

# CALABI-YAU BLACK HOLES

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## ABSTRACT

We have found the entropy of N=2 extreme black holes associated with general Calabi-Yau moduli space and prepotential  $F = d_{ABC} \frac{X^A X^B X^C}{X^0}$ . We show that for arbitrary  $d_{ABC}$  and black hole charges  $p$  and  $q$  the entropy-area formula depends on combinations of these charges and parameters  $d_{ABC}$ . These combinations are the solutions of the simple system of algebraic equations. We gave a few examples of particular Calabi-Yau moduli space for which this system has an explicit solution. For special case when one of black hole charges is equal to zero ( $p^0 = 0$ ) the solution always exists.

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# 1 Introduction

Recently supersymmetric black holes become a “natural” theoretical laboratory for string theory. Calculation of black hole entropy by counting microscopic configurations [1, 2, 3] and comparison of these results with one obtained by classical macroscopic calculations [4, 5] is a very powerful tool in this “laboratory”. In this paper we will focus on macroscopic calculations of the black hole entropy as the area of the black hole horizon. We will give a solution for N=2 extreme black holes associated with the general Calabi-Yau moduli space.

One of the most important properties of N=2 supersymmetric black holes in the general version of ungauged N=2 supergravity interacting with arbitrary number  $n_v$  of vector multiplets [6] has been established recently [5]. It was shown that the area of the extreme black hole horizon and moduli of vector multiplets near the horizon are functions of charges only.

$$A = A(z^i(p, q, d_{ijk}), p, q, d_{ijk}), \quad (1)$$

where  $A$  is an area of the black hole horizon,  $z^i(p, q, d_{ijk})$  are moduli fields,  $p$  and  $q$  are electrical and magnetic charges of black hole and  $d_{ijk}$  is a characteristic of Calabi-Yau (classical intersection numbers). Relation between charges and moduli for supersymmetric black holes near the horizon has the following form [4]:

$$\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \text{Re} \begin{pmatrix} 2i\bar{Z}L^\Lambda \\ 2i\bar{Z}M_\Lambda \end{pmatrix}, \quad (2)$$

where  $Z$  is the central charge and  $(L^\Lambda, M_\Lambda)$  are covariantly holomorphic sections.

Classical Calabi-Yau moduli space is described by the prepotential of the form:

$$F = d_{ABC} \frac{X^A X^B X^C}{X^0} + \text{corrections}. \quad (3)$$

Here  $d_{ABC}$  are the topological intersection numbers  $d_{ABC} \equiv \int J_A \wedge J_B \wedge J_C$  and  $J_A$  Kähler cone generators,  $J_A \in H^{1,1}(Y, \mathbf{Z})$ . Special coordinates  $z^A$  of Kähler “moduli space” are connected with  $X^A$  by:

$$z^A = \frac{X^A}{X^0}; \quad X^0 = 1; \quad A = 1, 2, \dots, \quad \Lambda = 0, 1, \dots \quad (4)$$

According to [7, 8, 9] this form of the Calabi-Yau prepotential can be extended by addition of an extra topological term determined by second Chern class  $c_2$  to the prepotential:

$$F_1(z) = \sum_A^h \frac{c_2 \cdot J_A}{24} z^A. \quad (5)$$

Symplectic invariant form of the Kähler potential is  $K = -\ln i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)$ , where

$$L^\Lambda = e^{K/2} X^\Lambda, \quad M_\Lambda = e^{K/2} F_\Lambda. \quad (6)$$

In terms of special coordinates the Kähler potential is:

$$K(z, \bar{z}) = -\log(-id_{ABC}(z - \bar{z})^A(z - \bar{z})^B(z - \bar{z})^C). \quad (7)$$

The central charge  $Z$  has a form [6]:

$$Z = e^{\frac{K(z, \bar{z})}{2}} (X^\Lambda q_\Lambda - F_\Lambda p^\Lambda). \quad (8)$$

In [9] it was proposed to introduce new variables:

$$Y^\Lambda \equiv \bar{Z} X^\Lambda. \quad (9)$$

Stabilization equations (2) in terms of these new variables become:

$$\begin{aligned} ip^\Lambda &= Y^\Lambda - \bar{Y}^\Lambda \\ iq_\Lambda &= F_\Lambda(Y) - \bar{F}_\Lambda(\bar{Y}). \end{aligned} \quad (10)$$

Mass of the double-extreme black hole and entropy then has a form:

$$\frac{S}{\pi} = M_{BPS}^2 = |Z|^2 = |Y^0|^2 \exp(-K(z, \bar{z})). \quad (11)$$

In this paper we will show the solution of equations (10) for most general form of  $d_{ABC}$  and arbitrary  $p$  and  $q$  charges. In Section 2 we will derive the expression for black hole entropy in terms of  $d_{ABC}$  and some combinations of  $p$  and  $q$ . In Section 3 solutions for moduli fields will be given and in Section 4 some examples of explicit solutions for entropy for particular Calabi-Yau intersection numbers will be considered.

## 2 Entropy of Calabi-Yau black holes.

Stabilization equations (10) and the entropy formula (11) lead to the general result for entropy of Calabi-Yau black hole with prepotential (3). For the most general case with  $p^0 \neq 0, q_0 \neq 0$  and arbitrary charges  $(p^A, q_A)$  the entropy of the black hole depends only on these charges and numbers  $d_{ABC}$ :

$$\frac{S}{\pi} = \frac{1}{3p^0} \sqrt{\frac{4}{3} (\Delta_A \tilde{x}^A)^2 - 9(p^0(p \cdot q) - 2D)^2}. \quad (12)$$

Here  $\Delta_A = 3D_A - p^0 q_A$ , and  $D_A = d_{ABC} p^B p^C$ ,  $D = d_{ABC} p^A p^B p^C$ ,  $p \cdot q = p^0 q_0 + p^A q_A$ . In combination

$$(\Delta_A \tilde{x}^A)^2 = ((3D_A - p^0 q_A) \tilde{x}^A)^2 \quad (13)$$

the variables  $\tilde{x}^A$  are the *real* solutions of algebraic system:

$$d_{ABI} \tilde{x}^A \tilde{x}^B = \Delta_I. \quad (14)$$

which is determined by intersection numbers of the Calabi-Yau manifold and black hole charges. This system might have no analytical solution in general case. The number of equations in (14)

is equal to the number of moduli fields  $n_v$ . Left-hand parts of each equations are quadratic forms with coefficients  $d_{AB(I)}$  and right-hand parts are arbitrary and depend on values of electric and magnetic charges of the black hole. In other words, we have  $n_v$  2-dimensional surfaces embedded in  $n_v$ -dimensional space and intersection of these surfaces is a solution of this system. It is not clear yet if there is any connection between the geometry of Calabi-Yau manifolds and the geometry of this picture. We considered a few examples of Calabi-Yau manifolds with different sets of  $d_{ABC}$ . For each of them we found the expression for the black hole entropy. These examples will be considered in part 4 of this paper.

Solution of equations (10) becomes much simpler if  $p^0 = 0$ . In this case we have the system of linear ( instead of quadratic ) equations and there is a general solution for moduli fields in terms of black hole charges. The entropy of the black hole is then given by the formula:

$$\frac{S}{\pi} = \sqrt{\frac{D}{3}(q_B D^B + 12q_0)} . \quad (15)$$

Here  $D_{AB} = d_{ABC} p^C$ ,  $D^{AB} = [D_{AB}]^{-1}$ , and  $D^A = D^{AB} q_B$ .

It is straightforward to generalize these results to the prepotential with additional topological term  $F_1 = \frac{c_2 J_A}{24} z^A$  .

$$F(X) = d_{ABC} z^A z^B z^C + \sum_A^h \frac{c_2 \cdot J_A}{24} z^A \quad (16)$$

In [9] it was shown that the addition of this topological term leads to simple transformation of charges  $q_\Sigma$ :

$$\begin{aligned} \tilde{q}_\Sigma &= q_\Sigma - W_{\Sigma\Lambda} p^\Sigma \\ W_{0A} &= \frac{c_2 J_A}{24}. \end{aligned} \quad (17)$$

Therefore in our expressions for entropy (12) and (15) we should use these new  $\tilde{q}_\Sigma$  connected with old charges by the equations:

$$\begin{aligned} \tilde{q}_0 &= q_0 - \frac{c_2 J_A}{24} p^A \\ \tilde{q}_A &= q_A - \frac{c_2 J_A}{24} p^0. \end{aligned} \quad (18)$$

### 3 Fixed Moduli solutions

Stabilization equations (10) define values of the moduli fields near the black hole horizon through the electric and magnetic charges of the black hole. From "  $p^\Lambda$  " equations (10) it follows immediately that  $Im Y^\Lambda = \frac{p^\Lambda}{2}$ . Therefore the second part of (10), "  $q_\Lambda$  " -equations, is a system of

equations on  $ReY^\Lambda$ . In new variables:

$$x^A = ReY^A - \frac{p^A ReY^0}{p^0}; \quad x^0 = ReY^0 \quad (19)$$

this system has a form:

$$2p^0 ReY^0 \Delta_C x^C = (p^0(p \cdot q) - 2D)(|Y^0|^2)^2 \quad (20)$$

$$d_{ABI} x^A x^B = \frac{\Delta_I}{3p^{02}} |Y^0|^2. \quad (21)$$

where  $|Y^0|^2 = (ReY^0)^2 + \frac{p^{02}}{4}$ . Solution of equations (20) and (21) can be expressed in terms of variables  $\tilde{x}^A$  from (14), in that case :

$$x^A = \tilde{x}^A \sqrt{\frac{|Y^0|^2}{3p^{02}}} \quad (22)$$

and

$$|Y^0|^2 = \frac{p^{02}}{3} \frac{(\Delta_C \tilde{x}^C)^2}{\frac{4}{3}(\Delta_C \tilde{x}^C)^2 - 9(p^0(p \cdot q) - 2D)^2}. \quad (23)$$

Finally the moduli fields  $z^A$  are:

$$z^A = \frac{3}{2} \frac{\tilde{x}^A}{p^0(\Delta_C \tilde{x}^C)} (p^0(p \cdot q) - 2D) + \frac{p^A}{p^0} - i \frac{3}{2} \frac{\tilde{x}^A}{(\Delta_C \tilde{x}^C)} \frac{S}{\pi} \quad (24)$$

where the entropy  $S/\pi$  is given by (12) and  $\tilde{x}^A$  are solutions of algebraic system:

$$d_{ABI} \tilde{x}^A \tilde{x}^B = \Delta_I. \quad (25)$$

In case  $p = 0$  equations the system (10) becomes very simple:

$$\begin{aligned} 6D_{AI} ReY^A &= q_I ReY^0 \\ 3D_{AB} ReY^A ReY^B - \frac{D}{4} &= -q_0 (ReY^0)^2 \end{aligned} \quad (26)$$

and the solution for moduli is:

$$z^A = \frac{D^A}{6} - i \frac{p^A}{2} \sqrt{\frac{q_B D^B + 12q_0}{3D}} = \frac{D^A}{6} - i \frac{p^A}{2} \frac{DS}{\pi} \quad (27)$$

were  $S/\pi$  is given by (15).

## 4 Examples.

We will present below some examples of the entropy formulas for particular CY manifolds.

In case of two moduli  $n_v = 2$  the algebraic system (14) has a general solution. Expression  $(\Delta_C \tilde{x}^C)^2$  in entropy formula (12) has the form:

$$\begin{aligned} (\Delta_C \tilde{x}^C)^2 &= \left( -2\Delta_A \Delta_B \Delta_C d_A^{BC} \det d_1 \det d_2 \right. \\ &+ (\det \hat{d}_{123})(\Delta_1 \Delta_A \Delta_B d_2^{AB} \det d_2 + \Delta_2 \Delta_A \Delta_B d_1^{AB} \det d_1) \\ &+ 2(-\Delta_1^2 \det d_2 - \Delta_2^2 \det d_1 + \Delta_1 \Delta_2 (\det \hat{d}_{123}))^{3/2} \left. \cdot ((\det \hat{d}_{123}) - 4\Delta_1 \Delta_2)^{-1} \right) \end{aligned} \quad (28)$$

were  $(\det \hat{d}_{123}) = d_{111}d_{222} - d_{112}d_{122}$  and

$$\det d_1 = \det(d_{1AB}) \neq 0, \quad \det d_2 = \det(d_{2AB}) \neq 0, \quad (29)$$

$$d_1^{AB} = [d_{1AB}]^{-1}, \quad d_2^{AB} = [d_{2AB}]^{-1}. \quad (30)$$

Even if these conditions are not satisfied and there is no inverse matrix  $d_1^{AB}$  or  $d_2^{AB}$  a solution can still exist. One of particular choices of Calabi-Yau intersection numbers with  $n_v = 2$  for which condition (29) does not work was given in [7]. In that case:

$$d_{111} = 8/6; \quad d_{112} = 4/6; \quad d_{122} = d_{222} = 0. \quad (31)$$

Solution of (14) exists and expression (13) has a form:

$$(\Delta_C \tilde{x}^C)^2 = 3/8 \Delta_2 (3\Delta_1 - 2\Delta_2)^2 \quad (32)$$

so that black hole entropy is:

$$\frac{S}{\pi} = \frac{1}{3p^0} \sqrt{\frac{\Delta_2}{2} (3\Delta_1 - 2\Delta_2)^2 - 9(p^0(p \cdot q) - 2D)^2}. \quad (33)$$

One of examples of model with  $n_v = 3$  is STU model studied in [10] with prepotential  $F(X) = d_{ABC} z^A z^B z^C$  and  $d_{ABC} = d_{123} = 1/6$ . Expressions for entropy and for moduli fields were already found in [11]. Here we show these expressions in terms of solutions of algebraic system (14) and  $\Delta_A$ :

$$(\Delta_C \tilde{x}^C)^2 = \frac{9}{2} \frac{\Delta_1 \Delta_2 \Delta_3}{d_{123}}. \quad (34)$$

and entropy (12) is

$$\frac{S}{\pi} = \frac{1}{p^0} \sqrt{\frac{2}{3d_{123}} \Delta_1 \Delta_2 \Delta_3 - (p^0(p \cdot q) - 2D)^2} \quad (35)$$

this expression coincides with expression for entropy of STU black holes in [11]. Expression for moduli is:

$$z^A = \frac{3(p^0(p \cdot q) - 2D) + 6p^A \Delta_A}{2p^0 \Delta_A} - i \frac{S}{2\pi \Delta_A} \quad (\text{no summation on } A) \quad (36)$$

which also coincides with expression for moduli from [11].

Some interesting examples of  $n_v = 3$  Calabi-Yau manifolds were given in [8]. Those examples of classical prepotentials correspond to Type II vacuum. Each of prepotentials are related with each other and with  $F_{II}^0 = STU + \frac{1}{3}U^3$  prepotential by linear transformations:

$$F_{II}^1 = \frac{4}{3}(z^1)^3 + (z^1)^2 z^2 + (z^1)^2 z^3 + z^1 z^2 z^3 \quad (37)$$

$$(z^1 = U, z^2 = T - U, z^3 = S - U); \quad (38)$$

$$F_{II}^2 = \frac{4}{3}(z^1)^3 + 2(z^1)^2 z^2 + z^1(z^2)^2 + (z^1)^2 z^3 + z^1 z^2 z^3, \quad (39)$$

$$(z^1 = U, z^2 = T - U, z^3 = S - T); \quad (40)$$

$$F_{II}^3 = \frac{4}{3}(z^1)^3 + \frac{2}{3}(z^1)^2 z^2 + \frac{1}{2}z^1(z^2)^2 + (z^1)^2 z^3 + z^1 z^2 z^3, \quad (41)$$

$$(z^1 = U, z^2 = T - U, z^3 = S - 1/2T - 1/2U). \quad (42)$$

Solution of algebraic equation (14) exists for each of these prepotentials and the entropy for  $F_{II}^0$  is:

$$\frac{S}{\pi} = \frac{1}{3p^0} \sqrt{\frac{4}{3} \left( \frac{3}{2} \Delta_3 ((\Delta_3)^2 + 12\Delta_1 \Delta_2) + \frac{3}{2} (W_0)^{\frac{3}{2}} \right) - 9(p^0(p \cdot q) - 2D)^2} \quad (43)$$

$$W_0 = ((\Delta_3)^2 - 4\Delta_1 \Delta_2), \quad (44)$$

expressions  $(\Delta_C \tilde{x}^C)^2$  for  $F_{II}^1$ ,  $F_{II}^2$  and  $F_{II}^3$  are:

$$(\Delta_C \tilde{x}^C)_{F_1}^2 = \frac{3}{2} (\Delta_1 - \Delta_2 - \Delta_3) ((\Delta_1 - \Delta_2 - \Delta_3)^2 + 12\Delta_2 \Delta_3) + \frac{3}{2} (W_1)^{\frac{3}{2}},$$

$$W_1 = ((\Delta_1 - \Delta_2 - \Delta_3)^2 - 4\Delta_2 \Delta_3). \quad (45)$$

$$(\Delta_C \tilde{x}^C)_{F_2}^2 = \frac{3}{2} (\Delta_1 - \Delta_2) ((\Delta_1 - \Delta_2)^2 + 12\Delta_3 (\Delta_2 - \Delta_3)) + \frac{3}{2} (W_2)^{\frac{3}{2}},$$

$$W_2 = ((\Delta_1 - \Delta_2)^2 - 4\Delta_3 (\Delta_2 - \Delta_3)). \quad (46)$$

$$(\Delta_C \tilde{x}^C)_{F_3}^2 = \frac{3}{2} (\Delta_1 - \Delta_2 - \frac{1}{2}\Delta_3) ((\Delta_1 - \Delta_2 - \frac{1}{2}\Delta_3)^2 + 12\Delta_3 (\Delta_2 - \frac{1}{2}\Delta_3))$$

$$+ \frac{3}{2}(W_3)^{\frac{3}{2}},$$

$$W_3 = \left( (\Delta_1 - \Delta_2 - \frac{1}{2}\Delta_3)^2 - 4\Delta_3(\Delta_2 - \frac{1}{2}\Delta_3) \right). \quad (47)$$

Solutions (45), (46) and (47) are connected with each other and with  $(\Delta_C \tilde{x}^C)^2$  solution for  $F_{II}^0$  by linear transformations of  $\Delta_A$ . For example, connection between (45) and (46) is :  $\{\Delta_1, \Delta_2, \Delta_3\} \rightarrow \{\Delta_1, (\Delta_2 - \Delta_3), \Delta_3\}$  and between (45) and (47) is :  $\{\Delta_1, \Delta_2, \Delta_3\} \rightarrow \{\Delta_1, (\Delta_2 - \frac{1}{2}\Delta_3), \Delta_3\}$ .

The last example of Calabi-Yau manifolds that we are going to consider here was again given in [8]. This model is connected with  $F = \frac{3}{8}U^3 + \frac{1}{2}UT^2 - \frac{1}{6}V_X^3$  by linear transformation:

$$z^1 = V_X, \quad z^2 = U - V_X, \quad z^3 = T - \frac{3}{2}U. \quad (48)$$

Classical prepotential in this case is:

$$\begin{aligned} F &= \frac{4}{3}(z^1)^3 + \frac{3}{2}(z^2)^3 + \frac{9}{2}(z^1)^2 z^2 + \frac{9}{2}z^1(z^2)^2 + \frac{3}{2}(z^1)^2 z^3 \\ &+ \frac{3}{2}(z^2)^2 z^3 + \frac{1}{2}z^1(z^3)^2 + \frac{1}{2}z^2(z^3)^2 + 3z^1 z^2 z^3. \end{aligned} \quad (49)$$

the solution for (12) is:

$$\frac{S}{\pi} = \frac{1}{3p^0} \sqrt{\frac{4}{3} \left( \sqrt{6(\Delta_2 - \Delta_1)^3} + \sqrt{\frac{2}{3}(\Delta_2 - 3\Delta_3)^3} - \sqrt{\frac{2}{3}(\Delta_2)^3} \right)^2 - 9(p^0(p \cdot q) - 2D)^2} \quad (50)$$

We have suppressed here the stabilized values of moduli fields although they are available for each case. The expressions for entropy of black holes are rather complicated, but they could be compared with entropy obtained by counting of microscopic degrees of freedom ( stringy microstates, "D-branes").

In this paper we have found the entropy-area formula of N=2 extreme black hole for most general case of Calabi-Yau moduli space. We have shown that the entropy and the moduli fields in general case depend on the solution of algebraic system of quadratic equations (14). We have found the entropy of the black hole for some particular examples of Calabi-Yau moduli space. Although it is not obvious that the existence of the solutions can be regarded as general property of Calabi-Yau prepotentials we hope that there is a connection between them. It will be very interesting to find such a connection.

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