{\sim}{p}' = \underset{\sim}{p} + \underset{\sim}{\eta} \quad . \tag{6.12b}$$

In a Cartesian coordinate system, $r_{\alpha} = r^{\alpha} = x^{\alpha}$, $p_{\alpha} = p^{\alpha} = r^{3+\alpha}$, $r'_{\alpha} = r'^{\alpha} = x'^{\alpha}$, and $p'_{\alpha} = p'^{\alpha} = r'^{3+\alpha}$. In curvilinear coordinate systems, these relations are no longer true. From Eqs. (6.12a) and (6.12b) we obtain, in general,

$$\zeta^\lambda = \underset{\sim}{e}^\lambda \cdot \underset{\sim}{r}' - r^\lambda = \left(r'^\lambda - r^\lambda\right) + \Gamma^\lambda_{\alpha\sigma}\, \xi^\alpha r^\sigma + \dots . \tag{6.12c}$$

and

$$\eta_\lambda = \underset{\sim}{e}_\lambda \cdot \underset{\sim}{p}' - p_\lambda = \chi_\lambda - \Gamma^\sigma_{\alpha\lambda}\, \xi^\alpha p_\sigma + \dots . \tag{6.12d}$$

Here, $\Gamma^\lambda_{\alpha\sigma}$ and $\Gamma^\sigma_{\alpha\lambda}$ are the 3-space Christoffel symbols[20,21] evaluated at the point $\underset{\sim}{r}$.

Taking the time-derivatives of Eqs. (6.12a) and (6.12b) we obtain

$$\underset{\sim}{u}' = \underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}} , \quad (\underset{\sim}{u}_0 \equiv \dot{\underset{\sim}{r}} ;\ \underset{\sim}{u}' \equiv \dot{\underset{\sim}{r}}') \tag{6.13a}$$

and

$$\underset{\sim}{F}' = \underset{\sim}{F}_0 + \dot{\underset{\sim}{\eta}} , \quad (\underset{\sim}{F}_0 \equiv \dot{\underset{\sim}{p}} ;\ \underset{\sim}{F}' \equiv \dot{\underset{\sim}{p}}') . \tag{6.13b}$$

Also,

$$(\dot{\zeta})^\lambda = \underset{\sim}{e}^\lambda \cdot \underset{\sim}{u}' - u_0^\lambda = \dot{\xi}^\lambda + \Gamma^\lambda_{\alpha\sigma}\, \xi^\alpha u_0^\sigma + \dots , \tag{6.13c}$$

and

$$(\dot{\eta})_\lambda = \underset{\sim}{e}_\lambda \cdot \underset{\sim}{F}' - F_{0\lambda} = \left(F'_\lambda - F_{0\lambda}\right) - \Gamma^\sigma_{\alpha\lambda}\, \xi^\alpha F_{0\sigma} + \dots . \tag{6.13d}$$

Equations (6.13a) and (6.13b) correspond to the two parts of the 6-dimensional vector equation,

$$\frac{d}{dt}\tilde{\underset{\sim}{r}}' = \frac{d}{dt}\tilde{\underset{\sim}{r}} + \frac{d}{dt}\tilde{\underset{\sim}{\zeta}} . \tag{6.14}$$

It should be noted that two corresponding equations are consistent but not the same equations.

From Eq. (6.12b), Eq. (6.13a), and the definitions (6.10a) and (6.10b) we may easily derive the relation between $\underset{\sim}{\eta}$ and $\dot{\underset{\sim}{\zeta}}$. This is

$$\underset{\sim}{\eta} = m\gamma_0 \dot{\underset{\sim}{\zeta}} + m(\gamma' - \gamma_0)(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}) + e(\underset{\sim}{A}' - \underset{\sim}{A}_0) . \tag{6.15}$$

This equation corresponds to Eq. (6.11) and has the advantage over the latter of having no explicit dependence on the metric tensor.

In many practical problems, $(\gamma' - \gamma_0)$ may be assumed negligibly small. This is often true in either the slightly relativistic or the extremely relativistic case. If so, the second term on the right of Eq. (6.15) may be omitted. Furthermore, if $\underset{\sim}{p}$ and $\underset{\sim}{p}'$ are the kinetic momentum, or if $\underset{\sim}{A}_0$ and $\underset{\sim}{A}'$ have only one component and may be absorbed into the scalar potential by a gauge transformation, then the third term may also be absent. In any case, if we denote by $\underset{\sim}{\eta}_K$ the kinetic part of $\underset{\sim}{\eta}$, then

$$\underset{\sim}{\eta}_K = m\gamma_0 \dot{\underset{\sim}{\zeta}} + m(\gamma' - \gamma_0)(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}) . \tag{6.16}$$

This equation may also be written as

$$\underset{\sim}{\eta}_K = \frac{m\left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right)}{\left[1 - \left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right)\cdot\left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right)c^{-2}\right]^{1/2}} - m\gamma_0\underset{\sim}{u}_0 , \tag{6.17a}$$

and conversely,

$$\dot{\underset{\sim}{\zeta}} = \frac{m\gamma_0\underset{\sim}{u}_0 + \underset{\sim}{\eta}_K}{m\left[1 + \left(m\gamma_0\underset{\sim}{u}_0 + \underset{\sim}{\eta}_K\right)\cdot\left(m\gamma_0\underset{\sim}{u}_0 + \underset{\sim}{\eta}_K\right)\left(mc\right)^{-2}\right]^{1/2}} - \underset{\sim}{u}_0 . \tag{6.17b}$$

VII. THE LORENTZ EQUATION

The vector $\underset{\sim}{u}'$ is the perturbed velocity of a certain particle at the phase point (x', p') and the time $t' = t$. The vector $\underset{\sim}{F}'$ is the perturbed force (canonical or kinetic) on this particle at the same phase point and the same time. Thus,

$$\underset{\sim}{u}' \equiv \underset{\sim}{u}(x', p', t) = \Sigma \underset{\sim}{u}(x, p, t) ,$$

i.e.,

$$\underset{\sim}{u}' = \Sigma \underset{\sim}{u} . \tag{7.1a}$$

Here, the Taylor operator Σ is defined in the 6-space (x^α, p_α) by Eq. (2.6). Similarly,

$$\underset{\sim}{F}' \equiv \underset{\sim}{F}(x', p', t) = \Sigma \underset{\sim}{F}(x, p, t),$$

i.e.,

$$\underset{\sim}{F}' = \Sigma \underset{\sim}{F} . \tag{7.1b}$$

Using these relations we obtain from Eq. (6.13a) and Eq. (6.13b)

$$\dot{\underset{\sim}{\zeta}} = \Sigma \underset{\sim}{u} - \underset{\sim}{u}_0 ; \tag{7.2a}$$

$$\dot{\underset{\sim}{\eta}} = \Sigma \underset{\sim}{F} - \underset{\sim}{F}_0 . \tag{7.2b}$$

Equation (7.2a) serves to define the different orders of the velocity vector, $\underset{\sim}{u} = \underset{\sim}{u}_0 + \underset{\sim}{u}_1 + \underset{\sim}{u}_2 + \ldots$, in terms of $\underset{\sim}{\zeta}$ and $\underset{\sim}{\eta}$. For example,

$$\underset{\sim}{u}_1 = \dot{\underset{\sim}{\zeta}}_1 - \xi_1^i \frac{\partial \underset{\sim}{u}_0}{\partial x^i} , \tag{7.3a}$$

$$\underset{\sim}{u}_2 = \dot{\underset{\sim}{\zeta}}_2 - \xi_2^i \frac{\partial \underset{\sim}{u}_0}{\partial x^i} - \xi_1^i \frac{\partial \underset{\sim}{u}_1}{\partial x^i} - \frac{1}{2!} \xi_1^i \xi_1^k \frac{\partial^2 \underset{\sim}{u}_0}{\partial x^i \partial x^k} , \tag{7.3b}$$

etc. Similarly, Eq. (7.2b) defines the different orders of the vector $\underset{\sim}{F}$ and, in conjunction with Eq. (6.15), represents the Lorentz equation of motion. Since

$$\dot{\underset{\sim}{\eta}} = \frac{d}{dt}\left[m\gamma'\left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right)\right] + \Sigma \frac{d}{dt} e\underset{\sim}{A} - \frac{d}{dt}\left(m\gamma_0 \underset{\sim}{u}_0 + e\underset{\sim}{A}_0\right) , \tag{7.4}$$

Eq. (7.2b) becomes

$$\frac{d}{dt}\left[m\gamma'\left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right)\right] - \frac{d}{dt}\left(m\gamma_0 \underset{\sim}{u}_0\right) = \Sigma\left(\underset{\sim}{F} - \frac{d}{dt}\, e\underset{\sim}{A}\right) - \left(\underset{\sim}{F}_0 - \frac{d}{dt}\, e\underset{\sim}{A}_0\right). \qquad (7.5)$$

The 3-vectors $\underset{\sim}{F}$ and $d(e\underset{\sim}{A})/dt$ are evaluated in the usual manner. We have

$$\underset{\sim}{F} = -\frac{\partial eV}{\partial \underset{\sim}{r}} + \left(\frac{\partial}{\partial \underset{\sim}{r}}\, e\underset{\sim}{A}\right) \cdot \underset{\sim}{u} \quad ;$$

$$\frac{d}{dt}(e\underset{\sim}{A}) = \left(\frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \frac{\partial}{\partial \underset{\sim}{r}}\right) e\underset{\sim}{A} \quad .$$

These yield

$$\underset{\sim}{F} - \frac{d}{dt}\, e\underset{\sim}{A} = e\left(\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B}\right). \qquad (7.6)$$

Using this and Eq. (7.2a), we obtain from Eq. (7.5)

$$\frac{d}{dt}\left[m\gamma'\left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right)\right] = e\left[\Sigma\underset{\sim}{E} + \left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right) \times \Sigma\underset{\sim}{B}\right]. \qquad (7.7)$$

The scalar product of this vector equation and the vector $(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}})$ is quite simple. This is

$$\frac{d}{dt}\left(m\gamma' c^2\right) = \left(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}}\right) \cdot \Sigma e\underset{\sim}{E} \quad . \qquad (7.8)$$

The Lorentz equation in its present form may easily be separated into different orders, if γ' may be approximated by γ_0. In any case,

$$\gamma' = \left[1 - \left(u'^2/c^2\right)\right]^{-1/2}$$

$$= \gamma_0\left[1 + \left(\gamma_0^2/c^2\right)\dot{\underset{\sim}{\zeta}} \cdot \underset{\sim}{u}_0 + \ldots\right], \qquad (7.9a)$$

$\left(\gamma_0^2/c^2\right)\dot{\underset{\sim}{\zeta}} \cdot \underset{\sim}{u}_0 \ll 1$, and alternatively

$$\gamma' = \left[1 + \left(\underset{\sim}{p}'_K \cdot \underset{\sim}{p}'_K/m^2c^2\right)\right]^{1/2}$$

$$= \gamma_0\left[1 + \left(\underset{\sim}{\eta}_K \cdot \underset{\sim}{p}_K/m^2\gamma_0^2c^2\right) + \ldots\right], \qquad (7.9b)$$

$\underset{\sim}{\eta}_K \cdot \underset{\sim}{p}_K \ll (m\gamma_0 c)^2$.

Also,

$$\frac{\gamma'}{\gamma_0} = 1 + \frac{\delta\gamma}{\gamma_0} = 1 + \frac{\left(\eta_K - m\gamma_0\dot{\zeta}\right)\cdot p_K}{\left(p_K + m\gamma_0\dot{\zeta}\right)\cdot p_K} \quad . \tag{7.9c}$$

The first-order parts of Eqs. (7.7) and (7.8) are as follows:

$$\frac{d}{dt}\left[m\gamma_0\dot{\zeta}_1 + m(\delta\gamma)_1 u_0\right] = e\left[\left(E_1 + \zeta_1\cdot\frac{\partial}{\partial r}E_0\right) + u_0\times\left(B_1 + \zeta_1\cdot\frac{\partial}{\partial r}B_0\right) + \dot{\zeta}_1\times B_0\right] \cdot \tag{7.10}$$

$$\frac{mc^2}{e}\frac{d}{dt}(\delta\gamma)_1 = u_0\cdot\left(E_1 + \zeta_1\cdot\frac{\partial}{\partial r}E_0\right) + \dot{\zeta}_1\cdot E_0 \quad . \tag{7.11}$$

These equations are, as they should be, precisely the same as obtained before.[10] In Ref. 10, γ' is represented by Eq. (7.9a). When ζ_1 satisfies Eq. (7.10) and η_1 is obtained from Eq. (6.15), the pair of Hamilton's equations

$$u_1^\alpha \equiv (\dot{x}^\alpha)_1 = \partial\mathcal{H}_1/\partial p_\alpha \quad \text{and} \quad (\dot{p}_\alpha)_1 = -\partial\mathcal{H}_1/\partial x^\alpha$$

will also be satisfied. Then, as proved earlier, $[(1/J)(dJ/dt)]_1$ will vanish. This may also be shown directly by evaluating $(d/dt)(\tilde{\nabla}\cdot\tilde{\zeta}_1)$. This is

$$\frac{d}{dt}\left(\frac{\partial}{\partial\tilde{r}}\cdot\tilde{\zeta}_1\right) = \frac{d}{dt}\left(\frac{\partial}{\partial r}\cdot\zeta_1 + \frac{\partial}{\partial p}\cdot\eta_1\right) ,$$

i.e.,

$$\frac{d}{dt}\left(\frac{\partial}{\partial\tilde{r}}\cdot\tilde{\zeta}_1\right) = \left(\frac{\partial}{\partial r}\cdot u_1 + \frac{\partial}{\partial p}\cdot F_1\right) + \left(\zeta_1\cdot\frac{\partial}{\partial r} + \eta_1\cdot\frac{\partial}{\partial p}\right)\left(\frac{\partial}{\partial r}\cdot u_0 + \frac{\partial}{\partial p}\cdot F_0\right) , \tag{7.12}$$

where u_1 is as given by Eq. (7.3a) and F_1 is obtained from Eq. (7.2b),

$$F_1 = \dot{\eta}_1 - \left(\zeta_1\cdot\frac{\partial}{\partial r} + \eta_1\cdot\frac{\partial}{\partial p}\right)F_0 \quad . \tag{7.13}$$

Since

$$\frac{\partial}{\partial r}\cdot u + \frac{\partial}{\partial p}\cdot F = 0$$

because of Hamilton's equations, $(d/dt)(\tilde{\nabla}\cdot\tilde{\zeta}_1) = 0$, i.e., $[(1/J)(dJ/dt)]_1 = 0$. Therefore, $(d\psi/dt)_1 = 0$.

VIII. CANONICAL VS. KINETIC PHASE SPACE

In this section we will discuss the kinetic phase space explicitly and compare this with the canonical case. Sections II, III, and IV apply also to the case where $p = p_K$, the kinetic momentum. The proof of Eq. (5.10) in Section V remains the same. The difference between the two cases appears when Hamiltonian functions are used and when the vector potential is present.

In kinetic phase space the Hamiltonians have the following expressions:

$$\mathcal{H}_0(x, p, t) = eV_0 + mc^2\left(1 + \frac{1}{m^2c^2}\, p^\alpha p_\alpha\right)^{1/2} . \tag{8.1}$$

$$\mathcal{H}(x', p', t) = eV' + mc^2\left(1 + \frac{1}{m^2c^2}\, p'^\alpha p'_\alpha\right)^{1/2} \tag{8.2}$$

$$= \Sigma\mathcal{H}(x, p, t) . \tag{8.3}$$

Here, p_α and p'_α are kinetic momentum components. Σ is the Taylor operator which, as defined by Eq. (2.6), may be written as

$$\Sigma = \exp\left(\xi^\alpha \frac{\partial}{\partial x^\alpha} + \chi_\alpha \frac{\partial}{\partial p_\alpha}\right) .$$

One of the two canonical equations remains the same,

$$\dot{x}'^\alpha = \partial\mathcal{H}(x', p', t)/\partial p'_\alpha , \tag{8.4}$$

but the other one must be replaced by

$$-\dot{p}'_\alpha = \frac{\partial}{\partial x'^\alpha}\mathcal{H}(x', p', t) + \frac{\partial eA'_\alpha}{\partial t} + \dot{x}'^\beta\left(\frac{\partial eA'_\alpha}{\partial x'^\beta} - \frac{\partial eA'_\beta}{\partial x'^\alpha}\right) . \tag{8.5a}$$

Here,

$$\frac{\partial}{\partial x'^\alpha}\mathcal{H}(x', p', t) = \frac{\partial eV'}{\partial x'^\alpha} - \Gamma'^\mu_{\alpha\beta}\, p'_\mu\, \dot{x}'^\beta , \tag{8.5b}$$

in which $\Gamma'^\mu_{\alpha\beta}$ denotes the 3-space Christoffel symbol evaluated at the point $\underset{\sim}{r}'$.

Equation (8.4) is obvious; Eq. (8.5a) follows from the fact that, in kinetic phase space, the Lorentz equation is

$$\begin{aligned} \dot{\underset{\sim}{p}}' &= e(\underset{\sim}{E}' + \underset{\sim}{u}' \times \underset{\sim}{B}') \\ &= -e\left[\frac{\partial}{\partial \underset{\sim}{r}'} V' + \dot{\underset{\sim}{A}}' - \left(\frac{\partial}{\partial \underset{\sim}{r}'} \underset{\sim}{A}'\right) \cdot \underset{\sim}{u}'\right] . \end{aligned} \tag{8.6}$$

Thus we again have

$$\frac{\partial \dot{p}'_\alpha}{\partial p'_\alpha} = - \frac{\partial}{\partial p'_\alpha} \frac{\partial}{\partial x'^\alpha} \mathcal{H}(x', p', t)$$

and

$$\frac{\partial \dot{x}'^i}{\partial x'^i} = \frac{\partial \dot{x}'^\alpha}{\partial x'^\alpha} + \frac{\partial \dot{p}'_\alpha}{\partial p'_\alpha} = 0 ,$$

because

$$\frac{\partial \dot{x}'^\beta}{\partial p'_\alpha} = \frac{\partial \dot{x}'^\alpha}{\partial p'_\beta} = \frac{1}{m\gamma'}\left(g'^{\alpha\beta} - \frac{1}{c^2} \dot{x}'^\alpha \dot{x}'^\beta\right) \tag{8.7a}$$

and

$$\frac{\partial}{\partial p'_\alpha}\left[\dot{x}'^\beta\left(\frac{\partial A'_\alpha}{\partial x'^\beta} - \frac{\partial A'_\beta}{\partial x'^\alpha}\right)\right] = \frac{1}{2}\left(\frac{\partial A'_\alpha}{\partial x'^\beta} - \frac{\partial A'_\beta}{\partial x'^\alpha}\right)\left(\frac{\partial \dot{x}'^\beta}{\partial p'_\alpha} + \frac{\partial \dot{x}'^\alpha}{\partial p'_\beta}\right) = 0 . \tag{8.7b}$$

It then follows that, if the Lorentz equation is satisfied,

$$\frac{1}{J}\frac{dJ}{dt} = 0 \quad \text{and} \quad \frac{d\psi}{dt} = 0 .$$

Although these may be concluded from other simpler arguments, it seems instructive to compare the underlying mathematics for both cases.

There is another feature which distinguishes the kinetic from the canonical phase space. This concerns the velocity vector, which differs from the kinetic momentum vector only by a scalar factor.

$$\underset{\sim}{u} = \underset{\sim}{p}/m\gamma = \underset{\sim}{p}/\left(m^2 + p^\alpha p_\alpha/c^2\right)^{1/2} .$$

Thus,

$$u^{\alpha} = \frac{\partial}{\partial p_{\alpha}} \mathcal{H}(x, p, t) = \frac{\partial}{\partial p_{\alpha}} \mathcal{H}_0(x, p, t) ,$$

i.e.,

$$\underset{\sim}{u}(x, p, t) = \underset{\sim}{u}_0(x, p, t) , \qquad (8.8)$$

because

$$\mathcal{H} - \mathcal{H}_0 = eV - eV_0$$

and this is independent of $\underset{\sim}{p}$. In kinetic phase space, $\underset{\sim}{u}_1 = \underset{\sim}{u}_2 = \ldots\ldots = 0$. In this respect, the velocity components u^{α} or u_{α} behave like coordinate variables x^{α} and p_{α}, and the kinetic phase space (x^{α}, p_{α}) is akin to the velocity phase space (x^{α}, u_{α}).

From Eq. (7.2a) we obtain, in view of Eq. (8.8),

$$\dot{\underset{\sim}{\zeta}} = (\Sigma - 1)\, \underset{\sim}{u}_0 . \qquad (8.9)$$

Thus, Eq. (6.16) becomes

$$\underset{\sim}{\eta} = \underset{\sim}{\eta}_K = m\gamma_0 \dot{\underset{\sim}{\zeta}} + m(\gamma' - \gamma_0)\, \Sigma \underset{\sim}{u}_0 . \qquad (8.10)$$

The Lorentz equation (7.7) or (8.6) may then be written as

$$(d/dt)\left[m\gamma'(\underset{\sim}{u}_0 + \dot{\underset{\sim}{\zeta}})\right] = \Sigma\, e(\underset{\sim}{E} + \underset{\sim}{u}_0 \times \underset{\sim}{B}) . \qquad (8.11)$$

The first-order part of this equation is, as it should be, the same as Eq. (7.10).

In canonical phase space $(\underset{\sim}{p} = \underset{\sim}{p}_K + e\underset{\sim}{A})$,

$$\left(\frac{d\psi}{dt}\right)_1 = \frac{\partial \psi_1}{\partial t} + \left[\psi_1, \mathcal{H}_0\right] + \left[\psi_0, \mathcal{H}_1\right]$$

$$= \left(\frac{\partial}{\partial t} + \frac{\partial \mathcal{H}_0}{\partial p_{\alpha}} \frac{\partial}{\partial x^{\alpha}} - \frac{\partial \mathcal{H}_0}{\partial x^{\alpha}} \frac{\partial}{\partial p_{\alpha}}\right)\psi_1 + \left(\frac{\partial \mathcal{H}_1}{\partial p_{\alpha}} \frac{\partial}{\partial x^{\alpha}} - \frac{\partial \mathcal{H}_1}{\partial x^{\alpha}} \frac{\partial}{\partial p_{\alpha}}\right)\psi_0 . \qquad (8.12)$$

In kinetic phase space ($\underset{\sim}{p} = \underset{\sim}{p}_K$) ,

$$\left(\frac{d\psi}{dt}\right)_1 = \left[\frac{\partial}{\partial t} + u_0^\alpha \frac{\partial}{\partial x^\alpha} + \left(\dot{p}_\alpha\right)_0 \frac{\partial}{\partial p_\alpha}\right]\psi_1 + \left(\dot{p}_\alpha\right)_1 \frac{\partial \psi_0}{\partial p_\alpha} \quad , \tag{8.13}$$

where

$$\left(\dot{p}_\alpha\right)_0 = \underset{\sim}{e}_\alpha \cdot e\left(\underset{\sim}{E}_0 + \underset{\sim}{u}_0 \times \underset{\sim}{B}_0\right) + \Gamma^\mu_{\alpha\beta}\, p_\mu\, u_0^\beta \tag{8.14a}$$

and

$$\left(\dot{p}_\alpha\right)_1 = \underset{\sim}{e}_\alpha \cdot e\left(\underset{\sim}{E}_1 + \underset{\sim}{u}_0 \times \underset{\sim}{B}_1\right) \tag{8.14b}$$

are the first two orders of

$$\dot{p}_\alpha = \left(\dot{p}\right)_\alpha + \Gamma^\mu_{\alpha\beta}\, p_\mu\, u^\beta \ . \tag{8.15}$$

As discussed earlier, $(d\psi/dt)_1 = 0$ if ψ_1 is as given by Eq. (3.6a), i.e.,

$$\psi_1 = -\frac{\partial}{\partial \underset{\sim}{\tilde{r}}} \cdot \left(\psi_0\, \underset{\sim}{\tilde{\xi}}_1\right),$$

and $\tilde{\xi}_1$ satisfies the condition required by the Liouville theorem $\left[(1/J)(dJ/dt)\right]_1 = 0$.

IX. AN EXAMPLE: MULTIPOLE OSCILLATIONS OF A BUNCHED BEAM

Lee, Mills, and Morton[22] have described a self-consistent solution of multipole oscillations of a bunched electron beam in connection with their work on storage-ring beam instabilities. The bunched beam is circular in cross section with radius $\underline{a}$ and travels along the axial z-direction with a constant velocity. For our present purpose, the beam may be assumed to be enclosed in a circular waveguide of radius $\underline{b}$, $b \gg a$, and of infinite length in the z-direction. In the unperturbed state, the distribution of the charged particles in the beam is supposed to be axially symmetric, and the particles execute simple harmonic motion of small amplitudes in radial directions with a certain characteristic frequency ω_0 determined by the electromagnetic field acting on the beam. The transverse momentum of any particle is assumed to be negligibly small in comparison with its longitudinal momentum. Their problem is to determine whether such a beam may become unstable with respect to transverse oscillations of multipole symmetry. We will first describe their formulation of the unperturbed problem in kinetic phase-space, and then use a specific displacement vector to derive ψ_1 from which the charge and current densities, ρ_1 and $\underset{\sim}{i}_1$, are obtained. As discussed in previous sections, if the displacement vector satisfies the Lorentz equation, then ψ_1 satisfies the Boltzmann-Vlasov equation, and vice versa.

The unperturbed state is characterized by the Hamiltonian $\mathcal{H}_0$ and the charge distribution function ψ_0.

$$\mathcal{H}_0 = \frac{1}{2} M \omega_0^2 \left(x^2 + y^2\right) + mc^2 \left(1 + \frac{\underset{\sim}{p} \cdot \underset{\sim}{p}}{m^2 c^2}\right)^{1/2} . \tag{9.1}$$

$$\psi_0 = \left(eN/2\pi^2 a^2\right) \cdot f\left(z - v_0 t\right) \delta\left(p_z - p_{z0}\right) \cdot \frac{1}{M} \delta\left[\frac{1}{2M}\left(p_x^2 + p_y^2\right) + \frac{1}{2} M \omega_0^2 \left(x^2 + y^2 - a^2\right)\right] . \tag{9.2}$$

In these equations, all quantities other than the independent variables (x, p, t) are constants; $M = m\left(1 - v_0^2/c^2\right)^{-1/2}$ being approximately the relativistic mass, $p_{z0} = Mv_0$ and ω_0 is certain characteristic frequency which defines the scalar potential such that

$$e V_0 = (1/2) M \omega_0^2 (x^2 + y^2) \ . \tag{9.3}$$

The two δ-functions are Dirac functions. The function $f(z - v_0 t)$ describes the degree of longitudinal bunching and is so normalized that N represents the total number of charged particles in the whole system,

$$\int_{-\infty}^{\infty} f(z)\, dz = 1 \ .$$

The constant $\underline{a}$ will be shown to be the radius of the beam.

Assuming that $\underset{\sim}{A}_0 = 0$, we obtain from $\mathcal{H}_0$

$$\dot{p}_x = - M\omega_0^2 x \ , \qquad \dot{p}_y = - M\omega_0^2 y \ , \qquad \dot{p}_z = 0 \ , \tag{9.4}$$

and

$$\dot{x}^\alpha = \left(p^\alpha/m\right) \left(1 + \frac{\underset{\sim}{p} \cdot \underset{\sim}{p}}{m^2 c^2}\right)^{-1/2} \ . \tag{9.5}$$

It can then be shown that

$$\frac{d\psi_0}{dt} = \left(\frac{\partial}{\partial t} + \dot{x}^\alpha \frac{\partial}{\partial x^\alpha} + \dot{p}_\alpha \frac{\partial}{\partial p_\alpha}\right) \psi_0 = 0$$

if $\dot{x}^\alpha$ is replaced by p^α/M, which is a valid approximation as long as $\omega_0^2 a^2 \ll c^2$.

From ψ_0 we obtain the charge density by integrating over the whole momentum space.

$$\rho_0(\underset{\sim}{r}, t) = \int \psi_0 (x, p, t) (dp)^3$$

$$= (eN/\pi a^2) f(z - v_0 t) \, U(a - \kappa) \ , \tag{9.6}$$

where $\kappa = (x^2 + y^2)^{1/2}$ and $U(a - \kappa)$ is the Heaviside unit-step function. Thus, the unperturbed beam is a circular beam of radius $\underline{a}$. The charge density inside the beam $(\kappa < a)$ is uniform with respect to any transverse direction. Individual particles travel with a constant velocity v_0 along the z-direction and perform small harmonic motion on the transverse plane with a restoring force-constant $M\omega_0^2$ according to Eqs. (9.4). Similarly, the unperturbed current density is

$$i_0^{\alpha}(\underset{\sim}{r}, t) = \int \dot{x}^{\alpha} \psi_0 (x, p, t)(dp)^3 \quad ,$$

i.e.,

$$\begin{aligned} \underset{\sim}{i}_0 &= \underset{\sim}{a}_z v_0 \rho_0(\underset{\sim}{r}, t) \left[1 + \left(\omega_0^2/c^2\right)\left(a^2 - \kappa^2\right)\right]^{-1/2} \\ &\cong \underset{\sim}{a}_z v_0 \rho_0(\underset{\sim}{r}, t) \quad , \qquad \text{if} \quad \omega_0^2 a^2 \ll c^2 \quad . \end{aligned} \tag{9.7}$$

The electromagnetic fields $\underset{\sim}{E}_0$ and $\underset{\sim}{B}_0$ are the resultant fields arising from both the beam itself and whatever external sources there may exist. These fields are the solutions of Maxwell's equations for the prescribed beam $(\underset{\sim}{i}_0, \rho_0)$ under certain boundary conditions which are required to account for the presence of material boundaries and external sources. The fields $(\underset{\sim}{E}_0, \underset{\sim}{B}_0)$ should satisfy the Lorentz equation,

$$e\left(\underset{\sim}{E}_0 + \underset{\sim}{u}_0 \times \underset{\sim}{B}_0\right) = - M\omega_0^2\left(\underset{\sim}{a}_x x + \underset{\sim}{a}_y y\right) \quad , \tag{9.8}$$

in order that $\mathcal{H}_0$ and ψ_0, as given above, may characterize the unperturbed state correctly.

When this beam-field system is perturbed, individual particles will execute orbits slightly different from the unperturbed ones. This will give rise to a small current- and charge-density $(\underset{\sim}{i}_1, \rho_1)$, which will induce a weak field $(\underset{\sim}{E}_1, \underset{\sim}{B}_1)$, which in turn would act on the beam to give rise to small current and charge densities. The act of perturbation upon perturbation proceeds indefinitely. A physically realizable (self-consistent) first-order solution is represented by the quantities $(\underset{\sim}{i}_1, \rho_1)$ and $(\underset{\sim}{E}_1, \underset{\sim}{B}_1)$, when they simultaneously satisfy the Lorentz and Maxwell equations under appropriate supplementary conditions.

Now we consider an oscillatory perturbation of certain 2ℓ-pole symmetry represented by the following displacement vector in 3-space :

$$\underset{\sim}{\zeta}_1 = \epsilon\, e^{-j\omega t}\left[\underset{\sim}{e}_\kappa\, \kappa \cos \ell\phi - \underset{\sim}{e}_\phi\, (2/\ell) \sin \ell\phi\right] . \tag{9.9}$$

In this equation, ϵ denotes a small dimensionless parameter ($\epsilon \ll 1$) and ω an unknown frequency to be determined; (κ, ϕ, z) are the usual cylindrical coordinates, and $(\underset{\sim}{e}_\kappa, \underset{\sim}{e}_\phi, \underset{\sim}{e}_z)$ the three covariant base vectors. It is to be noted that $\underset{\sim}{\zeta}_1$ is independent of the momentum variables and $\underset{\sim}{\nabla} \cdot \underset{\sim}{\zeta}_1 = 0$. Taking the time-derivative of Eq. (9.9) and denoting $\epsilon\, e^{-j\omega t} \equiv \tilde{\epsilon}$, we obtain

$$\begin{aligned}\dot{\underset{\sim}{\zeta}}_1 = {} & \underset{\sim}{e}^\kappa \tilde{\epsilon}\left[-j\omega\kappa \cos \ell\phi + \dot{\kappa} \cos \ell\phi + \left(\frac{2}{\ell} - \ell\right)\kappa\dot{\phi} \sin \ell\phi\right] \\ & + \underset{\sim}{e}^\phi \tilde{\epsilon}\left[j\omega\kappa^2 \frac{2}{\ell} \sin \ell\phi - \kappa^2\dot{\phi} \cos \ell\phi - \kappa\dot{\kappa}\, \frac{2}{\ell} \sin \ell\phi\right] .\end{aligned} \tag{9.10}$$

The momentum-part η_α of the displacement vector $\tilde{\underset{\sim}{\zeta}}$ does not contribute to the charge density ρ. This may be seen easily by integrating the function ψ over the momentum space to obtain ρ.

$$\begin{aligned}\rho(\underset{\sim}{r}, t) &= \int \psi(x, p, t)(dp)^3 = \int \Omega\, \psi_0(x, p, t)(dp)^3 \\ &= \rho_0(\underset{\sim}{r}, t) + \int (\Omega - 1)\psi_0 (x, p, t)(dp)^3 .\end{aligned}$$

The part of the integrand $(\Omega - 1)\psi_0$, which contains η_α, is a complete divergence in the momentum space. The volume integral of a divergence expression may be transformed to a surface integral and therefore vanishes because ψ_0 vanishes on the surface at $|\underset{\sim}{p}| = \infty$.

Despite its null effect on ρ, the vector $\underset{\sim}{\eta}$ is required for the representation of ψ. According to Eq. (8.10),

$$\underset{\sim}{\eta}_1 = m\gamma_0\, \dot{\underset{\sim}{\zeta}} + m(\delta\gamma)_1\, \underset{\sim}{u}_0 ,$$

i.e.,

$$\underset{\sim}{\eta}_1 = m\gamma_0 \dot{\underset{\sim}{\zeta}}_1 \cdot \left(\underset{\sim}{\mathcal{I}} + \frac{\gamma_0^2}{c^2} \underset{\sim}{u}_0 \underset{\sim}{u}_0 \right) , \tag{9.11a}$$

i.e.,

$$\underset{\sim}{\eta}_1 \cdot \left(\underset{\sim}{\mathcal{I}} - \frac{\underset{\sim}{u}_0 \underset{\sim}{u}_0}{c^2} \right) = m\gamma_0 \dot{\underset{\sim}{\zeta}}_1 \quad . \tag{9.11b}$$

The second term on the left of Eq. (9.11b) is negligible if $\underset{\sim}{\eta}_1 \cdot \underset{\sim}{u}_0 \ll |\eta_1| \cdot |u_0|$. Under this condition, $\underset{\sim}{\eta}_1 = m\gamma_0 \dot{\underset{\sim}{\zeta}}_1$, i.e.,

$$\underset{\sim}{\eta}_1 = \tilde{\epsilon} \left\{ \underset{\sim}{e}^{\kappa} \left[- j\omega M \kappa \cos \ell\phi + p_\kappa \cos \ell\phi + \left(\frac{2}{\ell} - \ell \right) \frac{p_\phi}{\kappa} \sin \ell\phi \right] \right.$$

$$\left. + \underset{\sim}{e}^{\phi} \left[j\omega M \kappa^2 \frac{2}{\ell} \sin \ell\phi - p_\phi \cos \ell\phi - \kappa p_\kappa \frac{2}{\ell} \sin \ell\phi \right] \right\} \quad . \tag{9.12}$$

The momentum-part of $\left(\tilde{\underset{\sim}{\nabla}} \cdot \tilde{\underset{\sim}{\zeta}}_1 \right)$ is

$$\frac{\partial}{\partial \underset{\sim}{p}} \cdot \underset{\sim}{\eta}_1 = \frac{\partial}{\partial p_\alpha} \eta_{1\alpha} \quad . \tag{9.13}$$

This equation applies to any spatial coordinate system. From Eq. (9.12) we obtain

$$\frac{\partial}{\partial \underset{\sim}{p}} \cdot \underset{\sim}{\eta}_1 = 0 \, .$$

Since $\underset{\sim}{\zeta}_1$ is independent of $\underset{\sim}{p}$,

$$\frac{\partial}{\partial \underset{\sim}{r}} \cdot \underset{\sim}{\zeta}_1 = \underset{\sim}{\nabla} \cdot \underset{\sim}{\zeta}_1 = 0 \, .$$

Therefore,

$$\tilde{\underset{\sim}{\nabla}} \cdot \tilde{\underset{\sim}{\zeta}}_1 = \frac{\partial}{\partial \underset{\sim}{r}} \cdot \underset{\sim}{\zeta}_1 + \frac{\partial}{\partial \underset{\sim}{p}} \cdot \underset{\sim}{\eta}_1 = 0$$

and $[(1/J)(dJ/dt)]_1 = 0$, whatever the unknown frequency ω is. (If $\underset{\sim}{\zeta}_1$ is not independent of $\underset{\sim}{p}$, then

$$\frac{\partial}{\partial \underset{\sim}{r}} \cdot \underset{\sim}{\zeta}_1 = \left(\underset{\sim}{\nabla} \cdot \underset{\sim}{\zeta}_1 \right) \Big|_{p_\alpha \text{ fixed}} + \Gamma^\sigma_{\lambda\mu} \, p_\sigma \frac{\partial \zeta_1^\mu}{\partial p_\lambda} \, .)$$

This result may easily be verified by transforming $\underset{\sim}{\zeta}_1$ and $\underset{\sim}{\eta}_1$ to refer to Cartesian coordinates. We may also evaluate the divergence of the 6-vector $\underset{\sim}{\tilde{\zeta}}_1$ directly. The latter is

$$\underset{\sim}{\tilde{\zeta}}_1 = \tilde{\varepsilon}\Bigg[\underset{\sim}{\tilde{e}}_1 \, \kappa \cos \ell\phi - \underset{\sim}{\tilde{e}}_2 \, \frac{2}{\ell} \sin \ell\phi + \underset{\sim}{\tilde{e}}_4\left(- j\,\omega M \,\kappa \cos \ell\phi + p_\kappa \cos \ell\phi - \frac{p_\phi}{\kappa} \,\ell \sin \ell\phi\right) + \underset{\sim}{\tilde{e}}_5\left(j\omega M \,\kappa^2 \, \frac{2}{\ell} \sin \ell\phi\right)\Bigg] \, . \tag{9.14}$$

Thus,

$$\underset{\sim}{\tilde{\nabla}} \cdot \underset{\sim}{\tilde{\zeta}}_1 = \frac{1}{\sqrt{\tilde{g}}} \, \frac{\partial}{\partial x^i} \left(\zeta_1^i \sqrt{\tilde{g}}\right) = \frac{\partial}{\partial x^i} \, \zeta_1^i = 0 \; ,$$

because the 6-dimensional metric determinant $\tilde{g}$ is unity. Here, we may note that $\zeta_1^{3+\alpha} \neq \eta_{1\alpha}$ except in Cartesian coordinates.

The first-order distribution function is given in terms of $\underset{\sim}{\zeta}_1$ and $\underset{\sim}{\eta}_1$ as follows:

$$\psi_1 = - \frac{\partial}{\partial \underset{\sim}{r}} \cdot \left(\psi_0 \underset{\sim}{\zeta}_1\right) - \frac{\partial}{\partial \underset{\sim}{p}} \cdot \left(\psi_0 \underset{\sim}{\eta}_1\right) \; ,$$

i.e.,

$$\psi_1 = - \underset{\sim}{\zeta}_1 \cdot \frac{\partial \psi_0}{\partial \underset{\sim}{r}} - \underset{\sim}{\eta}_1 \cdot \frac{\partial \psi_0}{\partial \underset{\sim}{p}} \; . \tag{9.15}$$

Since $d\psi_0/dt = 0$ and $\underset{\sim}{\tilde{\nabla}} \cdot \underset{\sim}{\tilde{\zeta}}_1 = 0$, this ψ_1 satisfies the Liouville equation $(d\psi/dt)_1 = 0$ which, however, should not be confused with the Boltzmann-Vlasov equation. Both equations are given by Eq. (8.13), but they differ in the coefficient $(\dot{p}_\alpha)_1$. In the Boltzmann-Vlasov equation this coefficient is given by Eq. (8.14b); in the other

$$(\dot{p}_\alpha)_1 = (\dot{p})_{1\alpha} = \underset{\sim}{e}_\alpha \cdot \underset{\sim}{F}_1$$

and $\underset{\sim}{F}_1$ is given by Eq. (7.13).

It is quite plausible, though not assuredly, to believe that this ψ_1 has the appropriate functional form to enable one to calculate reliably $(\underset{\sim}{i}_1, \rho_1)$ and $(\underset{\sim}{E}_1, \underset{\sim}{B}_1)$. When the latter field intensities are calculated, one may then set up the Lorentz equation or the Boltzmann-Vlasov equation in order to determine the yet unknown frequency ω. Whether the oscillation will grow or be damped depends on the imaginary part of this complex frequency.

Using Eqs. (9.9), (9.12), and (9.15) we obtain

$$\begin{aligned} \psi_1(x, p, t) = \tilde{\epsilon}\left(eN/2\pi^2 a^2\right) f\left(z - v_0 t\right) \delta\left(p_z - p_{z0}\right) \cdot \\ \Bigg\{ \frac{p_y^2 - p_x^2}{M^2}\left(\cos 2\phi \cos \ell\phi + \frac{\ell}{2} \sin 2\phi \sin \ell\phi\right) \\ + \frac{2 p_x p_y}{M^2}\left(\sin 2\phi \cos \ell\phi - \frac{\ell}{2} \cos 2\phi \sin \ell\phi\right) \\ - \omega_0^2 \left(x^2 + y^2\right) \cos \ell\phi \\ + \frac{j\omega}{M}\left[\left(y p_y + x p_x\right) \cos \ell\phi + \left(y p_x - x p_y\right) \frac{2}{\ell} \sin \ell\phi\right] \Bigg\} \cdot \\ \delta'\left[\frac{1}{2M}\left(p_x^2 + p_y^2\right) + \frac{1}{2} M\omega_0^2 \left(x^2 + y^2 - a^2\right)\right]. \end{aligned} \tag{9.16}$$

Here, $\delta'[w] = (d/dw)\delta[w]$. Then,

$$\rho_1 = \int \psi_1 (dp)^3 = -\frac{\partial}{\partial \underset{\sim}{r}} \cdot \left[\int \psi_0 \underset{\sim}{\zeta}_1 (dp)^3\right] . \tag{9.17}$$

Since $\underset{\sim}{\zeta}_1$ is independent of $\underset{\sim}{p}$ in our present case,

$$\rho_1 = -\frac{\partial}{\partial \underset{\sim}{r}} \cdot \left(\rho_0 \underset{\sim}{\zeta}_1\right) \tag{9.18}$$

$$= \tilde{\epsilon}\,(eN/\pi a)\, f\,(z - v_0 t) \cos \ell\phi \;\delta(a - \kappa) . \tag{9.19}$$

Noting that in kinetic phase-space $\underset{\sim}{u}_1 = 0$ and

$$\dot{\underset{\sim}{\zeta}}_1 = \left(\underset{\sim}{\zeta}_1 \cdot \frac{\partial}{\partial \underset{\sim}{r}} + \underset{\sim}{\eta}_1 \cdot \frac{\partial}{\partial \underset{\sim}{p}}\right) \underset{\sim}{u}_0 ,$$

we obtain the current density

$$\underset{\sim}{i}_1 = \int \underset{\sim}{u}_0 \psi_1 \,(dp)^3 = -\int \underset{\sim}{u}_0 \left[\frac{\partial}{\partial \underset{\sim}{r}} \cdot \left(\psi_0 \underset{\sim}{\zeta}_1\right) + \frac{\partial}{\partial \underset{\sim}{p}} \cdot \left(\psi_0 \underset{\sim}{\eta}_1\right)\right] (dp)^3$$

$$= \int \left[\psi_0 \dot{\underset{\sim}{\zeta}}_1 - \frac{\partial}{\partial \underset{\sim}{r}} \cdot \left(\psi_0 \underset{\sim}{\zeta}_1 \underset{\sim}{u}_0\right)\right] (dp)^3 .$$

Since $d\psi_0/dt = 0$, the integrand may be transformed as follows:

$$\psi_0 \dot{\underset{\sim}{\zeta}}_1 - \frac{\partial}{\partial \underset{\sim}{r}} \cdot \left(\psi_0 \underset{\sim}{\zeta}_1 \underset{\sim}{u}_0\right) = \frac{d}{dt}\left(\psi_0 \underset{\sim}{\zeta}_1\right) - \frac{\partial}{\partial \underset{\sim}{r}} \cdot \left(\psi_0 \underset{\sim}{\zeta}_1 \underset{\sim}{u}_0\right)$$

$$= \frac{\partial}{\partial t}\left(\psi_0 \underset{\sim}{\zeta}_1\right) + \frac{\partial}{\partial \underset{\sim}{r}} \cdot \left(\psi_0 \underset{\sim}{u}_0 \underset{\sim}{\zeta}_1 - \psi_0 \underset{\sim}{\zeta}_1 \underset{\sim}{u}_0\right) + \frac{\partial}{\partial \underset{\sim}{p}} \cdot \left(\psi_0 \underset{\sim}{F}_0 \underset{\sim}{\zeta}_1\right)$$

$$- \psi_0 \underset{\sim}{\zeta}_1 \left(\frac{\partial}{\partial \underset{\sim}{r}} \cdot \underset{\sim}{u}_0 + \frac{\partial}{\partial \underset{\sim}{p}} \cdot \underset{\sim}{F}_0\right) .$$

Hence, in view of the relation $\frac{\partial}{\partial \underset{\sim}{r}} \cdot \underset{\sim}{u}_0 + \frac{\partial}{\partial \underset{\sim}{p}} \cdot \underset{\sim}{F}_0 = 0$,

$$\underset{\sim}{i}_1 = \int \left[\frac{\partial}{\partial t}\left(\psi_0 \underset{\sim}{\zeta}_1\right) + \frac{\partial}{\partial \underset{\sim}{r}} \cdot \left(\psi_0 \underset{\sim}{u}_0 \underset{\sim}{\zeta}_1 - \psi_0 \underset{\sim}{\zeta}_1 \underset{\sim}{u}_0\right)\right] (dp)^3 . \tag{9.20}$$

Thus,

$$\underset{\sim}{i}_1 = \frac{\partial}{\partial t}\left(\rho_0 \underset{\sim}{\zeta}_1\right) + \frac{\partial}{\partial \underset{\sim}{r}} \cdot \left(\underset{\sim}{i}_0 \underset{\sim}{\zeta}_1 - \underset{\sim}{\zeta}_1 \underset{\sim}{i}_0\right) . \tag{9.21}$$

This result agrees with the corresponding expression obtained previously.[9,10]

On substitution of Eq. (9.9), $\underset{\sim}{i}_1$ becomes

$$\underset{\sim}{i}_1 = \tilde{\epsilon}\,(eN/\pi a)\, f\left(z - v_0 t\right)\Big[-(j\omega/a)\, U\,(a - \kappa)\,\left(\underset{\sim}{e}_\kappa \kappa \cos \ell\phi - \underset{\sim}{e}_\phi\, 2\ell^{-1} \sin \ell\phi\right) + \underset{\sim}{e}_z v_0 \cos \ell\phi\; \delta(a - \kappa)\Big] . \tag{9.22}$$

This expression of $\underset{\sim}{i}_1$ is the same as obtained by direct integration of the integral $\int \underset{\sim}{u}_0 \psi_1 (dp)^3$, in which ψ_1 is as given by Eq. (9.16) and $\underset{\sim}{u}_0$ is replaced by $\underset{\sim}{p}/M$.

Both Eq. (9.19) and Eq. (9.22) agree with the corresponding expressions considered by Lee et al.[22] The quadrupole case is particularly simple. When $\ell = 2$, we have

$$\underset{\sim}{\zeta}_1 = \tilde{\epsilon} (\underset{\sim}{a}_x x - \underset{\sim}{a}_y y) , \quad (9.23a)$$

$$\underset{\sim}{\eta}_1 = \tilde{\epsilon} (\underset{\sim}{a}_x p_x - \underset{\sim}{a}_y p_y) - j\omega M \underset{\sim}{\zeta}_1 , \quad (9.23b)$$

$$\psi_1 = \tilde{\epsilon} \left(eN/2\pi^2 a^2\right) f\left(z - v_0 t\right) \delta \left(p_z - p_{z0}\right) \cdot \left[M^{-2}\left(p_y^2 - p_x^2\right) - \omega_0^2\left(x^2 - y^2\right) + j\omega M^{-1}\left(x p_x - y p_y\right)\right] \cdot \delta' \left[\frac{1}{2M}\left(p_x^2 + p_y^2\right) + \frac{1}{2} M\omega_0^2 \left(x^2 + y^2 - a^2\right)\right] , \quad (9.23c)$$

$$\rho_1 = \tilde{\epsilon} \, (eN/\pi a) \, f\left(z - v_0 t\right) \cos 2\phi \; \delta(a - \kappa) , \quad (9.23d)$$

and

$$\underset{\sim}{i}_1 = - j \omega \rho_0 \underset{\sim}{\zeta}_1 + \underset{\sim}{a}_z \rho_1 v_0 . \quad (9.23e)$$

As discussed earlier, from ρ_1 and $\underset{\sim}{i}_1$ one obtains $\underset{\sim}{E}_1$ and $\underset{\sim}{B}_1$. The unknown complex frequency ω, which determines the stability of the multipole oscillation, may then be obtained by two alternative procedures, one using the Boltzmann-Vlasov equation and the other the Lorentz equation. In most problems there are some simplifying approximations. For the present example, $e(\underset{\sim}{E}_0 + \underset{\sim}{u}_0 \times \underset{\sim}{B}_0)_\perp$ is replaced by $- \underset{\sim}{\nabla}_\perp e V_0 = - \underset{\sim}{e}_\kappa M \omega_0^2 \kappa$ and $\underset{\sim}{\eta}_1 = (m\gamma' \underset{\sim}{u}' - m\gamma_0 \underset{\sim}{u}_0)_1$ by $M \dot{\underset{\sim}{\zeta}}_1$. The two results obtained by these alternative methods should agree with each other to the desired degree of accuracy. If not, one or the other calculation may be improved upon, whichever is more convenient.

X. SUMMARY AND DISCUSSION

The perturbed distribution function $\psi(x, p, t)$ may always be represented by $\Omega\psi_0(x, p, t)$ in terms of the unperturbed function ψ_0 and a 6-dimensional displacement vector $\tilde{\underset{\sim}{\zeta}}(x, p, t)$ or $\xi^i(x, p, t)$ which defines the Lagrange operator Ω according to Eq. (2.9). The total time-derivative $d\psi_0/dt$ should vanish, because ψ_0 represents a physically realizable state. When this is true and the displacement vector satisfies the Liouville theorem, Eq. (5.5), $d\psi/dt$ will also vanish.

The solution of an electrodynamical problem must not only satisfy the Liouville equation $d\psi/dt = 0$, but also be consistent with the Lorentz equation. If $\tilde{\underset{\sim}{\zeta}}$ is such that the latter equation is satisfied, then $\psi = \Omega\psi_0$ will satisfy the former, but not conversely. On the other hand, the Boltzmann-Vlasov equation is a combination of the two; it may be used instead of the Lorentz equation, and vice versa.

Using vector notation we may write the 6-dimensional displacement in several equivalent forms. In the Cartesian system $(\bar{x}^\mu, \bar{p}_\mu)$,

$$\tilde{\underset{\sim}{\zeta}} = \tilde{\underset{\sim}{a}}_i \bar{\zeta}^i = \tilde{\underset{\sim}{a}}_\alpha \bar{\zeta}^\alpha + \tilde{\underset{\sim}{a}}_{3+\alpha}\, \bar{\eta}_\alpha \quad . \tag{10.1}$$

Here, the base vectors $\tilde{\underset{\sim}{a}}_i$ are orthogonal unit vectors$\left(\tilde{\underset{\sim}{a}}_i = \tilde{\underset{\sim}{a}}^i,\ \tilde{\underset{\sim}{a}}_i \cdot \tilde{\underset{\sim}{a}}_k = \delta^k_i\right)$; $\bar{\eta}_\alpha \equiv \bar{\zeta}^{3+\alpha}$. In the curvilinear coordinate system (x^α, p_α), $x^\alpha = x^\alpha(\bar{x}^\mu)$, $p_\alpha = p_\alpha(\bar{x}^\mu, \bar{p}_\mu)$,

$$\tilde{\underset{\sim}{\zeta}} = \tilde{\underset{\sim}{e}}_i \zeta^i = \tilde{\underset{\sim}{e}}_\alpha\left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\ \bar{\zeta}^\mu\right) + \underset{\sim}{e}_{3+\alpha}\left(\frac{\partial p_\alpha}{\partial \bar{x}^\mu}\ \bar{\zeta}^\mu + \frac{\partial p_\alpha}{\partial \bar{p}_\mu}\ \bar{\eta}_\mu\right) . \tag{10.2}$$

Here, $\tilde{\underset{\sim}{e}}_i \cdot \tilde{\underset{\sim}{e}}^k = \delta^k_i$ and $\tilde{\underset{\sim}{e}}_i \tilde{\underset{\sim}{e}}^i = \tilde{\underset{\sim}{a}}_i \tilde{\underset{\sim}{a}}^i = \underset{\sim}{\mathcal{I}}$. It may further be noted that

$$\tilde{\underset{\sim}{e}}_\alpha = \frac{\partial \bar{x}^i}{\partial x^\alpha}\ \tilde{\underset{\sim}{a}}_i = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\ \tilde{\underset{\sim}{a}}_\mu + \frac{\partial \bar{p}_\mu}{\partial x^\alpha}\ \tilde{\underset{\sim}{a}}_{3+\mu}$$

and

$$\tilde{e}_{3+\alpha} = \frac{\partial \bar{x}^i}{\partial p_\alpha} \tilde{a}_i = \frac{\partial \bar{p}_\mu}{\partial p_\alpha} \tilde{a}_{3+\mu} \quad .$$

These yield:

$$\tilde{e}_\alpha \cdot \tilde{a}_\lambda = \frac{\partial \bar{x}^\lambda}{\partial x^\alpha} = \frac{\partial p_\alpha}{\partial \bar{p}_\lambda} = e_\alpha \cdot a_\lambda \quad ,$$

$$\tilde{e}_\alpha \cdot \tilde{a}_{3+\lambda} = \frac{\partial \bar{p}_\lambda}{\partial x^\alpha} = - \frac{\partial p_\alpha}{\partial \bar{x}^\lambda} \quad ,$$

$$\tilde{e}_{3+\alpha} \cdot \tilde{a}_\lambda = 0 \quad ,$$

and

$$\tilde{e}_{3+\alpha} \cdot \tilde{a}_{3+\lambda} = \frac{\partial \bar{p}_\lambda}{\partial p_\alpha} = \frac{\partial x^\alpha}{\partial \bar{x}^\lambda} = e^\alpha \cdot a^\lambda \quad .$$

Corresponding to the two parts of $\tilde{\zeta}$ are the 3-vectors ζ and η .

$$\zeta = a_\alpha \bar{\zeta}^\alpha = e_\alpha \zeta^\alpha \quad . \tag{10.3a}$$

$$\eta = a^\alpha \bar{\eta}_\alpha = e^\alpha \eta_\alpha \quad . \tag{10.3b}$$

These two vectors must satisfy a consistency relation given by Eq. (8.10) in the case of kinetic phase-space and by Eq. (6.15) or Eq. (6.16) in the case of canonical phase-space.

The canonical space is conceptually simpler because x^α and p_α are conjugate variables used in the Hamilton equations. On the other hand, the kinetic space is algebraically simpler, because in this space $u = u_0$. Here, the vector potential A does not appear in the Hamiltonian expression; one of the canonical equations must be modified to account for the presence of A [See Eq. (8.5a)].

Lagrange and Taylor operators are most convenient to use for perturbation calculations. They give rise to the desired series expansions, the first few terms of which may be obtained easily. In kinetic phase-space the Lorentz equation satisfied by ζ and η is, according to Eqs. (7.2b) and (8.6),

$$e(E_0 + u_0 \times B_0) + \dot{\eta} = \Sigma\, e(E + u_0 \times B),$$

or, alternatively,

$$e(E + u_0 \times B) = \Omega J\left[e(E_0 + u_0 \times B_0) + \dot{\eta}\right],$$

where J is the Jacobian given by Eq. (2.4) and $\Omega J = \Sigma^{-1}$. From either of these two forms, it may easily be proved that the first-order part of the Lorentz equation is as given by Eq. (7.10).

When the relativistic mass $m\gamma$ may be assumed constant for the whole system of particles, as assumed in the last section, the kinetic phase-space (x^α, p_α) is essentially a velocity phase-space (x^α, u_α). When γ is not a constant, the velocity space must be bounded because $|u|<c$. The displacement vector $\tilde{\zeta}$ in the latter phase-space may be discussed in the same manner as that in the kinetic phase-space. If we again use ζ and η to correspond to the two parts of $\tilde{\zeta}$, $\zeta = r' - r$ and $\eta = u' - u$, then the consistency relation has the very simple form $\eta = \dot{\zeta}$.

In the velocity phase-space, the equations of motion are the Euler-Lagrange equations derived from the Lagrangian expression,

$$\mathcal{L}(x,u,t) = mc^2 + p \cdot u - \mathcal{H}(x,p,t) = mc^2\left[1 - (1 - u \cdot u/c^2)^{1/2}\right] + eA \cdot u - eV. \tag{10.4}$$

The equations are as follows:

$$p_\alpha = \left.\frac{\partial \mathcal{L}}{\partial u^\alpha}\right|_{x^\sigma} \qquad \text{or} \qquad p^\alpha = \left.\frac{\partial \mathcal{L}}{\partial u_\alpha}\right|_{x^\sigma},$$

i.e.,

$$p = m\gamma u + eA \equiv \frac{\partial \mathcal{L}}{\partial u}; \tag{10.5a}$$

$$\dot{p}_\alpha = \left.\frac{\partial \mathcal{L}}{\partial x^\alpha}\right|_{u^\sigma} = \left.\frac{\partial \mathcal{L}}{\partial x^\alpha}\right|_{u_\sigma} + \Gamma^\lambda_{\alpha\beta}\left(p_\lambda u^\beta + u_\lambda p^\beta\right) ,$$

i.e.,

$$(\dot{p})_\alpha = \left.\frac{\partial \mathcal{L}}{\partial x^\alpha}\right|_{u^\sigma} - \Gamma^\lambda_{\alpha\beta}\; p_\lambda u^\beta \quad ,$$

i.e.,

$$\dot{\underset{\sim}{p}} = - \underset{\sim}{\nabla} e V + (\underset{\sim}{\nabla} e \underset{\sim}{A}) \cdot \underset{\sim}{u} \quad . \tag{10.5b}$$

Here, the Lagrangian expression plays a role similar to that played by the Hamiltonian expression in the canonical phase-space.

Any one of the different phase spaces may be derived from any other one by an appropriate transformation of coordinates. The charge distribution function is a scalar density; it transforms as a relative scalar of weight one.[20] Let $\psi^*(x,u,t)$ be this function in velocity phase-space,

$$\rho(\underset{\sim}{r}, t) = \int \psi^* \,(x,u,t)(du)^3 \quad , \tag{10.6}$$

then

$$\psi^*(x,u,t) = \left|\partial p_{K\alpha}/\partial u_\sigma\right| \psi(x,p_K, t) \quad , \tag{10.7a}$$

where the Jacobian of coordinate transformation is

$$\left|\partial p_{K\alpha}/\partial u_\sigma\right| = m^3 \gamma^5 \quad . \tag{10.7b}$$

While $d\psi/dt = 0$ for a physically realizable system, $d\psi^*/dt \neq 0$ unless γ is a constant. The Liouville equation becomes, in velocity phase-space,

$$\frac{d}{dt}\left(\frac{\psi^*}{m^3 \gamma^5}\right) = 0 \quad , \qquad \text{i.e.,} \quad \frac{d\psi^*}{dt} - 5\,\frac{\dot{\gamma}}{\gamma}\,\psi^* = 0 \quad . \tag{10.8}$$

Since $\psi^* = \Omega^* \psi^*_0$, Ω^* being the corresponding Lagrange operator, and

$$\frac{d\psi^*_0}{dt} - 5\,\frac{\dot{\gamma}_0}{\gamma_0}\,\psi^*_0 = 0 \quad ,$$

we obtain from Eq. (10.8) and the equation corresponding to Eq. (5.3) :

$$5 \frac{\dot{\gamma}}{\gamma} \Omega^* \psi_0^* = \Omega^* \left(5 \frac{\dot{\gamma}_0}{\gamma_0} \psi_0^* - \frac{\psi_0^*}{J^*} \frac{dJ^*}{dt} \right) . \qquad (10.9)$$

Here J^* is the Jacobian of transformation from the unperturbed to the perturbed coordinates in velocity phase spaces,

$$J^* = \left| \partial \left(x'^{\alpha}, u'_{\alpha} \right) \Big/ \partial \left(x^{\sigma}, u_{\sigma} \right) \right| .$$

Operating on both sides of Eq. (10.9) from the left with $J^* \Sigma^*$ and recalling that $J^* \Sigma^* \Omega^* = 1$, we obtain, after canceling out the factor ψ_0^*,

$$\frac{1}{J^*} \frac{dJ^*}{dt} = 5 \left(\frac{\dot{\gamma}_0}{\gamma_0} - \frac{\dot{\gamma}'}{\gamma'} \right) . \qquad (10.10)$$

This equation implies that $J^* = \left(\gamma_0 / \gamma' \right)^5$.

As shown in Appendix B, Eq. (10.10) agrees with the general formula Eq. (5.10). In a velocity phase-space, the first-order displacement vector $\tilde{\zeta}_1$ should, therefore, satisfy

$$\frac{d}{dt} \left[\tilde{\nabla} \cdot \tilde{\zeta}_1 + 5 \left(\log \frac{\gamma'}{\gamma_0} \right)_1 \right] = 0$$

in order that the Liouville equation (10.8) may be satisfied to the first order. This equation may further be transformed, thus,

$$\frac{d}{dt} \left[\frac{1}{\gamma_0} \tilde{\nabla} \cdot \left(\gamma_0 \tilde{\zeta}_1 \right) - \frac{4}{5} \tilde{\nabla} \cdot \tilde{\zeta}_1 \right] = 0 . \qquad (10.11)$$

The spatial and non-spatial variables in different phase spaces have, so far, been considered tacitly to be the genuine spatial and momentum or velocity coordinates. When canonical transformations are applied as in the Hamiltonian treatment of dynamics, the new variables X^{α} and P_{α} are functions of both the spatial (x^{α}) and the non-spatial (p_{α}) variables. No new coordinate, X^{α} or P_{α}, is a pure spatial or a pure non-spatial one. To call X^{α} or P_{α} a spatial or non-spatial coordinate is a matter of nomenclature.[19] Let the new kind of phase spaces be called non-separable and the other kind separable. The 6-dimensional analyses discussed in this paper are applicable, without restriction, to both kinds of phase spaces. On the other hand, the 3-dimensional formulas are applicable only in separable phase spaces. Even in the latter spaces, any vector and any tensor must obey the law of 6-dimensional point transformation of coordinates. According to Eq. (10.2) the non-spatial components of a 6-vector in a separable curvilinear phase-space are not the same as those obtained from the non-spatial Cartesian components by a 3-dimensional transformation of spatial coordinates.

In separable phase spaces the momentum part of the displacement vector $\underset{\sim}{\eta}$ does not enter into the expression of the charge density $\rho(\underset{\sim}{r}, t)$, nor of the current density $\underset{\sim}{i}(\underset{\sim}{r}, t)$. The Lorentz equation (7.7) is also independent of $\underset{\sim}{\eta}$. As long as no collision between particles is contemplated, one may, perhaps, question the advisability of using the Boltzmann-Vlasov equation of seven independent variables $(x^{\alpha}, p_{\alpha}, t)$ instead of the Lorentz equation of four space-time variables. This question can hardly be answered because each method has its merits and shortcomings. We may, however, note that the Boltzmann-Vlasov equation is a scalar equation which may represent an easier approach for certain problems than to solve the vector Lorentz equation directly. A deeper insight into the physical aspects of a certain problem may always be gained by considering the charge distribution function and the various equations based on it. Such knowledge would be very helpful in justifying the use of certain simplifying approximations which may be needed.

Strictly speaking, the spatial part of the displacement vector $\underset{\sim}{\zeta}$ in a separable phase-space may depend on both the spatial and the momentum variables (x^{α} and p_{α}). In Section IX, $\underset{\sim}{\zeta}_1$ is taken to be independent of p_{α}. This assumption is, perhaps, generally possible if the random part of the unperturbed velocity $(\underset{\sim}{u}_0 - \langle \underset{\sim}{u}_0 \rangle)$ is at most a first-order small quantity. Then, it is permissible to substitute $\langle u_0 \rangle \equiv v_0(r, t)$ for u_0 in Eqs. (7.10) and (7.11). Then ζ_1 which is determined by these equations must be a function of (r, t). In fact, $\zeta = \zeta(r, t)$ if the velocity function u_0 of the system of particles may properly be described by an Eulerian velocity-field $u_0(r,t)$.

The usual equation of continuity (4.8) is shown to be the limiting form of a general formula, Eq. (4.7). The latter formula bears some resemblance in form to the Fokker-Planck equation.[6, 23, 24] This is not surprising, because the implication is simply that the transition probability in the Fokker-Planck equation could well be a scalar density in the 3-velocity space.

In the absence of collisions Eq. (4.8) is valid, as long as ψ represents the density distribution of something which is conserved. No other dynamic principle is involved. About two decades ago, Vlasov[2] noted the generality and the flexibility of this equation and applied it in his well-known work to varied subjects, such as the theory of crystals, electron plasma, striations in metals, and high frequency electron-beam tubes. He took the equation of continuity in 6-space as an obvious extension of the one in 3-space, and has even indicated the feasibility of further extending it to 9- and higher 3ℓ-dimensional spaces $(x^\alpha, \dot{x}^\alpha, \ddot{x}^\alpha, \dots)$.

In the presence of collisions caused by sharply varying short-range forces between particles, the homogeneous Liouville equation becomes invalid. One must then use the celebrated Boltzmann equation, namely,

$$\frac{d\psi}{dt} = \left(\frac{\partial \psi}{\partial t}\right)_{\text{coll.}} .$$

The collision term[1,25,26] serves as a source term to maintain the conservation law. We shall proceed no further, because short-range forces are considered to be outside the realm of classical electrodynamics.

ACKNOWLEDGEMENTS

The author is indebted to many of his colleagues, especially Drs. R. H. Helm, M. J. Lee, and P. L. Morton for helpful discussions. He is very grateful to Dr. H. Chang and Dr. K. G. Dedrick, both of Stanford Research Institute, for discussing many mathematical aspects of this subject.

APPENDIX A

CONCERNING THE EQUATION OF CONTINUITY

In the unperturbed state, the equation of continuity satisfied by $\psi_0(x^i, t)$ is given by

$$\frac{\partial \psi_0}{\partial t} + \frac{\partial}{\partial x^i}\left(\psi_0 \dot{x}^i\right) = 0 \quad , \qquad (i = 1,2,\ldots, 6) . \tag{A.1}$$

This may be written as

$$\frac{\partial}{\partial x^i}\left(\psi_0 u_0^i\right) = 0 \quad , \qquad (i = 1,2, \ldots, 7) \ , \tag{A.2}$$

where $x^7 = t$, $\dot{x}^7 = 1$, and $u_0^i = \dot{x}^i$. As noted in Section IV, $\psi_0 \tilde{\underset{\sim}{u}}_0$ is a vector density and $\partial(\psi_0 u_0^i)/\partial x^i$ a density divergence. The equation of continuity (A.2) states that the density divergence of $\psi_0 \tilde{\underset{\sim}{u}}_0$ in 7-space must vanish.

In the perturbed state, the corresponding scalar is $\partial(\psi u^i)/\partial x^i$. This may be transformed by using the perturbation representation $\psi = \Omega \psi_0$, Ω being the Lagrange operator defined by Eq. (2.9) with $\xi^7 = 0$.

$$\frac{\partial}{\partial x^i}\left(\psi u^i\right) = u^i \frac{\partial}{\partial x^i}\left(\Omega \psi_0\right) + \left(\Omega \psi_0\right) \frac{\partial u^i}{\partial x^i}$$

$$= \frac{d}{dt}\left(\Omega \psi_0\right) + \left(\Omega \psi_0\right) \Sigma^{-1} \frac{\partial u'^i}{\partial x'^i} \quad .$$

The last term is equal to

$$\Omega\left(\psi_0 \frac{\partial u'^i}{\partial x'^i}\right) = \Omega\left(\psi_0 \frac{\partial \dot{x}'^i}{\partial x'^i}\right)$$

in view of the adjoint property of the inverse Taylor operator [Eq. (6.8a) in Ref.9]. The total time-derivative of $\Omega\psi_0$ may be transformed according to Eq. (5.3) and

Eq. (5.10), thus

$$\frac{d}{dt}\left(\Omega\psi_0\right) = \Omega\left[\frac{d\psi_0}{dt} - \psi_0\left(\frac{\partial \dot{x}'^i}{\partial x'^i} - \frac{\partial \dot{x}^i}{\partial x^i}\right)\right] .$$

Using these relations we obtain

$$\frac{\partial}{\partial x^i}\left(\psi u^i\right) = \Omega\left(\frac{d\psi_0}{dt} + \psi_0 \frac{\partial \dot{x}^i}{\partial x^i}\right) ,$$

i.e.,

$$\frac{\partial}{\partial x^i}\left(\psi u^i\right) = \Omega\left[\frac{\partial}{\partial x^i}\left(\psi_0 u_0^i\right)\right] . \qquad \text{(A.3)}$$

This proves that the density divergence indeed transforms like a scalar density from the unperturbed to the perturbed state. If ψ_0 satisfies the equation of continuity, so must $\psi = \Omega\psi_0$ satisfy the corresponding equation $\partial(\psi u^i)/\partial x^i = 0$ regardless of the vanishing or non-vanishing of dJ/dt.

APPENDIX B

LIOUVILLE'S THEOREM IN VELOCITY PHASE-SPACE

In canonical and kinetic phase spaces, Liouville's theorem states that $dJ/dt = 0$, $J = \partial(x', p')/\partial(x,p)$ being the Jacobian of transformation from the unperturbed to the perturbed coordinates. In the velocity phase-space, dJ^*/dt is given by Eq. (10.10) and, in general, does not vanish. Here $J^* = \partial(x', u')/\partial(x,u)$. It seems instructive to derive Eq. (10.10) from Eq. (5.10) in curvilinear coordinates. The latter equation gives the general expression of dJ/dt in any phase-space.

In the velocity phase-space, $\dot{x}'^{\alpha} = u'^{\alpha}$ and $\dot{x}'^{3+\alpha} = \dot{u}'_{\alpha}$. Thus

$$\frac{\partial \dot{x}'^{i}}{\partial x'^{i}} = \left.\frac{\partial u'^{\alpha}}{\partial x'^{\alpha}}\right|_{u'_{\sigma}} + \left.\frac{\partial \dot{u}'_{\sigma}}{\partial u'_{\sigma}}\right|_{x'^{\alpha}} . \tag{B.1}$$

The last term in this equation may be evaluated by using the following relation:

$$(\dot{p}')_{\alpha} = m\gamma'\left(\dot{u}'_{\alpha} - \Gamma'^{\lambda}_{\alpha\beta}\, u'_{\lambda}\, u'^{\beta}\right) + m\dot{\gamma}' u'_{\alpha} ,$$

i.e.,

$$\dot{u}'_{\alpha} = \frac{1}{m\gamma'}\left[(\dot{p}')_{\alpha} - m\dot{\gamma}' u'_{\alpha}\right] + \Gamma'^{\lambda}_{\alpha\beta}\, g'^{\beta\tau}\, u'_{\lambda} u'_{\tau} . \tag{B.2}$$

This equation is based on the definition $p'_{\alpha} = m\gamma' u'_{\alpha}$. Differentiating $\dot{u}'_{\alpha}$ with respect to u'_{σ} and contracting, we obtain from Eq. (B.2)

$$\begin{aligned}\left.\frac{\partial \dot{u}'_{\sigma}}{\partial u'_{\sigma}}\right|_{x'^{\alpha}} &= -\frac{1}{m\gamma'^{2}}\frac{\partial \gamma'}{\partial u'_{\sigma}}\left[(\dot{p}')_{\sigma} - m\dot{\gamma}' u'_{\sigma}\right] \\ &+ \frac{1}{m\gamma'}\frac{\partial}{\partial u'_{\sigma}}\left[(\dot{p}')_{\sigma} - m\dot{\gamma}' u'_{\sigma}\right] \\ &+ \Gamma'^{\lambda}_{\sigma\beta}\, g'^{\beta\tau}\frac{\partial}{\partial u'_{\sigma}}\left(u'_{\lambda} u'_{\tau}\right) .\end{aligned} \tag{B.3}$$

Since

$$\gamma' = \left(1 - g'^{\beta\tau} u'_\beta u'_\tau / c^2\right)^{-1/2} \quad , \tag{B.4a}$$

we have

$$\partial\gamma'/\partial u'_\sigma = \left(\gamma'^3/c^2\right) u'^\sigma \tag{B.4b}$$

and

$$\dot{\gamma}' = \left(1/mc^2\right) u'^\sigma (\dot{p}')_\sigma \quad . \tag{B.4c}$$

We also have

$$(\dot{p}')_\sigma = e E'_\sigma - u'^\lambda \left(\frac{\partial}{\partial x'^\lambda} e A'_\sigma - \frac{\partial}{\partial x'^\sigma} e A'_\lambda\right) \tag{B.5a }$$

and

$$\partial(\dot{p}')_\sigma / \partial u'_\sigma = 0 \quad . \tag{B.5b}$$

On substitution of these relations Eq. (B.3) becomes:

$$\left.\frac{\partial \dot{u}'_\sigma}{\partial u'_\sigma}\right|_{x'^\alpha} = -5\frac{\dot{\gamma}'}{\gamma'} + u'_\sigma\left(\Gamma'^\sigma_{\alpha\beta}\, g'^{\alpha\beta} + \Gamma'^\alpha_{\alpha\beta}\, g'^{\beta\sigma}\right) ,$$

i.e.,

$$\left.\frac{\partial \dot{u}'_\sigma}{\partial u'_\sigma}\right|_{x'^\alpha} = -5\frac{\dot{\gamma}'}{\gamma'} - u'_\sigma \frac{\partial g'^{\alpha\sigma}}{\partial x'^\alpha}$$

$$= -5\frac{\dot{\gamma}'}{\gamma'} - \left.\frac{\partial u'^\alpha}{\partial x'^\alpha}\right|_{u'_\sigma} \quad .$$

Thus,

$$\partial \dot{x}'^i / \partial x'^i = -5\,\dot{\gamma}'/\gamma' \quad . \tag{B.6}$$

Similarly, the corresponding equation for the unperturbed state is

$$\partial \dot{x}^i / \partial x^i = -5 \, \dot{\gamma}_0 / \gamma_0 \quad . \tag{B.7}$$

Hence, from Eq. (5.10), i.e.,

$$\frac{1}{J^*} \frac{dJ^*}{dt} = \frac{\partial \dot{x}'^i}{\partial x'^i} - \frac{\partial \dot{x}^i}{\partial x^i} \quad , \tag{B.8}$$

we obtain Eq. (10.10),

$$\frac{1}{J^*} \frac{dJ^*}{dt} = 5 \left(\frac{\dot{\gamma}_0}{\gamma_0} - \frac{\dot{\gamma}'}{\gamma'} \right) \quad .$$

REFERENCES

1. L. Boltzmann, Lectures on Gas Theory, English Ed., translated by S.G. Brush (University of California Press, Berkeley and Los Angeles, 1964). The original German edition was first published in 1896.
2. A. A. Vlasov, Many-Particle Theory and its Application to Plasma (Gordon and Breach Science Publishers, Inc., New York, 1961). Original work in Russian first published in 1950.
3. L. Spitzer, Physics of Fully Ionized Gases, pp. 94-98 (Interscience Publishers, Inc. New York, 1956).
4. Yu. L. Klimontovich, The Statistical Theory of Non-Equilibrium Processes in a Plasma, translated by H. S. H. Massey and O. M. Blunn, First English Ed., Chap. II (The M.I.T. Press, Cambridge, Mass., 1967).
5. K. M. Watson, Phys. Rev. 102, 12-19 (1956).
6. S. Chandrasekhar, Plasma Physics, lecture notes compiled by S.K. Treben, pp. 11-13 (The University of Chicago Press, 1960).
7. A. I. Khinchin, Mathematical Foundations of Statistical Mechanics, translated from the Russian by G. Gamow, First American Ed., pp. 15-19 (Dover Publications, Inc., New York, 1949).
8. J. R. Pierce, Traveling-Wave Tubes (D. Van Nostrand Co., Princeton, New Jersey, 1950).
9. P. A. Sturrock, Ann. Phys. (N. Y.) 4, 306 (1958).
10. E. L. Chu, Ann. Phys. (N. Y.) 9, 76 (1960).
11. K. G. Dedrick and R. N. Wilson, J. Math. Phys. 3, 97 (1962).
12. K. G. Dedrick and E. L. Chu, Arch. Ratl. Mech. Anal., 16, 385 (1964).
13. S. T. Beliaev and G. I. Budker, Soviet Phys. Doklady 1, 318 (1956).

14. J. L. Synge, The Relativistic Gas (North-Holland Publishing Co., Amsterdam, 1957).

15. R. W. Lindquist, Ann. Phys. (N. Y.) 37, 487 (1966).

16. E. T. Whittaker, Analytical Dynamics, 4th Ed., Chap. X, XI, and XII (Dover Publications, New York, 1944).

17. L. Nordheim and E. Fues, Die Hamilton-Jacobische Theorie der Dynamik, Handbuch der Phys., Vol. 5, Chap. 3, pp. 91-130 (verlag von Julius Springer, Berlin, 1927).

18. H. Goldstein, Classical Mechanics, Chap. 8 (Addison-Wesley Press, Inc., Cambridge, Mass., 1951).

19. L. D. Landau and E. M. Lifshitz, Mechanics, translated from the Russian by J. B. Sykes and J. S. Bell, Chap. VII (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1960).

20. O. Veblen, Invariants of Quadratic Differential Forms, Cambridge Tracts in Math. and Math. Phys., No. 24, First Ed.(1927), Chap. II, III, and IV (Cambridge University Press, 1962).

21. E. Madelung, Die Mathematischen Hilfsmittel des Physikers, Third German Ed., pp. 135-138 (Dover Publications, New York, 1943).

22. M. J. Lee, F. E. Mills, and P. L. Morton, "Throbbing Beam Transverse Resistive Instabilities in Circular Accelerators and Storage Rings," SLAC Report No. 76, August 1967, Stanford Linear Accelerator Center, Stanford University, Stanford, California.

23. S. A. Rice and P. Gray, The Statistical Mechanics of Simple Liquids, Chap. 4 (Interscience Publishers, a division of John Wiley and Sons, Inc., New York, 1965).

24. Ta-You Wu, Kinetic Equations of Gases and Plasmas, pp. 51-54 (Addison-Wesley Publishing Co., Palo Alto, California, 1966).

25. S. Chapman and T. G. Cowling, The Mathematical Theory of Non-Uniform Gases (Cambridge University Press, 1952); first printed in 1939.

26. A. Sommerfeld, Thermodynamics and Statistical Mechanics, translated from the German by J. Kestin, Chap. V (Academic Press, Inc., New York, 1956).