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Counting Schwarzschild and Charged Black Holes

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We review the arguments that fundamental string states are in one to one correspondence with black hole states. We demonstrate the power of the assumption by showing that it implies that the statistical entropy of a wide class of nonextreme black holes occurring in string theory is proportional to the horizon area. However, the numerical coefficient relating the area and entropy only agrees with the Bekenstein–Hawking formula if the central charge of the string is six which does not correspond to any known string theory. Unlike the current D-brane methods the method used in this paper is applicable for the case of Schwarzschild and highly non-extreme charged black holes.

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1. Introduction

A number of years ago one of us speculated [1] that the statistical entropy of a black hole could be computed by counting the states of free strings. At the time the focus was on Schwarzschild black holes. In order to make a correspondence between the string and black hole states it was necessary to postulate a large mass–renormalization of the string spectrum when the coupling is turned on. While this mass shift is intuitively expected, it is quantitatively difficult to compute. However, it was soon realized by Sen [2] that the same logic could be applied to BPS black holes for which no mass renormalization can occur. Since then the program of counting the states of weakly coupled string theory and relating the degeneracy to BPS black hole entropy has succeeded brilliantly [3] [4][5][6]. Here we would like to return to the Schwarzschild case and describe a quantitative method for relating strings and black holes.

Consider the degeneracy of a free (neutral) string at mass level $M^2 = 8N_L = 8N_R$ (where $\alpha' = 1/2$). Standard methods give a degeneracy for large $N_{L,R}$,

$$d(N) = \exp 2\pi \left[\sqrt{\frac{N_L c_L}{6}} + \sqrt{\frac{N_R c_R}{6}} \right] \tag{1.1}$$

where $c_{L,R}$ are constants equal to (24,24) for bosonic strings, (12,12) for type II strings and (12,24) for heterotic strings. The entropy then satisfies

$$S_{string} = \log d(N) = 2\pi \left[\sqrt{\frac{N_L c_L}{6}} + \sqrt{\frac{N_R c_R}{6}} \right] \tag{1.2}$$

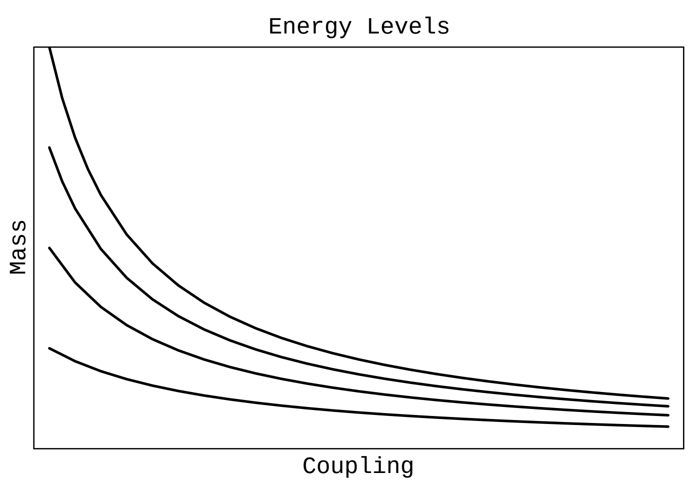
On the other hand the Bekenstein-Hawking entropy of a Schwarzschild black hole is given by

$$S_{BH} = 4\pi M^2 G_N \tag{1.3}$$

Obviously for large M the quantum states of a Schwarzschild black hole are much denser than those of a free string at the same mass. In order to understand how a correspondence can exist let us consider what happens to the free string when the coupling constant g is turned on. Obviously the mass of the state begins to vary due to interactions. In particular the long range gravitational interaction will begin to decrease the mass as the negative potential energy increases. In fact no matter how small g is, sufficiently massive strings will undergo large gravitational corrections. For example a string with level number satisfying $\sqrt{N} > g^{-4}$ will have a size smaller than its Schwarzschild radius and will certainly be subject to large corrections. Let us then consider the evolution of the mass of the string state as g is turned up from zero to its final value. On very general grounds the mass levels will be analytic functions of the parameter g. In general they will become slightly complex since black holes are unstable but the width of a typical Schwarzschild black hole is small, of order its inverse mass. In any case there should be no obstacle to following the real part of the mass of a given state that begins at string level N. It is obvious that the negative gravitational energy will cause the levels to become more dense. If the levels become dense enough then they can reproduce the level density implied by eq. (1.3). For example a formula like

$$M^2 = \frac{4N}{l_s^2(1+\sqrt{N}g^2/(64\sqrt{2}\pi))} \tag{1.4}$$

would turn the string degeneracy at $g = 0$ into the black hole degeneracy when $\sqrt{N}g^2 >> 1$ (see figure). However there does not seem to be much hope of following the masses into the highly nonperturbative region of black holes.



Evolution of the energy levels of a fundamental string as a function the coupling constant g.

It is clear that almost all string states lie within their Schwarzschild radius and must evolve into black holes as g is turned on. There is also a less well known argument that almost all uncharged black holes evolved from states of a single free string[1]. Consider what happens to a typical uncharged black hole in a large box as g is slowly turned off. It must evolve into some state of free string theory in the box, although not necessarily a single string (Other objects such as D-branes and solitons become infinitely massive as $g \to 0$). However, it has been known since the earliest days of string theory, that of the states of free string theory with a given mass the overwhelmingly most numerous are single strings. Thus we expect that almost all black holes evolve back to single string states. The purpose of this paper is to describe a strategy for confirming this hypothesis and that the Bekenstein–Hawking entropy just reflects the level density of the original strings. Our results are independent of the details of the compactified six dimensional space. Furthermore they apply in all dimensions less than or equal to 10.

2. Strategy

The strategy we will employ is the same one that has proved successful in studying D-brane systems in type II string theory just above extremality [4] [12] [13]. In these references the properties of Hawking radiation and absorption at low energies were studied by using perturbation theory around the weakly coupled limit in which black holes become perturbative D-brane systems. Although it is not entirely clear why perturbation theory should work, it seems that for wave lengths much longer than the Schwarzschild radius, black holes behave like weakly coupled systems. In particular, the quantities that have been successfully computed have the following features in common. First, they refer to very long wavelength. Second, when expressed in terms of the entropy and discrete quantum numbers of the black hole, the semiclassical expressions for these quantities are simple positive powers of the coupling constant. An example of such a quantity is the absorption cross section for scalar particles in the limit of vanishing frequency.

We begin with neutral systems. Let us assume that almost all black holes originate from single string states. In that case a given black hole can be labeled by level numbers $N_{L,R}$ and its entropy will be

$$S = 2\pi\sqrt{\frac{c}{6}}(\sqrt{N}_L + \sqrt{N}_R) \tag{2.1}$$

We will consider the independent quantity that specifies the black hole to be the entropy S. Now suppose we are interested in the quantity Q which can be computed in semiclassical black hole theory. In general, when expressed in terms of S and g, $Q(S,g)$ will not be a power series but as we shall see certain quantities are. In this case we can also hope to compute the same quantities in string perturbation theory as a function of $\sqrt{N}$ and g. If the correspondence between black holes and strings is correct the expressions should agree. Notice that this strategy circumvents the need to calculate the mass shift. As we mentioned earlier, this strategy has only been tested for very low energy quantities.

The quantity that we shall concentrate on is the area of the black hole. Consider the semiclassical Bekenstein–Hawking relation

$$S = \frac{A}{4G_N} \tag{2.2}$$

Let us rewrite it as a formula for the area A.

$$A = 4G_N S = 4\left(\frac{\kappa^2}{8\pi}\right) 2\pi\sqrt{\frac{c}{6}}(\sqrt{N}_L + \sqrt{N}_R) \tag{2.3}$$

Evidently the area of the black hole is a perturbative quantity of order g^2 and should be computable in string perturbation theory! This is a point that has been emphasized by Maldacena[7] in the context of BPS black holes.

Now area is not one of the quantities that one normally thinks of computing in string perturbation theory. String theory is set up for the computation of scattering amplitudes and decay rates. Therefore if we want to proceed we must find an expression for the area in terms of on-shell matrix elements. A number of possibilities come to mind. For example, the low energy limit of the absorption cross section for a massless particle to excite a black hole is known to be proportional to the horizon area [8][9]. In fact the cross section for a scalar particle at vanishing incident energy is exactly equal to the horizon area[10].

A completely equivalent definition can be given in terms of the low energy power spectrum $P(\omega)$ of the Hawking radiation emitted by the black hole. Unruh has calculated the power spectrum for massless minimally coupled scalars and found for $\omega << 1/R_{BH}$

$$P(\omega) = \frac{\omega^2 T}{2\pi^2} A_H \tag{2.4}$$

where ω is the frequency of the emitted quanta, T is the temperature of the emitter and A_H is the horizon area. For our purposes we regard (2.4) as a definition of the area.

$$A = \lim_{\omega \to 0} \frac{2\pi^2 P(\omega)}{\omega^2 T} \tag{2.5}$$

Defined in this way A is identical to the low energy limit of the absorption cross section.

Our strategy will be to compute the temperature and the power spectrum of a very highly excited string in powers of g^2. From (2.3) it follows that if we form the combination in (2.5) the higher orders beyond order g^2 should vanish in the limit of large mass and the order g^2 term should satisfy (2.2). In this way we would derive the area of a black hole of entropy S from the counting of levels of a quantum system. Exactly this type of calculation has been successfully done in the D-brane theory of near extreme black holes. [11][12][13].

3. Schwarzschild Black Holes and Strings

The temperature of a highly excited weakly coupled neutral string is easy to compute to leading order in perturbation theory. The entropy of the free string is proportional to its mass M. Using eq. (2.1) the first law gives

$$\beta = \frac{1}{T} = \frac{dS}{dM} = \frac{\pi}{2}\left[\sqrt{\frac{c_L}{3}} + \sqrt{\frac{c_R}{3}}\right] \tag{3.1}$$

This is just the Hagedorn temperature at which a very weakly coupled string will radiate. Although we have not calculated the perturbative corrections to the temperature there is no reason for them to be absent. Thus the temperature of a string at large level number should have a perturbation expansion of the form

$$T = T_{Hagedorn} - g^2 F(N) + ... \tag{3.2}$$

The luminosity $P(\omega)$ is more complicated and will be calculated in terms of decay rates. Obviously the decay rates and therefore $P(\omega)$ vanish for $g = 0$. The leading term is order

g^2. Therefore when calculating the area to order g^2 we only need the temperature to leading order.

A classical black hole solution represents a statistical ensemble of states. The initial state of a free string that we wish to consider should also be a statistical ensemble defined by introducing a thermal density matrix which is peaked at states with the desired mass. Recalling the first quantized expression for the mass.

$$M^2 = 8N_L = 8N_R = 4N \tag{3.3}$$

we are led to a density matrix of the form

$$\rho = Z^{-1} exp(-\beta_L^* N_L - \beta_R^* N_R) \tag{3.4}$$

where Z is defined so that $Tr\rho = 1$. It should be noted that β^* is not the inverse of the real temperature of the system. It is a dimensionless parameter, which is chosen to fix the average value of M^2. One finds that

$$\beta_{L,R}^* = \frac{dS}{dN_{L,R}} = \sqrt{\frac{\pi^2 c_{L,R}}{3N_{L,R}}} \tag{3.5}$$

Now consider the emission of a scalar particle by a typical member of the ensemble. Let us choose the particular scalar that corresponds to the component g_{56} of the graviton. The vertex operator for this scalar is (using the conventions of [11])

$$V(k) = \int \frac{4\sqrt{2}\kappa}{\pi}[\partial_+ X^5 \partial_- X^6 + \partial_- X^5 \partial_+ X^6 + fermion \quad terms] e^{ikX} d^2\sigma \tag{3.6}$$

where the derivatives refer to world sheet light cone coordinates and the momentum k is a null vector in the four dimensional uncompactified Minkowski space. If the momentum k is much smaller than l_s^{-1} in the rest frame of the decaying string then the fermionic term and the factor e^{ikx} in the vertex operator can be ignored except for the center of mass contribution which when integrated out provides a momentum conserving delta function.

The usual oscillator representation for the $X's$ leads to the expressions

$$\partial_+ X^\mu = \sum \alpha_n^\mu e^{-2in\sigma_+}$$

$$\partial_- X^\mu = \sum \tilde{\alpha}_n^\mu e^{-2in\sigma_-} \tag{3.7}$$

The matrix element for the decay of a state $|i\rangle$ which we take to be at rest, to a state $|f\rangle$ by colliding a right moving X^5 with a left moving X^6 and emitting a scalar g_{56} has the form

$$\mathcal{M} = \frac{4\sqrt{2}\kappa}{\pi} \langle i| \int \sum_{n,m} \alpha_n^5 \tilde{\alpha}_m^6 e^{-2i(n-m)\sigma} d\sigma |f\rangle \delta^4(p_i + k - p_f) \tag{3.8}$$

where the vertex function no longer contains the factor e^{ikX}. The delta function constrains the on-shell momenta of the initial and final string. An analogous expression comes from the second term in (3.6). In practice, if the mass of the initial and final strings are much larger than the energy ω carried by the scalar then the only effect of the delta function is to constrain the masses according to

$$M_i = M_f + \omega \tag{3.9}$$

Let us assume that the initial and final strings are at levels N and $N - \delta N$. Then using $M^2 = 4N$ we find

$$\omega = \delta M = \frac{2n}{M} \tag{3.10}$$

where $n = \delta N$. The process of decay is now seen to have a simple intuitive structure. The vertex operator may be averaged over the world sheet coordinate σ and becomes

$$V = 4\sqrt{2}\kappa \sum (\alpha_n^5 \tilde{\alpha}_n^6 + \tilde{\alpha}_n^5 \alpha_n^6) \tag{3.11}$$

It describes the annihilation of two oppositely moving quanta on the string with mode number n. The energy is carried off by the scalar whose energy is constrained to satisfy eq. (3.10).

To obtain the decay rate per $d\omega$ we square the amplitude, average over the initial thermal distribution and multiply by the density of resonances $M/4$ (3.10)

$$Tr\rho \sum_f |\mathcal{M}|^2 = 32\kappa^2 n^2 \frac{1}{(e^{\beta_L^* n} - 1)(e^{\beta_R^* n} - 1)} 2(\frac{1}{2M})^2 \frac{M}{4} \tag{3.12}$$

where the factor 2 comes from the two terms in (3.11), and $(1/2M)^2$ from the relativistic normalization of the initial and final states. The luminosity $P(\omega)$, in the low frequency limit, is given by Fermi's golden rule to be

$$P(\omega) = \frac{8\kappa^2}{M} \frac{\omega^2}{2\pi\beta_L^*\beta_R^*} \tag{3.13}$$

where we have used eq. (3.12) and the fact that $\beta_{L,R}^* n << 1$.

This has to be compared with the classical result for luminosity which we use as a definition of the area of the black hole horizon

$$P(\omega) = \frac{\omega^2}{2\pi^2\beta} A_H \tag{3.15}$$

from which we find

$$A_H = \frac{64\pi^2\beta G_N}{M\beta_L^*\beta_R^*}$$

where we have used $\kappa^2 = 8\pi G_N$.

The value of β to be used in (3.15) is the lowest order expression given by eq. (3.5). Expressing $\sqrt{N}$ in terms of the entropy, from eqs. (3.14) and (3.15) we get

$$4G_N S\sqrt{\frac{36}{c_L c_R}} = A_H \tag{3.16}$$

Two things are apparent from eq (3.16). The first is that the entropy and area are indeed proportional. Moreover, as we will see in the next section, this proportionality extends to black holes in arbitrary dimension as well as arbitrary charges and angular momentum. The numerical proportionality factor is always the same.

The second point is that the proportionality factor is not the correct Bekenstein Hawking factor unless $c_L c_R = 36$. Unfortunately this value does not correspond to to any fundamental string theory. The meaning of this result is very unclear but it suggests that there may be some kind of nonperturbative "renormalization" of c in the environment of a horizon .

4. Charged and Rotating Black Holes

We can extend the above calculations to charged and rotating black holes. The classical absorption cross section for charged black holes in arbitrary dimensions was calculated in [10] and was shown to be equal to the area. The absorption cross section for a four dimensional Kerr black hole was calculated in [14] and was also shown to be equal to the area. We will assume that this result generalizes to arbitrary charged rotating black holes and so the classical relation (3.15) continues to hold.

In general we can write

$$S = S_L + S_R \tag{4.1}$$

where S_L^2 is linear in N_L and S_R^2 is linear in N_R. For example for charged strings we have

$$S_L^2 = \left(2\pi\sqrt{\frac{c_L}{6}}\right)^2 N_L \tag{4.2}$$

and for rotating bosonic strings with $J \sim N_L$

$$S_L^2 = \left(2\pi\sqrt{\frac{c_L}{6}}\right)^2 (N_L - J) \tag{4.3}$$

with similar equations for S_R. Also $M^2 = 8N_L + Q_L^2 = 8N_R + Q_R^2$.

Now we can repeat the calculation of the previous section. The only changes are in the $\beta^*_{L,R}$ and the expressions for M and S. In particular, we find

$$A_H = (64\pi^2 G_N S)(\frac{\beta}{M\beta^*_L\beta^*_R S}) \tag{4.4}$$

We will evaluate the expression in the second parenthesis and show that it is independent of the charges and angular momentum if and only if $c_L = c_R$. The relation between area and entropy will therefore be identical to the one in the previous section. We have

$$\beta^*_{L,R} = \frac{dS_{L,R}}{dN_{L,R}} = \frac{\left(2\pi\sqrt{\frac{c_{L,R}}{6}}\right)^2}{2S_{L,R}} \tag{4.5}$$

$$\beta = \frac{dS}{dM} = \frac{dS_L}{dM} + \frac{dS_R}{dM} \tag{4.6}$$

Now

$$\frac{dS_L}{dM} = (\frac{dS_L}{dN_L})(\frac{dN_L}{dM}) = \frac{\left(2\pi\sqrt{\frac{c_{L,R}}{6}}\right)^2 M}{8S_L} \tag{4.7}$$

and so

$$\beta = M(\frac{\left(2\pi\sqrt{\frac{c_L}{6}}\right)^2}{8S_L} + \frac{\left(2\pi\sqrt{\frac{c_R}{6}}\right)^2}{8S_R}) \tag{4.8}$$

If we now evaluate the expression in parenthesis in eqn(4.4) we find that it is independent of all charges and angular momenta only if $c_L = c_R = c$. For this case we find

$$\frac{\beta}{M\beta_L^*\beta_R^* S} = \frac{1}{2\left(2\pi\sqrt{\frac{c}{6}}\right)^2} \tag{4.9}$$

Thus, as we anticipated, the relation between the entropy and area is unaffected by the presence of charges and angular momentum for $c_L = c_R = c$. We therefore reproduce the Bekenstein-Hawking entropy for $c = 6$ as in the previous section.

The generalization to arbitrary dimensions is a consequence of the independence of the string scattering amplitude on dimension and the equality of the phase space factors for the string and the classical calculation. So the correspondence of area and entropy should hold in any dimension.

5. Conclusions

We have seen that the low energy absorption cross section σ for scalars from fundamental strings satisfies

$$\sigma_{abs} = 8\pi G_N \sqrt{\frac{cN}{6}}$$

Combining this with a fact and an assumption leads to a proportionality between entropy and area. The fact is that the absorption cross section at $\omega \to 0$ for any black hole is the area of the classical horizon. The assumption is that the levels of a free string are in one to one correspondence with the levels of black holes that evolve from the strings as g is increased. If this is true, we may replace $\sqrt{\frac{cN}{6}}$ for the string with the entropy of the black hole ensemble. Thus, the assumption leads to a relation between entropy and area. However, the precise perturbative calculation of the numerical proportionality factor does not agree with the Bekenstein-Hawking value unless $c = 6$. We do not understand the meaning of this but it suggests that there may be some kind of "renormalization" of c in the environment of a black hole.

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