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We describe the one-dimensional SASE FEL instability using the wake approach. First, we obtain an expression for the longitudinal 1-D wake in a helical undulator. We then show that taking into account the retardation effect in the Vlasov equation with the proper wake leads to the correct result for the FEL instability, in agreement with the traditional theory.

INTRODUCTION

Coherent instabilities arise when the electromagnetic field produced by an electron beam, interacts with the environment, generating new fields which act back on the electrons. It is conventional to describe such phenomena by using the Vlasov equation, with the electromagnetic forces represented by a wakefield [1]. In the high-gain free-electron laser (FEL), the radiation emitted by an electron beam passing through a long undulator acts back on the electrons. This interaction is often described using the Vlasov-Maxwell equations [2–4]. In this paper, we discuss the FEL using the wakefield approach and emphasize the necessity of including the effects of retardation [5].

WAKE IN THE UNDULATOR

Consider a helical undulator with the undulator parameter K. A beam of density n_0 (per cubic cm) and relativistic factor γ propagates along the axis of the undulator. The transverse velocity $\boldsymbol{v}_\perp$ of a particle in the undulator is equal to

$$\boldsymbol{v}_\perp = v_\perp \left(\boldsymbol{e}_x \cos k_w s + \boldsymbol{e}_y \sin k_w s\right) \,, \qquad (1)$$

where

$$v_\perp = c\frac{K}{\gamma} \,,$$

with $k_w = 2\pi/\lambda_w$, λ_w the undulator period, $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ the unit vectors in x and y directions, respectively, and s the distance along the undulator. The longitudinal velocity of the particle in the undulator is $v_z \approx c[1-(1+K^2)/(2\gamma^2)]$, corresponding to the longitudinal gamma-factor γ_z

$$\gamma_z = \left(1 - \frac{v_z^2}{c^2}\right)^{-1/2} = \frac{\gamma}{\sqrt{1+K^2}} \,.$$

To derive a wake in 1-D theory we consider an infinitely thin sheet of electrons in the $x-y$ plane with the charge density σ per unit area. The longitudinal position of the sheet is $s = v_z t$. Due to the transverse motion, the sheet radiates electromagnetic field. To find the radiation field, we first calculate it in the beam frame. In this frame, the sheet rotates with the frequency $\gamma_z \omega_w$, where $\omega_w = ck_w$, and its transverse velocity is

$$\hat{\boldsymbol{v}}_\perp = \gamma_z v_\perp \left[\boldsymbol{e}_x \cos(\gamma_z \omega_w \hat{t}) + \boldsymbol{e}_y \sin(\gamma_z \omega_w \hat{t})\right] \,, \qquad (2)$$

where the hat indicates variables in the beam frame. The sheet radiates two circularly polarized plane electromagnetic waves—one in the direction of the beam propagation, and the other in the opposite direction—with equal amplitudes and the frequency $\gamma_z \omega_w$. From the symmetry of the problem, the directions of the magnetic field vector in these waves at the location of the sheet ($s = v_z t$) are opposite. To find the amplitude of the magnetic field, we use Ampere's law:

$$2\hat{\boldsymbol{H}}(t)|_{\text{sheet}} = \frac{4\pi}{c}\sigma \hat{\boldsymbol{v}}_\perp \times \boldsymbol{e}_z \,,$$

from which it follows that the amplitude $\hat{H}_0$ of the field is

$$\hat{H}_0 = \frac{2\pi}{c}\sigma \hat{v}_\perp = \frac{2\pi}{c}\gamma_z \sigma v_\perp \,.$$

Note that the amplitude of the electric field $\hat{E}_0$ is also equal to $\hat{H}_0$. Returning to the lab frame we find that the frequency of the wave propagating in the forward direction is

$$\omega_0 = 2\gamma_z^2 \omega_w = 2\frac{\gamma^2}{1+K^2}\omega_w \,, \qquad (3)$$

and the amplitudes of the electric and magnetic fields are

$$E_0 = H_0 = 2\gamma_z \hat{H}_0 = \frac{4\pi}{c}\sigma v_\perp \gamma_z^2 \,.$$

The electric field in this wave is

$$\begin{aligned} \boldsymbol{E}(s,t) = -E_0 [&\boldsymbol{e}_x \cos(\omega_0(t - s/c)) \\ &+ \boldsymbol{e}_y \sin(\omega_0(t - s/c))] \,. \end{aligned} \qquad (4)$$

The magnetic and electric fields in the backward wave, in the limit $\gamma_z \gg 1$, are much smaller than H_0, and we neglect them below in the calculation of the wake.

To calculate the longitudinal wake, we consider a test sheet of particles, with a unit charge per unit area, moving in front of the source sheet in the undulator, at distance z, with the same velocity given by Eq. (1). For the test sheet we have $s = v_z t + z$. The radiated electromagnetic wave will exert a force on the test sheet, and the work of the force per unit time (and per unit area) is

$$\boldsymbol{E}\left(s, \frac{s-z}{v_z}\right) \cdot \boldsymbol{v}_\perp(s) \approx -\frac{4\pi}{c}\sigma v_\perp^2 \gamma_z^2 \cos(\omega_0 z/v_z) = -4\pi\sigma \frac{K^2}{1+K^2}\cos(\omega_0 z/v_z)\,, \quad (5)$$

where we used Eqs. (2), (4), and (3). If we define the longitudinal wake $w(s,z)$ as the energy loss of the test sheet per unit area per unit length of path and per unit σ, then

$$w(s,z) = \begin{cases} 2\kappa \cos\left(\frac{\omega_0 z}{v_z}\right), & \text{for } 0 < z < \frac{c-v_z}{c}s \\ \kappa, & \text{for } z = 0 \\ 0, & \text{otherwize} \end{cases}, \quad (6)$$

where the loss factor κ is:

$$\kappa = 2\pi \frac{K^2}{1+K^2}\,. \quad (7)$$

This wake is localized in front of the sheet because the radiated wave overtakes the particles. Positive wake corresponds to the energy loss, and negative wake means an energy gain. Note that the wake is a function of two variables: the distance z between the source and the test sheets, and the current position s of the source.

The product $\kappa\sigma^2$ is the spontaneous radiation emitted per unit area per unit length of path. It is interesting to note, that usually in accelerators the longitudinal wake is associated with the longitudinal component of the electric field E_z, with the energy gain for the test particle given by $eE_z v_z$. In undulator, as expressed by Eq. (5), the work is done by the transverse component of the electric field coupled with the wiggling motion of the particle.

VLASOV EQUATION

Having derived the longitudinal wake, we can now apply the standard formalism of accelerator theory to describe dynamics of the beam [1]. The one-dimensional Vlasov equation in a coasting beam approximation is

$$\frac{\partial f}{\partial s} - \eta\delta\frac{\partial f}{\partial z} - \frac{r_0}{\gamma}\frac{\partial f}{\partial \delta} \times \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} d\delta' w(s, z-z') f(\delta', z', s) = 0\,, \quad (8)$$

where η is the slip factor per unit length, $\delta = \Delta\gamma/\gamma$ is the energy deviation relative to the nominal value γmc^2, and $r_0 = e^2/mc^2$ is the classical electron radius. The distribution function f is normalized so that $\int f dz d\delta$ gives the particle density (per cm^3).

It turns out, however, that the standard form of the Vlasov equation (8) should be corrected by taking into account the retardation effect in the last term on the left hand side[1]:

$$\int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} d\delta' w(s, z-z') f(\delta', z', s) \to \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} d\delta' w(s, z-z') f\left(\delta', z', s - c\frac{z-z'}{c-v_z}\right)\,. \quad (9)$$

Indeed, the wake that is generated at coordinate z' moves relative to the beam with the velocity $c-v_z$, and if it reaches the point z at time t, it should have been emitted at position $s - c(z-z')/(c-v_z)$. Taking into account that in the undulator

$$\eta = -\frac{1+K^2}{\gamma^2}\,, \quad (10)$$

we obtain

$$\frac{\partial f}{\partial s} + \frac{\delta(1+K^2)}{\gamma^2}\frac{\partial f}{\partial z} - \frac{r_0}{\gamma}\frac{\partial f}{\partial \delta}\int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} d\delta' \times w(s, z-z') f\left(\delta', z', s - c\frac{z-z'}{c-v_z}\right) = 0\,. \quad (11)$$

It is convenient to introduce new variables: $\bar{s} = k_w s$ and $\theta = \omega_0 z/v_z$ and consider f as a function of $\bar{s}$ and θ. We linearize Eq. (11) assuming that $f = f_0(\delta) + f_1(\delta, z, s)$ with $|f_1| \ll f_0$. Using notation $f_0(\delta) = n_0 h(\delta)$, we find [2–4]

$$\frac{\partial f_1}{\partial \bar{s}} + 2\delta\frac{\partial f_1}{\partial \theta} - (2\rho)^3 h'(\delta) \int_{\theta-\bar{s}}^{\theta} d\theta' \times \int_{-\infty}^{\infty} d\delta' \cos(\theta-\theta') f_1(\delta', \theta', \bar{s} - \theta + \theta') = 0\,, \quad (12)$$

where ρ is the Pierce parameter [6] given by

$$(2\rho)^3 = \frac{2n_0\kappa c r_0}{k_w \gamma \omega_0} = \frac{2\pi K^2 r_0 n_0}{\gamma^3 k_w^2}\,, \quad (13)$$

and we have used the relation

$$\frac{k_0}{k_w} = \frac{v_z}{c-v_z}\,.$$

FEL DISPERSION RELATION

We introduce a new variable $\bar{s}' = \bar{s} - \theta + \theta'$, and rewrite Eq. (12) in the following form

$$\frac{\partial f_1}{\partial \bar{s}} + 2\delta\frac{\partial f_1}{\partial \theta} - (2\rho)^3 h'(\delta) \int_{0}^{\bar{s}} ds' \times \int_{-\infty}^{\infty} d\delta' \cos(\bar{s}-\bar{s}') f_1(\delta', \theta - \bar{s} + \bar{s}', \bar{s}') = 0\,. \quad (14)$$

Assume sinusoidal modulation of the distribution function with frequency ω, $f_1 \propto e^{i\omega z/c} = e^{i(1+\nu)\theta}$, where $\nu = (\omega-\omega_0)/\omega_0$. We then define functions Φ_ν and K_ν such that

$$\begin{aligned} f_1(\delta, \theta, \bar{s}) &= e^{i(1+\nu)\theta}\Phi_\nu(\delta, \bar{s})\,, \\ K_\nu(\bar{s}) &= e^{-i(1+\nu)\bar{s}}\cos\bar{s}\,. \end{aligned}$$

[1] A similar modification of the Vlasov equation is invoked in the derivation if a single-mode CSR instability in rings, [5].

Then Eq. (14) takes the form

$$\frac{\partial \Phi_\nu}{\partial \bar{s}} + 2i\delta(1+\nu)\Phi_\nu = (2\rho)^3 h'(\delta) \int_0^{\bar{s}} d\bar{s}' K_\nu(\bar{s}-\bar{s}') \times \int_{-\infty}^{\infty} d\delta' \Phi_\nu(\delta', \bar{s}') = 0\,. \tag{15}$$

Laplace transforming Eq. (15) we find

$$-\Phi_\nu(\delta,0) + [\beta + 2i\delta(1+\nu)]\tilde{\Phi}_\nu(\delta,\beta) = (2\rho)^3 h'(\delta) \tilde{K}_\nu(\beta) \int_{-\infty}^{\infty} d\delta' \tilde{\Phi}_\nu(\delta',\beta)\,, \tag{16}$$

where

$$\begin{aligned} \tilde{\Phi}_\nu(\delta,\beta) &= \int_0^\infty d\bar{s} e^{-\beta\bar{s}} \Phi_\nu(\delta,\bar{s})\,, \\ \tilde{K}_\nu(\beta) &= \int_0^\infty d\bar{s} e^{-\beta\bar{s}} K_\nu(\bar{s}) \\ &= \frac{1}{2}\left[\frac{1}{\beta+i\nu} + \frac{1}{\beta+i\nu+2i}\right]. \end{aligned} \tag{17}$$

Dividing Eq. (16) by $\beta + 2i\delta(1+\nu)$ and integrating over δ yields

$$\int_{-\infty}^{\infty} d\delta \tilde{\Phi}_\nu(\delta,\beta) = \frac{\int_{-\infty}^{\infty} d\delta \frac{\Phi_\nu(\delta,0)}{\beta+2i\delta(1+\nu)}}{1 - (2\rho)^3 \tilde{K}_\nu(\beta) \int_{-\infty}^{\infty} d\delta \frac{h'(\delta)}{\beta+2i\delta(1+\nu)}}\,.$$

The dispersion relation that defines the frequency ν of modes is given by zeros of the denominator on the right hand side of this equation:

$$(2\rho)^3 \tilde{K}_\nu(\beta) \int_{-\infty}^{\infty} d\delta \frac{h'(\delta)}{\beta + 2i\delta(1+\nu)} = 1\,. \tag{18}$$

Rapid growth will be seen to correspond to $|\nu| \lesssim 2\rho$ and $\beta \sim 2\rho$. The second term in expression for $\tilde{K}_\nu$ on Eq. (17) is not resonant and can be neglected, which gives

$$\frac{1}{2}(2\rho)^3 \frac{1}{\beta+i\nu} \int_{-\infty}^{\infty} d\delta \frac{h'(\delta)}{\beta+2i\delta} = 1\,,$$

where we neglected ν relative to unity in the denominator of the integrand of Eq. (18).

For a cold beam, $h(\delta) = \delta(\delta)$ (where the first δ stands for the delta-function), and we obtain

$$\beta^2(\beta + i\nu) = i(2\rho)^3 \tag{19}$$

in agreement with conventional result of the FEL theory [2–4].

DISCUSSION

In order to clarify the effect of retardation introduced in Eq. (9), we will analyze here the dispersion relation that would result from using the Vlasov equation Eq. (8), without retardation. In this analysis, we will neglect the s dependance of the wake in Eq. (6). Linearizing Eq. (8), and assuming the dependence $f_1(\delta, z, s) \propto e^{\beta k_w s + i(1+\nu)\omega_0 z/c}$ we obtain a well known Keil-Schnell dispersion relation [1] for a coasting beam instability:

$$\frac{cr_0}{\gamma} Z\left(\frac{\omega_0(1+\nu)}{c}\right) \int_{-\infty}^{\infty} \frac{d\delta(\partial f_0/\partial\delta)}{\beta k_w - i\eta\delta(1+\nu)\omega_0/c} = 1\,, \tag{20}$$

where the impedance $Z(k)$ is related to the wake by the following equation

$$Z(k) = \frac{1}{c}\int_{-\infty}^{\infty} w(z) e^{-ikz} dz\,.$$

For a cold beam, with the distribution function $f_0 = n_0\delta(\delta)$, Eq. (20) reduces to

$$-\frac{i\eta n_0 r_0 \omega_0 (1+\nu)}{\gamma\beta^2 k_w^2} Z\left(\frac{\omega_0(1+\nu)}{c}\right) = 1\,. \tag{21}$$

To illustrate our point, we will use the wake given by $w = 2\kappa\cos(\omega_0 z/v_z)$ for arbitrary $z > 0$ (that is we neglect the condition $0 < z < \frac{c-v_z}{c}s$ in Eq. (6)). We then find

$$Z(k) = \frac{2\kappa}{c}\int_0^\infty \cos\left(\frac{\omega_0 z}{c}\right) e^{-ikz} dz \approx \frac{i\kappa}{c}\frac{1}{k - \omega_0/c}\,, \tag{22}$$

where we left only the resonant term, dominant when kc is close to the FEL frequency ω_0. Substituting Eq. (22) into Eq. (21) and using equations Eq. (7) and (10) for κ and η, and Eq. (13) for ρ we find

$$\beta^2 = \frac{(2\rho)^3}{\nu}\,. \tag{23}$$

According to this dispersion relation, the quantity β diverges when $\nu \to 0$. This result is due to the fact that $\nu = 0$ corresponds to the exact resonance with the wake, when the impedance $Z = \infty$. Comparing with the correct dispersion relation Eq. (19), we see that the retardation effectively detunes and broadens the resonance in Eq. (23) changing $\nu \to \nu - i\beta$, and effectively eliminates the divergence at the resonance.

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