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# THROBBING BEAM TRANSVERSE RESISTIVE INSTABILITIES IN CIRCULAR ACCELERATORS AND STORAGE RINGS

by

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# ABSTRACT

Finite conductivity of the vacuum chamber wall can cause unstable transverse oscillations of the center of charge or oscillations of the transverse cross section of a beam of charged particles. The former case, which can be characterized by a dipole oscillation, has been studied extensively by others.[1,2,3] In this work, a study has been conducted of the cross-sectional oscillations of a nearly circular beam centered in a circular pipe, and a self-consistent solution has been obtained for both monopole and multipole oscillations. Dispersion relations for the oscillation frequencies have been found, and conditions for stability have been deduced. For multipole instabilities the growth rates and thresholds are close to those obtained for the dipole instability,[1,2] differing only by a geometrical factor; whereas, for the monopole instability, the growth rates are so small that the oscillations in present accelerators and storage rings will be suppressed by the physical processes such as interaction with the residual gas, radiation damping, etc. In all cases, the growth rate is proportional to the number of particles in the beam and inversely proportional to the square root of the wall conductivity. It is shown that in the absence of Landau damping a longitudinally continuous beam is always unstable against the development of transverse waves having a phase velocity close to $(n - m\nu_0)\,\Omega R$, where $\nu_0$ is the number of betatron oscillations per revolution, $\Omega$ is the revolution frequency, R is the radius of the machine, n is an integer greater than $m\nu_0$ with 2m the multipole number (m = 1 for dipole and m = 2 for quadrupole, etc.). A condition for stable multipole oscillations for a single bunched beam is found to be $n < \nu_0 < \left(n + \frac{1}{2m}\right)$. It is known[1,2] that unstable oscillations can be Landau damped by having a sufficiently large spread in the betatron oscillation frequency $\nu_0\,\Omega$. A criterion for the spread required to damp unstable quadrupole oscillations of a bunched beam is shown to be, in the limit of low energy, dependent upon the particle density, energy and the betatron oscillation frequency, but not upon the conductivity. For the case of equal horizontal and vertical betatron frequencies monopole and multipole oscillations can exist independently. However, for the case of a large difference between the two frequencies, it is found that coupling exists between monopole and quadrupole oscillations, and the motion of the beam is characterized by two normal modes such that the monopole and quadrupole oscillations are in phase in one mode and 180° out of phase in the other.

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## I. INTRODUCTION

In this work the transverse coherent resistive instabilities in circular accelerators and storage rings are investigated for a nearly circular beam centered in a vacuum tank of circular cross section. The tank walls are taken to be non-perfectly conducting. The transverse motion of a beam in an accelerator can be described by oscillations of the center of charge and transverse size of the beam. These oscillations can be characterized by a combination of monopole, dipole, quadrupole, sextupole, etc., oscillations.

In the past few years, a number of particle accelerators have exhibited dipole-type transverse coherent instabilities.[4-6] In 1965 Laslett, Neil and Sessler[1] showed theoretically the possibility for a longitudinal continuous beam of charged particles to have unstable dipole oscillations. This theory was later extended to bunched beams by Courant and Sessler,[2] and Dikanskii Skrinskii.[3] More recently, instabilities that may be coherent oscillations of the size of the beam have been observed.[7-9] The purpose of this work is to develop a theory for the transverse instabilities of the beam cross section called the throbbing beam instability.[10]

The idea of resistive wall instabilities comes conceptually from the theory of the resistive wall amplifier.[11] In 1953 Birdsall, Brewer and Haeil[12] predicted theoretically and demonstrated experimentally the possibility of amplification of longitudinal density fluctuations in an electron beam by the resistance in the surrounding walls. The occurrence of this phenomenon in particle accelerators was studied by Neil and Sessler[13] in 1965.

In order to understand physically the wake fields, which are responsible for the resistive instabilities, consider the following argument due to Robinson.[14] As the beam passes through a given point along the machine, a surface current is induced on the wall. Subsequently, if the conductivity of the wall is finite, this current diffuses into the metal which gives rise to the wake fields. This process can best be illustrated with a simple situation. We will examine the currents in the wall for the case of perfect and non-perfect wall conductivity. Take the case of a pulse of charged particles traveling parallel to an infinite metallic plane as shown in Fig. 1.1c. Imagine that the pulse of particles is made up of two semi-infinite beams, one positive and one negative, as shown in Figs. 1.1a and 1.1b. The image charges and currents are shown in the same figures for the case of a perfectly conducting wall. Because the wall conductivity is infinite no current can exist inside the metal, and the induced currents stay on the surface of the wall. The wall currents and charges due to the (+) and (-) beams have the same magnitudes but opposite signs; by superposition, they cancel each other in the region behind the pulse as shown in Fig. 1.1c. Hence, no current is left in the wall for the case of a perfectly conducting wall. However, if the wall conductivity is finite, the surface currents can diffuse toward the inside of the metal. The diffusion of the image currents of the (+) and (-) beams is shown in Figs. 1.2a and 1.2b. Because these image currents are turned on at different times, the wall current corresponding to the (+) beam has diffused farther into the metal than that of the (-) beam at the same point along the wall. This gives rise to currents in the wall in the region behind the pulse as shown in Fig. 1.2c. Near the wall surface the currents are positive and inside the metal the currents are negative. Hence, in the presence of wall resistance, there are wall currents left behind a pulse of charged particles. These currents provide the sources for the wake fields.[15]

Now let us consider the situation of a pulse of particles circulating in a circular accelerator or storage ring. The particles in the pulse execute betatron oscillation[16] about some closed orbit as shown in Fig. 1.3. Consider the case where the bunch moves as a whole, then each particle in the bunch experiences

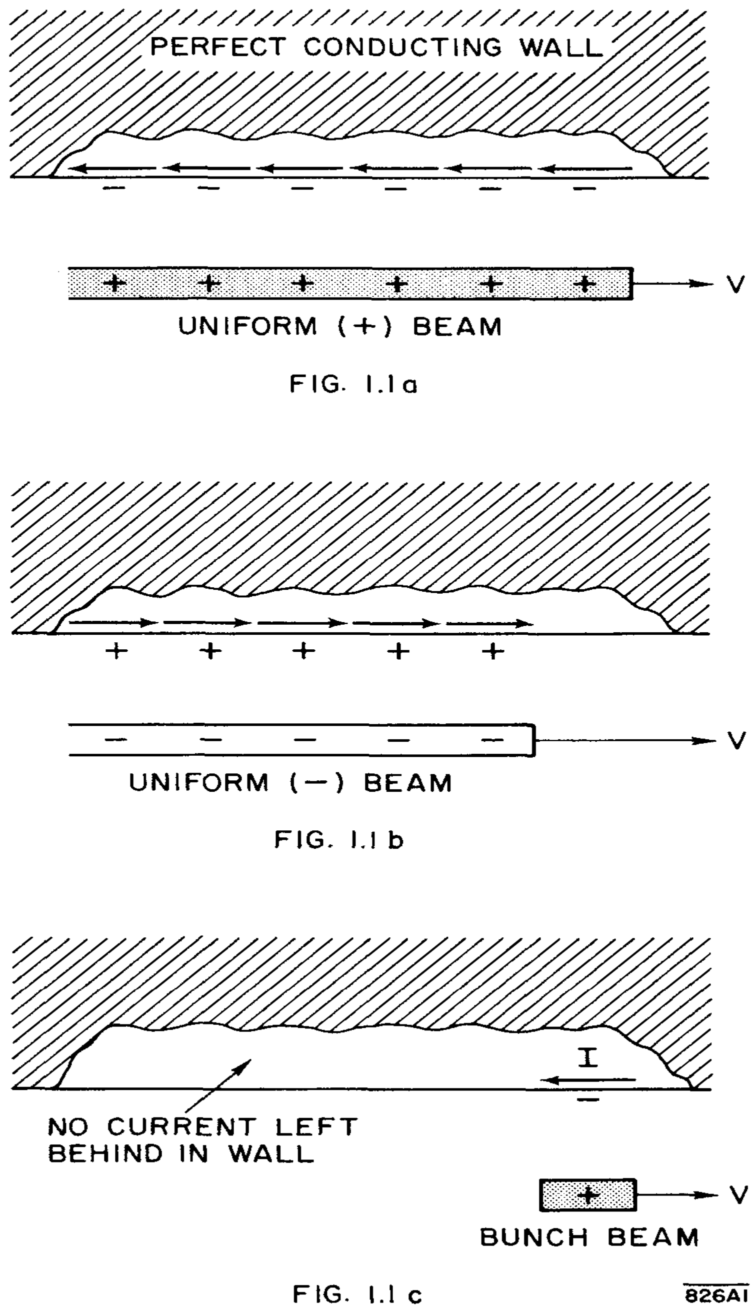


Fig. 1.1. An illustration of resistive wall effects for a perfectly conducting wall: (a) A semi-infinite (+) beam, (b) a semi-infinite (-) beam, and (c) a bunched (+) beam.

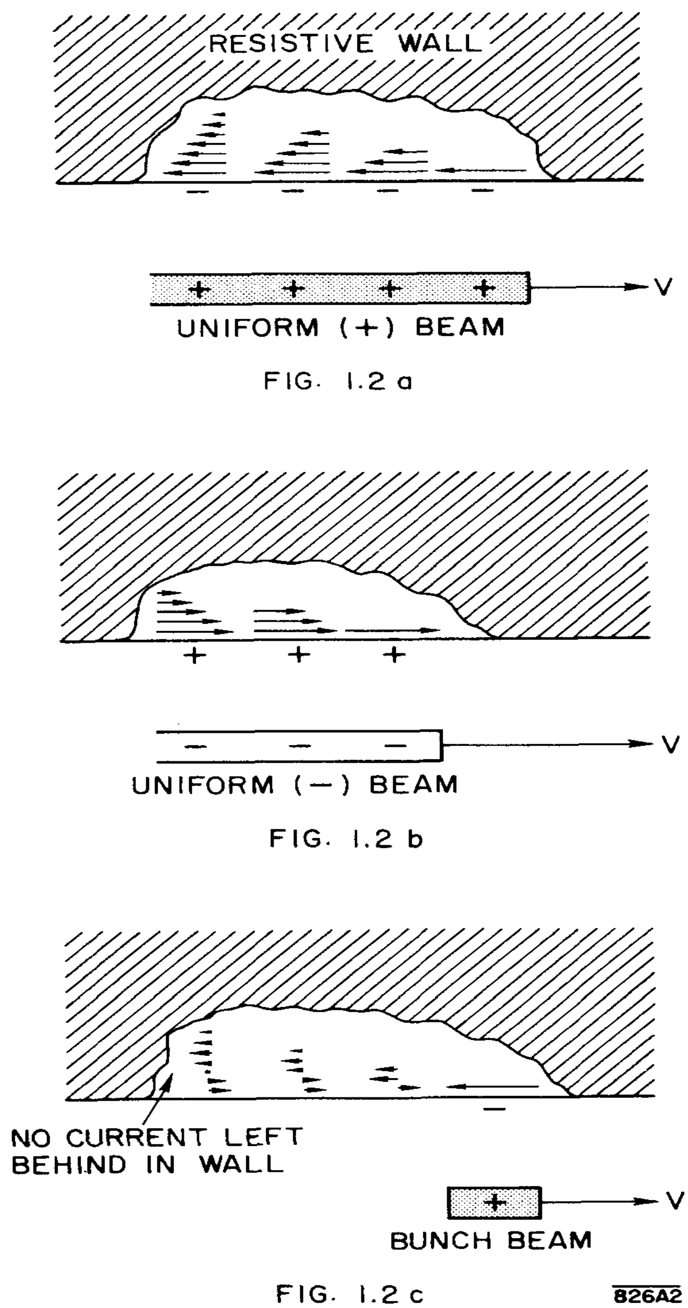


Fig. 1.2. An illustration of resistive wall effects for a lossy wall: (a) A semi-infinite (+) beam, (b) a semi-infinite (–) beam, and (c) a bunched (+) beam.

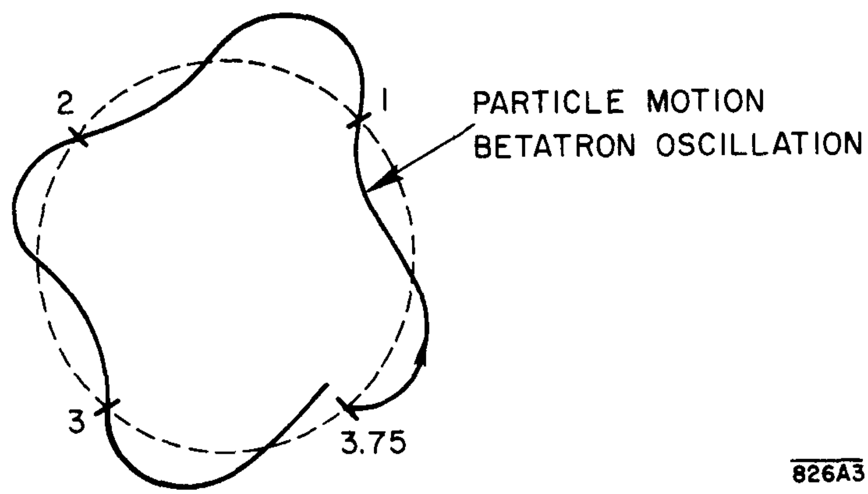


Fig. 1.3. Betatron oscillation about a closed orbit of a particle in a circulating beam.

the wake fields produced by the motion of the bunch in its previous revolutions. The nature of the wake fields depends on the type of oscillation of the pulse, e.g., monopole, dipole, quadrupole, etc., and the frequency of oscillation $\omega$. The relative phase of the field to that of the particle motion depends on the number of betatron oscillations, $\nu$, per revolution. Hence, it is natural to expect that for a given type of oscillation some relationship exists between the quantities $\omega$ and $\nu$. The objective of this study is to find this relationship and to deduce from it the conditions for stable oscillation and the values of the growth rates for each type of oscillation. From the knowledge of the stable conditions, future machines can be designed with the proper value of $\nu$ to overcome the resistive instabilities.

There are other means for suppressing resistive instabilities.[17] A successful method for suppressing transverse dipole instabilities has been the use of feedback.[18,19] The growth rate of the instabilities is generally of the order of milliseconds.[1] In this method the transverse motion of the beam is detected electronically, the signal is amplified and then fed back to the beam in such a phase as to damp the oscillation.

Another means for suppressing resistive instabilities is the mechanism of Landau damping.[20,21] This mechanism always relies on a spread in some parameter of the beam. The spread in the parameter may be introduced artificially into the beam via some machine nonlinearities. Usually for a given spread of the parameter, instabilities are suppressed up to some maximum beam intensity.[22] When this level of intensity is exceeded, the beam becomes unstable and a loss of current results. This mechanism of damping is investigated in detail for the quadrupole oscillation.

Our detailed investigation for monopole and multipole transverse oscillations was suggested by the work on the transverse dipole instabilities of Laslett, Neil, and Sessler.[1] The solutions obtained are self-consistent[23] based on a small signal model. In particular, for the dipole oscillation our result gives the same expression of growth rate and the same condition of stability as those given by others.[1,2] The method used in the analysis is outlined in the next section.

## II. METHOD OF ANALYSIS

The charge and current densities are taken to be

$$\rho = \rho_0 + \rho_1 \tag{2.1a}$$

and

$$\vec{J} = \vec{J}_0 + \vec{J}_1 \tag{2.1b}$$

with the perturbations $\rho_1$ and $\vec{J}_1$ assumed to be small compared with the unperturbed densities $\rho_0$ and $\vec{J}_0$. Then Maxwell's equations are solved for the fields due to $\rho$ and $\vec{J}$. The transverse position of each particle in the beam can be determined by the equations of motion containing the forces due to these fields and the known initial conditions. From the knowledge of the position of every particle in the beam, in principle, the charge and current densities of the beam can be constructed. A self-consistent solution is obtained when the charge and current densities so constructed are the same as those assumed in Eq. (2.1). A more convenient method to obtain this self-consistent solution is to find the particle distribution function in phase space that gives the assumed charge and current densities. The latter method is used in this paper, and an outline of the method is given in Fig. 2.1.

Following the outline given in Fig. 2.1, we first characterize the monopole and multipole oscillations of a beam, uniform or bunched, by some assumed charge and current densities, and present the equations of motion for the particles in Section III.

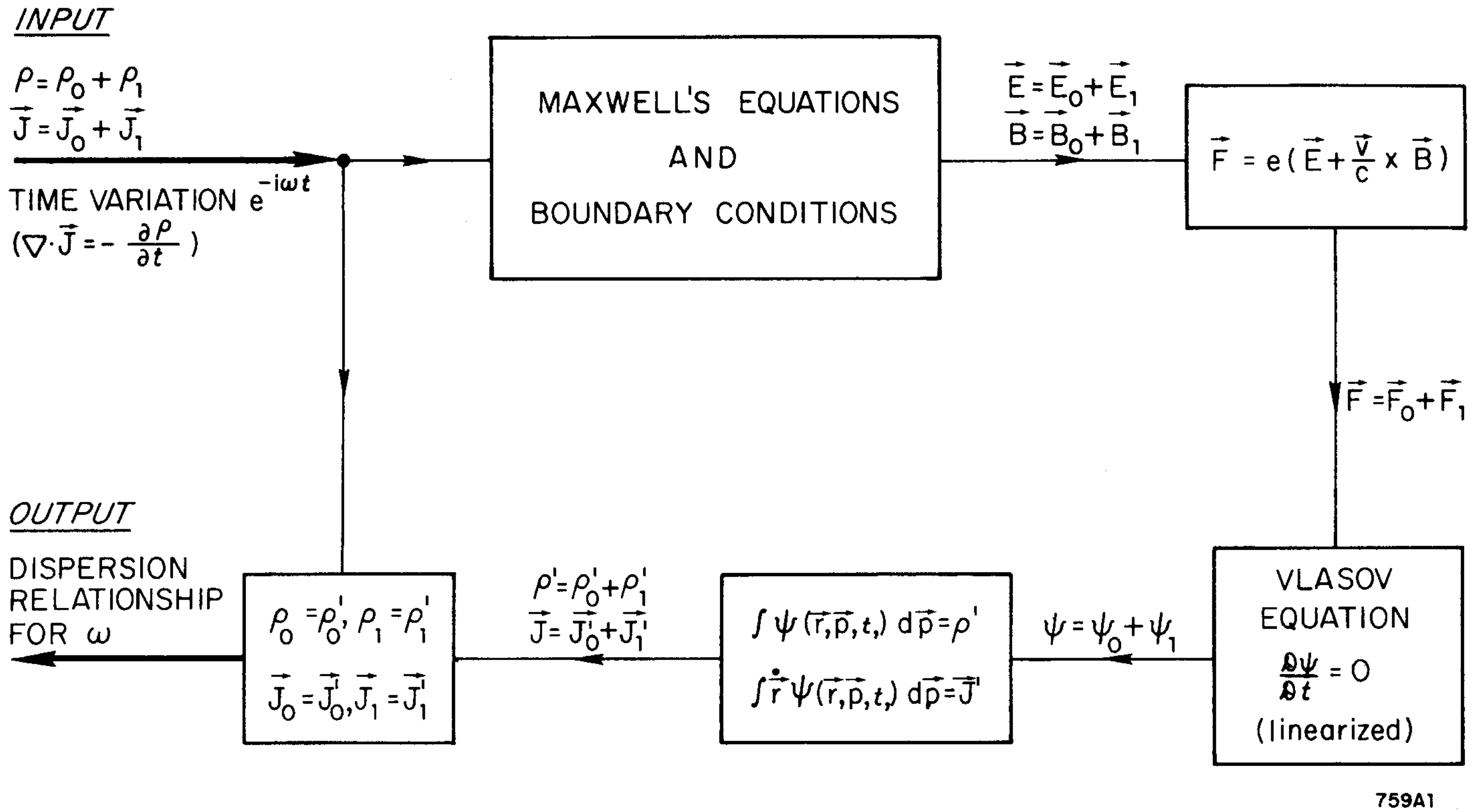


Fig. 2.1. Flow chart of the method used for finding a self-consistent solution.

The body of the analysis is contained in Section IV, where the Vlasov equation combined with the equations of motion is solved for a self-consistent particle distribution and the dispersion relation for the oscillation frequencies is obtained. This dispersion relation is analyzed in Section V, culminating in the determination of the growth times and the stability criteria for the oscillations.

The effects of Landau damping on quadrupole oscillations resulting from a spread in the amplitude of oscillations are considered in Section VI, and the effect of unequal horizontal and vertical betatron frequencies for the particles is investigated in Section VII.

## III. MONOPOLE AND MULTIPOLE CHARGE OSCILLATIONS

In this section the monopole and multipole oscillations of a uniform or bunched beam inside a metallic vacuum chamber are characterized by some simple models. As the major curvature of the vacuum chamber has little influence on the calculation of the fields,[1] the chamber is taken to be a straight pipe of radius b. The particles in the beam are taken to be moving longitudinally in the z-direction, along the axis of the pipe, with a constant velocity v.

The unperturbed beam is taken as uniform in the transverse cross section over a circle of radius a, with the center of the beam fixed along the pipe axis as shown in Fig. 3.1. Thus, the charge and current densities of the unperturbed beam are:

$$\rho_0 = \frac{e\lambda}{\pi a^2} H(a - r) \qquad (3.1)$$

and

$$\vec{J}_0 = v\,\rho_0\,\hat{e}_z \qquad (3.2)$$

where H(x) is the Heaviside unit step function. For the uniform beam $e\lambda$ is the charge per unit length, while for the bunched beam the charge per unit length is

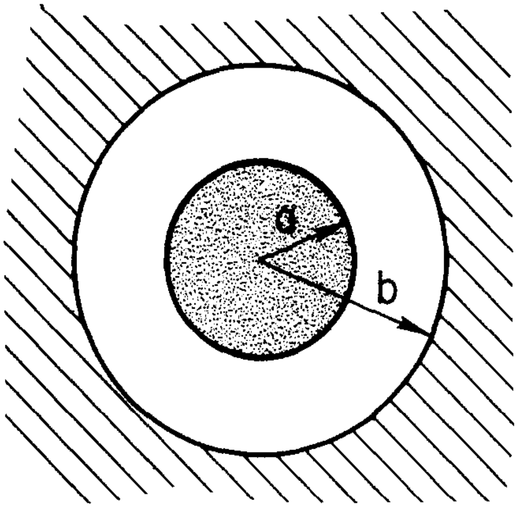


Fig. 3.1. The geometry of an unperturbed beam and vacuum tank.

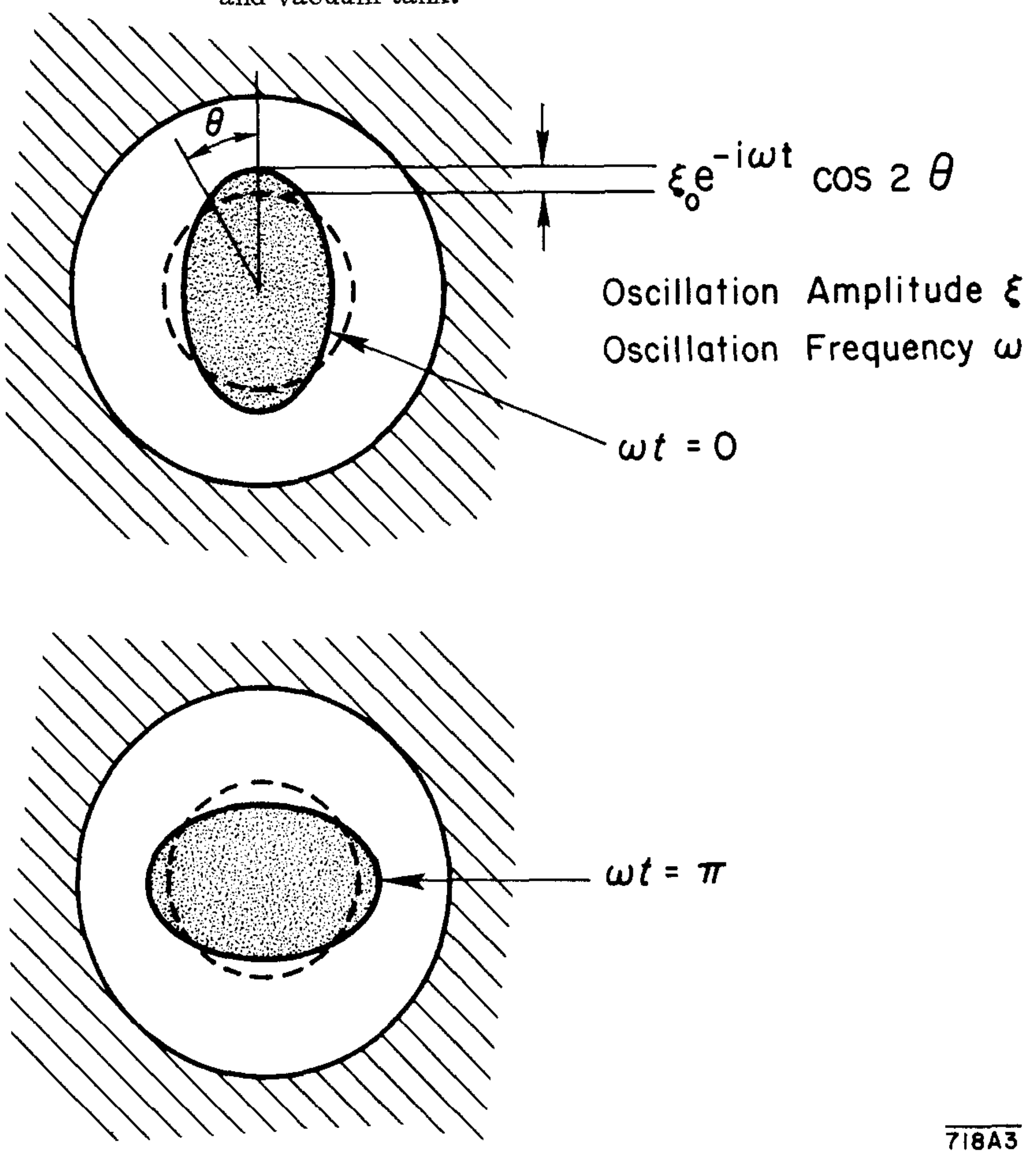


Fig. 3.2. The geometry of a perturbed beam having quadrupole oscillation.

$e\lambda = eNf(z - vt)$ with the function f(x) normalized such that eN is the total charge in the bunch, i.e.,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 .$$

In the perturbed beam the radius varies as $(a + \xi)$ for monopole oscillations and $(a + \xi \cos m\theta)$ for multipole oscillations, with the perturbation amplitude $\xi$ given by

$$\xi = \begin{cases} \xi_0 e^{i(kz - \omega t)} & \text{(continuous beam)} \quad (3.3a) \\ \xi_0 e^{-i\omega t} \; . & \text{(bunched beam)} \quad (3.3b) \end{cases}$$

The multipole number is 2m, e.g., m = 1 for dipole oscillations and m = 2 for quadrupole oscillations. The change in the beam cross section in time is illustrated in Fig. 3.2 for the quadrupole case. As a consequence of the perturbation, to first order in $\xi$ the charge density can be written as

$$\rho = \rho_0 + \xi \rho_1 \qquad (3.4)$$

where

$$\rho_1 = \begin{cases} \dfrac{e\lambda}{\pi a^2} \left[ \delta(a - r) - \dfrac{2}{a} H(a - r) \right]^* & \text{(monopole)} \quad (3.5a) \\ \dfrac{e\lambda}{\pi a^2} \delta(a - r) \cos m\theta \; . & \text{(multipole)} \quad (3.5b) \end{cases}$$

---

*This is a common and convenient mathematical approximation for a uniform dilation of the beam cross section in which the physical perturbation is characterized by an equivalent surface charge distribution on a boundary of constant radius. A similar approximation is used for the multipole cases.

Similarly, the current density can be written as

$$\vec{J} = \vec{J}_0 + \xi \vec{J}_1 \tag{3.6}$$

with

$$\vec{J}_1 = \begin{cases} \dfrac{e\lambda}{\pi a^2} \left\{ i(kv - \omega) \dfrac{r}{a} H(a-r)\, \hat{e}_r + v\left[\delta(a-r) - \dfrac{2}{a} H(a-r)\right] \hat{e}_z \right\} & \text{(monopole)} \quad (3.7a) \\ \\ \dfrac{e\lambda}{\pi a^2} \left\{ i(kv - \omega) \left(\dfrac{r}{a}\right)^{m-1} H(a-r) \left[\cos m\theta\, \hat{e}_r - \sin m\theta\, \hat{e}_\theta\right] \right. & \text{(multipole)} \quad (3.7b) \\ \qquad \left. + v\, \delta(a-r) \cos m\theta\, \hat{e}_z \right\} \quad . & (k = 0 \text{ for bunched beam}) \end{cases}$$

It is easily verified that $\vec{J}_1$ satisfies the continuity equation.*

For a particle in the beam, the equations of motion are

$$\begin{aligned} \dot{p}_x &= -\omega_0^2 x + \xi \frac{F_{x1}}{m_0 \gamma} \quad , \quad \dot{x} = p_x \quad , \\ \dot{p}_y &= -\omega_0^2 y + \xi \frac{F_{y1}}{m_0 \gamma} \quad , \quad \dot{y} = p_y \quad , \\ \dot{p}_z &= 0 \quad , \quad \text{and} \qquad \dot{z} = v \quad , \end{aligned} \tag{3.8}$$

where $p_x$, $p_y$ and $p_z$ are the conjugate momenta, $m_0$ is the particle rest mass, and $\gamma$ is equal to $\sqrt{1 - \beta^2}$ with $\beta = v/c$. The contribution of both the external fields and the electromagnetic fields due to $\rho_0$ and $\vec{J}_0$ is included in the quantity $\omega_0^2$,** while the contribution of the electromagnetic fields produced by

*The continuity equation does not determine $\vec{J}_1$ uniquely; however, this is the form of $\vec{J}_1$ that will yield a self-consistent solution. It is a pleasure to thank Dr. E. L. Chu for valuable discussions on this point.

**Since the particles are assumed to have an angular revolution frequency $\Omega$, $\omega_0 = \nu_0 \Omega$ where $\nu_0$ is the number of betatron oscillations per revolution. For the case of a single particle beam, the value of $\nu_0$ is determined by the periodicity of the focusing elements in the machine. The effect of space charge is to shift this value of $\nu_0$ by an amount which is proportional to the charge density of the beam. Hence, the value of $\omega_0$ in general depends on both the design of the machine and the value of the beam current.

$\rho_1$ and $\vec{J}_1$ is contained in the perturbing terms, $F_{x1}(x, y)$ and $F_{y1}(x, y)$.

To find the perturbing forces, Maxwell's equations are solved for the fields due to $\rho_1$ and $\vec{J}_1$ with appropriate boundary conditions at the vacuum chamber wall. The forces are calculated from the fields $\vec{E}_1$ and $\vec{B}_1$, using the formula

$$F_{x1}\hat{e}_x + F_{y1}\hat{e}_y = e\left(\vec{E}_1 + \frac{\vec{v}}{c} \times \vec{B}_1\right)_{\text{transverse}} . \tag{3.9}$$

The calculation of $F_{x1}$ and $F_{y1}$ is presented in the Appendix.

The set of equations of motion can be written in an alternate form by eliminating the variables $p_x$, $p_y$ and $p_z$:

$$\ddot{x} + \omega_0^2 x = \xi \frac{F_{x1}(x,y)}{m_0\gamma} \tag{3.10a}$$

$$\ddot{y} + \omega_0^2 y = \xi \frac{F_{y1}(x,y)}{m_0\gamma} \tag{3.10b}$$

and

$$\ddot{z} = 0 . \tag{3.10c}$$

As seen from these equations the particle betatron oscillation frequencies of x and y motions, $\omega_{0x}$ and $\omega_{0y}$, are taken to be $\omega_0$. In a later section, however, this restriction is removed and the effect of $\omega_{0x} \neq \omega_{0y}$ is considered for monopole and quadrupole oscillations.

The transverse position of each particle in the beam can be found by solving the equations of motion with some known initial conditions. If the position of every particle in the beam is known, the charge and current densities of the beam can, in principle, be constructed. A self-consistent solution is obtained when the charge and current densities constructed are the same as those originally assumed (i.e., $\rho$ and $\vec{J}$). A more convenient method to obtain this self-consistent solution is to find the particle distribution function in phase space, as outlined in Section II.

IV. SELF-CONSISTENT DISTRIBUTION FUNCTIONS IN PHASE SPACE

In this section we proceed to find the self-consistent particle distribution functions in phase space $\psi(x, y, z, p_x, p_y, p_z, t)$, which give rise to the charge and current densities assumed in Section III.

4.1. Solving the Linearized Vlasov Equation

In general, the particle distribution function satisfies the Vlasov equation[24,25]

$$\frac{\partial \psi}{\partial t} + \dot{x}\frac{\partial \psi}{\partial x} + \dot{y}\frac{\partial \psi}{\partial y} + \dot{z}\frac{\partial \psi}{\partial z} + \dot{p}_x\frac{\partial \psi}{\partial p_x} + \dot{p}_y\frac{\partial \psi}{\partial p_y} + \dot{p}_z\frac{\partial \psi}{\partial p_z} = 0 . \quad (4.1)$$

In analogy to Eq. (3.4) $\psi$ is written as:

$$\psi = \left[\psi_0(x,y,p_x,p_y) + \xi\psi_1(x,y,p_x,p_y)\right]\lambda\,\delta(p_z - p_{z0}) , \quad (4.2)$$

where $\psi_0$ is the self-consistent particle distribution function corresponding to an unperturbed beam, i.e.,

$$e\lambda\int\psi_0\,\delta(p_z - p_{z0})\,d^3p = \rho_0 \quad (4.3)$$

and

$$e\lambda\int\dot{\vec{r}}\,\psi_0\,\delta(p_z - p_{z0})\,d^3p = \vec{J}_0 . \quad (4.4)$$

When Eq. (3.8) is substituted for the time derivatives of the coordinates and momenta and Eq. (4.2) is substituted for $\psi$ into the Vlasov equation, for the bunched beam, we obtain to first order in $\xi$

$$\left[p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} - \omega_0^2 x\frac{\partial}{\partial p_x} - \omega_0^2 y\frac{\partial}{\partial p_y}\right]\psi_0 = 0 \quad (4.5)$$

and

$$\left[-i\omega + p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} - \omega_0^2 x\frac{\partial}{\partial p_x} - \omega_0^2 y\frac{\partial}{\partial p_y}\right]\psi_1 = -\left[\frac{F_{x1}}{m_0\gamma}\frac{\partial}{\partial p_x} + \frac{F_{y1}}{m_0\gamma}\frac{\partial}{\partial p_y}\right]\psi_0 . \quad (4.6)$$

The quantity $(-\omega)$ is replaced by $(kv - \omega)$ for a continuous beam.

A self-consistent solution for $\psi_0$ [26] that satisfies Eqs. (4.3), (4.4), and (4.5) is

$$\psi_0 = \frac{\lambda}{2\pi^2 a^2}\,\delta\left[\mathcal{H}_0(x, y, p_x, p_y) - \frac{\omega_0^2 a^2}{2}\right] \tag{4.7}$$

where $\mathcal{H}_0$ is the Hamiltonian of the unperturbed beam:

$$\mathcal{H}_0 = \frac{1}{2}\left(p_z^2 + p_y^2\right) + \frac{\omega_0^2}{2}\left(x^2 + y^2\right) \; . \tag{4.8}$$

Substituting $\psi_0$ from Eq. (4.7) into Eq. (4.6), we obtain the linearized Vlasov equation for $\psi_1$

$$[\mathcal{L}]\psi_1 = -\frac{\lambda}{2\pi^2 a^2}\left(\frac{F_{x1}}{m_0\gamma}\,p_x + \frac{F_{y1}}{m_0\gamma}\,p_y\right)\delta'\left[\mathcal{H}_0 - \frac{\omega_0^2 a^2}{2}\right] \tag{4.9}$$

where

$$[\mathcal{L}] = \left[-i\,\omega + p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} - \omega_0^2 x\frac{\partial}{\partial p_x} - \omega_0^2 y\frac{\partial}{\partial p_y}\right].$$

In the Appendix, it is shown that $F_{x1}(x,y)$ and $F_{y1}(x,y)$ can be derived from a potential function $U_0$

$$\frac{F_{x1}}{m_0\gamma} = \xi K(\omega)\frac{\partial}{\partial x}\,U_0\left[r(x,y),\;\theta(x,y)\right] \tag{4.10a}$$

and

$$\frac{F_{y1}}{m_0\gamma} = \xi K(\omega)\frac{\partial}{\partial y}\,U_0\left[r(x,y),\;\theta(x,y)\right] \tag{4.10b}$$

where $K(\omega)$ is a complex constant which is determined by the parameters of the beam and the machine and

$$U_0(r,\theta) = \begin{cases} \dfrac{r^2}{2} & \text{(monopole)} \quad (4.11a) \\[2ex] \dfrac{r^m}{m}\cos m\theta \; . & \text{(multipole)} \quad (4.11b) \end{cases}$$

For convenience, the dependence of the force on the transverse position of the particle is given by the function $U_0$, whereas the dependence on the size of the beam, the velocity, energy and density of the particles, the radius of the chamber, the wall conductivity, and the frequency of oscillation $\omega$ is included in K. (See Appendix Eqs. 48-50.)

The expressions for $F_{x1}$ and $F_{y1}$ are substituted into the Vlasov equation, yielding

$$[\mathcal{L}]\psi_1 = -\frac{\lambda K(\omega)}{2\pi^2 a^2}\,\delta'\left[\mathcal{H}_0 - \frac{\omega_0^2 a^2}{2}\right]\left[p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y}\right] U_0 \qquad (4.12)$$

which can be written more compactly in terms of the operators $\mathcal{L}_+$ and $\mathcal{L}_-$ as

$$\left[-i\omega - \omega_0^2\mathcal{L}_- + \mathcal{L}_+\right]\psi_1 = -\frac{\lambda K(\omega)}{2\pi^2 a^2}\,\delta'\left[\mathcal{H}_0 - \frac{\omega_0^2 a^2}{2}\right]\mathcal{L}_+ U_0 \qquad (4.13)$$

with

$$[\mathcal{L}_-] = \left[x\frac{\partial}{\partial p_x} + y\frac{\partial}{\partial p_y}\right] \qquad (4.14)$$

and

$$[\mathcal{L}_+] = \left[p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y}\right]. \qquad (4.15)$$

Since $U_0$ has a simple form when expressed in polar coordinates, it is convenient to rewrite the linearized Vlasov equation above in terms of the coordinates:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

and

$$p = \sqrt{p_x^2 + p_y^2}$$

$$\phi = \tan^{-1}\left(\frac{p_y}{p_x}\right)$$

Then the operators in the Vlasov equation become

$$\left[\mathcal{L}_-\right] = r\left[\cos\,(\theta-\phi)\,\frac{\partial}{\partial p} + \sin\,(\theta-\phi)\,\frac{1}{p}\,\frac{\partial}{\partial\phi}\right] \tag{4.16}$$

and

$$\left[\mathcal{L}_+\right] = p\left[\cos\,(\theta-\phi)\,\frac{\partial}{\partial r} - \sin\,(\theta-\phi)\,\frac{1}{r}\,\frac{\partial}{\partial\theta}\right]\ . \tag{4.17}$$

In order to solve the linearized Vlasov equation (Eq. 4.13), it is useful to define the function $U_k(r,\theta)$ for positive integer k

$$\mathcal{L}_+ U_{k-1} = U_k \tag{4.18}$$

with $U_0$ given by Eq. (4.11).

It follows from the above definition that for monopole oscillations,

$$U_\ell = \begin{cases} p^\ell r^{2-\ell}\,\dfrac{1}{(2-\ell)!}\,\cos\left[\ell\,(2-\ell)\,(\phi-\theta)\right], & \ell \le 2 \\ 0\ , & \ell > 2 \end{cases} \tag{4.19a}$$

and for multipole oscillations,

$$U_\ell = \begin{cases} p^\ell r^{m-\ell}\,\dfrac{(m-1)!}{(m-\ell)!}\,\cos\left[(m-\ell)\,\theta + \ell\phi\right], & \ell \le m \\ 0\ . & \ell > m \end{cases} \tag{4.19b}$$

Operating on $U_\ell$ with $\mathcal{L}_-$, we obtain useful relationships for $\ell \ge 1$

$$\mathcal{L}_- U_\ell = \begin{cases} \ell\,(2-\ell+1)\,U_{\ell-1} & \text{(monopole)} \\ \ell\,(m-\ell+1)\,U_{\ell-1}\ . & \text{(multipole)} \end{cases} \tag{4.20}$$

It can be seen from Eqs. (4.18) and (4.20) that the set of functions $U_\ell$ for $\ell \ge 0$, form a closed set with respect to the operators $\mathcal{L}_-$, $\mathcal{L}_+$ and $i\omega$.

Because of this property,

$$\psi_1 = \sum_{\ell} b_\ell U_\ell \delta' \left[\mathcal{H}_0 - \frac{\omega_0^2 a^2}{2}\right] \tag{4.21}$$

where $b_\ell$ is the unknown constant coefficient to be determined. To find $b_\ell$, this series for $\psi_1$ is substituted into Eq. (4.13), and the fact that

$$\left[-\omega_0^2 \mathcal{L}_- + \mathcal{L}_+\right] \delta' \left[\mathcal{H}_0 (x, y, p_x, p_y) - \frac{\omega_0^2 a^2}{2}\right] = 0 \tag{4.22}$$

is used to obtain

$$\sum_{\ell=0}^{m} b_\ell \left[- i\omega - \omega_0^2 \mathcal{L}_- + \mathcal{L}_+\right] U_\ell + \frac{\lambda K(\omega)}{2\pi^2 a^2} \mathcal{L}_+ U_0 = 0 \quad . \tag{4.23}$$

The relationship between $\mathcal{L}_\pm$ and $U_\ell$ as given by Eqs. (4.18) and (4.20) is used in the above equation to obtain

$$\sum_{\ell=0}^{m} \left[b_{\ell-1} - i\omega b_\ell - \omega_0^2 (m-\ell)(\ell+1)\right] U_\ell + \frac{\lambda K(\omega)}{2\pi^2 a^2} U_1 = 0 \tag{4.24}$$

with $b_{-1}$ taken to be zero. To satisfy this equation, the values for the b's are chosen such that the coefficients of all of the $U_\ell$'s are zero, and the following (m+1) linear equations relating the b's are obtained

$$\begin{bmatrix} -i\omega & -m\omega_0^2 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 1 & -i\omega & -2(m-1)\omega_0^2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & -i\omega & -3(m-2)\omega_0^2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -i\omega & -m^2\omega_0^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i\omega \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\lambda K(\omega)}{2\pi^2 a^2} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{4.25}$$

Thus, it is possible to solve for $b_\ell$ in terms of $\omega$ and the other parameters of the beam and the machines. In particular, solving for $b_0$ we find

$$b_0 = -\frac{\lambda K(\omega)}{2\pi^2 a^2} \frac{\begin{vmatrix} 0 & -m\omega_0^2 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 1 & -i\omega & -2(m-1)\omega_0^2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & -i\omega & -3(m-2)\omega_0^2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -i\omega & -m^2\omega_0^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i\omega \end{vmatrix}}{\begin{vmatrix} -i\omega & -m\omega_0^2 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 1 & -i\omega & -2(m-1)\omega_0^2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & -i\omega & -3(m-2)\omega_0^2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -i\omega & -m^2\omega_0^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i\omega \end{vmatrix}} \tag{4.26}$$

The solution of the Vlasov Eq. (4.13) can be found by substituting the values $b_\ell$ into Eq. (4.21).

### 4.2. Making the Solution Self-Consistent

Thus far, no attempt has been made to insure that $\psi_1$ is a self-consistent solution. In this section, the self-consistent requirement is imposed on $\psi_1$. We proceed first to calculate the charge and current densities $\rho_1'$ and $\vec{J}_1'$ corresponding to $\psi_1$; then we require that they be the same as $\rho_1$ and $\vec{J}_1$, as given by Eqs. (3.5) and (3.7). We have

$$\rho_1' = e\lambda \sum_{\ell=0}^{m} b_\ell \int_0^{2\pi}\int_0^{\infty} U_\ell \, \delta'\left(\frac{p^2}{2} + \frac{r^2}{2} - \frac{\omega_0^2 a^2}{2}\right) \, p\,dp\,d\phi \; . \tag{4.27}$$

The integrals in Eq. (4.27) can be evaluated quite simply by noting that the integral of $U_\ell$ over $\phi$ is zero whenever $U_\ell$ is a function of $\phi$. It can be seen from Eq. (4.19) that the only terms independent of $\phi$ are $U_0$ for multipole oscillations, and $U_0$ and $U_2$ for monopole oscillations.

The above integrations are performed, yielding

$$\rho_1' = \begin{cases} -\dfrac{\pi e a}{\omega_0^2} \, \delta(a - r) \, b_0 - 4\pi H(a - r) \, b_2 & (4.28a) \\[2ex] -\dfrac{2\pi e a^{m-1}}{\omega_0^2 m} \, \delta(a - r) \cos m\theta \, b_0 & (4.28b) \end{cases}$$

where use has been made of the integral

$$\int_0^{\infty} g(u) \, \delta'(u - f) \, du = -g(0)\,\delta(f) - g'(f)\,H(f) \quad .$$

From Eq. (4.25) we find that for monopole oscillations $b_0$ and $b_2$ are related by

$$b_2 = -\frac{b_0}{2\,\omega_0^2} \; . \tag{4.29}$$

By inserting this expression into Eq. (4.28a), $\rho_1'$ can be expressed in terms of only $b_0$. Thus

$$\rho_1' = \begin{cases} -\dfrac{\pi e a}{\omega_0^2} \left[\delta(a - r) - \dfrac{2}{a} H(a - r)\right] b_0 & \text{(monopole)} \quad (4.30a) \\[3ex] -\dfrac{2\pi e a^{m-1}}{\omega_0^2 m} \, \delta(a - r) \cos m\theta \, b_0 \; . & \text{(multipole)} \quad (4.30b) \end{cases}$$

For the solution $\psi_1$ to be self-consistent, it is necessary that

$$\rho_1' = \rho_1$$

$$= \begin{cases} \dfrac{\lambda e}{\pi a^2}\left[\delta(a-r) - \dfrac{2}{a} H(a-r)\right] & \text{(monopole)} \\[2ex] \dfrac{\lambda e}{\pi a^2}\, \delta(a-r) \cos m\theta\,, & \text{(multipole)} \end{cases}$$

which implies

$$b_0 = \begin{cases} -\dfrac{\lambda \omega_0^2}{\pi^2 a^3} & \text{(monopole)} \quad (4.31a) \\[2ex] -\dfrac{m \lambda \omega_0^2}{2\pi^2 a^{m+1}}\,. & \text{(multipole)} \quad (4.31b) \end{cases}$$

Similarly, we can calculate the current $\vec{J}_1'$ from $\psi_1$. We find that Eq. (4.31) is the only condition required for both $\rho_1 = \rho_1'$ and $\vec{J}_1 = \vec{J}_1'$.

The frequency of the oscillation, $\omega$, has not yet been specified. To find $\omega$, the two expressions for $b_0$ given by Eqs. (4.26) and (4.31) are equated to obtain the following dispersion relation, which relates $\omega$ with the other parameters of the beam and the machine:

$$\frac{a^{m-1}K(\omega)}{m\,\omega_0^2}\;\frac{\begin{vmatrix} 0 & -m\omega_0^2 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 1 & -i\omega & -2(m-1)\omega_0^2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & -i\omega & -3(m-2)\omega_0^2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -i\omega & -m^2\omega_0^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i\omega \end{vmatrix}}{\begin{vmatrix} -i\omega & -m\omega_0^2 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 1 & -i\omega & -2(m-1)\omega_0^2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & -i\omega & -3(m-2)\omega_0^2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -i\omega & -m^2\omega_0^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i\omega \end{vmatrix}} = 1 \qquad (4.32)$$

Dispersion relationships for various values of m are given in Table 4.1.

TABLE 4.1

Some Dispersion Relationships for the Oscillation Frequency $\omega$

| Type of Oscillation | Value of m | Dispersion Relation for $\omega$ ($-\omega$ is replaced by $(k-\omega)$ for continuous beam) |
|---|---|---|
| monopole | 2 | $4\omega_0^2 - \omega^2 = aK(\omega)$ |
| dipole | 1 | $\omega_0^2 - \omega^2 = K(\omega)$ |
| quadrupole | 2 | $4\omega_0^2 - \omega^2 = aK(\omega)$ |
| sextupole | 3 | $\dfrac{\left(9\omega_0^2 - \omega^2\right)\left(\omega_0^2 - \omega^2\right)}{3\omega_0^2 - \omega^2} = a^2 K(\omega)$ |
| octupole | 4 | $\dfrac{\left(16\omega_0^2 - \omega^2\right)\left(4\omega_0^2 - \omega^2\right)}{\left(10\omega_0^2 - \omega^2\right)} = a^3 K(\omega)$ |

The value of $\omega$ can be obtained from the dispersion relation if the machine and beam parameters are known. With $\omega$ known and its value substituted into the b's in Eq. (4.21), a self-consistent solution $\psi_1$ is explicitly determined.

## V. CONSEQUENCES OF THE DISPERSION RELATIONSHIPS

In this section the dispersion relationships given in Table 4.1 of the preceding section are analyzed. The frequencies of oscillation $\omega$ are determined in terms of the parameters of the beam and the machine. Conditions for stable oscillations are deduced and values of the growth rates for unstable oscillations are found.

As an illustration of the procedure used, a detailed analysis of the dispersion relationship for quadrupole beams is given. Since the method of analysis is the same for all cases of oscillations, only the final results are presented for monopole beams and other multipole beams.

For a small perturbation, the frequency of oscillation for quadrupole case will be close to the natural oscillation frequency $2\omega_0$. The dispersion relationship

$$4\omega_0^2 - (kv - \omega)^2 = aK(\omega) \tag{5.1}$$

can be solved for values of $\omega$ near $2\omega_0$. For a storage ring or circular accelerator the values of k are restricted to $k = n\Omega/v$ with $\Omega$ the revolution frequency, n a positive integer for the uniform beam and equal to zero for the bunched beam. It is convenient to write $\omega_0 = \nu_0 \Omega$ with $\nu_0$ the unperturbed number of betatron oscillations per revolution, so that we obtain for the dispersion relationship:

$$4\nu_0^2\,\Omega^2 - (n\Omega - \omega)^2 = a\left[K_r(\omega) + iK_i(\omega)\right], \tag{5.2}$$

where $K_r$ and $K_i$ denote the real and imaginary parts of K. In practice

$$a\sqrt{K_r^2 + K_i^2} \ll \nu_0\,\Omega$$

so that two of the roots of Eq. (5.2) for $\omega$ are:

$$\omega = \left(n \pm 2\nu_0\right)\Omega \mp \frac{a}{4\nu_0\Omega}\left[K_r\left(n\Omega \pm 2\nu_0\,\Omega\right) + iK_i\left(n\Omega \pm 2\nu_0\,\Omega\right)\right] .$$

For the case of the uniform beam the sign of the imaginary part of $K(\omega)$ is determined by the sign of $\omega$, so that

$$\omega = \left(n \pm 2\nu_0\right)\Omega \mp \frac{aK_r\left(\left|n\Omega \pm 2\nu_0\Omega\right|\right)}{4\nu_0\,\Omega} \mp i\,\mathrm{Sign}\left(n \pm 2\nu_0\,\Omega\right)\frac{aK_i\left(\left|n\Omega \pm 2\nu_0\,\Omega\right|\right)}{4\nu_0\,\Omega} , \tag{5.3}$$

with the upper sign representing a fast wave, the lower sign representing a slow wave,[1] and $K_i\left(\left|n\Omega \pm 2\nu_0\,\Omega\right|\right) > 0$. Since the motion is damped when the imaginary part of $\omega$ is negative, the fast wave is always damped, while the slow wave is damped only for $n < 2\nu_0$. However, for $n > 2\nu_0$ the slow wave grows exponentially with an e-folding time $\tau$ given by

$$\tau = \frac{4\nu_0 \Omega}{a K_i\left(\left|n\Omega - 2\nu_0 \Omega\right|\right)} \quad . \tag{5.4}$$

For the bunched beam $K_r(\omega) = K_r(-\omega)$ and $K_i(\omega) = -K_i(-\omega)$ so that the dispersion relation is given by

$$\omega = \pm\left[2\nu_0 \Omega - \frac{a K_r\left(2\nu_0 \Omega\right)}{4\nu_0 \Omega}\right] - \frac{a K_i\left(2\nu_0 \Omega\right)}{4\nu_0 \Omega} \quad . \tag{5.5}$$

Hence, oscillation is damped if $K_i\left(2\nu_0 \Omega\right) > 0$ and grows exponentially for $K_i\left(2\nu_0 \Omega\right) < 0$ with an e-folding time $\tau$ given by

$$\tau = -\frac{4\nu_0 \Omega}{a K_i\left(2\nu_0 \Omega\right)} \quad . \tag{5.6}$$

The frequencies of oscillations $\omega$ and e-folding times $\tau$ have been calculated for the other cases following the procedure given above. We find that unstable monopole oscillations have very long growth times (many years) and thus impose no practical limitations on the design of accelerators and storage rings.[27] However, for unstable multipole oscillations the growth times are short enough to be of practical importance. Henceforth, we restrict our attention to only multipole oscillations.

To find the condition for stable multipole oscillations, we note that the perturbation has been assumed to vary as $e^{-i\omega t}$, so that for stability $\mathrm{Im}\,\omega < 0$ or $K_i > 0$.

In particular, for stable quadrupole oscillations $K_i(2\nu_0 \Omega) > 0$. For a sufficiently narrow bunch $\left(\frac{\nu_0 L}{\pi R} \ll 1\right)$ the effect of local fields can be neglected so that $K_i\left(2\nu_0 \Omega\right)$ is directly proportional to $\mathrm{Im}\, G\left(2\nu_0\right)$ with the sum G given by[2]

$$G(x) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{n}}\, e^{i 2 n \pi x} \quad . \tag{5.7}$$

A plot of the sum G(x) is given in Fig. 5.1. The n-th term in G gives the contribution of the wake fields to K produced by the beam in its n-th previous turn in the machine. The amplitude of the wake fields decreases as $1/\sqrt{n}$ and the relative phase between the wake fields and the particle oscillation varies as $2\nu_0(2n\pi)$. It can be shown[2] that the sign of Im G is determined by the first term in the sum over n of G. Hence, the condition for stable quadrupole oscillation of a single bunched beam is given by:

$$\sin\left(4\pi\nu_0\right) > 0$$

or

$$n < 2\nu_0 < \left(n + \frac{1}{2}\right) \tag{5.8}$$

with n an integer.

Since the contribution to $K_i$ from the local fields is positive as seen from Eq. (51b) of the Appendix, the effect of the local fields is to widen the region of stable oscillation given above.

A similar analysis has been made for the other multipole cases. The results are summarized in Table 5.1.

TABLE 5.1

Conditions for Stable Multipole Oscillations for a Single Bunched Beam

| Types of Oscillations | Approximate Frequency of Oscillations $(\omega_0 = \nu_0\,\Omega)$ | Stable Conditions for a Bunched Beam |
|---|---|---|
| dipole | $\omega_0$ | $n < \nu_0 < n + 1/2$ |
| quadrupole | $2\omega_0$ | $n < 2\nu_0 < n + 1/2$ |
| sextupole | $\omega_0$ | $n < \nu_0 < n + 1/2$ |
| | $3\omega_0$ | $n' < 3\nu_0 < n' + 1/2$ |
| octupole | $2\omega_0$ | $n < 2\nu_0 < n + 1/2$ |
| | $4\omega_0$ | $n' < 4\nu_0 < n' + 1/2$ |

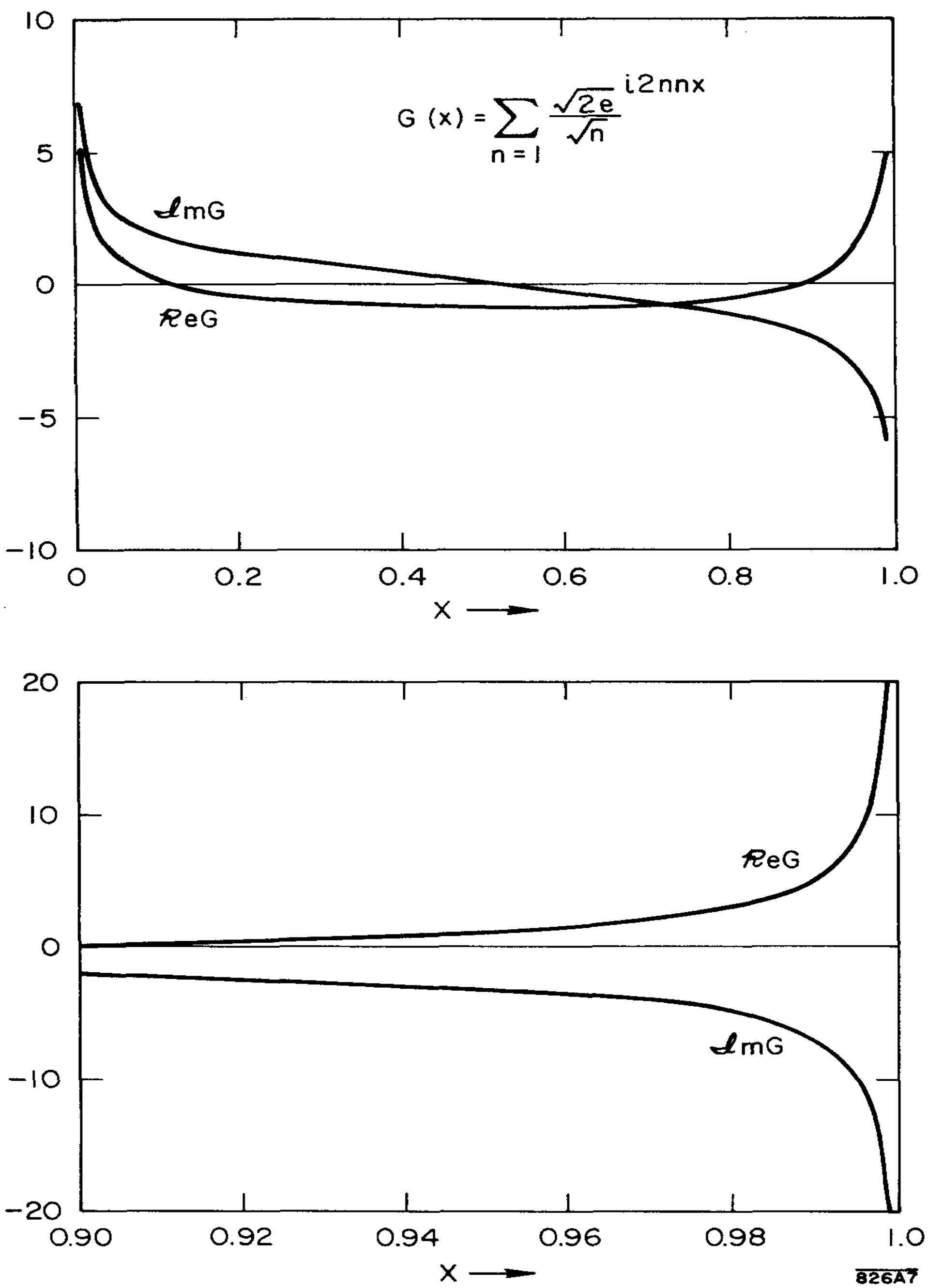


Fig. 5.1. Values of the sum G(x) for $0 < x < 1$.
The sum is defined by Eq. (5.7).

For unstable quadrupole oscillations the maximum growth rate $1/\tau_2$ is related to the maximum growth rate of the unstable dipole oscillations of Laslett, Neil, and Sessler[1] $1/\tau_1$ by:

$$\frac{1}{\tau_2} = \left(\frac{a}{b}\right)^2 \frac{1}{\tau_1} \tag{5.9}$$

In general, the actual growth rate is less than $(1/\tau)$ calculated from Eq. (5.6) because in the analysis thus far all of the particles have been assumed to have the same unperturbed frequency $\omega_0$, i.e., Landau damping has not been considered. The effects of Landau damping will be discussed in Section VI.

## VI. LANDAU DAMPING OF QUADRUPOLE BUNCHED BEAM

It is known that unstable beam oscillations can be suppressed by Landau damping due to a spread in the betatron oscillation frequency $\omega_0$.[1,2] In this section, we examine the effect of this stabilizing mechanism on unstable quadrupole oscillations of a bunch beam. A criterion for the spread in the frequency $\omega_0$ required for stabilization is obtained.

In the following analysis the charge density of the unperturbed beam is assumed to vary with radius. The unperturbed beam can be thought of as being made up of a sum of many beams each with a constant radial density inside of radius a. Then the charge density can be written as

$$\rho_0(r, z) = \mathscr{D}_0(r) f(z - vt) \tag{6.1a}$$

with the transverse dependence of the charge density given by

$$\mathscr{D}_0(r) = \int_0^\infty \frac{e\, n(a) H(a - r)\, da}{\pi a^2} \tag{6.1b}$$

where $n(a)\,da$ is the number of particles inside a beam with uniform transverse cross section density of radius a and

$$\int_0^{\infty} n(a)\, da = N \, . \qquad (6.2)$$

In particular, for a beam of uniform transverse cross section density

$$n(a) = N\delta(a - r) \, .$$

In the perturbed beam, we assume the beam radius of each of the uniform cross section beams varies as $\left(a + \xi(a) \cos 2\theta\right)$ with

$$\xi(a) = \xi_0(a)\, e^{-i\omega t} \qquad (6.3)$$

for quadrupole oscillation of a bunched beam. Then the transverse charge density of the beam is given by

$$\mathscr{D}(r) = \mathscr{D}_0(r) + \mathscr{D}_q(r) \qquad (6.4)$$

where

$$\mathscr{D}_q(r) = \int_0^{\infty} \frac{e\,\xi(a)\, n(a)\, \delta(a - r) \cos 2\theta}{\pi a^2}\, da$$

$$= \frac{e\,\xi(r)\, n(r)}{\pi r^2} \cos 2\theta \, . \qquad (6.5)$$

The expressions for the charge densities can be derived directly from those previously obtained for a uniform cross section beam (Eqs. 3.1 and 3.5b) by replacing N with n(a)da and integrating over a. By the same superposition principle, the perturbing force fields can be found from Eq. (4.10) yielding

$$\frac{F_{x1}}{m_0\gamma} = \int_0^{\infty} \xi(a)\, \overline{K}(a)\, da \, \frac{\partial}{\partial x}\, U_0(r, \theta) \qquad (6.6a)$$

and

$$\frac{F_{y1}}{m_0\gamma} = \int_0^\infty \xi(a)\overline{K}(a)\, da \frac{\partial}{\partial y} U_0(r, \theta) \tag{6.6b}$$

where $\overline{K}(a)$ is the value of K given by Eq. (51b) of the Appendix, with N replaced by n(a).*

We find that a self-consistent solution of the Vlasov equation is

$$\psi(x,y,z,p_x,p_y,p_z) = \left[\psi_0(x,y,p_x,p_y) + \psi_1(x,y,p_x,p_y)\right] f(z-vt)\delta(p_z - p_{z0}) \tag{6.7}$$

where $\psi_0$ and $\psi_1$ are given by Eqs. (4.7) and (4.21) with N replaced by n(a)da and integrating over a. The dispersion relationship for the oscillation frequencies is

$$\xi(a)\left(4\omega_0^2 - \omega^2\right) = a\int_0^\infty \xi(u)\overline{K}(u)\, du\ . \tag{6.8}$$

Dividing Eq. (6.8) by the quantity $\left(4\omega_0^2 - \omega^2\right)$, multiplying the result by $\overline{K}(a)da$ and integrating over a, we obtain a more useful form of the dispersion

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*The term K is defined for an observation point inside a uniform cross section beam. For an observation point outside of the beam, K must be modified to include an additional term which is independent of the wall conductivity. Thus, for a nonuniform beam Eq. (6.6) gives only a portion of the total force fields. The inclusion of this additional term, however, will not significantly affect the result of this section. Furthermore, since $\omega_0 = \nu_0\,\Omega$ and both $\nu_0$ and $\Omega$ are functions of the betatron oscillation amplitude a and the particle energy $\gamma$, a spread in both a and $\gamma$ can contribute to a spread in $\omega_0$. The exact value of the lower limit on a spread in $\omega_0$ necessary for Landau damping depends on the distribution of particles in a and $\gamma$. In Ref. 1 a spread in either a or $\gamma$ has been treated. The result of this section is in agreement with the result of Ref. 1, namely, if the spread in $\omega_0$ is large compared to the shift in the coherent oscillation frequency $|Ka/4\,\omega_0|$, the unstable coherent oscillations are Landau damped.

relationship

$$\int_0^{\infty} \frac{a\overline{K}(a)}{4\,\omega_0^2(a) - \omega^2}\, da = 1 \tag{6.9}$$

which is independent of the perturbation amplitudes. It will be assumed that the frequency $\omega_0$ depends on a.

For a small perturbation, $\omega \approx 2\omega_0$ so that we may write

$$4\omega_0^2 - \omega^2 \approx 4\omega_0\left(2\,\omega_0(a) - \omega\right) . \tag{6.10}$$

Let

$$\omega_0(a) \approx \nu_0\,\Omega + \omega_0'\, a \tag{6.11}$$

then Eq. (6.10) becomes

$$4\,\omega_0^2 - \omega^2 \approx 4\nu_0\,\Omega\,\left(2\,\nu_0\Omega - \omega + 2\omega_0'\,a\right) \tag{6.12}$$

where

$$\omega_0' = \frac{\partial\omega_0}{\partial a} .$$

From Eq. (6.11) the spread in the particle oscillation frequencies in the beam is given by

$$\Delta\left(2\omega_0\right) = \left(2\omega_0 - 2\nu_0\,\Omega\right)_{\text{maximum}} = 2\,\omega_0'\,a_m \tag{6.13}$$

where $a_m$ is the maximum value of a. Hence, for a given value of $\omega_0'$ the frequency spread in the beam is proportional to $a_m$.

Since $\mathscr{D}_0(r)$ and n(a) are related by Eq. (6.1b), differentiation of $\mathscr{D}_0(r)$ with respect to r gives

$$n(a) = -\pi a^2 \mathscr{D}_0'(a) . \tag{6.14}$$

Inserting Eqs. (6.12) and (6.14) into Eq. (6.9), we find for the dispersion integral

$$\int_0^{\infty} \frac{\mathscr{D}'_0(a)\left[C_1 b^4 + \left(-C_1+C_2+C_3\right) a^4 + \left(D_2+D_3\right) a^4\right]}{a - a_1(\omega)}\, da = 2\,\omega'_0 \qquad (6.15)$$

where

$$a_1(\omega) = \frac{\omega - 2\nu_0\Omega}{2\,\omega'_0} \qquad (6.16)$$

$$C_1 = \frac{\sqrt{\pi}\, r_0 C^2}{2\nu_0 \Omega L \gamma^3 b^4}$$

,

$$\left.\begin{matrix} C_2 \\ \\ C_3 \\ \\ D_2 \\ \\ D_3 \end{matrix}\right\} = - \frac{r_0 \beta^2 C^3}{2\nu_0 \Omega \gamma b^5 \sqrt{\pi\beta C\sigma L}} \left\{\begin{matrix} \Gamma\left(\frac{1}{4}\right) \\ \\ \sqrt{\frac{L}{R}}\, \mathrm{Re}\, G\left(2\nu_0\right) \\ \\ \frac{\nu L}{R}\, \Gamma\left(\frac{3}{4}\right) \\ \\ \sqrt{\frac{L}{R}}\, \mathrm{Im}\, G\left(2\nu_0\right) \end{matrix}\right.$$

and

$$G(x) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{n}}\, e^{i2n\pi x} .$$

We will be interested only in the case where the oscillations are unstable in the absence of Landau damping. It may be recalled that for the case of uniform beam the stability condition depends on the sign of $\mathrm{Im}\,\omega$ or $\mathrm{Im}\,K$. In particular, unstable quadrupole oscillations occur when $\mathrm{Im}K\left(2\nu_0\Omega\right) < 0$, i.e., $\left(D_2 + D_3\right) > 0$. Moreover, the shift in the oscillation frequency $\omega$ is proportional to $\mathrm{Re}K\left(2\nu_0\Omega\right)$ which can either be positive or negative depending on the relative magnitudes of the constants $C_1$, $C_2$, and $C_3$. For a proton accelerator with a low $\gamma$, $C_1$ is dominant. Hence $\Delta\omega > 0$, where

$$\Delta\omega = \mathrm{Re}\,\omega - 2\nu_0\Omega \quad . \tag{6.17}$$

However, for an electron storage ring having a high $\gamma$, $C_2$ is dominant and therefore $\Delta\omega < 0$. The cases of high and low $\gamma$ will be studied separately.

The mechanism effective in stabilizing the beam has been characterized by the parameter $\omega_0'$ whose value depends on the nonlinearity in the machine.[1] As defined by Eq. (6.16) the shift of the oscillation frequency due to the perturbation is given by

$$\Delta\omega = 2\omega_0' a$$

where $\Delta\omega$ is the frequency shift. Note that $a_1$ is the amplitude at which the change in twice the betatron frequency, $\Delta\left(2\omega_0\right)$, equals the frequency shift $\Delta\omega$. We are interested in finding the values of $a_1$ which satisfy the above dispersion relationships for a given value of $\omega_0'$. In particular, we wish to find the value of $a_1$ at the stability limit. Since $a_1$ is real at the stability limit, the integrand in Eq. (6.15) is singular so that care must be taken in evaluating the integral.

To define the integral for real $a_1$, first we note that for oscillation varying as $e^{-i\omega t}$ with

$$\omega = 2\nu_0\Omega + 2\omega_0' a \ , \tag{6.18}$$

unstable oscillations occur when $2\omega_0' \mathrm{Im}\, a_1 > 0$. In the limit $\mathrm{Im}\, a_1 \longrightarrow 0$ the pole $a_1$ in the complex a-plane approaches the real axis from above for $\omega_0' > 0$ and from below for $\omega_0' < 0$. Thus for the case of $\omega_0' > 0$ we consider $a_1$ to have an infinitesimal positive imaginary part, so that it lies just above the path of integration as shown in Fig. 6.1a with an equivalent path of integration shown in Fig. 6.1b. The opposite is true of $\omega_0' < 0$. For Landau damping it is important that $a_1$ be less than $a_m$.

With this in mind the dispersion integral can be evaluated for real $a_1$ which results in

$$P\int_0^{a_m} \frac{\mathscr{D}_0'(a)\left[C_1 b^4 + (-C_1 + C_2 + C_3)a^4\right]}{a - a_1}\, da \mp \pi \mathscr{D}_0'(a_1)(D_2 + D_3)a_1^4 = 2\omega_0' \qquad (6.19a)$$

and

$$P\int_0^{a_m} \frac{\mathscr{D}_0'(a)(D_2 + D_3)a^4}{a - a_1}\, da \pm \pi \mathscr{D}_0'(a_1)\left[C_1 b^4 + (-C_1 + C_2 + C_3)a_1^4\right] = 0 \qquad (6.19b)$$

where P indicates the principal value. The upper and lower signs correspond to the case of $\omega_0' > 0$ and $\omega_0' < 0$, respectively.

As a reasonable approximation[1] of the unperturbed transverse charge density we take

$$\mathscr{D}_0(a) = \frac{6eN}{\pi a_m^4}\,(a_m - a)^2\, H(a_m - a)\ . \qquad (6.20)$$

Differentiating $\mathscr{D}_0$ gives

$$\mathscr{D}_0'(a) = -\frac{12eN}{\pi a_m^4}(a_m - a)H(a_m - a)\ . \qquad (6.21)$$

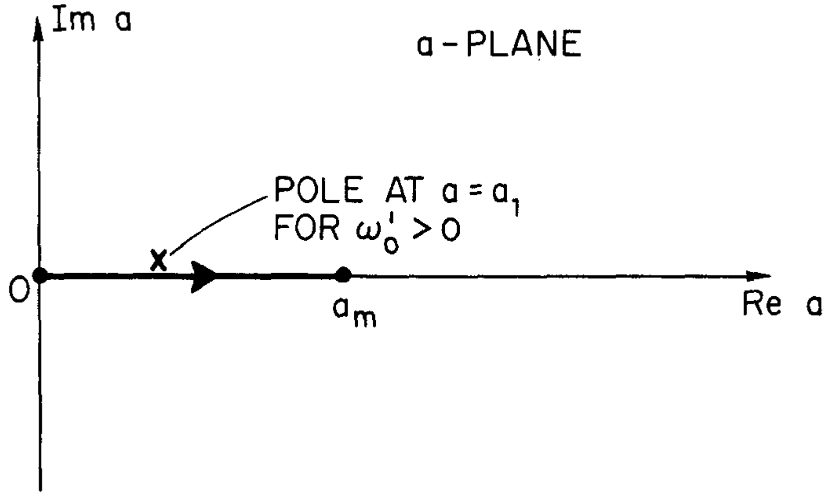


Fig. 6.1a. Actual path of integration for the dispersion integral. The integral is given by Eq. (6.19).

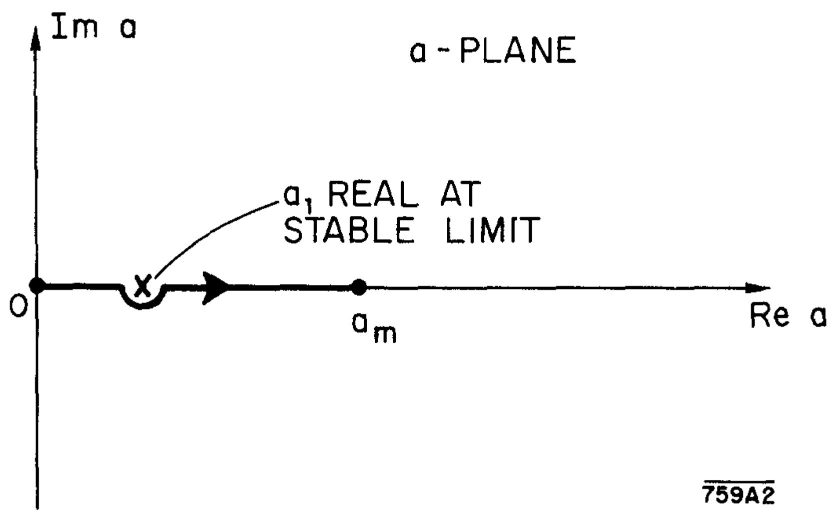


Fig. 6.1b. Equivalent path of integration for the evaluation of the dispersion integral.

Substituting Eq. (6.21) into Eqs. (6.19) and evaluating the integrals, we obtain for $\omega_0' > 0$:

$$(a_m - a_1)\left\{\left[C_1 b^4 + (-C_1 + C_2 + C_3) a_1^4\right] \ln\left(\frac{a_m - a_1}{a_1}\right) + (-C_1 + C_2 + C_3)\, g(a_1, a_m) - \pi (D_2 + D_3)\, a_1^4\right\}$$

$$= -\frac{\pi \omega_0' a_m^4}{6N} + C_1 b^4 a_m + (-C_1 + C_2 + C_3)\frac{a_m^5}{5} \qquad (6.22a)$$

and

$$a_1^4 \ln\left(\frac{a_m - a_1}{a_1}\right) + g(a_1, a_m) = -\frac{\pi\left[C_1 b^4 + (-C_1 + C_2 + C_3)\, a_1^4\right]}{D_2 + D_3} \qquad (6.22b)$$

where

$$g(a_1, a_m) = \frac{(a_m - a_1)^4}{4} + \frac{4(a_m - a_1)^3 a_1}{3} + 3(a_m - a_1)^2 a_1^2 + 4(a_m - a_1) a_1^3 - \frac{25}{12} a_1^4 . \qquad (6.23)$$

In general, solving for the unknown parameters $a_1$ and $a_m$ from Eq. (6.22) may be difficult. But, for the special cases of high or low $\gamma$, the solutions can be readily found.

When $\gamma$ is low $C_1$ is large compared to the other constants and since $(D_2 + D_3) > 0$, the right-hand side of Eq. (4.22b) is a large negative number and thus $a_1 \approx a_m$. Hence, the spread in $2\omega_0$ required for Landau damping is equal to the value of the frequency shift, i.e., $\Delta(2\omega_0) \approx \Delta\omega$. Setting $a_1$ equal to $a_m$ in Eq. (6.22a) yields

$$\frac{a_m^4}{5} + \frac{\pi \omega_0' a_m^3}{6NC_1} = b^4 . \qquad (6.24)$$

For the case of high $\gamma$, $C_2$ is the dominant term and the right-hand side of

Eq. (6.22b) is a large positive number. Therefore,

$$\frac{a_m}{a_1} = \left(-\frac{4\pi C_2}{D_2 + D_3}\right)^{\frac{1}{4}} \tag{6.25}$$

so that $a_m \gg a_1$. The spread in $2\omega_0$ required for Landau damping in this case is much larger than the frequency shift

$$\Delta(2\omega_0) = -\left(\frac{4\pi C_2}{D_2 + D_3}\right)^{\frac{1}{4}} \Delta\omega . \tag{6.26}$$

In both cases the required frequency spread for Landau damping is directly proportional to the frequency shift due to the perturbation. In the limit of low $\gamma$, the criterion for the spread required is independent of the wall conductivity. This result agrees with the results of Laslett, Neil and Sessler[1] who have treated the problem in much more detail for the case of dipole oscillations.

When the expression for $C_1$ is substituted into Eq. (6.24) and the result is simplified, we find for the case of low $\gamma$ the number of particles in one bunch at the stable limit is given by

$$N_{max} = \frac{\sqrt{\pi}\, L \gamma^3 \nu_0 \Omega^2}{3 r_0 c^2} \left[\frac{a_m^2 b^4}{b^4 - \frac{a_m^4}{5}}\right] \Delta\nu_0 \tag{6.27}$$

with the betatron frequency spread in the beam given by $\Delta\nu_0$. Thus the criterion for damping unstable quadrupole oscillation for a machine having low $\gamma$ is

$$\Delta\nu_0 \geq \frac{3 N r_0 c^2}{\sqrt{\pi}\, L \gamma^3 \nu_0 \Omega^2} \left[\frac{b^4 - \frac{a_m^4}{5}}{a_m^2 b^4}\right] . \tag{6.28}$$

Similarly, for a case of high $\gamma$ the stability criterion is

$$\Delta\nu_0 \geq \frac{3 N r_0 c^3 \beta^2 a_m^2 \, \Gamma(1/4)}{20\pi\gamma b^5 \nu_0 \Omega^2 \sqrt{\pi\beta C\sigma L}} \tag{6.29}$$

or the maximum number of particles in a stable bunch is

$$N_{max} = \frac{20\pi\gamma b^5 \nu_0 \Omega^2 \sqrt{\pi\beta C\sigma L}}{3 r_0 c^3 \beta^2 a_m^2 \, \Gamma(1/4)} \, \Delta\nu_0 \; . \tag{6.30}$$

## VII. THE EFFECT OF BETATRON FREQUENCY SPLITS

In the analysis of the preceding sections, the betatron frequencies of the particles $\omega_{0x}$ and $\omega_{0y}$ have been assumed to be equal. As a consequence of this assumption, oscillations of the different ordered poles do not affect one another. However, if $\omega_{0x}$ and $\omega_{0y}$ are not equal, coupling may occur between different types of oscillations so that an oscillation of one type may excite an oscillation of a different type. As an example of this occurrence, the effect of $\omega_{0x} \neq \omega_{0y}$ on monopole and quadrupole oscillations is considered in this section. It is found that coupling exists between monopole and quadrupole oscillations and the motion of the beam is characterized by two normal modes such that the monopole and quadrupole oscillations are in phase in one mode and 180° out of phase in the other.

In a perturbed beam, we take for the charge and current densities for a bunched beam

$$\rho = \rho_0 + \xi_m \rho_m + \xi_q \rho_q \tag{7.1}$$

and

$$\vec{J} = \vec{J}_0 + \xi_m \vec{J}_m + \xi_q \vec{J}_q \tag{7.2}$$

where

$$\xi_{\substack{m\\q}} = \xi_{0\substack{m\\q}} e^{-i\omega t} . \tag{7.3}$$

$\rho_0$ is the charge density for an unperturbed beam as given in Eq. (3.1), $\rho_m$ and $\rho_q$ are the perturbing charge densities as given by Eq. (3.5) with m for monopole and q for quadrupole, $\vec{J}_0$, $\vec{J}_m$ and $\vec{J}_q$ are the corresponding current densities as given by Eqs. (3.2) and (3.7).

For a particle in the beam, the equations of motion are written as

$$\dot{p}_x = -x + \frac{F_{xm}}{m_0 \gamma \omega_{0x}^2} + \frac{F_{xq}}{m_0 \gamma \omega_{0x}^2} , \quad \dot{x} = \omega_{0x}^2 p_x , \tag{7.4}$$

$$\dot{p}_y = -y + \frac{F_{ym}}{m_0 \gamma \omega_{0y}^2} + \frac{F_{yq}}{m_0 \gamma \omega_{0y}^2} , \quad \dot{y} = \omega_{0y}^2 p_y , \tag{7.5}$$

$$\dot{p}_z = 0 , \qquad \text{and} \qquad \dot{z} = v , \tag{7.6}$$

where $p_x$, $p_y$ and $p_z$ are the conjugate momenta.*

In analogy to Eq. (4.2), we take $\psi$ as

$$\psi = \left[\psi_0(x,y,p_x,p_y) + \xi_m \psi_m(x,y,p_x,p_y) + \xi_q \psi_q(x,y,p_x,p_y)\right] \lambda \delta(p_z - p_{0z}) \tag{7.7}$$

where $\psi_0$ is the self-consistent particle distribution function corresponding to an unperturbed beam which satisfies the conditions given in Eqs. (4.3) and (4.4).

Substituting $\psi$ into the Vlasov equation, Eq. (4.1), we obtain

$$\mathcal{L}_1 \psi_0 = 0 , \tag{7.8}$$

*It may be noted that the conjugate momenta as defined here differ from those given on p. 12 by some normalization constants.

and

$$\left[-i\omega+\mathcal{L}_1\right](\xi_m\psi_m+\xi_q\psi_q) = -\frac{1}{m_0\gamma}\left[\frac{F_{xm}+F_{xq}}{\omega_{0x}^2}\frac{\partial}{\partial p_x}+\frac{F_{ym}+F_{yq}}{\omega_{0y}^2}\frac{\partial}{\partial p_y}\right]\psi_0 \; , \qquad (7.9)$$

where

$$\left[\mathcal{L}_1\right] = \left[\omega_{0x}^2 p_x \frac{\partial}{\partial x} + \omega_{0y}^2 p_y \frac{\partial}{\partial y} - x\frac{\partial}{\partial p_x} - y\frac{\partial}{\partial p_y}\right] . \qquad (7.10)$$

A self-consistent solution $\psi_0$ is

$$\psi_0 = \frac{\lambda\,\omega_{0x}\,\omega_{0y}}{2\pi^2 a^2}\,\delta\left[\mathcal{H}_0\,(x,y,p_x,p_y) - \frac{a^2}{2}\right] \qquad (7.11)$$

with

$$\mathcal{H}_0 = \frac{\omega_{0x}^2 p_x^2 + \omega_{0y}^2 p_y^2}{2} + \frac{x^2+y^2}{2} \; . \qquad (7.12)$$

Substituting $\psi_0$ into Eq. (7.9) gives the linearized Vlasov equation for $\psi_1$:

$$\left[-i\omega+\mathcal{L}_1\right](\xi_m\,\psi_m+\xi_q\,\psi_q) =$$

$$-\frac{\lambda\,\omega_{0x}\,\omega_{0y}}{2\pi^2 a^2}\left[(\xi_m K_m+\xi_q K_q)\,x\,p_x+(\xi_m K_m-\xi_q K_q)\,y\,p_y\right]\delta'\left(\mathcal{H}_0-\frac{a^2}{2}\right) \qquad (7.13)$$

where

$$\frac{F_{xm}}{m_0\gamma} = \xi_m K_m x \; , \qquad \frac{F_{ym}}{m_0\gamma} = \xi_m K_m y \; , \qquad (7.14)$$

$$\frac{F_{xq}}{m_0\gamma} = \xi_q K_q x \; , \qquad \frac{F_{yq}}{m_0\gamma} = -\xi_q K_q y \qquad (7.15)$$

with the values of the constants $K_m$ and $K_q$ given in the Appendix.

A self-consistent solution $\psi_{\substack{m\\q}}$ which satisfies the above linearized Vlasov equations is found to be

$$\psi_{\substack{m\\q}} = \frac{\lambda \omega_{0x} \omega_{0y}}{\pi^2 a^3} \left[ \frac{\omega_{0x}^2 p_x^2 \pm \omega_{0y}^2 p_y^2}{2} - \frac{x^2 \pm y^2}{2} + \frac{i\omega(x p_x \pm y p_y)}{2} \right] \delta' \left( \mathcal{H}_0 - \frac{a^2}{2} \right) \qquad (7.16)$$

with the oscillation frequency and amplitudes obeying the relationships:

$$\left(\omega^2 - 4\,\omega_{0x}^2 + aK_m\right) \xi_m + \left(\omega^2 - 4\,\omega_{0x}^2 + aK_q\right) \xi_q = 0 \qquad (7.17a)$$

and

$$\left(\omega^2 - 4\,\omega_{0y}^2 + aK_m\right) \xi_m - \left(\omega^2 - 4\,\omega_{0y}^2 + aK_q\right) \xi_q = 0 \ . \qquad (7.17b)$$

In order to have nontrivial solutions for the oscillation amplitudes $\xi_m$ and $\xi_q$ the oscillation frequency must satisfy the dispersion relationship below:

$$\begin{vmatrix} \left(\omega^2 - 4\,\omega_{0x}^2 + aK_m\right) & \left(\omega^2 - 4\,\omega_{0x}^2 + aK_q\right) \\ \left(\omega^2 - 4\,\omega_{0y}^2 + aK_m\right) & -\left(\omega^2 - 4\,\omega_{0y}^2 + aK_q\right) \end{vmatrix} = 0 \ . \qquad (7.19)$$

In solving for $\omega$, we will assume that the frequency shift due to the perturbation is small compared to the split in the frequencies, i.e.,

$$4\left|\left(\omega_{0x}^2 - \omega_{0y}^2\right)\right| \gg a\left|K_t\right| \ .$$

where

$$K_t = K_m + K_q \ .$$

Since, in the absence of perturbations, the natural frequencies are $2\,\omega_{0x}$ and $2\,\omega_{0y}$ the values of the frequencies of oscillation for small perturbations are close to $2\,\omega_{0x}$ and $2\,\omega_{0y}$. Hence, we solve for the frequency $(\omega_1)$ near $2\,\omega_{0x}$ and obtain

$$\omega_1 = 2\,\omega_{0x} - \frac{aK_t\,(2\,\omega_{0x})}{8\,\omega_{0x}} \ . \qquad (7.20a)$$

Similarly, for the frequency $(\omega_2)$ near $2\,\omega_{0y}$ ,

$$\omega_2 = 2\,\omega_{0y} - \frac{a K_t\left(2\,\omega_{0y}\right)}{8\,\omega_{0y}} \, . \tag{7.20b}$$

When the values of $\omega_1$ and $\omega_2$ are substituted into Eq. (7.17) and the perturbation amplitudes are solved, we find that for the normal mode amplitudes corresponding to $\omega_1$, $\xi_m = \xi_q$, and for the normal mode corresponding to $\omega_2$, $\xi_m = -\xi_q$. Thus, we find that as a consequence of large frequency splits, monopole oscillations are excited by quadrupole oscillations and vice versa. The motion of the beam can be decomposed into two normal modes which are made up by monopole and quadrupole oscillations. The monopole and quadrupole oscillations are in phase in one mode and 180° out of phase in the other. The energy of each mode is split equally between the monopole and quadrupole oscillations.

A similar analysis can be carried out for the case with small frequency splits $\left(\omega_{0x} \approx \omega_{0y}\right)$, i.e.,

$$4\left|\left(\omega_{0x}^2 - \omega_{0y}^2\right)\right| \ll a\left|K_t\right| \, .$$

Under this condition, we obtain the normal mode frequencies

$$\omega_m = 2\,\omega_0 - \frac{a K_m\left(2\,\omega_0\right)}{4\,\omega_0} \tag{7.21a}$$

and

$$\omega_q = 2\,\omega_0 - \frac{a K_q\left(2\,\omega_0\right)}{4\,\omega_0} \tag{7.21b}$$

$\left(\text{with } \omega_0 \approx \omega_{0x} \approx \omega_{0y}\right)$ , which agree with the dispersion relations given in Section V. The corresponding normal mode amplitudes are given by $\xi_m$ arbitrary and $\xi_q = 0$ for $\omega = \omega_m$, and $\xi_q$ arbitrary and $\xi_m = 0$ for $\omega = \omega_q$. This result shows that for this case the normal mode oscillations are either pure monopole or pure quadrupole oscillations.

Since the perturbations due to monopole oscillations are much weaker than those due to quadrupole oscillations, we have $K_t \approx K_q$ so that for the case of large frequency splits

$$\omega_1 \approx 2\omega_{0x} - \frac{a K_q (2\omega_{0x})}{8\omega_{0x}} \tag{7.22a}$$

and

$$\omega_2 \approx 2\omega_{0y} - \frac{a K_q (2\omega_{0y})}{8\omega_{0y}} . \tag{7.22b}$$

The conditions of stability for the normal modes are similar to those for pure quadrupole oscillations

$$n < 2\nu_{0x} < n + \frac{1}{2} \tag{7.23a}$$

and

$$n' < 2\nu_{0y} < n' + \frac{1}{2} \tag{7.23b}$$

where n and n' are integers, $\nu_{0x}\Omega = \omega_{0x}$, and $\nu_{0y}\Omega = \omega_{0y}$.

By comparing Eqs. (7.21) and (7.22), we find that for the large frequency splits the maximum growth rate is one-half the growth rate for a pure quadrupole oscillation. This is because for large frequency splits the energy is shared equally between the monopole and quadrupole components of each mode. However, the amount of energy feedback to the oscillations by the monopole component is negligible compared to the amount of feedback by the quadrupole component. Thus, the energy contained in the monopole component does not contribute to the unstable growth. Furthermore, the frequency shift is one-half the frequency shift for a pure quadrupole oscillation so that the frequency spread required for Landau damping is one-half of that required for damping a pure quadrupole oscillation.

## VIII. SUMMARY

The possibility for both longitudinally continuous and longitudinally bunched beams in circular accelerators to have unstable transverse beam oscillations has been demonstrated theoretically. The oscillations of a nearly circular beam centered in a vacuum tank of circular cross section can be characterized by monopole, dipole, quadrupole, etc., oscillations. For the case of equal horizontal and vertical betatron frequencies these oscillations can exist independently, with no coupling between the various poles.

Dispersion relationships which relate the oscillation frequency $\omega$ to the number of betatron oscillations per revolution $\nu_0$ have been found for monopole and multipole oscillations. It follows from these dispersion relationships that in the absence of Landau damping a longitudinally continuous beam is always unstable against the development of waves having a phase velocity close to $\left(n - m\nu_0\right)\Omega R$, where $\Omega$ is the revolution frequency, R is the radius of the machine and n is an integer greater than $m\nu_0$ with 2m the multipole number (m = 1 for dipole, m = 2 for quadrupole, etc.). A longitudinal single bunched beam is always stable against multipole oscillations of order m provided $n < \nu_0 < \left(n + \frac{1}{2m}\right)$. It follows from this condition that for $\nu_0$ just above an integer all multipole oscillations are stable for a single bunched beam.

For multipole instabilities the growth rates and thresholds are close to those obtained for the dipole instability, differing only by a geometrical factor which depends on the radius of the tank b and the radius of the beam a. For example, the maximum growth rate $\frac{1}{\tau_2}$ for the unstable quadrupole oscillations is related to the maximum growth rate $\frac{1}{\tau_1}$ for the unstable dipole oscillations by

$$\frac{1}{\tau_2} = \left(\frac{a}{b}\right)^2 \frac{1}{\tau_1} \quad . \tag{8.1}$$

The growth rate for the dipole instability is generally of the order of milliseconds. For monopole oscillations, however, the growth rates are typically years.

The maximum growth rate for unstable quadrupole oscillations is given by:

$$\tau_2 = \begin{cases} \dfrac{4\nu_0\Omega}{a K_i\left(n\Omega - 2\nu_0\Omega\right)} & \begin{array}{c} n > 2\nu_0 \\ \text{(continuous beam)} \end{array} \quad (8.2a) \\[2ex] \dfrac{-4\nu_0\Omega}{a K_i\left(2\nu_0\Omega\right)} & \text{(bunched beam)} \quad (8.2b) \end{cases}$$

with $K_i = \mathrm{Im}\,K$

and

$$K = \begin{cases} -\dfrac{2 r_0 \lambda c^2\left(b^4 - a^4\right)}{\gamma^3 b^4 a^3} + (1+i)\dfrac{4 r_0 \lambda \beta^2 c^3}{\gamma b^5}\left[\dfrac{1}{2\pi\sigma\left(n - 2\nu_0\right)\Omega}\right]^{\frac{1}{2}} & \begin{array}{c} n > 2\nu_0 \\ \text{(continuous beam)} \end{array} \quad (8.3a) \\[3ex] -\dfrac{2 r_0 N\left(b^4 - a^4\right) c^2}{\pi^{\frac{1}{2}} L \gamma^3 b^4 a^3} + \dfrac{2 N r_0 \beta^2 c^3 a}{\gamma\pi b^5\sqrt{\pi\beta c\sigma L}}\left[\Gamma\left(\frac{1}{4}\right) + i\frac{2\nu_0 L}{R}\Gamma\left(\frac{3}{4}\right)\right] & \\[3ex] + \dfrac{2 r_0 N \beta^2 c^3 a}{\pi\gamma b^5\sqrt{\pi\beta c\sigma R}}\, G\left(2\nu_0\right) & \text{(bunched beam)} \quad (8.3b) \end{cases}$$

where

$$G(x) = \sum_{n=1}^{\infty}\sqrt{\frac{2}{n}}\, e^{i 2 n \pi x} \tag{8.4}$$

$r_0$ = classical radius of the particles

L = length of the bunch

R = radius of the machine

$\nu_0$ = number of betatron oscillations per revolution

$\sigma$ = conductivity of the pipe

$\Omega$ = revolution frequency of the particles

N = total number of particles per bunch

$\lambda$ = number of particles per unit length

c = velocity of light

n = positive integer greater than $2\nu_0$

$\Gamma$ = complete gamma function

The value of the sum G can be obtained from Fig. 5.1.

In general, a spread in both the betatron oscillation amplitude a and the particle energy $\gamma$ can contribute to a spread in the betatron oscillation frequency $\omega_0$. The exact value of the lower limit on a spread in $\omega_0$ necessary for Landau damping depends on the distribution of particles in a and $\gamma$.

For a bunched beam with no energy spread, the criterion for stability by Landau damping due to a spread in the betatron oscillation frequency against unstable quadrupole oscillations is given by:*

$$\Delta\nu_0 \geq \begin{cases} \dfrac{3 N r_0 c^2}{\sqrt{\pi} L \gamma^3 \nu_0 \Omega^2} \left[ \dfrac{b^4 - \frac{a_m^4}{5}}{a_m^2 b^4} \right] & \text{(low } \gamma\text{)} \quad (8.5a) \\[2em] \dfrac{3 N r_0 c^3 \beta^2 \; a_m^2 \Gamma\left(\frac{1}{4}\right)}{20 \pi \gamma b^5 \nu_0 \Omega^2 \sqrt{\pi \beta c \sigma L}} & \text{(high } \gamma\text{)} \quad (8.5b) \end{cases}$$

*This result applies only to the case treated in Section VI. For other variations of $\omega_0$ with amplitude or other particle distribution functions, some modifications of these equations are required. However, these equations are in agreement with the general result that if the betatron oscillation frequency spread $\Delta(2\omega_0)$ is large compared to the coherent tune shift $|Ka/4\omega_0|$, then the unstable oscillations are Landau damped. See Ref. 1 for a more complete discussion of this point for the case of dipole oscillation.

or the maximum number of particles per bunch is given by:

$$N_{max} = \begin{cases} \dfrac{\sqrt{\pi}\, L \gamma^3 \nu_0 \Omega^2}{3 r_0 c^2} \left[ \dfrac{a_m^2 b^4}{b^4 - \dfrac{a_m^4}{5}} \right] \Delta\nu_0 & \text{(low } \gamma\text{)} \quad (8.6a) \\[2em] \dfrac{20 \pi \gamma b^5 \nu_0 \Omega^2 \sqrt{\pi \beta c \sigma L}}{3 r_0 c^3 \beta^3 a_m^3 \Gamma\left(\frac{1}{4}\right)} \Delta\nu_0 & \text{(high } \gamma\text{)} \quad (8.6b) \end{cases}$$

where $a_m$ is the maximum value of a.

For the case of a large difference between the horizontal and vertical betatron oscillation frequencies, it is found that coupling exists between monopole and quadrupole oscillations. The motion of the beam can be characterized by two normal modes such that the monopole and quadrupole oscillations are in phase in one mode and 180° out of phase in the other.

## 8.1. Numerical Example

In most accelerators and storage rings, the cross sections of the beam and vacuum chamber are not circular. Therefore, the theory must be extended before it can be rigorously applied. However, for the purpose of illustration, we calculate the conditions for unstable quadrupole oscillations, the maximum growth rate and the spread in $\nu_0$ necessary to Landau damp these unstable oscillations in the proposed SLAC Storage Ring. We take

| | |
|---|---|
| $R = 3 \times 10^3$ cm | $N = 4 \times 10^{12}$ per bunch |
| $L = 3 \times 10$ cm | $\sigma = 0.4 \times 10^{18}$ sec$^{-1}$ |
| $\nu_0 = 5.25$ | $a = 1$ cm |
| $\gamma = 6 \times 10^3$ | $b = 5$ cm |

We find that because of the local damping fields the condition $K_i(2\nu_0\Omega) < 0$ restricts the range of unstable oscillations to $5.41 < \nu_0 < 5.5$ or $5.91 < \nu_0 < 6.0$, and the maximum growth rate is given by

$$\frac{1}{\tau_2} \approx -\left(0.6 + 0.46\,\mathrm{Im}\,G\left(2\nu_0\right)\right) \quad \sec^{-1} \quad .$$

To stabilize the unstable quadrupole oscillations with Landau damping, we would need a spread in the tune within the beam of $\Delta\nu_0 \approx 0.5 \times 10^{-6}$.

APPENDIX: Solving for the Perturbing Fields

In this section, we solve Maxwell's equations for the fields due to the perturbing sources $\rho_1$ and $\vec{J}_1$ given in Section III:

$$\rho_1 = \begin{cases} \dfrac{e\lambda}{\pi a^2}\left[\delta(a-r) - \dfrac{2}{a}H(a-r)\right] & \text{(monopole)} \quad (1a) \\[2ex] \dfrac{e\lambda}{\pi a^2}\,\delta(a-r)\cos m\theta\,, & \text{(multipole)} \quad (1b) \end{cases}$$

$$\vec{J}_1 = \begin{cases} \dfrac{e\lambda}{\pi a^2}\left\{ i(kv-\omega)\left(\dfrac{r}{a}\right)H(a-r)\hat{e}_r + v\left[\delta(a-r) - \dfrac{2}{a}H(a-r)\right]\hat{e}_z \right\} & (2a) \\ \qquad \text{(monopole)} & \\[2ex] \dfrac{e\lambda}{\pi a^2}\left\{ i(kv-\omega)\left(\dfrac{r}{a}\right)^{m-1}H(a-r)\left[\cos m\theta\,\hat{e}_r - \sin m\theta\,\hat{e}_\theta\right] \right. & \\ \qquad \left. + v\,\delta(a-r)\cos m\theta\,\hat{e}_z \right\}, \quad \text{(multipole)} & (2b) \end{cases}$$

where H(x) is the Heaviside step function. For the uniform beam $e\lambda$ is the charge per unit length, while for the bunched beam $e\lambda = eNf(z-vt)$ with the function f(x) normalized such that eN is the total charge in the bunch, i.e.,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \qquad (3)$$

and k = 0 for bunched beams.

In order to illustrate the method of solution, a detailed derivation of the fields for multipole bunched beams is given. Since the method of solution is the same for continuous beams and for monopole bunched beams, only the final expressions for the forces are given for these cases.

In solving for the fields, it is useful to employ the following transforms.[15]

For the longitudinal particle distribution f, the transform $\tilde{f}(k)$ is defined by

$$f(z - vt) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{ik(z-vt)} dk \qquad (4)$$

and the transform field component $\tilde{E}_z$ is defined by

$$E_{z1}(t, z - vt) = \cos\theta \, e^{-i\omega t} \int_{-\infty}^{\infty} \tilde{E}_{z1}(k,\omega) e^{ik(z-vt)} dk . \qquad (5)$$

Similar definitions are used for the transform of the other field components, where the theta dependent and the frequency dependent terms are explicitly introduced in the definitions to simplify the subsequent expressions for the transformed field components. The components $E_{r1}$, $E_{z1}$, and $B_{\theta 1}$, are proportional to $\cos m\theta$, while $E_{\theta 1}$, $B_{r1}$, and $B_{z1}$ are proportional to $\sin m\theta$.

By using Maxwell's equations and Ohm's law the field components can be expressed in terms of $\tilde{E}_{z1}$ and $\tilde{B}_{z1}$. Inside the vacuum chamber ($r < b$) we have

$$\nu^2 \tilde{E}_{r1} = ik \frac{\partial \tilde{E}_{z1}}{\partial r} + \frac{ikm(\beta+\beta_w)}{r} \tilde{B}_{z1} - \frac{i4\pi k(\beta+\beta_w)}{c} \tilde{J}_{r1} \;, \qquad (6a)$$

$$\nu^2 \tilde{B}_{\theta 1} = ik(\beta+\beta_w) \frac{\partial \tilde{E}_{z1}}{\partial r} + \frac{ikm}{r} \tilde{B}_{z1} - \frac{i4\pi k}{c} \tilde{J}_{r1} \;, \qquad (6b)$$

$$\nu^2 \tilde{E}_{\theta 1} = -\frac{ikm}{r} \tilde{E}_{z1} - ik(\beta+\beta_w) \frac{\partial \tilde{B}_{z1}}{\partial r} - \frac{i4\pi k(\beta+\beta_w)}{c} \tilde{J}_{\theta 1} \;, \qquad (6c)$$

$$\nu^2 \tilde{B}_{r1} = \frac{ikm(\beta+\beta_w)}{r} \tilde{E}_{z1} + ik \frac{\partial \tilde{B}_{z1}}{\partial r} + \frac{i4\pi k}{c} \tilde{J}_{\theta 1} \;, \qquad (6d)$$

where $\beta_w = \frac{\omega}{kc}$ and $\nu^2 = -k^2 \left[1 - (\beta+\beta_w)^2\right]$.

The components $\widetilde{E}_{z1}$ and $\widetilde{B}_{z1}$ satisfy the differential equations:

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \left(\nu^2 - \frac{m}{r^2}\right)\right]\begin{pmatrix}\widetilde{E}_{z1}\\ \\ \widetilde{B}_{z1}\end{pmatrix} = \begin{pmatrix}\frac{i4eNk\widetilde{f}}{a^2}\left(1-\beta^2-\beta\beta_w\right)\delta(a-r)\\ \\ \frac{i4eNk\widetilde{f}\beta_w}{a^2}\;\delta(a-r)\end{pmatrix} . \quad (7)$$

Similarly, inside the metal walls (r > b)

$$\lambda^2\widetilde{E}_{r1} = \frac{ik}{r}\frac{\partial\widetilde{E}_{z1}}{\partial r} + \frac{ikm\left(\beta+\beta_w\right)}{r}\;\widetilde{B}_{z1} \quad , \qquad (8a)$$

$$\lambda^2\widetilde{B}_{\theta 1} = \left[ik\left(\beta+\beta_w\right) - 4\pi\sigma/c\right]\frac{\partial\widetilde{E}_{z1}}{\partial r} + \frac{ikm}{r}\widetilde{B}_{z1} \quad , \qquad (8b)$$

$$\lambda^2\widetilde{E}_{\theta 1} = -\frac{ikm}{r}\widetilde{E}_{z1} - ik\left(\beta+\beta_w\right)\frac{\partial\widetilde{B}_{z1}}{\partial r} \quad , \qquad (8c)$$

$$\lambda^2\widetilde{B}_{r1} = \left[ik\left(\beta+\beta_w\right) - 4\pi\sigma/c\right] m\frac{\widetilde{E}_{z1}}{r} + ik\frac{\partial\widetilde{B}_{z1}}{\partial r} \quad , \qquad (8d)$$

with $\lambda^2 = \nu^2 + (4\pi ik\sigma/c)\left(\beta+\beta_w\right)$, and the differential equations for $\widetilde{E}_{z1}$ and $\widetilde{B}_{z1}$ given by:

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \left(\lambda^2 - \frac{m}{r^2}\right)\right]\begin{pmatrix}\widetilde{E}_{z1}\\ \\ \widetilde{B}_{z1}\end{pmatrix} = 0 . \qquad (9)$$

In addition, to satisfy Eqs. (6) through (9) the transformed fields must also satisfy the proper boundary conditions at r = a, and r = b. At r = a the components $\widetilde{B}_{z1}$, $\widetilde{E}_{z1}$, $\widetilde{E}_{\theta 1}$ and $\widetilde{B}_{r1}$ are continuous while

$$\widetilde{B}_{\theta 1}(r=a^+) - \widetilde{B}_{\theta 1}(r=a^-) = 4\pi/c\int_{a^-}^{a^+}\widetilde{j}_{z1}\,dr = 4\pi\beta N_0\xi\widetilde{f} \quad , \qquad (10)$$

and

$$\widetilde{E}_{r1}\left(r=a^{+}\right) - \widetilde{E}_{r1}\left(r=a^{-}\right) = 4\pi \int_{a^{-}}^{a^{+}} \widetilde{\rho}_1 \, dr = 4\pi N_0 \xi \widetilde{f} \quad . \tag{11}$$

These jump conditions at the surface of the beam are the consequence of characterization of the charge oscillation by an equivalent surface charge distribution on a boundary of constant radius. At $r = b$ the components $\widetilde{B}_{z1}$, $\widetilde{E}_{z1}$, $\widetilde{E}_{\theta 1}$, $\widetilde{B}_{\theta 1}$ and $\widetilde{B}_{r1}$ are continuous.

It can be seen from Eqs. (7) and (9) that the solutions for $\widetilde{E}_{z1}$ and $\widetilde{B}_{z1}$ are Bessels functions of order m. However, it has been shown in Ref. 15 that for oscillation frequencies considerably below cutoff and wall conductivity sufficiently high, the fields in the regions of interest can be obtained by using the approximation that $|\nu b| \ll 1$ and $|\lambda b| \gg 1$. Under these assumptions, the equations for $\widetilde{E}_{z1}$ and $\widetilde{B}_{z1}$ become

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{m^2}{r^2}\right]\begin{pmatrix}\widetilde{E}_{z1}\\ \\ \widetilde{B}_{z1}\end{pmatrix} = \begin{pmatrix}\dfrac{i4eNk\widetilde{f}}{a^2}\left(1-\beta^2-\beta\beta_w\right)\delta(a-r)\\ \\ \dfrac{i4eNk\widetilde{f}\beta_w}{a^2}\,\delta(a-r)\end{pmatrix} \quad (r < b) \tag{12}$$

and

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \lambda^2\right]\begin{pmatrix}\widetilde{E}_{z1}\\ \\ \widetilde{B}_{z1}\end{pmatrix} = 0 \quad . \qquad (r > b) \tag{13}$$

The general solutions for the above equations are $r^{\pm m}$ for $r < b$, and $e^{\pm i\lambda r}$ for $r > b$.

Since $\tilde{E}_{z1}$ and $\tilde{B}_{z1}$ are finite at $r = 0$, for $r < a$

$$\tilde{E}_{z1} = C_1 r^m \tag{14a}$$

and

$$\tilde{B}_{z1} = D_1 r^m \ . \tag{14b}$$

Combining Eqs. (2b), (14), and (6), we obtain for $r < a$

$$\tilde{E}_{\theta 1} = -\frac{m}{\nu^2} ikC_1 r^{m-1} - \frac{m}{\nu^2} ik(\beta + \beta_w) D_1 r^{m-1} + \frac{\xi 4eNk^2 \beta_w (\beta + \beta_w)}{\nu^2 a^{m+1}} \tilde{f} r^{m-1} \tag{15a}$$

$$\tilde{B}_{\theta 1} = \frac{m}{\nu^2} ik(\beta + \beta_w) C_1 r^{m-1} + \frac{m}{\nu^2} ikD_1 r^{m-1} - \frac{\xi 4eNk^2 \beta_w}{\nu^2 a^{m+1}} \tilde{f} r^{m-1} \tag{15b}$$

$$\tilde{E}_{r1} = -\tilde{E}_{\theta 1}$$

$$\tilde{B}_{r1} = \tilde{B}_{\theta 1} .$$

For the region $a < r < b$ the solutions can be written as

$$\tilde{E}_{z1} = C_1 r^m + C_1' \left( r^m - \frac{a^{2m}}{r^m} \right) \tag{16a}$$

and

$$\tilde{B}_{z1} = D_1 r^m + D_1' \left( r^m - \frac{a^{2m}}{r^m} \right) . \tag{16b}$$

These equations satisfy the continuity requirement for $\tilde{E}_{z1}$ and $\tilde{B}_{z1}$ at $r = a$.

In order to satisfy Eq. (7) and the boundary condition at $r = a$, the derivatives of $\widetilde{E}_{z1}$ and $\widetilde{B}_{z1}$ must be discontinuous at $r = a$, i.e.,

$$\left.\frac{\partial \widetilde{E}_{z1}}{\partial r}\right|_{a^-}^{a^+} = i\, \frac{4eNk\widetilde{f}}{a^2}\left[1 - \beta\left(\beta + \beta_w\right)\right] \tag{17a}$$

and

$$\left.\frac{\partial \widetilde{B}_{z1}}{\partial r}\right|_{a^-}^{a^+} = i\, \frac{4eNk\widetilde{f}\beta_w}{a^2} \ . \tag{17b}$$

It can be seen that these jump conditions and the boundary conditions are satisfied if we choose

$$C_1' = \frac{ik\left(1 - \beta^2 - \beta\beta_w\right)}{2ma^{m-1}}\, G_1 \tag{18a}$$

and

$$D_1' = \frac{ik\beta_w}{2ma^{m-1}}\, G_1 \tag{18b}$$

with

$$G_1 = \frac{4eN\widetilde{f}}{a^2} \ . \tag{19}$$

Thus for $a < r < b$, the solutions are

$$\widetilde{E}_{z1} = \left[C_1 + \frac{ik\left(1 - \beta^2 - \beta\beta_w\right)}{2ma^{m-1}}\, G_1\right] r^m - \frac{ik\left(1 - \beta^2 - \beta\beta_w\right)a^{m+1}}{2m}\, G_1 r^{-m} \tag{20a}$$

and

$$\widetilde{B}_{z1} = \left[D_1 + \frac{ik\beta_w}{2ma^{m-1}}\, G_1\right] r^m - \frac{ik\beta_w a^{m+1}}{2m}\, G_1 r^{-m} \ . \tag{20b}$$

The other field components can be found by substituting the above expressions for $\tilde{E}_{z1}$ and $\tilde{B}_{z1}$ into Eq. (15). The result gives for $a < r < b$

$$\tilde{E}_{\theta 1} = -\left[\frac{ikmC_1}{\nu^2} + \frac{ikm(\beta+\beta_w)D_1}{\nu^2}\right] r^{m-1} - \frac{G_1}{2}\left(\frac{r}{a}\right)^{m-1}\left[\frac{1-\beta^2-\beta\beta_w}{1-(\beta+\beta_w)^2}\right] + \frac{G_1}{2}\left(\frac{a}{r}\right)^{m+1} \tag{21a}$$

$$\tilde{B}_{\theta 1} = \left[\frac{ikm(\beta+\beta_w)C_1}{\nu^2} + \frac{ikmD_1}{\nu^2}\right] r^{m-1} + \frac{G_1}{2}\left(\frac{r}{a}\right)^{m-1}\left[\frac{\beta+2\beta_w-\beta(\beta+\beta_w)^2}{1-(\beta+\beta_w)^2}\right] + \frac{G_1}{2}\left(\frac{a}{r}\right)^{m+1}\beta \ . \tag{21b}$$

$$\tilde{E}_{r1} = -\tilde{E}_{\theta 1}$$

and

$$\tilde{B}_{r1} = \tilde{B}_{\theta 1} \ .$$

The general solutions for $r > b$ are $e^{\pm i\lambda r}$, so that for a wall with infinite thickness we take

$$\tilde{E}_{z1} = \left\{\left[C_1 + \frac{ik(1-\beta^2-\beta\beta_w)}{2ma^{m-1}}G_1\right] b^m - \frac{ik(1-\beta^2-\beta\beta_w)a^{m+1}}{2mb^m}G_1\right\}\frac{e^{i\lambda r}}{e^{i\lambda b}} \tag{22a}$$

and

$$\tilde{B}_{z1} = \left\{\left[D_1 + \frac{ik\beta_w}{2ma^{m-1}}G_1\right] b^m - \frac{ik\beta_w a^{m+1}}{2mb^m}G_1\right\}\frac{e^{i\lambda r}}{e^{i\lambda b}} \tag{22b}$$

with $\mathrm{Im}\,\lambda > 0$. By this construction the continuity requirement for $\tilde{E}_{z1}$ and $\tilde{B}_{z1}$ at $r = b$ is satisfied, and $\tilde{E}_{z1}$ and $\tilde{B}_{z1}$ go to zero at $r = \infty$.

As before, the other field components can be obtained by substituting the above expressions for $\widetilde{E}_{z1}$ and $\widetilde{B}_{z1}$ into Eq. (8). In particular, the result gives for $r > b$

$$\widetilde{E}_{\theta 1} = -\frac{ikme^{i\lambda r}}{b\lambda^2 e^{i\lambda b}}\left\{\left[C_1 + \frac{ik\left(1-\beta^2-\beta\beta_w\right)}{2\,m\,a^{m-1}}\,G_1\right]b^m - \frac{ik\left(1-\beta^2-\beta\beta_w\right)a^{m+1}}{2\,m\,b^m}\,G_1\right\}$$

$$+\frac{k\left(\beta+\beta_w\right)e^{i\lambda r}}{\lambda e^{i\lambda b}}\left\{\left[D_1 + \frac{ik\beta_w}{2\,m\,a^{m-1}}\,G_1\right]b^m - \frac{ik\beta_w a^{m+1}}{2\,m\,b^m}\right\} \qquad (23a)$$

and

$$\widetilde{B}_{\theta 1} = -\frac{i4\pi\sigma e^{i\lambda r}}{\lambda c e^{i\lambda b}}\left\{\left[C_1 + \frac{ik\left(1-\beta^2-\beta\beta_w\right)}{2\,m\,a^{m-1}}\,G_1\right]b^m - \frac{ik\left(1-\beta^2-\beta\beta_w\right)a^{m+1}}{2\,m\,b^m}\,G_1\right\}$$

$$+\frac{ikme^{i\lambda r}}{b\lambda^2 e^{i\lambda b}}\left\{\left[D_1 + \frac{ik\beta_w}{2\,m\,a^{m-1}}\,G_1\right]b^m - \frac{ik\beta_w a^{m+1}}{2\,m\,b^m}\,G_1\right\}. \qquad (23b)$$

The constants $C_1$ and $D_1$ are determined by the other boundary conditions at $r = b$. Imposing the conditions that $\widetilde{B}_{\theta 1}$ and $\widetilde{E}_{\theta 1}$ be continuous at $r = b$ gives the following two equations relating $C_1$ and $D_1$ which are valid to first order in $\sigma^{-1/2}$:

$$a_{11}C_1 + a_{12}D_1 = \alpha_1$$

$$a_{21}C_1 + a_{22}D_1 = \alpha_2$$

where

$$a_{11} = \left[\frac{i4\pi\sigma b^m}{\lambda c} + \frac{ik(\beta+\beta_w)\,mb^m}{\nu^2 b}\right]$$

$$a_{12} = \frac{ikmb^m}{\nu^2 b}$$

$$a_{21} = \frac{i k m b^{m}}{\nu^{2} b}$$

$$a_{22} = k(\beta+\beta_{w})\left[\frac{i m b^{m}}{\nu^{2} b} + \frac{b^{m}}{\lambda}\right]$$

$$\alpha_{1} = \frac{4\pi\sigma k a}{2\lambda c m} G_{1}\left(1-\beta^{2}-\beta\beta_{w}\right)\left[\frac{b^{m}}{a^{m}} - \frac{a^{m}}{b^{m}}\right] - \frac{a\beta}{2b} G_{1}\left[\frac{b^{m}}{a^{m}} - \frac{a^{m}}{b^{m}}\right]$$

$$-\left(\frac{b}{a}\right)^{m-1} G_{1}\left[\frac{\beta_{w}}{1-(\beta+\beta_{w})^{2}}\right]$$

$$\alpha_{2} = -\frac{i k^{2} a G_{1}}{2\lambda m} \beta_{w}(\beta+\beta_{w})\left[\frac{b^{m}}{a^{m}} - \frac{a^{m}}{b^{m}}\right] - \frac{a G_{1}}{2b}\left[\frac{b^{m}}{a^{m}} - \frac{a^{m}}{b^{m}}\right]$$

$$-\left(\frac{b}{a}\right)^{m-1} G_{1}\left[\frac{\beta_{w}(\beta+\beta_{w})}{1-(\beta+\beta_{w})^{2}}\right] .$$

Solving for $C_1$ and $D_1$, we obtain

$$C_{1} = -\frac{ik\left(1-\beta^{2}-\beta\beta_{w}\right)G_{1}\left(b^{2m}-a^{2m}\right)}{2 m a^{m-1} b^{2m}} - \frac{k\beta(\beta+\beta_{w})G_{1} a^{m+1}}{\lambda b^{2m+1}} \qquad (24a)$$

and

$$D_{1} = -\frac{i k\beta_{w} G_{1}\left(b^{2m}+a^{2m}\right)}{2m a^{m-1} b^{2m}} + \frac{k\beta G_{1} a^{m+1}}{\lambda b^{2m+1}} \qquad (24b)$$

which may be inserted into Eqs. (22) and (23) to yield

$$\widetilde{E}_{z1} = \frac{iG_1\left(b^{2m} - a^{2m}\right)}{2ma^{m-1}b^{2m}}\left(\frac{k}{\gamma^2} - \frac{\omega\beta}{c}\right)r^m - \left(1 - i\,\mathrm{Sign}\,\kappa\right)\left|\kappa\right|^{\frac{1}{2}} \frac{G_1\beta^2 a^{m+1}}{\sqrt{2\mathcal{R}}\,b^{2m}}\, r^m \quad (25a)$$

$$\widetilde{B}_{z1} = -\frac{i\omega G_1\left(b^{2m} + a^{2m}\right)}{2mca^{m-1}b^{2m}}\, r^m + \frac{G_1\beta a^{m+1}}{\sqrt{2\mathcal{R}}\,b^{2m+1}}\left[\left(1 - i\,\mathrm{Sign}\,\kappa\right)\left|\kappa\right|^{\frac{1}{2}} + \frac{i\omega}{\beta c}\left(1 + i\,\mathrm{Sign}\,\kappa\right)\left|\kappa\right|^{-\frac{1}{2}}\right] \quad (25b)$$

$$\widetilde{E}_{\theta 1} = \frac{G_1\left(b^{2m} - a^{2m}\right)}{2a^{m-1}b^{2m}}\, r^{m-1} \quad (26a)$$

and

$$\widetilde{B}_{\theta 1} = -\frac{G_1\left(b^{2m} - a^{2m}\right)}{2a^{m-1}b^{2m}}\, r^{m-1} - \frac{mG_1\beta a^{m+1}}{\sqrt{2\mathcal{R}}\,b^{2m+1}}\left(1 + i\,\mathrm{Sign}\,\kappa\right)\left|\kappa\right|^{-\frac{1}{2}} r^{m-1} \quad (26b)$$

where

$$\kappa = \frac{k\left(\beta + \beta_w\right)}{\beta} \quad (27)$$

$$\mathcal{R} = \frac{4\pi\beta\sigma}{c} \; .$$

Knowing the fields, the transforms of the force components can be calculated from the following expressions:

$$\widetilde{F}_{r1}(k) = -e\left(\widetilde{E}_{\theta 1} + \beta\widetilde{B}_{\theta 1}\right)e^{-i\omega t}\cos m\theta \quad (28a)$$

and

$$\widetilde{F}_{\theta 1}(k) = e\left(\widetilde{E}_{\theta 1} + \beta\widetilde{B}_{\theta 1}\right)e^{-i\omega t}\sin m\theta \quad (28b)$$

since $\widetilde{E}_{r1} = -\widetilde{E}_{\theta 1}$ and $\widetilde{B}_{r1} = \widetilde{B}_{\theta 1}$. Thus, the force components can be found by taking the transforms of Eq. (28).

Before we embark on finding the fields, we note the total force fields come from two sources, local fields and wake fields. The effect of the motion of the front of the bunch on the rear of the bunch is given by the local fields, which are dependent on the form of the longitudinal particle distributions $f(z - vt)$ or $\tilde{f}(k)$. The wake fields, on the other hand, are fields at distances $|z - vt|$ large compared to the length of the bunch and hence not dependent on the form of f. Because of this property, we will calculate the forces due to these fields separately.

Wake Fields (multipole bunched beam)

Since the wake fields are not dependent on the form of f, we take for f a delta function in $(z - vt)$ which corresponds to $\tilde{f} = \frac{1}{2\pi}$. Substituting $\frac{1}{2\pi}$ for $\tilde{f}$ into Eq. (5) and using the table of integrals given in Ref. 15 to perform the integrations in the inversion of the transformed quantities, we obtain the following expressions for the wake fields of a multipole bunched beam:

$$E_{z1} = \frac{\xi G_1 \beta^2 a^{m+1} \sqrt{\pi}}{\sqrt{\mathcal{R}}\, b^{2m+1}} \frac{S(z,t)}{|z-vt|^{3/2}} e^{-i\omega z/\beta c} \cos m\theta \, r^m \tag{29a}$$

$$E_{r1} = E_{\theta 1} = 0$$

$$B_{z1} = -\frac{\xi 2eNa^{m-1}}{\sqrt{\pi\mathcal{R}}\, b^{2m+1}} \left[\frac{S(z,t)}{|z-vt|^{3/2}} - 2\left(\frac{i\omega}{\beta c}\right)\frac{S(z,t)}{|z-vt|^{1/2}}\right] e^{-i\omega z/\beta c} \sin m\theta \, r^m \tag{29b}$$

$$B_{r1} = -\frac{\xi 4eNm\beta a^{m-1}}{\sqrt{\pi\mathcal{R}}\, b^{2m+1}} \frac{S(z,t)}{|z-vt|^{1/2}} e^{-i\omega z/\beta c} \sin m\theta \, r^{m-1} \tag{30}$$

$$B_{\theta 1} = B_{r1} \frac{\cos m\theta}{\sin m\theta} \tag{31}$$

where $S(z,t)$ is defined as unity for $z < vt$ and zero for $z > vt$. We note that $\tilde{E}_{r1}$ and $\tilde{E}_{\theta 1}$ are zero because higher order terms in $|\nu b|$ have been neglected and their transforms are analytic. The dominant contributions to the transverse force fields come from $\tilde{B}_{r1}$ and $\tilde{B}_{\theta 1}$.

Thus, the force components along the x and y directions are given by

$$F_{x1} = F_{r1} \cos\theta - F_{\theta 1} \sin\theta \tag{32a}$$

and

$$F_{y1} = F_{r1} \sin\theta + F_{\theta 1} \cos\theta \tag{32b}$$

with

$$F_{r1} = e\beta B_{\theta 1} \tag{33a}$$

and

$$F_{\theta 1} = -F_{r1} \frac{\sin m\theta}{\cos m\theta} \quad . \tag{33b}$$

Local Force-Fields (multipole bunched beam)

It has been pointed out that the local force fields are dependent on the type of particle distributions in the z direction. In the calculation which follows, two types of longitudinal distributions will be assumed, a Gaussian distribution and a uniform distribution. Since the bunch is assumed to move as a whole, the force on a particle within the bunch is given by the total force acting on the bunch averaged over the number of particles in the bunch.

To calculate the total force on a bunch, first we note from Eqs. (26) and (28) that the transforms of the transverse forces are given by

$$\tilde{F}_{r1}(k) = \tilde{g}_m(k)\tilde{f}(k)\, r^{m-1} e^{-i\omega t} \cos m\theta \tag{34a}$$

and

$$\tilde{F}_{\theta 1}(k) = -\tilde{g}_m(k)\tilde{f}(k)\, r^{m-1} e^{-i\omega t} \sin m\theta \tag{34b}$$

where

$$\tilde{g}_m(k) = -\frac{\xi 2e^2 N\left(b^{2m} - a^{2m}\right)}{\gamma^2 b^{2m} a^{m+1}} + \left[\frac{1 + i\,\mathrm{Sign}\left(k + \frac{\omega}{\beta c}\right)}{\left|k + \frac{\omega}{\beta c}\right|^{1/2}}\right]\left(\frac{c}{2\pi\beta\sigma}\right)^{\frac{1}{2}} \frac{\xi 2e^2 N m\beta^2 a^{m-1}}{b^{2m+1}} \ . \tag{35}$$

The average local perturbed force on a particle of the bunch is given by

$$\mathfrak{F}_r = \int_{-\infty}^{\infty} F_{r1}(z) f(z)\, dz \tag{36}$$

where $F_{r1}$ is the inverse transform of $\widetilde{F}_{r1}(k)$ given by

$$F_{r1}(z) = \int_{-\infty}^{\infty} \widetilde{F}_{r1}(k)\, e^{i(kz - \omega t)}\, dk \tag{37}$$

and the function f is related to the transform $\widetilde{f}(k)$ by

$$f(z) = \int_{-\infty}^{\infty} \widetilde{f}(k)\, e^{ikz}\, dk\,. \tag{38}$$

Therefore, Eq. (36) becomes

$$\mathfrak{F}_r = A_m r^{m-1} e^{-i\omega t} \cos m\,\theta \tag{39}$$

where

$$A_m = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{g}_m(k)\, \widetilde{f}(k)\, \widetilde{f}(k')\, e^{i(k+k')z}\, dz\, dk\, dk'\,. \tag{40}$$

Note that

$$\int_{-\infty}^{\infty} e^{i(k+k')z}\, dz = 2\pi\,\delta(k+k')\,,$$

so that

$$A_m = \int_{-\infty}^{\infty} 2\pi\, \widetilde{g}(k)\, \widetilde{f}(k)\, \widetilde{f}(-k)\, dk\,. \tag{41}$$

Since radiation damping produces a Gaussian density distribution in the longitudinal direction, we assume

$$f(z) = \sqrt{\frac{2}{\pi}}\ \frac{1}{L}\ e^{-2z^2/L^2} . \tag{42}$$

For this distribution function, the standard deviation equals L/2. The Fourier transform of f is

$$\widetilde{f}(k) = \frac{1}{2\pi} e^{-k^2 L^2/8} . \tag{43}$$

Thus, by combining Eqs. (41) and (43) we obtain

$$A_m = -\int_{-\infty}^{\infty} \frac{\xi e^2 N \left(b^{2m} - a^{2m}\right)}{\pi \gamma^2 b^{2m} a^{m+1}}\ e^{-k^2 L^2/4}\, dk$$

$$+ \frac{\xi e^2 N m \beta^2 a^{m-1}}{\pi b^{2m+1}} \sqrt{\frac{c}{2\pi\beta\sigma}} \int_{-\infty}^{\infty} \frac{(1 + i\,\mathrm{Sign}\,\kappa)}{|\kappa|^{1/2}}\, e^{-\left(\kappa - \frac{\omega}{\beta c}\right)\frac{L^2}{4}}\, d\kappa . \tag{44}$$

The second integral in Eq. (44) may be evaluated for the case where $(\omega L/\beta c) \ll 1$. With this assumption we obtain to lowest order in $(\omega L/\beta c)$

$$A_m = -\frac{\xi 2 e^2 N \left(b^{2m} - a^{2m}\right)}{\sqrt{\pi}\ L \gamma^2 b^{2m} a^{m+1}}$$

$$+ \frac{\xi e^2 N m \beta^2 a^{m-1}}{\sqrt{L}\ \pi b^{2m+1}} \sqrt{\frac{c}{\pi\beta\sigma}} \left[\Gamma\left(\tfrac{1}{4}\right) + i\,\frac{\omega L}{\beta c}\,\Gamma\left(\tfrac{3}{4}\right)\right] \tag{45}$$

where $\Gamma(x)$ is the complete gamma function.

Thus for a bunch of length L with a Gaussian density distribution in the longitudinal direction, the average local perturbed forces acting on a particle of the bunch are

$$\mathfrak{F}_r = A_m r^{m-1} e^{-i\omega t} \cos m\theta \tag{46a}$$

$$\mathfrak{F}_\theta = -A_m r^{m-1} e^{-i\omega t} \sin m\theta \tag{46b}$$

with $A_m$ given by Eq. (45). The force components along the x and y directions are given by Eq. (32).

In order to see the effect of the form of longitudinal distribution f on the local force-fields, we also calculated the average force for a uniform distribution. For this distribution

$$A_m = -\frac{\xi 2 e^2 N \left(b^{2m} - a^{2m}\right)}{\sqrt{\pi}\, L \gamma^2 b^{2m} a^{m+1}}$$

$$+ \frac{\xi e^2 N m \beta^2 a^{m-1}}{\sqrt{L}\, \pi b^{2m+1}} \sqrt{\frac{c}{\pi \beta \sigma}} \left[\frac{8}{3}\sqrt{\pi} + \frac{i\omega L}{\beta c} \; \frac{8}{15}\sqrt{\pi}\right], \tag{47}$$

which is only slightly different from $A_m$ given by Eq. (45) for a Gaussian distribution.

## A Summary of Results

The perturbing force fields for monopole and other multipole cases have been calculated using the method just given for the quadrupole case. The results are summarized in this section.

We find that, in general, $F_{x1}(x,y)$ and $F_{y1}(x,y)$ are derivable from a potential function $U_0$

$$\frac{F_{x1}}{m_0\gamma} = \xi K(\omega) \frac{\partial}{\partial x} U_0\left[r(x,y), \theta(x,y)\right] \tag{48a}$$

and

$$\frac{F_{y1}}{m_0\gamma} = \xi K(\omega) \frac{\partial}{\partial y} U_0\left[r(x,y), \theta(x,y)\right] \tag{48b}$$

where $K(\omega)$ is a constant which is determined by the parameters of the beam and the machine, and

$$U_0(r,\theta) = \begin{cases} \dfrac{r^2}{2} & \text{(monopole)} \quad (49a) \\[2ex] \dfrac{r^m}{m} \cos m\theta \ . & \text{(multipole)} \quad (49b) \end{cases}$$

For the uniform beam we obtain for the monopole and multipole oscillations[28,29]:

$$K = -\frac{4 r_0 \lambda c^2}{\gamma^3 a^3} + (1 + i\,\mathrm{Sign}\,\omega) \frac{r_0 \lambda a}{c b} \left|\frac{\omega}{2\pi\sigma}\right|^{\frac{1}{2}} |\omega| \left(k - \frac{\omega\beta}{c}\right)^2 \quad \text{(monopole)} \tag{50a}$$

$$K = -\frac{2 r_0 \lambda c^2 (b^{2m} - a^{2m})}{\gamma^3 b^{2m} a^{m+1}} + (1 + i\,\mathrm{Sign}\,\omega) \frac{2 m r_0 \lambda \beta^2 c^3 a^{m-1}}{\gamma b^{2m+1}} \left|\frac{1}{2\pi\sigma\omega}\right|^{\frac{1}{2}} \quad \text{(multipole)} \tag{50b}$$

and for the bunched beam we obtain:

$$K = -\frac{4 r_0 N c^2}{\sqrt{\pi}\gamma^3 a^3 L} + \frac{r_0 N a \beta^2 c^3}{\pi b \gamma \sqrt{\pi \beta c \sigma L^9}} \left\{ \left[ \frac{\Gamma\left(\frac{9}{4}\right)}{\gamma^4} - \frac{\nu^2 L^2 \Gamma\left(\frac{9}{4}\right)}{R^2 \gamma^2} \right. \right. \qquad \text{(monopole)} \quad (51a)$$

$$\left. + \frac{\nu^2 L^2 \Gamma\left(\frac{5}{4}\right)}{4R^2} \right] + i \frac{\nu L}{R} \left. \left[ \frac{\Gamma\left(\frac{11}{4}\right)}{\gamma^4} - \frac{\Gamma\left(\frac{7}{4}\right)}{\gamma^2} + \frac{\nu^2 L^2 \Gamma\left(\frac{7}{4}\right)}{4R^2} \right] \right\}$$

$$+ \frac{3 r_0 N a \beta^2 c^3}{32 \pi b \gamma \sqrt{\beta c \sigma R^9}} \sum_{n=1}^{\infty} \left[ \frac{35}{4\gamma^4 (2\pi n)^{9/2}} - i \frac{5\nu}{\gamma^2 (2\pi n)^{7/2}} - \frac{\nu^2}{(2\pi n)^{5/2}} \right] e^{i 2\pi n \nu}$$

$$K = -\frac{2 r_0 N c^2 \left(b^{2m} - a^{2m}\right)}{\sqrt{\pi}\gamma^3 b^{2m} a^{m+1} L} + \frac{m r_0 N a^{m-1} \beta^2 c^3}{\pi b^{2m+1} \gamma \sqrt{\pi \beta c \sigma L}} \left[ \Gamma\left(\tfrac{1}{4}\right) + i \frac{\nu L}{R} \Gamma\left(\tfrac{3}{4}\right) \right] \text{(multipole)} \quad (51b)$$

$$+ \frac{2 m r_0 N a^{m-1} \beta^2 c^3}{\pi b^{2m+1} \gamma \sqrt{\beta c \sigma R}} \sum_{n=1}^{\infty} \frac{e^{i 2\pi n \nu}}{(2\pi n)^{1/2}}$$

where

$r_0$ = classical radius of the particles

L = length of the bunch

R = radius of the machine

$\nu$ = $\omega/\Omega$

$\sigma$ = conductivity of the pipe

and $\Gamma$(x) is the complete gamma function.

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