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Using beam echo to recover transverse emittance

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Abstract. If the beam is injected into the ring with an offset a, it undergoes betatron oscillation. After the oscillation decoheres, the beam transverse emittance increases by $\Delta\varepsilon$. To avoid this emittance increase one typically uses a feedback (or damper) that takes out the oscillation before it damps down. We show that using echo one can recover a fraction of $\Delta\varepsilon$ long after the beam oscillation decoheres.

INTRODUCTION

Beam echo has been introduced in accelerator physics in Refs. [1, 2]. The echo effect in a circular accelerator could be observed in a situation where the beam in the ring is deflected (or injected) off-orbit at time $t = 0$, causing its centroid to undergo betatron oscillation. After this oscillation has completely damped out due to a betatron frequency spread in the beam, the beam is excited again by a quadrupole kick at $t = \tau$. This kick does not produce any visible displacement of the beam at that time, but it turns out that close to time $t = \tau_{\text{echo}} = 2\tau$ the beam centroid undergoes transient betatron oscillations with an amplitude that is a fraction of the initial beam offset.

Originally the echo was predicted and calculated for the transverse betatron oscillation of the beam. It was pointed out later that the echo can also be observed in the longitudinal direction [4].

Experimentally, for the first time, the echo was measured in Fermilab on Antiproton Accumulator for a coasting beam [5]. Detailed studies of the longitudinal echo were later performed at CERN [6]. Theory of the longitudinal echo was further advanced in Refs. [6, 7].

The echo effect is based on the fact that the decoherence is not a true damping — it can be somewhat reversed and the initial perturbation of the beam can be partially restored. At the same time, it turns out that the echo is very sensitive to the diffusion in the phase space. In CERN experiments the diffusion coefficient of order of 10^{-13} s^{-1} corresponding to the longitudinal emittance doubling time ~ 100 days!

In this paper we explore another aspect of the time-reversing property of the echo. If the beam is injected with an offset (or the offset arises due to deflection of the beam), the beam emittance increases after the oscillation decoheres. This increase, however, is not fully irreversible, and we will show that part of it can be recovered. Although the fraction of the recovered emittance is relatively small, it is remarkable, that the recovering can be carried out long after the oscillation decoheres (assuming that there are no diffusion effects on the time interval of interest).

ECHO THEORY

Let us assume, for concreteness, that the beam offset a is due to beam deflection that happens at time $t = 0$. The offset is much smaller than the transverse beam size, $a \ll \sigma_0$. For description of particle's motion in the ring, we will use dimensionless variables x and p related to the transverse deviation from the equilibrium orbit X by the following relations:

$$x(s) = \frac{X(s)}{\sqrt{\varepsilon_0\beta(s)}}\,, \qquad p(s) = \beta(s)\frac{dx(s)}{ds}\,, \tag{1}$$

where ε_0 is the beam emittance. The variables x and p are the conjugate variables of Hamiltonian motion; we will also use the action-angle variables J and ϕ: $J = (x^2+p^2)/2$, $\phi = -\text{ArcTan}(p/x)$.

We will assume that, due to nonlinearity of the lattice, the tune is a function of the amplitude of the betatron oscillations, or action, J:

$$\nu = \nu_0 + \Delta\nu J\,, \tag{2}$$

where ν_0 is the linear tune, and $\Delta\nu$ represents the tune spread in the beam.

The beam evolution is described in terms of the distribution function $\psi(x,p,t)$, or equivalently, in angle-phase representation, $\psi(J,\phi,t)$. The dimensionless beam emittance, measured in the units of the equilibrium emittance ε_0, is defined as

$$\varepsilon = \int_0^\infty dJ \int_0^{2\pi} J\psi(J,\phi,t)d\phi\,. \tag{3}$$

The equilibrium distribution function ψ_0 of the beam depends only on the action, and for a Gaussian distribution, $\psi_0(J) = (2\pi)^{-1}\exp(-J)$. The equilibrium beam emittance calculated for this distribution function with the help of Eq. (3) is equal to unity. To simplify the notation we will measure time in units of inverse betatron frequency ω_β^{-1}.

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The echo calculation is carried out in four steps.

1. Beam deflection. It is performed as a dipole kick that translates the beam distribution function along the p axes, $p \to p+\alpha$, where α is the dimensionless offset, $\alpha = a/\sqrt{\varepsilon_0 \beta}$. The kick changes the distribution function from ψ_0 to ψ_1,

$$\psi_1(p,x) = \psi_0(p-\alpha,x)\,. \tag{4}$$

In terms of the action-angle variables, the distribution function after the kick, which we denote ψ_2, is given by the following relation

$$\psi_2(J,\phi) = \psi_1(p(J,\phi),x(J,\phi))\,. \tag{5}$$

Using Eq. (3) it is easy to calculate that for small α the emittance increase of the beam after the dipole kick is

$$\Delta\varepsilon = \frac{\alpha^2}{2}\,. \tag{6}$$

2. Free betatron oscillation during time τ. The oscillation conserves the action and advances the angle, $\phi \to \phi + \tau(\nu_0 + \Delta\nu J)$, and results in the new distribution function ψ_3 at time τ:

$$\psi_3(J,\phi,\tau) = \psi_2(J,\phi - \tau(\nu_0 + \Delta\nu J))\,. \tag{7}$$

Calculation of the average offset of the beam with this distribution function reveals decoherence of the betatron oscillation on the time scale equal to the inverse betatron frequency spread in the beam, $\Delta\nu^{-1}$.

3. Quadrupole kick at time τ. We assume that $\tau\Delta\nu \gg 1$, which means that the quadrupole kick is applied after the initial betatron oscillation decoheres. The kick generates a distribution function ψ_5 which we calculate in two steps: first transforming from $J-\phi$ to $x-p$ variables,

$$\psi_4(x,p,\tau) = \psi_3(J(p,x),\phi(p,x),\tau)\,, \tag{8}$$

and then applying the transformation $p \to p + qx$:

$$\psi_5(p,x,\tau) = \psi_4(p-qx,x,\tau)\,, \tag{9}$$

where q is the dimensionless strength of the kick equal to the ratio of the beta-function at the location of the quad to the focal length of the quadrupole, $q = \beta/F$.

4. The final step is to allow a free betatron oscillation for the time interval $\tau + \Delta t$. It gives the distribution function ψ_{echo} which is calculated via an intermediate function ψ_6

$$\psi_6(J,\phi,\tau) = \psi_5(p(J,\phi),x(J,\phi),\tau)\,, \tag{10}$$

$$\psi_{\mathrm{echo}}(J,\phi,\tau,\Delta t) = \psi_6(J,\phi - (\tau+\Delta t)(\nu_0 + \Delta\nu J),\tau)\,. \tag{11}$$

In the analytical approach we used the following approximations that greatly simplify the analysis:

- We assumed that $\alpha \ll p,x$ and expanded ψ in Taylor series keeping terms of order of α and α^2 only. This assumption means that the initial beam offset is much smaller than the transverse beam size.
- We assumed that the quadrupole kick is weak, $q \ll 1$, and expanded ψ keeping only linear terms in q. Moreover, it turns out that the echo signal is proportional to the product $q\tau$, so we kept only those terms in which q is multiplied by τ.
- We used Fourier decomposition in angle ϕ and kept only terms $\propto e^{in\phi}$ with $n = 0, \pm 1$, because only those terms contribute to the beam offset ($n = \pm 1$) and the emittance ($n = 0$).

With these simplifying assumptions, calculations can be performed with the help of symbolic capabilities of Mathematica [10]; the notebook with the code can be obtained from the author's web site [11]. As a result, the part of the distribution function $\Delta\psi_{\mathrm{echo}}$ that is responsible for the echo signal takes a very simple form:

$$\Delta\psi_{\mathrm{echo}}(J,\phi,\tau,\Delta t) = \tag{12}$$
$$-\quad \frac{a\sqrt{J}}{\sqrt{2\pi}} e^{-J} \sin(\nu_0\Delta t + \Delta\nu_0 J\Delta t - \phi)\mathrm{J}_1\left(Jq\tau\Delta\nu\right)\,,$$

where J_1 is the Bessel function.

RECOVERING EMITTANCE

The idea of recovering beam emittance is based on the application, at some time Δt (measured from the echo time $\tau_{\mathrm{echo}} = 2\tau$) another dipole kick of amplitude ξ and phase ϕ, so that $p \to p + \xi\sin\phi$ and $x \to x + \xi\cos\phi$. The parameters of the kick ξ and ϕ as well as the application time Δt should be chosen so as to minimize the beam emittance after the kick.

The contribution to the beam emittance due to this last kick comes from two sources. First, perturbation of the equilibrium part of the distribution function gives the emittance *increase* $\xi^2/2$ analogously to the original term Eq. (6). However, this increase is offset by the perturbation of the echo term Eq. (12), which, for a small kick can be calculated as

$$\delta\psi_{\mathrm{echo}} = \xi\cos\phi\,\frac{\partial\Delta\psi_{\mathrm{echo}}}{\partial x} + \xi\sin\phi\,\frac{\partial\Delta\psi_{\mathrm{echo}}}{\partial p}\,. \tag{13}$$

As a result, the change of the beam emittance $\Delta\varepsilon_1$ is

$$\Delta\varepsilon_1 = \frac{\xi^2}{2} + \Delta\varepsilon_{\mathrm{echo}}\,, \tag{14}$$

where $\Delta\varepsilon_{\text{echo}}$ is the change of the emittance generated by the distribution function Eq. (13),

$$\Delta\varepsilon_{\text{echo}} = \int_0^\infty dJ \int_0^{2\pi} J\delta\psi_{\text{echo}} d\phi\,. \qquad (15)$$

Calculations show that the minimum value of $\Delta\varepsilon_1$ is negative, which means that emittance decreases as a result of the kick. It is achieved for $\Delta t = 0$, $\phi = \pi/2$, when

$$\xi = 0.51\alpha\,, \qquad q\tau = \frac{0.43}{\Delta\nu}\,, \qquad (16)$$

and is equal to

$$\Delta\varepsilon_{1\,\min} = -0.26\frac{\alpha^2}{2}\,. \qquad (17)$$

The conditions $\Delta t = 0$, $\phi = \pi/2$ mean that the emittance restoring beam offset should be applied at the moment of the maximum echo signal τ_{echo}. Of course, one can achieve the same result by applying a dipole kick of proper magnitude a quarter of the betatron period earlier. The strength of the kick, as Eq. (16) shows, should be such that it would produce aproximately 50% of the initial beam offset. In addition, the optimal value of the quadrupole strength given by Eq. (16) is required related to the tune spread in the beam and the delay τ between the first (dipole) kick and the second (quadrupole) one.

The emittance decrease Eq. (17) constitutes approximately a quarter of the initial increase Eq. (6) due to original beam offset. Hence the beam is still left with three quarters of the initial $\Delta\varepsilon$.

In conclusion, in this paper we demonstrated another aspect of the time-reversing with the beam echo. We showed that using the echo effect to regenerate the beam dipole signal long after the original oscillation decoheres and applying a proper dipole kick at the time of the echo, one can partially compensate for the initial beam emittance increase. Unfortunately, the compensation is apparently too small to be of practical interest. We hope, however, that this demonstration indicates new unusual properties of the echo phenomenon, which can find applications in the future.

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REFERENCES

1. G. Stupakov, SSCL Report 579 (1992).
2. G. Stupakov, S. Kauffmann, SSCL Report 587 (1992).
3. G. Stupakov, A.W. Chao. PAC 97.
4. N. Mahale et al., SSCL-N-817, 1993.
5. L.K. Spenzouris, J.-F. Ostigy, P.L. Colestock, PRL 76, 620 (1996).
6. O. Bruning, CERN SL/95-83 (AP), 1995.
7. E. Shaposhnikova, CERN SL/Note 95-125 (RF), 1995.
8. O. Bruning et al, CERN SL-MD Note 217, 1996.
9. O. Bruning et al, EPAC 96, 1996.
10. S. Wolfram, *The Mathematica Book* (Wolfram Media/Cambridge University Press, 1999), 4th ed.
11. http://www.slac.stanford.edu/∼stupakov/codes.shtml