STU Black Holes and String Triality

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We found double-extreme black holes associated with the special geometry of the Calabi-Yau moduli space with the prepotential $F = STU$. The area formula is STU-moduli independent and has $[SL(2, \mathbf{Z})]^3$ symmetry in space of charges. The dual version of this theory without prepotential treats the dilaton S asymmetric versus T, U-moduli. We display the dual relation between new (STU) black holes and stringy $(S|TU)$ black holes using particular $Sp(8, \mathbf{Z})$ transformation. The area formula of one theory equals the area formula of the dual theory when expressed in terms of dual charges.

We analyse the relation between (STU) black holes to string triality of black holes: $(S|TU)$, $(T|US)$, $(U|ST)$ solutions. In democratic STU-symmetric version we find that all three S and T and U duality symmetries are non-perturbative and mix electric and magnetic charges.

I. INTRODUCTION

The non-perturbative properties of the future fundamental theory manifests themselves in the duality properties of the area formulae of the supersymmetric black holes horizon. The universal entropy-area formula of supersymmetric black holes is given by the central charge extremized in the moduli space Z_{fix} and depends only on quantized charges. The universal formula obtained by Ferrara and one of the authors [1] is $S = \frac{A(p,q)}{4} = \pi |Z_{\text{fix}}|^\alpha$ with $\alpha = 2$ (3/2) for $d = 4$ ($d = 5$).

This universal formula has various implementations in different theories. A particularly rich class of area formulae may be expected to exist in N=2 supersymmetric theories which are characterized by different choice of the holomorphic prepotential and/or symplectic sections. A beautiful interplay between the geometry of special Kähler manifolds [2]- [6] and space-time geometry of supersymmetric black holes has been discovered recently [7], [1], [8].

In this paper we will find the 4d double-extreme black holes in a class of N=2 theories with the prepotential $F = d_{ABC} \frac{X^A X^B X^C}{X^0}$ [3]. These theories with real symmetric constant tensors d_{ABC} are related to geometry occurring in 5-dimensional supergravity [9] where the term $\int d_{ABC} F^A \wedge F^B \wedge A^C$ is present in the action. These theories are also related to the special geometry of Calabi-Yau moduli spaces where d_{ABC} are the intersection numbers of the Calabi-Yau manifold, $t^A = \frac{X^A}{X^0}$ are the moduli fields of the Kähler class [10]. The theories of this class are also referred to as "very special geometry" [11] and "real special geometry" [12].

We will focus mostly on STU-symmetric model $F = \frac{X^1 X^2 X^3}{X^0}$, and will find the moduli-independent $[SL(2, \mathbf{Z})]^3$ symmetric area formula. The moduli of this theory are coordinates of the $\left(\frac{SU(1,1)}{U(1)}\right)^3$ manifold. Duality symmetry of this theory is $[SL(2, \mathbf{Z})]^3$. The dual partners of these black holes (where one of the moduli, e.g. S is singled out and whose imaginary part plays the role of string coupling) are already known [8]. The moduli in this version of the theory are coordinates of $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2) \times SO(2)}$ manifold. S-duality, or $SL(2, \mathbf{Z})$ symmetry associated with the S moduli in string theory has a non-perturbative character, whereas T and U dualities, related to $SO(2, 2)$ symmetry have perturbative character. Perturbative symmetries of string theory do not mix electric and magnetic charges. Stringy black holes treat one of the moduli on different footing than others. This is due to the fact that 11-dimensional supergravity has to be reduced to $d = 10$ first and this makes 11-th component of the metric or the dilaton, special. If however, we are looking for exact non-perturbative solution of 11-dimensional supergravity, we may expect some

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solutions where the radius of 11-th, 6-th and 5-th dimensions are all on equal footing. These are our STU black holes. They may be related to M, F, Y or whatever fundamental theory which is not the conventional theory of strings. To establish the relation between new (STU)-symmetric black holes and their dual $(S|TU)$ stringy partners is our main goal. In string triality picture [13] the role of S may be replaced by T or U but still there is one moduli different from the others and only one duality symmetry is non-perturbative whereas the other two are perturbative. We will find that all three S and T and U duality symmetries in "democratic" black hole solutions are non-perturbative. This is not too surprising: black holes are non-perturbative objects!

We will find that the area of the horizon in (STU)-symmetric theory equals the area of the horizon of the $(S|TU)$ dual theory

$$A^{(STU)}(p,q) = A^{(S|TU)}(\hat{p},\hat{q}) \ , \tag{1}$$

where the charges are related by particular $Sp(8,\mathbf{Z})$ duality transformation. This transformation has been found in [6] and relates the symplectic sections and charges in two theories.

The duality symmetry of this area formula is of an unusual form. The typical situation studied before was that the area as a function of charges was invariant under duality transformation.

$$A(p,q) = A(\hat{p},\hat{q}) \ . \tag{2}$$

For example, U-duality [1] invariant area formula is given by the quartic Cartan invariant of E_7 in d=4, [14], [1], [15], where the 2×28 unhatted (p,q) are charges before duality transformation and 2×28 hatted $(\hat{p},\hat{q})$ are charges after E_7 transformation. This duality transformation was a property of one specific theory: in this case, for example, N=8 supergravity in d=4. The equations of motion of this theory have hidden symmetry and it manifests itself in E_7 invariance of the area formula of the black holes of this theory with 1/8 of supersymmetry unbroken.

$$A^{N=8}(p,q) = A^{N=8}(\hat{p},\hat{q}) = 4\pi\sqrt{J(p,q)} = 4\pi\sqrt{J(\hat{p},\hat{q})} \ . \tag{3}$$

The new phenomenon which we observe here by studying the black holes in the framework of special geometry is the following. Black holes in two versions of the theory related by symplectic transformation have two different area formulae, when the area of the original version is expressed in terms of charges of original theory and the area of the transformed (dual) theory is expressed as a function of charges of dual theory. However, these two area formulae are related as in eq. (1). If one has the area in one theory and the transformation which defines the dual theory is known, the area can be found using (1). The reason for the area formulae to be different is that they carry different symmetries: $\left(\frac{SU(1,1)}{U(1)}\right)^3$ in one case and $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2)\times SO(2)}$ in the other case.

The paper is organized as follows. In Sec. 2 we discuss the basic equations [1] defining the double-extreme black holes of N=2 theory [8] and the values of moduli as functions of charges. We refer to these equations as "stabilization equations". The main property of these equations relevant to present investigation is that they are symplectic covariant. Therefore once the solution for moduli in terms of charges is known in one version of the theory, the dual solution can be found by applying the symplectic transformation to the known solution. We explain this for the case of ST[2,n] manifold, $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}$ symmetric theory which does not admit a prepotential and the dual version of it which admits the prepotential. We also explain that in both theories one has an option of solving for double-extreme black holes directly in each version, without using the information on the solution in the dual theory. Having obtained these two sets of double-extreme black holes one can check that the solutions are actually connected by symplectic transformation. Or one could use the solution available on one side and perform a relevant symplectic transformation to get the double-extreme black holes of the dual theory and the mass-area formula in terms of dual charges. One can verify that the transformed solution indeed solves the equations of the dual theory. In Sec. 3 we proceed with solving "stabilization equations" on the prepotential side and we consider the prepotential $F = \frac{X^1 X^2 X^3}{X^0}$. We derive the new mass-area formula for this theory. In Sec. 4 we show that alternative derivation of STU symmetric double-extreme black holes is possible: via symplectic transformation from the dual version of the theory without the prepotential, i.e. from the theory where S is not symmetric with T, U. In Sec. 5 these two sets of double-extreme black holes are studied from the perspective of string triality and the difference between the new and stringy black holes solutions is explained. In the Outlook we point out the implication of our new d=4 area formulae for Calabi-Yau moduli space and the corresponding d=5 area formulae. We also comment on string loop corrections and their possible effect on supersymmetric black holes and vice versa.

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[1] U-duality in the context of E_7-symmetry should not be confused with U-duality in the context of $SL(2,\mathbf{Z})$ symmetry related to U-moduli. Unfortunately, these two different dualities carry the same name in the current literature.

II. SYMPLECTIC COVARIANCE OF "STABILIZATION EQUATIONS"

Stabilization equations for n_v complex moduli of supersymmetric black holes in N=2 theory near the horizon have the following form [1]

$$\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \mathrm{Re} \begin{pmatrix} 2i\bar{Z}L^\Lambda \\ 2i\bar{Z}M_\Lambda \end{pmatrix} , \tag{4}$$

where the central charge [6]

$$Z(z, \bar{z}, q, p) = e^{\frac{K(z,\bar{z})}{2}}(X^\Lambda(z)q_\Lambda - F_\Lambda(z)p^\Lambda) = (L^\Lambda q_\Lambda - M_\Lambda p^\Lambda) \tag{5}$$

depends on moduli and on $2n_v+2$ conserved charges (p^Λ, q_Λ). (L^Λ, M_Λ) are covariantly holomorphic sections depending on moduli. For double-extreme black holes [8] with frozen moduli these equations implicitly define the frozen moduli as functions of charges.

Symplectic transformation acts on charges as well as on sections

$$\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} , \qquad\qquad \begin{pmatrix} \hat{X} \\ \hat{F} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} X \\ F \end{pmatrix} , \tag{6}$$

and provides the relation between the dual versions of the theory. Here

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n_v + 2, \mathbf{Z}) . \tag{7}$$

Stabilization equations are covariant under symplectic transformations

$$\begin{pmatrix} \hat{p}^\Lambda \\ \hat{q}_\Lambda \end{pmatrix} = \mathrm{Re} \begin{pmatrix} 2i\bar{Z}\hat{L}^\Lambda \\ 2i\bar{Z}\hat{M}_\Lambda \end{pmatrix} , \tag{8}$$

since the central charge is invariant.

$$Z = (L^\Lambda q_\Lambda - M_\Lambda p^\Lambda) = (\hat{L}^\Lambda \hat{q}_\Lambda - \hat{M}_\Lambda \hat{p}^\Lambda) . \tag{9}$$

and the sections are covariant.

We are interested in dual relation between black holes of two theories. The first one can be defined in terms of a prepotential [2]

$$F = \frac{1}{2}d_{ABC}t^A t^B t^C = S\,\eta_{\alpha\beta}\,t^\alpha t^\beta , \qquad X^0 = 1 , \tag{10}$$

where

$$t^1 = S, \qquad d_{ABC} = \begin{pmatrix} d_{1\alpha\beta} = \eta_{\alpha\beta} \\ 0\ \mathrm{otherwise} \end{pmatrix}, \qquad\qquad A, B, C = 1, 2, \ldots, n+1 , \tag{11}$$

and

$$\eta_{\alpha\beta} = \mathrm{diag}(+, -, - \ldots, -), \qquad \alpha, \beta = 2, \ldots, n+1. \tag{12}$$

This prepotential corresponds to the product manifold $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}$. The $\frac{SU(1,1)}{U(1)}$ coordinate is the axion-dilaton field S. The remaining n complex moduli t^i are special coordinates of the $\frac{SO(2,n)}{SO(2)\times SO(n)}$ manifold. In particular, when $n = 2$ we have

$$F = \frac{1}{2}d_{ABC}t^A t^B t^C = \frac{1}{2}S\left[(t^2)^2 - (t^3)^2\right] . \tag{13}$$

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[2]We are using here notation of [16].

3

This theory is defined by 3 complex moduli and 4 gauge groups and the corresponding manifold is $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2) \times SO(2)}$.

If we introduce the notation

$$t^2 \equiv \frac{1}{\sqrt{2}}(T+U) , \qquad t^3 \equiv \frac{1}{\sqrt{2}}(T-U) , \tag{14}$$

the prepotential becomes

$$F = STU . \tag{15}$$

This theory has the symmetry of the manifold $\left(\frac{SU(1,1)}{U(1)}\right)^3$, which corresponds to the embedding of $\left(\frac{SU(1,1)}{U(1)}\right)^2$ into $\frac{SO(2,2)}{SO(2) \times SO(2)}$. The STU symmetric theory with the cubic holomorphic prepotential (15) is associated with particular Calabi-Yau moduli space. It is related to the dual version of the theory via symplectic transformation [6]. We will study this relation in the context of black holes in Sec. 4.

There are two possibilities to find double-extreme black holes in the theory with the prepotential. Either directly solve the stabilization equations or perform the dual rotation from the known solution. In the next section we will first find the black holes directly in STU-symmetric model.

III. DOUBLE-EXTREME BLACK HOLES IN STU-MODEL

STU-model [17] is described by the prepotential:

$$F(X) = \frac{d_{ijk}X^i X^j X^k}{X^0} , \qquad i,j,k = 1,2,3, \tag{16}$$

For our case

$$F(X) = \frac{X^1 X^2 X^3}{X^0} . \tag{17}$$

Holomorphic section determined by that prepotential has a form: (X^Λ, F_Λ) with $F_\Lambda = \frac{\partial F}{\partial X^\Lambda}$ and $\Lambda = (0, i = 1,2,3)$. Special coordinates z^i are determined by

$$z^i = \frac{X^i}{X^0} , \qquad X^0 = 1 . \tag{18}$$

Corresponding Kähler potential is

$$K = -\log\left(-id_{ijk}(z-\bar{z})^i(z-\bar{z})^j(z-\bar{z})^k \right) . \tag{19}$$

and we will use also

$$z^1 = S, \quad z^2 = T , \quad z^3 = U . \tag{20}$$

In terms of special coordinates the holomorphic sections are given by:

$$X^\Lambda = \begin{pmatrix} 1 \\ z^1 \\ z^2 \\ z^3 \end{pmatrix}, \qquad F_\Lambda = \begin{pmatrix} -z^1 z^2 z^3 \\ z^2 z^3 \\ z^1 z^3 \\ z^1 z^2 \end{pmatrix} . \tag{21}$$

The stabilization equations are:

$$p^\Lambda = ie^{K/2}\left(\bar{Z}X^\Lambda - Z\bar{X}^\Lambda\right) , \tag{22}$$

$$q_\Lambda = ie^{K/2}\left(\bar{Z}F_\Lambda - Z\bar{F}_\Lambda\right) . \tag{23}$$

We can eliminate $\bar{Z}$ from these equations so that:

$$X^\Lambda q_\Sigma - p^\Lambda F_\Sigma = ie^{K/2}Z\left(\bar{X}^\Lambda F_\Sigma - X^\Lambda \bar{F}_\Sigma\right) . \tag{24}$$

4

This is the matrix equation we used in [8] to solve for the solution of frozen moduli. In what follows we will solve for z^1 as a function of charges. The solution for z^2 and z^3 can be obtained in an analogous way as a result of symmetry between the three moduli.

Here are the components ($(\Lambda, \Sigma) = (1,0), (0,1), (1,1), (2,3)$ and $(3,2)$ respectively) from the matrix equation (24) we need for the derivation of z^1:

$$q_0 + p^1 z^2 z^3 = i e^{K/2} Z (\bar{z}^1 \bar{z}^2 \bar{z}^3 - \bar{z}^1 z^2 z^3) \ , \tag{25}$$

$$q_1 - p^0 z^2 z^3 = i e^{K/2} Z (z^2 z^3 - \bar{z}^2 \bar{z}^3) \ , \tag{26}$$

$$q_1 z^1 - p^1 z^2 z^3 = i e^{K/2} Z (\bar{z}^1 z^2 z^3 - z^1 \bar{z}^2 \bar{z}^3) \ , \tag{27}$$

$$q_3 z^2 - p^2 z^1 z^2 = i e^{K/2} Z (\bar{z}^2 z^2 z^1 - z^2 \bar{z}^2 \bar{z}^1) \ , \tag{28}$$

$$q_2 z^3 - p^3 z^1 z^3 = i e^{K/2} Z (\bar{z}^3 z^3 z^1 - z^3 \bar{z}^3 \bar{z}^1) \ . \tag{29}$$

Using (25) and (26) we can eliminate the factor $i e^{K/2} Z$ and obtain

$$z^2 z^3 = \frac{q_1 \bar{z}^1 + q_0}{p^0 \bar{z}^1 - p^1} \ . \tag{30}$$

Using (27, 30) we can obtain a simple formula for $i e^{K/2} Z$

$$i e^{K/2} Z = \frac{(p^0 z^1 - p^1)}{(\bar{z}^1 - z^1)} \ . \tag{31}$$

Substituting (31) into (28) and (29) respectively we can express z^2 and z^3 in terms of z^1 and the charges only:

$$\bar{z}^2 = \frac{(p^2 z^1 - q_3)}{(p^0 z^1 - p^1)} \ , \quad \text{and} \quad \bar{z}^3 = \frac{(p^3 z^1 - q_2)}{(p^0 z^1 - p^1)} \ . \tag{32}$$

Finally, using (30) and the above equations to eliminate z^2 and z^3 we are getting a quadratic equation for z^1

$$(z^1)^2 + \frac{((p \cdot q) - 2p^1 q_1)}{(p^0 q_1 - p^2 p^3)} z^1 - \frac{(p^1 q_0 + q_1 q_2)}{(p^0 q_1 - p^2 p^3)} = 0 \ , \tag{33}$$

where

$$(p \cdot q) = (p^0 q_0) + (p^1 q_1) + (p^2 q_2) + (p^3 q_3) \equiv p^\Lambda q_\Lambda \ , \tag{34}$$

and the solution for z^1 moduli is

$$z^1 = \frac{((p \cdot q) - 2p^1 q_1)}{2(p^3 p^2 - p^0 q_1)} \mp i \frac{\sqrt{W}}{2(p^3 p^2 - p^0 q_1)} \ , \tag{35}$$

where

$$W(p^\Lambda, q_\Lambda) = -(p \cdot q)^2 + 4\big((p^1 q_1)(p^2 q_2) + (p^1 q_1)(p^3 q_3) + (p^3 q_3)(p^2 q_2)\big) - 4 p^0 q_1 q_2 q_3 + 4 q_0 p^1 p^2 p^3 \ . \tag{36}$$

The function $W(p^\Lambda, q_\Lambda)$ is symmetric under transformations: $p^1 \leftrightarrow p^2 \leftrightarrow p^3$ and $q_1 \leftrightarrow q_2 \leftrightarrow q_3$. Finally solution for all three complex modulus are:

$$z^i = \frac{((p \cdot q) - 2p^i q_i) \mp i \sqrt{W}}{2(3 d_{ijk} p^j p^k - p^0 q_i)} \ . \tag{37}$$

There is no summation over i in $p^i q_i$. For the solution to be consistent we have to require $W > 0$, otherwise the moduli are real and the Kähler potential is not defined

At this point the choice of signs in the imaginary part of the moduli is ambiguous. However, to preserve the obvious exchange symmetries, we want to choose common signs for all. In fact it turns out that only the "$-$" is consistent as we shall see.

With these expressions for the z^i the Kähler potential eq. (19) is easily computed. We find

5

$$e^{-K} = \pm \frac{W^{3/2}}{\omega_1 \omega_2 \omega_3} \ , \tag{38}$$

where

$$\omega_i = (3 d_{ijk} p^j p^k - p^0 q_i) \ . \tag{39}$$

It is also useful to calculate the product of three ω_i's which appears to be positive:

$$\omega_1 \omega_2 \omega_3 = \frac{1}{4} \left((p^0)^2 W + [2 p^1 p^2 p^3 - p^0 (p \cdot q)]^2 \right) > 0 \tag{40}$$

with $\Lambda = 0, 1, 2, 3$,

For the Kähler potential e^{-K} to be positive we have to pick up only one choice of sign for each imaginary part of the special coordinates in eq. (37), it has to be negative:

$$z^i = \frac{((p \cdot q) - 2 p^i q_i) - i \sqrt{W}}{2(3 d_{ijk} p^j p^k - p^0 q_i)} \qquad \Longrightarrow \qquad e^{-K} = \frac{W^{3/2}}{\omega_1 \omega_2 \omega_3} > 0 \ . \tag{41}$$

We can proceed now with the calculation of the central charge to find the black hole mass, which for double-extreme black holes is proportional to the area of the black hole horizon. We find that

$$e^K Z \bar{Z} = \frac{(p^0)^2 W + [2 p^1 p^2 p^3 - p^0 (p \cdot q)]^2}{4W} \ . \tag{42}$$

We deduce for the mass/area

$$Z \bar{Z} = M^2 = \frac{W^{3/2}}{\omega_1 \omega_2 \omega_3} \frac{(p^0)^2 W + [2 p^1 p^2 p^3 - p^0 (p \cdot q)]^2}{4W} \ , \tag{43}$$

which finally gives the beautiful result

$$Z \bar{Z} = M^2 = \frac{A}{4\pi} = \left(W(p^\Lambda, q_\Lambda) \right)^{1/2} \ . \tag{44}$$

This is a very nice and simple expression for the area which relies on the fact that the nominator of the second expression in (43) and the product of the ω_i cancel. Thus we have completely described the double-extreme black holes solutions with frozen moduli in the STU symmetric theory. The geometry is that of extreme Reissner- Nordström type with the mass/area formula, as function of quantized charges given in eq. (36).

$$ds^2 = \left((1 + \frac{[W(p,q)]^{1/4}}{r})^{-2} \right) dt^2 - \left((1 + \frac{[W(p,q)]^{1/4}}{r})^2 \right) d\vec{x}^2 \ . \tag{45}$$

It is instructive to remind that our mass/area formula has also a nice symplectic invariant form, as explained in [1], [8].

$$M^2 = -\frac{1}{2} \left(p^\Lambda, q_\Lambda \right) \begin{pmatrix} (\operatorname{Im}\mathcal{N} + \operatorname{Re}\mathcal{N}\operatorname{Im}\mathcal{N}^{-1}\operatorname{Re}\mathcal{N})_{\Lambda\Sigma} & (-\operatorname{Re}\mathcal{N}\operatorname{Im}\mathcal{N}^{-1})_\Lambda{}^\Sigma \\ (-\operatorname{Im}\mathcal{N}^{-1}\operatorname{Re}\mathcal{N})^\Lambda_\Sigma & (\operatorname{Im}\mathcal{N}^{-1})^{\Lambda\Sigma} \end{pmatrix}_{\text{fix}} \begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix}$$

$$= -\frac{1}{2} \left(p^\Lambda, q_\Lambda \right) \begin{pmatrix} (\operatorname{Im}F + \operatorname{Re}F\operatorname{Im}F^{-1}\operatorname{Re}F)_{\Lambda\Sigma} & (-\operatorname{Re}F\operatorname{Im}F^{-1})_\Lambda{}^\Sigma \\ (-\operatorname{Im}F^{-1}\operatorname{Re}F)^\Lambda_\Sigma & (\operatorname{Im}F^{-1})^{\Lambda\Sigma} \end{pmatrix}_{\text{fix}} \begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix} \ , \tag{46}$$

where the period matrix $\mathcal{N}$ as well as the second derivative of the prepotential F are functions of moduli which at the fixed point near the black hole horizon become functions of charges, as defined in eq. (41).

If we parametrize all 3 moduli in terms of axion-dilaton fields

$$z^i = a^i - i e^{-\eta_i} \tag{47}$$

where

$$a^i = \frac{((p \cdot q) - 2 p^i q_i)}{2 \omega_i} \ , \qquad e^{-\eta_i} = \frac{\sqrt{W}}{2 \omega_i} \ , \tag{48}$$

Káhler potential is

$$e^{-K} = -8 \operatorname{Im}S \operatorname{Im}T \operatorname{Im}U = 8 e^{-\eta_1} e^{-\eta_2} e^{-\eta_3} \ . \tag{49}$$

This parametrization is possible under the condition that all three combination of charges are positive,

$$\omega_i > 0 \ . \tag{50}$$

6

IV. DUAL ROTATION OF DOUBLE-EXTREME BLACK HOLES

The double-extreme black holes for this model without the prepotential for the general case of arbitrary n as well as for $n = 2$ have been found before [8]. The resume of this black hole for $n = 2$ is the following. Solution is defined in terms of 4 magnetic and 4 electric and charges $(\hat{p}^\Lambda, \hat{q}_\Lambda)$ and $\Lambda = 0, 1, 2, 3$. The frozen moduli are given by [3]

$$S = \frac{\hat{p} \cdot \hat{q} - i \left(\hat{p}^2 \hat{q}^2 - (\hat{p} \cdot \hat{q})^2\right)^{1/2}}{\hat{p}^2} \ , \tag{51}$$

$$T = \frac{\hat{X}^3 - \hat{X}^1}{\hat{X}^0 - \hat{X}^2} = \frac{\bar{S}(\hat{p}^3 - \hat{p}^1) - (\hat{q}^3 - \hat{q}^1)}{\bar{S}(\hat{p}^0 - \hat{p}^2) - (\hat{q}^0 - \hat{q}^2)} \ , \tag{52}$$

$$U = \frac{-\hat{X}^3 - \hat{X}^1}{\hat{X}^0 - \hat{X}^2} = \frac{\bar{S}(-\hat{p}^3 - \hat{p}^1) - (-\hat{q}^3 - \hat{q}^1)}{\bar{S}(\hat{p}^0 - \hat{p}^2) - (\hat{q}^0 - \hat{q}^2)} \ , \tag{53}$$

and the mass/area formula is

$$Z\bar{Z} = M^2 = \frac{A}{4\pi} = \left(\hat{p}^2 \hat{q}^2 - (\hat{p} \cdot \hat{q})^2\right)^{1/2} \ . \tag{54}$$

The symplectic transformation between the theory without the prepotential ("hatted" version) to the one with the prepotential ("unhatted" version) is [6]:

$$Sp(8, \mathbf{Z}) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \tag{55}$$

with

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{56}$$

Starting with the prepotential $F = STU$ in terms of special coordinates we have the holomorphic section:

$$X^\Lambda = \begin{pmatrix} 1 \\ S \\ T \\ U \end{pmatrix}, \qquad F_\Lambda = \begin{pmatrix} -STU \\ TU \\ SU \\ ST \end{pmatrix}. \tag{57}$$

After symplectic transformation defined in (6), (56) we get for hatted sections:

$$\hat{X}^\Lambda = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - TU \\ -(T + U) \\ -(1 + TU) \\ T - U \end{pmatrix}, \qquad \hat{F}_\Lambda = \begin{pmatrix} S\hat{X}^0 \\ S\hat{X}^1 \\ -S\hat{X}^2 \\ -S\hat{X}^3 \end{pmatrix} = S\eta_{\Lambda\Sigma} \hat{X}^\Sigma \ , \tag{58}$$

where the metric is $\eta_{\Lambda\Sigma} = (+ + --)$. This theory does not admit the prepotential [6].

We can now relate the known results of the version without a prepotential (for which we use variables with a hat) to the ones obtained here. From eqs. (6), (56) we find the transformation between $\hat{p}, \hat{q}$ and p, q to be:

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[3] We are choosing the negative sign for the imaginary part of S here for the sake of the dual rotation to the prepotential version, using the symplectic matrix (56).

$$\begin{bmatrix} \hat{p}^\Lambda \\ \hat{q}_\Lambda \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} p^0 - q_1 \\ -p^2 - p^3 \\ -p^0 - q_1 \\ p^2 - p^3 \\ p^1 + q_0 \\ -q_2 - q_3 \\ p^1 - q_0 \\ q_2 - q_3 \end{bmatrix} \tag{59}$$

This transformation gives us the relations:

$$\hat{p}^2 = 2(p^2 p^3 - p^0 q_1) = 2\omega_1 \ , \tag{60}$$

$$\hat{p} \cdot \hat{q} = p \cdot q - 2p^1 q_1 \ , \tag{61}$$

hence we find

$$S = z_1 \ , \tag{62}$$

where S is the first moduli field of the version without the prepotential, and (with a little more work) we have

$$\hat{p}^2 \hat{q}^2 - (\hat{p} \cdot \hat{q})^2 = W \ . \tag{63}$$

For T and U we get, using the relation between charges (59)

$$T = \frac{\bar{S}(\hat{p}^3 - \hat{p}^1) - (\hat{q}^3 - \hat{q}^1)}{\bar{S}(\hat{p}^0 - \hat{p}^2) - (\hat{q}^0 - \hat{q}^2)} = \frac{\bar{S}p^2 - q_3}{\bar{S}p^0 - p^1} = z_2 \ , \tag{64}$$

$$U = \frac{\bar{S}(-\hat{p}^3 - \hat{p}^1) - (-\hat{q}^3 - \hat{q}^1)}{\bar{S}(\hat{p}^0 - \hat{p}^2) - (\hat{q}^0 - \hat{q}^2)} = \frac{\bar{S}p^3 - q_2}{\bar{S}p^0 - p^1} = z_3 \ . \tag{65}$$

V. STRING TRIALITY AND STU BLACK HOLES

Our results allow for a comparison with the string triality picture as described in [13]. There, a six-dimensional string, described by the low energy action

$$I_6 = \frac{1}{2\kappa^2} \int d^6 x \sqrt{-G} e^{-\Phi} \left[R_G + G^{MN} \partial_M \Phi \partial_N \Phi - \frac{1}{12} G^{MQ} G^{NR} G^{PS} H_{MNP} H_{QRS} \right] \tag{66}$$

with $M, N = 0, .., 5$ was considered. This string might be a truncated version of a heterotic or a Type II string. Upon toroidal compactification to $D = 4$ one obtains a $N = 2$ supergravity theory coupled to three vector multiplets. The four-dimensional metric is related to the six-dimensional one by

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + A_\mu^m A_\nu^n G_{mn} & A_\mu^m G_{mn} \\ A_\nu^n G_{mn} & G_{mn} \end{pmatrix} \ , \tag{67}$$

where the space-time indices are $\mu, \nu = 0, 1, 2, 3$ and the internal indices are $m, n = 1, 2$. Two more vectors arise from the reduction of the B field.

One also finds six scalars, four of which are moduli of the 2-torus. We parametrize the internal metric and 2-form as

$$G_{mn} = e^{\eta_3 - \eta_2} \begin{pmatrix} e^{-2\eta_3} + a_3^2 & -a_3 \\ -a_3 & 1 \end{pmatrix} , \tag{68}$$

and

$$B_{mn} = a_2 \, \epsilon_{mn} \ . \tag{69}$$

η_1, the four-dimensional dilaton, is given by

$$e^{-\eta_1} = e^{-\Phi} \sqrt{\det G_{mn}} = e^{-(\Phi+\eta_2)} \ . \tag{70}$$

The sixth scalar is the axion a_1 which arises from dualization of the three-form field strength in four dimensions.

The scalars are typically combined into three complex scalars, which in notation suitable for our previous sections are:

$$\begin{aligned}
z_1 &= S = a_1 - ie^{-\eta_1} \ , \\
z_2 &= T = a_2 - ie^{-\eta_2} \ , \\
z_3 &= U = a_3 - ie^{-\eta_3} \ .
\end{aligned} \tag{71}$$

Here S is a dilaton-axion of the heterotic string, T and U are the Kähler form and the complex structure of the torus. These three scalars are obviously the ones considered so far in this paper. The four vectors are combined to a vector A_μ^a with $a = 1, 2, 3, 4$. Details can be found, e.g. , in [13]. In fact, the electric and magnetic charges can be put together to an $SP(8)$ vector as given in the earlier chapters.

The symmetry of this theory is $SL(2, Z) \times \frac{O(2,2,Z)}{O(2) \times O(2)}$. The $SL(2, Z)$ component is the famous S-duality, a conjectured non-perturbative symmetry of string theory. The second factor, which is just a product of two $SL(2, Z)$ plus their exchange, is related to perturbative T-duality symmetry. In the following, T-duality will denote the duality symmetry generated by the first $SL(2, Z)$, which acts on the Kähler form, whereas the second one is called U-duality and acts on the complex structure U. All three symmetries act on the scalars by

$$z_i \to \frac{a_i z_i + b_i}{c_i z_i + d_i} \tag{72}$$

with $a_i d_i - b_i c_i = 1$. The electric and magnetic charges transform as vectors under the three duality symmetries (where the $SP(8)$ vector has to be converted into an $SL(2)^3$ vector [13]).

This theory is precisely the one studied in [8]. In [13] it was found that the theory allows two (or five, according to taste) dual descriptions where the roles of S, T and U get interchanged. For example, S is the dilaton/axion field for the heterotic string, the Kähler form for the Type IIA string and the complex structure of the Type IIB string. However, all those theories were of the same type, in the sense that (at least in the truncated versions considered here) two symmetries were perturbative and one was non-perturbative.

This is easily seen by considering (for example) the four-dimensional heterotic Lagrangian (in the absence of axionic fields):

$$\mathcal{L} = \sqrt{-g} \left[R - \frac{1}{2} \sum (\partial \eta_i)^2 - \frac{1}{4} \left(e^{-\eta_1-\eta_2-\eta_3} F^1 F^1 + e^{-\eta_1-\eta_2+\eta_3} F^2 F^2 + e^{-\eta_1+\eta_2+\eta_3} F^3 F^3 + e^{-\eta_1+\eta_2-\eta_3} F^4 F^4 \right) \right]. \tag{73}$$

Clearly, $\eta_2 \to -\eta_2$ and $\eta_3 \to -\eta_3$ (accompanied by an exchange of a few field strengths) are off-shell symmetries, where as $\eta_1 \to -\eta_1$ requires dualizations of field strengths.

How does the STU-model considered in this paper tie in with those three (or six) string theories? Obviously, it cannot correspond to either of those, since it treats S, T and U on equal footing. This is already clear from the prepotential, but also the action gives some insights. It can be obtained from

$$\mathcal{L}_V = \text{Im}\mathcal{N}_{\Lambda\Sigma} \ \mathcal{F}^\Lambda \mathcal{F}^\Sigma + \text{Re}\mathcal{N}_{\Lambda\Sigma} \ \mathcal{F}^\Lambda \ {}^*\mathcal{F}^\Sigma. \tag{74}$$

We find

$$F_{\Lambda\Sigma} = \begin{pmatrix} 2d_{ijk} z^i z^j z^k & -3d_{mij} z^i z^j \\ -3d_{lij} z^i z^j & 6d_{mli} z^i \end{pmatrix} \ , \tag{75}$$

from which one can deduce

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i \frac{(\text{Im}F_{\Lambda\Omega})(\text{Im}F_{\Pi\Sigma}) X^\Omega X^\Pi}{(\text{Im}F_{\Omega\Pi}) X^\Omega X^\Pi} \ , \tag{76}$$

where $X^\Lambda = (1, z^1, z^2, z^3)$. We do not try to express $\mathcal{N}$ in full generality, however we note that the lower three by three matrix $\mathcal{N}_{ij}$ has the extremely simple form

$$\mathcal{N}_{ij} = \begin{pmatrix} -ie^{+\eta_1-\eta_2-\eta_3} & a_3 & a_2 \\ a_3 & -ie^{-\eta_1+\eta_2-\eta_3} & a_1 \\ a_2 & a_1 & -ie^{-\eta_1-\eta_2+\eta_3} \end{pmatrix} . \tag{77}$$

9

The vector part of the Lagrangian in the absence of any axion-like fields is then given by

$$\mathcal{L}_V = -\left(e^{-\eta_1-\eta_2-\eta_3}\,\mathcal{F}^0\mathcal{F}^0 + e^{+\eta_1-\eta_2-\eta_3}\,\mathcal{F}^1\mathcal{F}^1 + e^{-\eta_1+\eta_2-\eta_3}\,\mathcal{F}^2\mathcal{F}^2 + e^{-\eta_1-\eta_2+\eta_3}\,\mathcal{F}^3\mathcal{F}^3\right)\Big]. \tag{78}$$

Note that this Lagrangian has perfect exchange $(1 \leftrightarrow 2 \leftrightarrow 3)$ symmetry, but it is not invariant under any $\eta_i \to -\eta_i$ duality transformation (accompanied by the appropriate exchange of vector fields). Hence, the theory has neither S, T nor U duality (in the notation of [13]) realized off-shell!

We can also compare how S, T and U dualities (72) are realized as $SP(8)$ matrices. For this we go first in a basis $(S|TU)$ (heterotic string compactified on a two-torus). This can be done by the symplectic rotation

$$\mathcal{C}: \qquad (p^\Lambda, q_\Lambda) \to (\tilde{p}^\Lambda, \tilde{q}_\Lambda) = (p^0, -q_1, p^2, p^3 \,|\, q_0, p^1, q_2, q_3). \tag{79}$$

In this new basis the $SL(2,Z)$ transformations are realized by a matrix (55) with [18]

$$S \to \frac{aS+b}{cS+d} \;:\; \text{by}\;\; a\,A = d\,D = a\,d\,\delta_{ij}\,; \quad b\,B = c\,C = b\,c \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{80}$$

$$T \to \frac{aT+b}{cT+d} \;:\; \text{by}\;\; B = C = 0\,, \quad A = (D^T)^{-1} = \begin{pmatrix} d & 0 & c & 0 \\ 0 & a & 0 & -b \\ b & 0 & a & 0 \\ 0 & -c & 0 & d \end{pmatrix}, \tag{81}$$

$$U \to \frac{aU+b}{cU+d} \;:\; \text{by}\;\; B = C = 0\,, \quad A = (D^T)^{-1} = \begin{pmatrix} d & 0 & 0 & c \\ 0 & a & -b & 0 \\ 0 & -c & d & 0 \\ b & 0 & 0 & a \end{pmatrix}. \tag{82}$$

As we can see, in this basis only the S-duality is non-perturbative (exchange of electric with magnetic charges) whereas the T-and U-duality acts diagonal, i.e. exchange electric with electric and magnetic with magnetic quantum numbers. Finally, to get the transformation for our original charges we have to invert the transformation $\mathcal{C}$. Combining all symplectic transformations we find for our charges in the STU basis the transformations

$$\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \to \begin{pmatrix} d\,p^0 + c\,p^1 \\ b\,p^0 + a\,p^1 \\ d\,p^2 + c\,q_3 \\ d\,p^3 + c\,q_2 \\ a\,q_0 - b\,q_1 \\ -c\,q_0 + d\,q_1 \\ b\,p^3 + a\,q_2 \\ b\,p^2 + a\,q_3 \end{pmatrix}_{SL(2)_S} ; \quad \begin{pmatrix} d\,p^0 + c\,p^2 \\ d\,p^1 + c\,q_3 \\ b\,p^0 + a\,p^2 \\ d\,p^3 + c\,q_1 \\ a\,q_0 - b\,q_2 \\ b\,p^3 + a\,q_1 \\ -c\,q_0 + d\,q_2 \\ b\,p^1 + a\,q_3 \end{pmatrix}_{SL(2)_T} ; \quad \begin{pmatrix} d\,p^0 + c\,p^3 \\ d\,p^1 + c\,q_2 \\ d\,p^2 + c\,q_1 \\ b\,p^0 + a\,p^3 \\ a\,q_0 - b\,q_3 \\ b\,p^2 + a\,q_1 \\ b\,q_1 + a\,q_2 \\ -c\,q_0 + d\,q_3 \end{pmatrix}_{SL(2)_U} . \tag{83}$$

As expected, one finds that W as given in eq. (36) or the area of the horizon (mass) is invariant under these transformations (or $(SL(2))^3$ symmetric).

Let us turn back to the Lagrangians (73) and (78). Both lagrangians are on-shell equivalent, (73) corresponds to the $(S|TU)$ basis whereas (78) is our (STU) basis and the transformation $\mathcal{C}$ maps both. This transformation is a dualization of $\mathcal{F}_1$ and a renaming $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \to 1/2(F_1, F_3, F_4, F_2)$. Note, that under dualization the pre-factor of $\mathcal{F}_1$ gets inverted. This dualization makes all duality symmetries in the (STU) basis non-perturbative. The statement that in this symmetric basis none of the dualities is perturbative is equivalent to the statement that there is no basis in which all of dualities are perturbative. The most we can achieve is to make only one non-perturbative and two perturbative. We have considered the case where the S-transformation is non-perturbative. On equal footing we could take T or U. These three possibilities fix then the three underlying string theories (heterotic, type IIa or type IIb). When all three theories are symmetrised by going to the STU basis, immediately all dualities become non-perturbative.

These transformations can be nicely visualized in the form of a cube, as it was done in [13]. Figure 1 shows how the dualities transform the field strengths into each other and their duals in the (STU) model. Figure 1 and Figure 2 also illustrate the crucial difference with the $(S|TU)$-, $(T|US)$- and $(U|ST)$-models of [13] (Figure 2). In the STU theory, the fundamental field strengths (or electric charges) are located around $\mathcal{F}^0$, whereas in the $(S|TU)$, $(T|US)$ and $(U|ST)$ models the fundamental fields were located on one side of the cube, allowing 2 dualities to be perturbative.

The black holes with vanishing axions and finite scalars all have four charges. These charges must be located on four corners of the cube which are NOT connected by edges. Hence, the choices one has are q_0, p^1, p^2, p^3 (with product of 4 charges $q_0 p^1 p^2 p^3$ positive) and p^0, q_1, q_2, q_3 (with product of 4 charges $p^0 q_1 q_2 q_3$ negative), which is consistent with our results. In the S-, T-, or U- string picture one always needed two electric and two magnetic charges.
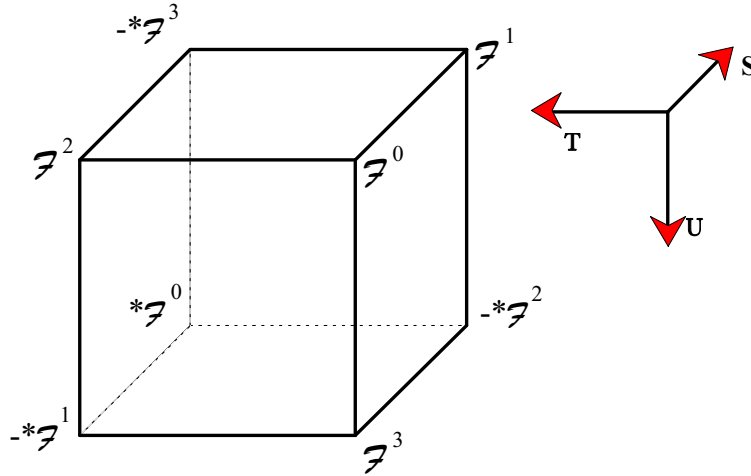

FIG. 1. Duality transformations in the STU-model. The fundamental field strengths are not located on one side.
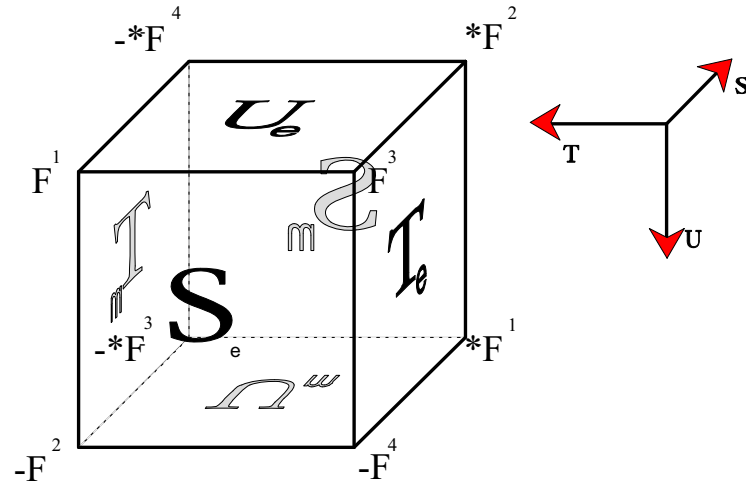

FIG. 2. Duality transformations in the S,T and U strings. The fundamental field strengths are located on one side and two duality symmetries are perturbative. The field strengths have different indices from Figure one, because the ones here are fundamental S-string fields.

VI. OUTLOOK

In conclusion, we have found a new type of d=4 supersymmetric black holes in the context of N=2 special geometry related to Calabi-Yau threefold. The main difference with the existing supersymmetric black holes is a completely *democratic treatment of all moduli of the theory.* This is due to our use of the version of special geometry with the prepotential $F = d_{ABC} \frac{X^A X^B X^C}{X^0}$ [3] where d_{ABC} are real symmetric constant tensors. A particular model of this type with $F = STU$ gives no preference to any of the moduli and therefore none of them can play a role of coupling constants. This makes the new (STU) black holes different from stringy $(S|TU)$-, $(T|US)$- and $(U|ST)$ black holes [8], [13] where one of the moduli (S in heterotic case, T in type IIa case and U in type IIb case) does play the role of the coupling constant.

One may try to relate our new d=4 black holes to d=5 supersymmetric black holes described in [1]. The area formula found there depends on symmetric tensor d_{ABC} as follows.

$$Z_{\text{fix}} = \sqrt{\left(d^{AB}(q)\right)^{-1} q_A q_B} \ , \qquad A \sim \left[\left(d^{AB}(q)\right)^{-1} q_A q_B\right]^{3/4} \ , \tag{84}$$

where $\left(d^{AB}(q)\right)^{-1} = \left(d^{AB}(t(z))|_{\partial_i Z=0}\right)^{-1}$ and $(d^{AB})^{-1}$ is the inverse of $d_{ABC}t^C$. Equation (84) applies in particular to eleven dimensional supergravity compactified on Calabi-Yau threefold.

The new result found in this paper for four-dimensional black holes is the value of the moduli t^A as the function of charges at the fixed point $\partial_i Z = 0$ for particular example of d_{ABC}. It can be used to find also the area of the five-dimensional black holes as the function of charges for this theory.

It remains to be seen if it is possible to address the issue of quantum corrections in string theory using extreme black holes of classical moduli spaces as the starting point. In this paper we have established a duality relation between stringy $(S|TU)$-, $(T|US)$- and $(U|ST)$ black holes and "democratic" STU black holes. Stringy black holes were known to be related to each other by the so-called triality in such a way that only one of S, T, U-dualities was non-perturbative [13]. The "democratic" black holes give us some new insights into the spectrum of states of the fundamental theory: all dualities there are non-perturbative.

VII. ACKNOWLEDGMENTS

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