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String Theory on Parallelizable PP-Waves

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Abstract

The most general parallelizable pp-wave backgrounds which are non-dilatonic solutions in the NS-NS sector of type IIA and IIB string theories are considered. We demonstrate that parallelizable pp-wave backgrounds are necessarily homogeneous plane-waves, and that a large class of homogeneous plane-waves are parallelizable, stating the necessary conditions. Such plane-waves can be classified according to the number of preserved supersymmetries. In type IIA, these include backgrounds preserving 16, 18, 20, 22 and 24 supercharges, while in the IIB case they preserve 16, 20, 24 or 28 supercharges. An intriguing property of parallelizable pp-wave backgrounds is that the bosonic part of these solutions are invariant under T-duality, while the number of supercharges might change under T-duality. Due to their α' exactness, they provide interesting backgrounds for studying string theory. Quantization of string modes, their compactification and behaviour under T-duality are studied. In addition, we consider BPS Dp-branes, and show that these Dp-branes can be classified in terms of the locations of their world volumes with respect to the background H-field.

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1 Introduction

More than a decade ago it was argued that pp-waves provide us with solutions of supergravity which are α'-exact [1]. It has also been shown that any solution of classical general relativity (GR) in a special limit known as the Penrose limit, generates a plane-wave geometry [2]. Plane-waves are a sub-class of pp-waves with planar symmetries. The process of taking the Penrose limit consists of finding a light-like geodesic, expanding the metric about that geodesic and scaling all the coordinates corresponding to the other directions properly [2, 3]. The Penrose limit has also been extended to supergravity solutions [4, 5]. However, not much attention was focused on these solutions until very recently, when it was realized that the string theory sigma model on a large class of plane-waves is solvable [7]. The recent interest has been further boosted by the work of Berenstein-Maldacena-Nastase, BMN, [8], where they noted that a specific plane-wave solution in the type IIB supergravity appears as the Penrose limit of $AdS_5 \times S^5$ geometry and hence is a maximally supersymmetric solution of type IIB theory [5]. Based on this observation they proposed a correspondence between certain operators of $\mathcal{N} = 4,\ D = 4$ super Yang-Mills theory and string theory on plane-waves. The BMN proposal has by now passed many crucial tests (see for e.g. [9], and references therein), and has been extended to many other cases with various numbers of supercharges [10].

It has also been argued that all ten dimensional parallelizable solutions of type IIA, IIB or heterotic supergravities are also α'-exact [11]. In the physics sense, parallelizable manifolds are those for which there exists a (parallelizing) torsion which makes the manifold flat.[1] Besides flat space which is trivially parallelizable, generically, parallelizable spaces are endowed with torsion and hence are usually ignored in the context of classical GR. However, they arise naturally in most string theories (with the exception of type I) where the field strength of the NS-NS two-form field $B_{\mu\nu}$ is interpreted as torsion in the target space.

Generally, in these parallelizable supergravity solutions only the metric and $B_{\mu\nu}$ fields are turned on, i.e., these are non-dilatonic solutions in the NS-NS sector of IIA or IIB supergravities. As interesting examples of such solutions we mention $AdS_3 \times S^7$ or $AdS_7 \times S^3$ [12] which are α'-exact, non-supersymmetric, (classically) stable solutions of type IIA or IIB theories. As we will prove in section 3, as a result of parallelizability, the supersymmetry variation of the gravitinos vanishes for 32 independent solutions and hence all the restrictions on the number of supercharges come from the dilatino variations.

[1] More precise definition of parallelizability would be presented in section 2.

In this work we will focus on pp-waves which are parallelizable and prove that in general the parallelizable pp-waves are necessarily of the form of homogeneous plane-waves and also show that a large class of homogeneous plane-wave geometries are parallelizable, including the geometries coming as Penrose limits of $AdS_5 \times S^5$ and $AdS_3 \times S^3$. We also argue that parallelizability survives the Penrose limit, and hence the Penrose limits of $AdS_{3,7} \times S^{7,3}$ are also parallelizable pp-waves. We show that general parallelizable pp-waves form a family of α'-exact supergravity solutions determined by four real parameters. Then we proceed with counting the number of supercharges for these parallelizable pp-waves. We show that the maximum number of supersymmetries for a parallelizable pp-wave is 28 which corresponds to a IIB plane-wave background with $U(4)$ rotational symmetry, a solution which has appeared previously in reference [13].

Performing a Michelson transformation [14], we study toroidal compactification of our solutions and their behaviour under T-duality. A remarkable property of parallelizable pp-waves is that the bosonic part of such solutions are invariant under T-duality; however, due to the change in the boundary conditions, the number supersymmetries of such T-dual solutions might be different. We recall that this property is a generic feature of all homogeneous plane-waves, regardless of whether they are parallelizable or not. In general, we show that a parallelizable pp-wave in type IIA with N_A supercharges is T-dual to the same geometry in type IIB with $N_B = 16$ or $N_B = 2N_A - 16$ supersymmetries. In addition, there exist $N_A = 20, 24$ solutions for which $N_A = N_B$. Therefore, for $N_A = 16, 20, 24$ there is the possibility of finding self T-dual solutions.

In section 6, we formulate string theory on the most general parallelizable pp-wave background and show that in light-cone gauge it is solvable with a particularly simple spectrum. We also study T-duality at the level of string theory. In our case, as a result of the existence of a non-zero NS-NS H-field, the right and left-movers of closed string modes enter differently. Specifically, only left (or right) movers appear in the zero-modes. This in particular leads to the peculiar property that the zero-mode in one direction behaves as the momentum mode of another direction. Hence, the string is probing a non-commutative cylinder where its fuzziness is inversely proportional to the B-field strength, as well as the light-cone momentum. We then turn to the question of D-branes in the parallelizable pp-wave backgrounds and briefly discuss which classes of BPS Dp-branes can exist in these backgrounds. A full analysis of these D-branes is postponed to future works.

The paper is organized as follows: Section 2 contains a review of needed material. We first review the definition of pp-waves, plane-waves and homogeneous plane-waves and fix our conventions and notations. We then state the definition of parallelizability and some facts about parallelizable manifolds. In section 3, we present implications of parallelizability for supergravity and its solutions. In section 4, we prove that all parallelizable pp-waves are homogeneous plane-waves and classify the parallelizable pp-waves by the amount of supersymmetry they preserve. In section 5, we discuss toroidal compactification of parallelizable pp-waves as well as their *invariance* under a large class of T-dualities. In section 6, we study superstring theory on parallelizable pp-waves and work out the bosonic and fermionic string spectrum (in the light-cone gauge). In section 7, we study T-duality on the string spectrum. In section 8, we briefly discuss Dp-branes on parallelizable pp-waves. The last section is devoted to concluding remarks and open questions.

2 Reviews

In this section we review, very briefly, some of the necessary facts about the two basic ingredients of this paper, pp-waves and parallelizability. For more detailed discussions of these topics the reader is referred to the literature [15, 16].

2.1 Review of pp-waves

A general class of space-times with interesting properties are given by *pp-waves*. They are defined as space-times which support a covariantly constant null Killing vector field v^μ,

$$\nabla_\mu v_\nu \;=\; 0 \;, \qquad v^\mu v_\mu \;=\; 0 \;. \tag{2.1}$$

In the most general form, they have metrics which can be written as

$$ds^2 \;=\; -2dudv - F(u,x^i)du^2 + 2A_j(u,x^i)dudx^j + g_{jk}(u,x^i)dx^j dx^k \;, \tag{2.2}$$

where $g_{jk}(u,x^i)$ is the metric on the space transverse to a pair of light-cone directions given by u,v and the coefficients $F(u,x^i)$, $A_j(u,x^i)$ and $g_{jk}(u,x^i)$ are constrained by (super-)gravity equations of motion. The pp-wave metric (2.2) has a null Killing vector given by $\frac{\partial}{\partial v}$ which is in fact covariantly constant by virtue of the vanishing of Γ^v_{vu}. The most useful pp-waves, and the ones generally considered in the literature, have $A_j = 0$ and are flat in the transverse directions, i.e. $g_{ij} = \delta_{ij}$, for which the metric becomes

$$ds^2 \;=\; -2dudv - F(u,x^i)du^2 + \delta_{ij}dx^i dx^j \;. \tag{2.3}$$

Existence of a covariantly null Killing vector field of the space-time implies that all the higher dimensional operators built from curvature invariants vanish and hence there are no α'-corrections to pp-waves of the form (2.3) which are solutions of classical supergravity [1].[2] For string theory, however, the existence of this null Killing vector leads to a definition of frequency (in light-cone gauge) which is conserved and as a result, the usual problem of non-flat space-times, namely the particle (string) creation, is not present.

A more restricted class of pp-waves, the *plane-waves*, are those admitting a *globally defined* covariantly constant null Killing vector field. One can show that for plane-waves $F(u,x^i)$ is quadratic in the x^i coordinates of the transverse space, but still can depend on the coordinate u, $F(u,x^i) = f_{ij}(u)x^i x^j$, so that the metric takes the form

$$ds^2 \;=\; -2dudv - f_{ij}(u)x^i x^j du^2 + \delta_{ij}dx^i dx^j \ . \tag{2.4}$$

Here f_{ij} is symmetric and by virtue of the only non-trivial condition coming form the equations of motion, its trace is related to the other field strengths present. For the case of vacuum Einstein equations, it is traceless.

There is yet a more restricted class of plane-waves, homogeneous plane waves, for which $f_{ij}(u)$ is a constant, hence their metric is of the form

$$ds^2 \;=\; -2dudv - \mu_{ij}x^i x^j du^2 + dx^i dx^i \ , \tag{2.5}$$

with μ_{ij} being a constant.[3]

2.2 Parallelizability

An n dimensional manifold M is said to be parallelizable if there exists a smooth section of the frame bundle, or equivalently, if there exist n smooth sections of the tangent bundle T(M), such that they are linearly independent at each point of M. More intuitively, a manifold is parallelizable if one can cover the whole manifold with a single non-degenerate coordinate system. In general, most manifolds are not parallelizable. However, group manifolds are always parallelizable. A well known result of K-theory due to Adams [19] is the classification

[2] In general, if $g_{ij}(u,x^i) \neq \delta_{ij}$ the statement about α'-exactness is not true and the transverse metric, g_{ij}, itself may receive α'-corrections, however, there are no extra corrections due to the wave part of the metric [17].

[3] This usage of the term homogeneous is not universal. For example, the term symmetric plane-wave has been used in [18] for this form of the metric, reserving homogeneous for a wider subclass of plane-waves.

of all parallelizable spheres. These consist only of S^1, S^3 and S^7. One can demonstrate the parallelizability of S^1 and S^3 by noting that they are group manifolds: S^1 is the group manifold of $U(1)$, while S^3 is the group manifold of $SU(2)$. However, S^7 which can be thought of as the octonions of unit norm, is not a group manifold because the octonionic algebra is non-associative and so the associativity property of groups is not satisfied. This result on parallelizable spheres is closely linked to the Hurwitz theorem [20], which gives a complete classification of the unital composition algebras (i.e. Hurwitz algebras), as the reals, the complex numbers, the quaternions and the octonions (or Cayley numbers), and the fact that S^1, S^3 and S^7 are topologically equivalent to the complex numbers, quaternions and octonions of unit norm, respectively. The non-parallelizability of S^2 is enshrined in the famous no hair theorem [22]. We should also note that, for the reason stated above, the Lorentzian version of S^3 and S^7, i.e., AdS_3 and AdS_7 are also parallelizable.

There is another definition of parallelizable manifolds due to Cartan-Schouten [21]: A manifold is called parallelizable if there exists a torsion which "flattens" the manifold, i.e. makes the Riemann curvature tensor vanish. We caution the reader that the definitions of parallelizability we have given here, which is the one assumed in the physics literature, is known in the mathematics literature as absolute parallelism, and is a stronger condition than the mathematical definition. In general, absolute parallelism implies parallelism, but not vice-versa.

Let us make explicit the decomposition of the connection into a Christoffel piece and a torsion contribution:

$$\hat{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + T^{\lambda}_{\mu\nu} \ , \tag{2.6}$$

where $\Gamma^{\lambda}_{\mu\nu}$ is symmetric in $\mu\nu$ indices and $T^{\lambda}_{\mu\nu}$ (torsion) is anti-symmetric. The curvature $\hat{R}_{\mu\nu\alpha\beta}$ may be decomposed in a similar way, into a piece which comes only from the Christoffel connection and the torsional contributions:

$$\hat{R}_{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha\beta} + \nabla_\alpha T_{\mu\nu\beta} - \nabla_\beta T_{\mu\nu\alpha} + T_{\mu\beta\rho} T^{\rho}_{\nu\alpha} - T_{\mu\alpha\rho} T^{\rho}_{\nu\beta} \ . \tag{2.7}$$

The parallelizability condition then simply becomes $\hat{R}_{\mu\nu\alpha\beta} = 0$. If the modified Ricci tensor

$$\hat{R}_{\mu\nu} \equiv R_{\mu\nu} + \nabla_\alpha T^{\alpha}_{\mu\nu} - T_{\mu\lambda\rho} T^{\lambda\rho}_{\nu} \tag{2.8}$$

is zero, the manifold is said to be Ricci-parallelizable. Note that the generalized Ricci tensor $\hat{R}_{\mu\nu}$ is not symmetric in its $\mu\nu$ indices. For a manifold to be Ricci-parallelizable, the symmetric and anti-symmetric parts of $\hat{R}_{\mu\nu}$ should both vanish, namely $\nabla_\alpha T^{\alpha}_{\mu\nu} = 0$ and $R_{\mu\nu} - T_{\mu\lambda\rho} T^{\lambda\rho}_{\nu} = 0$.

With the above definitions, the parallelizing torsion (in an orthonormal frame) for a group manifold is given by the structure constants of the group algebra [21].

3 Parallelizability and supergravity

Since in most string theories, a torsion field naturally arises, one may look for implications of parallelizability for supergravities and their solutions. Here we mainly focus on type II theories, however, most of our arguments can be used for heterotic theories as well.

First we recall that the NS-NS part of the supergravity action is of the form

$$S = \frac{1}{l_p^8} \int d^{10}x \sqrt{-g} e^{-2\phi} \left(R + 4(\nabla_\mu \phi)^2 - \frac{1}{12} H^2 \right) . \tag{3.1}$$

Since we are interested in solutions involving only the metric and torsion, we set the dilaton field ϕ to a constant. Then the supergravity equations of motion for the metric and $B_{\mu\nu}$ field are

$$R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\lambda} H_\nu^{\rho\lambda} = 0 \ , \tag{3.2a}$$

$$\nabla_\alpha (\sqrt{-g} H^\alpha_{\mu\nu}) = 0, \qquad H = dB \ . \tag{3.2b}$$

If we define $\frac{1}{2} H_{\alpha\mu\nu}$ as torsion $T_{\alpha\mu\nu}$, as we discussed in previous section, the supergravity equations for the metric and $B_{\mu\nu}$ fields are nothing but the Ricci-parallelizability condition. Hence all parallelizable manifolds (which are obviously also Ricci-parallelizable) satisfying the constant dilaton constraint

$$R = \frac{1}{12} H^2 \tag{3.3}$$

are solutions of supergravity.

For any supergravity solution one may wonder about g_s and α' exactness as well as (classical) stability. The parallelizable solutions which we are interested in are non-dilatonic and hence they are g_s independent. The α'-exacctness of these supergravity solutions were studied long ago in reference [11]. Computing the second α' contributions to the string theory β-functions, it was shown that such contributions are zero for parallelizable manifolds.[4] Therefore the parallelizable solutions of supergravity are exact up to order α'^2. It has been argued that this property is expected to remain to all orders in α' [11].

[4] Of course, as in any field theory with more than one coupling, the two loop renormalizations are scheme dependent, and according to reference [11] in a specific scheme such contributions are zero for parallelizable manifolds. For further details the reader is referred to that reference.

3.1 Parallelizability and supersymmetry

In this section we will demonstrate one of the interesting implications of parallelizability for supergravity solutions. Here we restrict ourselves to type II theories. Our conventions can be found in the appendix.

Theorem:

If we denote the supersymmetry variations of the gravitinos by $\delta\psi_\mu^\alpha,\ \alpha = 1, 2$, parallelizability implies that $\delta\psi_\mu^\alpha = 0$ has the maximal number of solutions (32) and thus does not lead to any restrictions on the number of supercharges. Therefore all the supersymmetry restricting conditions come from the dilatino variations.

To prove the above theorem, we make use of the supersymmetry variations for type IIA and IIB string theories, whose complete expression is presented in the appendix. For our parallelizable backgrounds, which do not depend on RR fields, the IIA and IIB expressions both reduce to

$$\hat{\mathcal{D}}_\mu \;=\; \partial_\mu + \frac{1}{4}\hat{\omega}_\mu \;, \tag{3.4}$$

where $\hat{\omega}_\mu$ is the torsional spin connection[5] and is given by

$$\hat{\omega}_\mu \;=\; \omega_\mu^{ab}\Gamma_{ab} + \frac{1}{2}\sigma_3\Gamma^{ab}H_{\mu ab} \;. \tag{3.5}$$

The Killing spinor equation reads

$$\delta\psi_\mu = \hat{\mathcal{D}}_\mu\epsilon \;=\; 0 \;. \tag{3.6}$$

The commutator of two covariant derivatives acting on a spinor ψ contains a contribution from the curvature and also a torsion piece multiplying a covariant derivative

$$[\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu]\,\epsilon \;=\; \hat{R}_{\mu\nu ab}\Gamma^{ab}\epsilon \;-\; T^\lambda_{\mu\nu}\,\hat{\mathcal{D}}_\lambda\,\epsilon \;. \tag{3.7}$$

Now we note that for parallelizable manifolds, by virtue of the vanishing of the curvature computed from the connection with parallelizing torsion, the Killing spinor equation implies

$$[\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu]\,\epsilon \;=\; -T^\lambda_{\mu\nu}\,\hat{\mathcal{D}}_\lambda\,\epsilon \;. \tag{3.8}$$

The left-hand side is zero for solutions of (3.6). That is, (3.8) is an integrability condition on the differential equation (3.6). We find that the integrability condition imposes no extra

[5] In string theory terminology ω_μ^{ab} only captures the torsion independent part of the connection and is completely determined by the Christoffel connection.

constraints on the solution (this is special to parallelizable manifolds). Note that the vanishing of the Riemann curvature tensor is central to this argument; Ricci parallelizability, i.e., the vanishing of the Ricci tensor computed with parallelizing torsion, does not suffice. Since the Killing spinor equation is a first order differential equation, it can be solved by introducing the ansatz,

$$\epsilon(x) \;=\; \mathcal{W}(x)\,\chi\;, \tag{3.9}$$

where

$$\mathcal{W}(x) \;=\; \mathcal{P}e^{-\frac{1}{4}\int^x \hat{\omega}\cdot dl}\;, \tag{3.10}$$

and χ is a 32 component spinor[6] and $\mathcal{P}$ denotes the path ordering symbol. Plugging the ansatz (3.9) into (3.6), the gravitino Killing equation reduces to

$$\partial_\mu \chi = 0\;. \tag{3.11}$$

That is, equation (3.6) is solved for any constant χ. This provides 32 independent solutions for both types IIA and IIB.

So far we have shown that for any parallelizable manifold, the Killing spinor equation arising from vanishing of the gravitino variation is satisfied for any spinor of the form (3.9,3.11) and the correct number of unbroken supersymmetries is only determined with the zero dilatino supersymmetry variation condition.[7] For a constant dilation background with the only non-vanishing flux the NS-NS field strength, in string frame, this condition is

$$\delta\lambda \;=\; -\frac{1}{4}\Gamma^{abc}H_{abc}\sigma^3\epsilon \;=\; 0\;. \tag{3.12}$$

Therefore, the associated condition for the existence of Killing spinors is $H_{abc}\Gamma^{abc}\,\epsilon^\alpha \;=\; 0$, for $\alpha = 1, 2$.

3.2 An explicit example: $AdS_{3,7} \times S^{7,3}$

Using the fact that the direct product of parallelizable manifolds is also parallelizable, we construct several parallelizable solutions of supergravities. As the first famous example we mention $AdS_3 \times S^3 \times M_4$ [23, 24]. The $AdS_3 \times S^3$ is parallelizable by virtue of being a group manifold. However, supergravity equations of motion for the dilaton, namely equation (3.3),

[6]This has an interpretation as a gauge transformation, where the Killing spinors are pure gauge when the connection is taken to be that with parallelizing torsion.

[7]This is in contrast to the usual situation with plane-waves, where the dilatino equation is redundant and the supersymmetries are determined only by the gravitino equation.

forces the AdS and sphere to have the same radii (this will become clearer in the next paragraph.) Therefore, this non-dilatonic solution is parallelizable if M_4 is parallelizable, and since the only parallelizable four dimensional manifold which is compatible with (3.3), is flat space, it is parallelizable if M_4 is R^4 or T^4. However, if M_4 is Ricci flat, e.g. $M_4 = K3$, this solution is only Ricci-parallelizable.

Another explicit example of a parallelizable supergravity background can be built out of $AdS_{3,7}$ and $S^{7,3}$, with a certain relation between the AdS and sphere radii. This is, as discussed previously, a non-dilatonic solution of the NS-NS sector of either type IIA or IIB supergravity. The supergravity equations of motion are given in (3.2). The dilaton equation of motion implies $R = \frac{H^2}{12}$, and consistency with the other equations then yields $R = H^2 = 0$. The example presented in [12] is $AdS_3 \times S^7$, given by the following background field configuration

$$ds^2 = R_1^2 \left(\frac{du^2}{u^2} + u^2 \left(-dt^2 + dx^2 \right) \right) , \tag{3.13a}$$

$$B_{tx} = R_1^2 u^2 , \tag{3.13b}$$

for the AdS_3 part, and

$$ds^2 = \frac{R_2^2}{4} \left(d\mu^2 + \frac{1}{4} sin^2\mu(\sigma_i - \Sigma_i)^2 + \lambda^2(\cos^2 \frac{\mu}{2}\sigma_i + \sin^2 \frac{\mu}{2}\Sigma_i)^2 \right) , \tag{3.14}$$

with

$$\sigma_1 = \cos\psi_1 d\theta_1 + \sin\psi_1 \sin\theta_1 d\phi_1 , \tag{3.15a}$$

$$\sigma_2 = -\sin\psi_1 d\theta_1 + \cos\psi_1 \sin\theta_1 d\phi_1 , \tag{3.15b}$$

$$\sigma_3 = d\psi_1 + \cos\theta_1 d\phi_1 , \tag{3.15c}$$

and the Σ_i given by the same relations as σ_i, but with $\psi_1, \theta_1, \phi_1 \to \psi_2, \theta_2, \phi_2$. For S^7, $\lambda^2 = 1$.[8] In the above, R_1 and R_2 are the radii of AdS_3 and S^7 respectively, while σ_i and Σ_i parametrize three-spheres. S^7 is the manifold spanned by unit octonions, where the octonion

[8] The round seven-sphere is the coset space $SO(8)/SO(7)$, with the standard metric induced from an isometric embedding in $\mathbb{R}^8$, making manifest its $SO(8)$ isometry. One may analogously define the squashed seven-sphere, topologically equivalent to S^7, as the distance sphere in the projective quaternionic plane, with metric derived from an isometric embedding. A derivation of this metric is presented in [16]. To yield an Einstein space, λ^2 can only take on two discrete values, with $\lambda^2 = 1$ corresponding to S^7 and $\lambda^2 = \frac{1}{5}$ giving the metric on the squashed S^7. An equivalent description of the squashed seven-sphere can be constructed by noting that the round seven-sphere can be described as a fiber-bundle, with base S^4 and fiber S^3. The squashing arises from a change of the S^3 radius over the base.

algebra[9] is

$$O_A\,O_B \;=\; -\delta_{AB} \;+\; f_{ABC}O_C \qquad (A=0,a,\hat{a})\;, \tag{3.16a}$$

$$f_{0a\hat{b}} \;=\; \delta_{ab}\;, \qquad f_{abc} \;=\; \epsilon_{abc}\;, \qquad f_{a\hat{b}\hat{c}} \;=\; -\epsilon_{abc}\;, \tag{3.16b}$$

where 0 corresponds to μ and the indices ranging over $a,b,c = 1,2,3$, and $\hat{a},\hat{b},\hat{c} = 4,5,6$ correspond to σ_i and Σ_i respectively. The components of the NS-NS three form field strength for this solution, in an orthonormal frame, are given by the octonionic structure constants as

$$H_{abc} \;=\; \frac{2}{R_2}\, f_{abc}\;, \tag{3.17}$$

and the Ricci tensor and square of the three-form are

$$R_{ab} \;=\; H^2_{ab} \;=\; \frac{6}{R_2^2}\,\delta_{ab}\;. \tag{3.18}$$

For the above AdS_3 and S^7 backgrounds to combine into a solution of ten dimensional supergravity, the total scalar curvature must vanish, imposing the relation between the two radii $\frac{R_2}{R_1} = \sqrt{7}$, recalling that the scalar curvature of the AdS_3 part is $R = -\frac{6}{R_1^2}$.

The round S^7 provides an example of a parallelizable solution, i.e., its modified Riemann tensor vanishes. However, a specific deformation of the round S^7, the squashed S^7 (S^7_q) with the metric given in (3.14) for $\lambda^2 = 1/5$, leads to a solution which is only Ricci-parallelizable. The torsion still has the form of (3.17), however, now the Ricci tensor and R_2 are related via $R_{ab} = \frac{54}{5R_2^2}\delta_{ab}$.[10] Then $AdS_3 \times S^7_q$ is a Ricci-parallelizable non-dilatonic solution of supergravity provided that $\frac{R_2}{R_1} = \sqrt{63/5}$, where R_1 is the AdS radius.

4 Parallelizable pp-waves

In this section we construct the most general pp-wave which is also parallelizable. Such solutions are "doubly α'-exact" backgrounds of string theory in the sense that they are both parallelizable and have a covariantly constant null Killing vector field. Here we first prove a theorem showing that all parallelizable pp-waves are homogeneous plane-waves. Then in the second part of this section we will classify all the parallelizable pp-waves by the number of their supersymmetries. In the last part of this section, as an example, we work out the

[9] A discussion of octonions, their algebra, and their appearance as the parallelizing torsion on S^7 can be found in [25, 26].

[10] Note that the Ricci tensor in (3.18) is derived from the connection without torsion.

Penrose limit of $AdS_{3,7}\times S^{7,3}$, and show that these solutions only preserve 16 (kinematical) supersymmetries.

4.1 Parallelizable pp-waves as homogeneous plane-waves

Theorem:

All parallelizable pp-waves are homogeneous plane-waves.

To prove this assertion, we begin with the most general ten dimensional pp-wave geometry whose metric is given in equation (2.2) where now $i,j=1,2,\cdots,8$. Of course, the functions F, A_i, g_{ij} are chosen to satisfy supergravity equations of motion. The next step is to write the most general NS-NS H field compatible with the covariantly constant Killing vector v^μ. This nails down the choices for the H-field to

$$H_{uij}=h_{ij}(u,x^i)\ , \tag{4.1}$$

and all the other components zero. Now, we are ready to impose the parallelizability conditions. We may start by imposing supergravity equations of motion. The Ricci curvature for the metric (2.2) may be found in [28]. Then we note that the only non-zero $g^{u\mu}$ components of the inverse metric is $g^{uv}=-1$, therefore

$$H_{\mu\alpha\beta}H_\nu^{\ \alpha\beta}=\delta_{u\mu}\delta_{u\nu}h_{ij}h_{kl}g^{ik}g^{jl}\ , \tag{4.2}$$

and hence all the components of the Ricci curvature except R_{uu} should be zero by virtue of the supergravity equations of motion or equivalently the Ricci-parallelizable conditions. However, first we note that $\hat{R}_{ijkl}=0$ implies that

$$R_{ijkl}=0\ \Longrightarrow\ g_{ij}(u,x^i)=g_{ij}(u)\ . \tag{4.3}$$

Before imposing other supergravity equations of motion we note that using (4.3), after a coordinate transformation, the metric can always be brought to the form

$$ds^2\ =\ -2dudv-\tilde{F}(u,x^i)du^2+2\tilde{A}_j(u,x^i)dudx^j+dx^idx^i\ .$$

Then $R_{ui}=0$ leads to

$$\partial_iF_{ij}=0,\qquad F_{ij}=\partial_{[i}\tilde{A}_{j]}\ , \tag{4.4}$$

and $R_{uu}=\frac{1}{4}H_{uij}H_u^{\ ij}$ results in

$$\frac{1}{2}\nabla^2\tilde{F}=\frac{1}{4}(F_{ij}F^{ij}+h_{ij}h^{ij})\ . \tag{4.5}$$

Note that the freedom in defining v coordinate leads to a $U(1)$ gauge symmetry in the definition of $\tilde{A}_i$ and equation (4.5) is written in the Lorentz gauge, $\partial_i A_i = 0$. Finally the equation of motion for the H field implies that

$$\partial_i h_{ij} = 0 \ . \tag{4.6}$$

Next we impose the parallelizability condition. It is easy to see that the only remaining non-trivial equations come from $\hat{R}_{uiuj} = 0$ and $\hat{R}_{uijk} = 0$. In order to avoid lengthy calculations we only summarize the results:

$$\begin{aligned} h_{ij}(u, x^i) &= h_{ij} = \text{constant} \ , \\ \tilde{F} &= \mu_{ij} x^i x^j \ , \quad \mu_{ij} = \text{constant} \ , \\ \tilde{A}_i &= \frac{1}{2} F_{ij} x^j \ , \quad F_{ij} = -F_{ji} = \text{constant} \ , \end{aligned} \tag{4.7}$$

and

$$\mu_{ij} = \frac{1}{4}(h_{ik} h_{jk} + F_{ik} F_{jk}) \ . \tag{4.8}$$

In summary, the most general parallelizable pp-wave is of the form

$$\begin{aligned} ds^2 &= -2dudv - \mu_{ij} x^i x^j du^2 + F_{ij} x^j du dx^j + dx^i dx^i \ , \\ H_{uij} &= h_{ij} = \text{constant} \ . \end{aligned}$$

It can be shown that we can still use rotations in the transverse space (x^i directions) to remove the $dudx^i$ terms in the metric. Note that such rotations will not change H_{uij}. We will further discuss this coordinate transformation in section 5. Therefore, the most general parallelizable pp-wave can also be written as[11]

$$\begin{aligned} ds^2 &= -2dudv - \mu_{ij} x^i x^j du^2 + dx^i dx^i \ , \\ H_{uij} &= h_{ij} = \text{constant} \ , \end{aligned} \tag{4.9}$$

with

$$\mu_{ij} = \frac{1}{4} h_{ik} h_{jk} \ . \tag{4.10}$$

This is of the form of a homogeneous plane-wave, completing the proof.

In general the converse may not hold, i.e., not all homogeneous plane-wave geometries are parallelizable. In order to see which homogeneous plane-waves are parallelizable, we note

[11] It may seem that geometries of the form $ppp-wave_d \times \mathcal{M}_{10-d}$, where $\mathcal{M}$ is a non-flat, parallelizable manifold and $ppp-wave_d$ is a parallelizable pp-wave of the kind we have discussed, should be included in our parallelizable pp-wave list. In fact since d can only be even, $d \geq 4$, the only possibility for $\mathcal{M}$ is $S^3 \times S^3$. Although such geometries are parallelizable, they do not satisfy the constant dilaton constraint (3.3), and hence are not non-dilatonic supergravity solutions.

that h_{ij} being an anti-symmetric 8×8 matrix, can always be brought to a block diagonal form by O(8) rotations, so that the only non-zero components are

$$h_{12} = -h_{21} = 2a_1 \ , \ h_{34} = -h_{43} = 2a_2 \ , \ h_{56} = -h_{65} = 2a_3 \ , \ h_{78} = -h_{87} = 2a_4 \ . \qquad (4.11)$$

Then eq. (4.10) yields

$$\mu_{ij} = diag(a_1^2, a_1^2, a_2^2, a_2^2, a_3^2, a_3^2, a_4^2, a_4^2) \ . \qquad (4.12)$$

Therefore, all homogeneous plane-waves given by the metric (2.5) are parallelizable if and only if μ_{ij} has doubly degenerate eigenvalues.

The above theorem can be understood more intuitively by noting that parallelizability, by definition, forces the covariantly constant null Killing vector field of the pp-wave, in our case ∂_v, to be globally defined. Furthermore, it forbids the torsion from having any non-trivial space-time dependence; leaving us with the homogeneous plane-waves as the only possibility.

We see from eq. (4.11), that in the most general case, a parallelizable pp-wave is completely determined by four real numbers, a_i. Depending on the values of a_i's, the O(8) rotational symmetry of the transverse space is broken to sub-groups, however, for the generic case (all a_i taking different values) there remains a $\mathrm{U}(1)^4$ symmetry. In the most symmetric case, where all a_i's are equal, we have a U(4) symmetry. The case with $a_3 = a_4 = 0$ and $a_1 = a_2$ arises from the Penrose limit of $AdS_3 \times S^3$ [30]. The plane-waves similar to those of eq. (4.9) have also been considered in [32, 33].

4.2 Supersymmetry counting and classification

In section 3 we showed that, for parallelizable backgrounds, the gravitino variations impose no constraints on the number of supersymmetries; hence, the number of unbroken supercharges is determined entirely by the variation of the dilatino (3.12). We write the two Majorana-Weyl spinors as two Majorana spinors as in the appendix, subject to the appropriate chirality conditions (A-1).

For the backgrounds we are considering, the non-zero spin connection components are $\omega_+{}^{-i} = \frac{1}{2}\mu^i{}_j x^j$. This together with the relations (A-7) implies that $\hat{\omega}^2 = 0$, and hence the expansion of $\mathcal{W}$ is linear in $\hat{\omega}_\mu$. We also have $[\mathcal{W}, \Gamma^+ \not{H}_+] = 0$. Therefore, the supersymmetry variation of the dilatino can be reduced to

$$\delta\lambda \ = \ -\frac{1}{4}\Gamma^{abc} H_{abc}\, \chi \ = \ 0 \qquad (4.13)$$

for $\chi^\alpha, \alpha = 1, 2$, constant Majorana spinors (3.9).

For these backgrounds, the field strength components with one leg along the light-cone direction x^+ are non-vanishing, and there are no purely transverse contributions. As a result, the dilatino variation always contains a Γ^+ acting on χ^α. In the most general parallelizable background (4.11), (4.13) imposes as a condition for the existence of Killing spinors the requirement

$$\Gamma^+ \not{H}_+ \chi^\alpha \;=\; 0, \tag{4.14}$$

with the two spinors subject again to their chirality conditions[12], and we have defined $\not{H}_+ \equiv H_{+ij}\Gamma^{ij}$.[13]

To analyze the dilatino variation, it is easiest to work in a specific basis adapted to the problem, and we choose the basis (A-6). We can write a general spinor as

$$\Psi \;=\; \begin{pmatrix} \phi \\ \omega \\ \xi \\ \lambda \end{pmatrix} \tag{4.15}$$

with $\phi, \omega, \xi, \lambda$, all eight component column vectors. The chirality conditions imply, in this basis,

$$\Gamma^{11}\,\Psi_R \;=+\Psi_R \;\Longrightarrow\; \Psi_R \;=\; \begin{pmatrix} \phi \\ 0 \\ 0 \\ \lambda \end{pmatrix} \qquad \textit{then} \qquad \Gamma^+\,\Psi_R \;\propto\; \begin{pmatrix} 0 \\ \lambda \\ 0 \\ 0 \end{pmatrix}\;, \tag{4.16}$$

for right handed spinors, and

$$\Gamma^{11}\,\Psi_L \;=-\Psi_L \;\Longrightarrow\; \Psi_L \;=\; \begin{pmatrix} 0 \\ \omega \\ \xi \\ 0 \end{pmatrix} \qquad \textit{then} \qquad \Gamma^+\,\Psi_L \;\propto\; \begin{pmatrix} \xi \\ 0 \\ 0 \\ 0 \end{pmatrix}\;, \tag{4.17}$$

for left handed spinors.

To analyze the Killing spinor equation, start with an arbitrary left-handed spinor Ψ_L, of the form (4.17), then subtract off components along the subspace projected out by Γ^+ (ω above). The remaining spinor is of the form $\tilde{\Psi}_L = (0\ 0\ \xi\ 0)$. Since Γ^+ commutes with $\not{H}_+$, we may add to $\tilde{\Psi}_L$ any arbitrary spinor components on the subspace projected out by Γ^+, and these do not contribute to the Killing spinor equation. These unconstrained components

[12] See (A-1).
[13] See (A-10) for an explicit realization of $\not{H}_+$.

provides a minimal set of Killing spinors, eight for each Majorana spinor above, for a total of 16. These are the standard or kinematical Killing spinors which exist on any plane-wave, so are always present after a Penrose-Gueven limit. The kinematical supercharges are thus in the kernel of Γ^+. Their presence is a direct consequence of the existence of a null Killing vector on the pp-wave space-time manifold.

Write $\not{H}_+$ as diag(A,B,A,B), with A and B themselves diagonal matrices with eigenvalues that are doubly degenerate (up to a sign). Now require $\not{H}_+\tilde{\Psi}_L = 0$. This amounts to the equation $A\xi = 0$. The entries of A are a set of four independent algebraic functions of the a_i from which the field strength is constructed (4.11), given by $c_i, i = 1, 2, 3, 4$, in equation (A-12). Some of these c_i may vanish identically for a given background, and the components of ξ acted on by these can be taken to be arbitrary. They provide additional Killing spinors beyond the standard kinematical ones. The components of ξ which are not projected out by the action of A in a given background must then vanish if Ψ_L is to provide a solution of the Killing spinor equation, and so do not contribute any additional supercharges.[14] A similar argument carries through for the case of right handed spinors, where the constraint becomes $B\lambda = 0$, and the four algebraic functions of interest are now $c_i, i = 5, 6, 7, 8$.

As an example, take the most symmetric non-trivial background, with $a_1 = a_2 = a_3 = a_4 \neq 0$, possessing a U(4) symmetry (whose $\mathbb{Z}_4$ center acts by interchanging the four planes). Let us assume that we have subtracted off the kinematical supercharges. For a left-handed spinor, the equations whose solutions gives rise to non-kinematical supercharges, are $c_i = 0, i = 1, 2, 3, 4$, with the c_i's given in (A-12). Of these, three are identically satisfied and one is not. As a result, 3/4 of ξ are unconstrained. For a right-handed spinor, the equations would be $c_i = 0, i = 5, 6, 7, 8$, and of these, none are identically satisfied, so all of λ must vanish for the dilatino variation to be zero. To each of these we may add 8 unconstrained supercharges which are projected out by the action of Γ^+. This leads to the following counting of supercharges: For type IIA, we have the standard 16, together with 6 from the left handed supersymmetry parameter and zero from the right handed one, for a total of 22 supercharges. In type IIB, if both spinors are left handed, we have the maximal of 28 supercharges for a non-trivial parallelizable background, while if both are right handed, we only have 16, which is the minimum. Clearly, for type IIA, the number of supercharges

[14]Notice that it is not possible, if all a_i are non-zero, to simultaneously have some $c_i = 0, i = 1, 2, 3, 4$ and another $c_j = 0, j = 5, 6, 7, 8$. If we had not subtracted off the kernel of Γ^+ (the kinematical supercharges ω), then $\not{H}_+\Psi_L = 0$ alone would yield two simultaneous equations, $A\xi = 0$ and $B\omega = 0$, but since ω was assumed arbitrary, $B = 0$ and so $c_j = 0, j = 5, 6, 7, 8$, which is possible only for all $a_i = 0$, contradicting the earlier assumption. This is another way to see that the presence of Γ^+ is central to the existence of any Killing spinors on non-trivial parallelizable pp-waves.

H_{+ij} components	Number of supercharges	Symmetry
$a_1 = a_2 = a_3 = a_4 = 0$ (flat space)	32	O(8)
$a_1 = \pm a_2, a_3 = a_4 = 0$	24	U(2) $\times$ O(4)
$a_1 = \pm a_2 = \pm a_3 = \pm a_4$	22	U(4)
$a_1 = \pm a_2, a_3 = \pm a_4$	20	$\mathrm{U}(2)^2 \times \mathbb{Z}_2$
$a_1 = a_2 + a_3, a_4 = 0$	20	$\mathrm{U}(1)^4 \times \mathbb{Z}_2$
$a_1 = \pm a_2 \pm a_3 \pm a_4$	18	$\mathrm{U}(1)^4 \times \mathbb{Z}_3$
$a_1 = a_2 = a_3, a_4$ arbitrary	16	U(1) $\times$ U(3)
a_1, a_2, a_3, a_4 arbitrary	16	$\mathrm{U}(1)^4 \times \mathbb{Z}_4$

Table 1: Classification of various parallelizable backgrounds in non-chiral type IIA.

must be one of $(16, 18, 20, 22, 24)$, while for type IIB the allowed number of supercharges is $(16, 20, 24, 28)$. The sensitivity to chirality appears since the non-zero components of right/left handed spinors appearing in (4.16) and (4.17) are required to satisfy different sets of algebraic equations, whose form is evident in (A-11).

Various configurations of the field strength, with their degrees of supersymmetry, are presented in tables 1 and 2, up to relabeling of the coordinates and parity. In both tables, unless otherwise stated, the a_i are assumed non-zero and not to be equal. We also state the symmetry of the transverse part of the given background, up to translations which do not depends on the form of the constant field strength, and parity. Since $16, 20, 24$ supercharges are possible in both IIA and IIB, there exists the possibility of finding self T-dual backgrounds for these numbers of preserved supersymmetries. This is discussed in section 5. The Penrose-Gueven limit of the $AdS_3 \times S^7$ example, presented in section 3.2, has a background symmetry of U(1) $\times$ U(3). The maximal symmetry for a parallelizable pp-wave is U(4) and the minimal is $\mathrm{U}(1)^4 \times \mathbb{Z}_3$.

Flipping the sign of one of the $a_i^\prime s$ corresponds to a parity operation on the two dimensional plane whose field strength components a_i determines, since the field H_{abc} is odd under parity. This parity acts by changing the sign of one of the coordinates on the plane; changing both amounts to a rotation. This parity operation also interchanges the chirality of the spinors, and so the number of supercharges is insensitive to this sign in IIA. In type IIB, this interchange corresponds to the exchange of two solutions in table 2, taking solutions with both left handed spinors to both right handed and vice-versa. For a few exceptional cases with some a_i vanishing,[15] the IIB theory of one chirality has the same amount of supersymmetry as the IIB theory of opposite chirality, and so the number of supercharges,

[15]See footnote 13.

H_{+ij} components	Both left-handed	Both right-handed
$a_1 = a_2 = a_3 = a_4 = 0$ (flat space)	32	32
$a_1 = \pm a_2 \neq 0, a_3 = a_4 = 0$	24	24
$a_1 = a_2 = a_3 = a_4 \neq 0$	28	16
$-a_1 = a_2 = a_3 = a_4 \neq 0$	16	28
$a_1 = -a_2 = -a_3 = a_4 \neq 0$	28	16
$a_1 = \pm a_2 \neq 0, , a_3 = a_4 = 0$	24	24
$a_1 = a_2 \neq 0, a_3 = a_4 \neq 0$	24	16
$a_1 = -a_2 \neq 0, a_3 = a_4 \neq 0$	16	24
$a_1 = -a_2 \neq 0, a_3 = -a_4 \neq 0$	24	16
$a_1 = a_2 = a_3 \neq a_4$	16	16

Table 2: Classification of various parallelizable backgrounds in chiral type IIB

even for type IIB, is not sensitive to parity. An example is the background $a_1 = \pm a_2, a_3 = a_4 = 0$.

4.3 Penrose limit of $AdS_{3,7} \times S^{7,3}$

Given any supergravity solution one can take the Penrose limit and obtain a plane-wave geometry. We may start with a parallelizable supergravity solution. Then, it is easy to show that parallelizability survives the Penrose limiting procedure, and hence after the Penrose limit we find a parallelizable pp-wave , which is necessarily of the form given in (4.9, 4.10). A well-known example is the Penrose limit of $AdS_3 \times S^3 \times \mathbb{R}^4$ [8, 30], which leads to a parallelizable pp-wave with $a_1 = a_2$, $a_3 = a_4 = 0$ in the notation of eq. (4.11).

As another example, we work out the Penrose limit of $AdS_{3,7} \times S^{7,3}$ solutions discussed earlier in section 3.2. To begin with, for definiteness, let us consider the $AdS_3 \times S^7$ case. To take the Penrose limit, it is more convenient to write the metric in the AdS global coordinates:

$$ds^2 = R_1^2[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\phi^2] + R_2^2[\cos^2\psi d\theta^2 + d\psi^2 + \sin^2\psi d\Omega_5^2] \tag{4.18a}$$

$$H_{\tau\rho\phi} = 2R_1^2 \cosh\rho \sinh\rho \tag{4.18b}$$

where

$$\frac{R_2^2}{R_1^2} \equiv r^2 = 7 \tag{4.19}$$

and the other H-field components along S^7 given in equation (3.17). Now we take the

$R_1 \to \infty$ limit together with

$$\rho = \frac{y}{R_1} \quad , \quad \psi = \frac{x}{R_2} \ ,$$
$$u = \tau + r\theta \quad , \quad v = \frac{1}{2R_1^2}(\tau - r\theta) \ , \tag{4.20}$$

where x, y, u, v, and all the other coordinates are kept fixed. Then the metric (4.18) becomes

$$ds^2 = -2dudv - \left(\frac{1}{r^2} \sum_{i=1}^{6} {x_i}^2 + \sum_{a=1}^{2} {y_a}^2 \right) du^2 \ + \sum_{i=1}^{6} \ dx^i dx^i + \sum_{a=1}^{2} \ dy^a dy^a \ , \tag{4.21a}$$
$$H_{+12} = 2 \ , \ H_{+34} = H_{+56} = H_{+78} = \frac{2}{r} \ . \tag{4.21b}$$

This solution in the notation of (4.11) is $a_1 = 1, \ a_2 = a_3 = a_4 = \frac{1}{r}$. One can show that the Penrose limit of $AdS_7 \times S^3$ is the same as (4.21).

Using our previous arguments, one can show that the parallelizable pp-wave (4.21) only preserves the 16 kinematical supersymmetries, while the original solution before the Penrose limit was not supersymmetric at all. As shown in [6], for a background to preserve more than half the maximal supersymmetry, it must have a trivial dilaton dependence. These are generic behaviours of supergravity solutions. The Penrose-Gueven limit can never destroy any supersymmetry, but might enhance it [5]. Starting with a non-supersymmetric solution, we arrive at half supersymmetric ones. Another example is the Penrose limit of $AdS_5 \times T^{p,q}, \ p \neq q$ [10].

One might wonder whether Ricci-parallelizability is enhanced to parallelizability under the Penrose limit. This is not true in general. As an example, if we start with $AdS_3 \times S^7_q$ as a Ricci-parallelizable solution, it remains only Ricci-parallelizable after the Penrose limit.

5 T-duality on parallelizable pp-waves

In this section we study T-duality on the parallelizable pp-waves introduced in previous sections. In general, in order to perform T-duality, we first need to compactify the manifold and for (toroidal) compactifications of any manifold we require translations along the compactification directions to be (space-like) isometries of the manifold. The existence of translation isometry along a space-like direction for the parallelizable pp-wave as written in eq. (4.9), is not manifest. However, the plane-wave solutions generically possess non-linearly realized symmetries and there is a chance that in a proper coordinate system some isometries which are hidden may become manifest. This is indeed the case. To see this, following Michelson [14], let us consider the "rotating" frame

$$\begin{aligned} X^1 &= x^1 \cos(a_1 u) - x^2 \sin(a_1 u) \ , \\ X^2 &= x^2 \cos(a_1 u) + x^1 \sin(a_1 u) \ , \end{aligned} \tag{5.1}$$

leaving all the other coordinates unchanged, i.e.

$$U = u \ , \quad \tilde{V} = v \ , \quad X^i = x^i \quad i \geq 3 \ . \tag{5.2}$$

The metric in the new coordiantes becomes

$$\begin{aligned} ds^2 &= -2dUd\tilde{V} + dX^1dX^1 + dX^2dX^2 + \sum_{i=3}^{8} dX^idX^i \\ &- [a_2^2(X_3^2 + X_4^2) + a_3^2(X_5^2 + X_6^2) + a_4^2(X_7^2 + X_8^2)]dU^2 - 2a_1(X^1dX^2 - X^2dX^1)dU \ , \end{aligned} \tag{5.3}$$

while the H-field remains invariant, i.e. $H_{UX^1X^2} = 2a_1$. It is evident that we can make the same transformation for all the other coordiantes and remove the dU^2 term of the metric completely; then we get a metric which has $dUdX^i$ term proportional to $F_{ij}X^j$. The translational symmetry is not manifest yet, to see that let us redefine[16]

$$V = \tilde{V} - a_1 X^1 X^2 \ ,$$

and the metric becomes

$$\begin{aligned} ds^2 &= -2dUdV + \sum_{i=1}^{8} dX^idX^i \\ &\quad -[a_2^2(X_3^2 + X_4^2) + a_3^2(X_5^2 + X_6^2) + a_4^2(X_7^2 + X_8^2)]dU^2 + 4a_1X^2dX^1dU \ . \end{aligned} \tag{5.4}$$

Now we are ready to compactify X^1 on a circle and use Bucher's rules to perform T-duality [34]

$$\begin{aligned} g^T_{\mu\nu} &= g_{\mu\nu} - \frac{1}{g_{11}}\left(g_{\mu 1}g_{\nu 1} - B_{\mu 1}B_{\nu 1}\right) \\ B^T_{\mu\nu} &= B_{\mu\nu} + 2\frac{1}{g_{11}}\left(g_{\mu 1}B_{\nu 1} - B_{\mu 1}g_{\nu 1}\right) \\ g^T_{\mu 1} &= \frac{B_{\mu 1}}{g_{11}} \ , \ B^T_{\mu 1} = \frac{g_{\mu 1}}{g_{11}} \ , \end{aligned} \tag{5.5}$$

[16] We remind the reader that, as far as the metric is concerned, to make the isometry along a direction, say X^1, manifest, the frame we choose can be rotating clockwise or counter-clockwise. However, due to the presence of a B-field in our case, there is a preferred orientation.

In our case[17]

$$B_{\mu 1} = g_{\mu 1} = 2a_1 \; X^2 \delta_{\mu U} \; . \tag{5.6}$$

Choosing $g_{11} = 1$, i.e., sitting at the self-dual radius, we find

$$\begin{aligned} B^T_{\mu\nu} = B_{\mu\nu} \quad &, \quad g^T_{\mu\nu} = g_{\mu\nu}, \\ B^T_{\mu 1} = g_{\mu 1} \quad &, \quad g^T_{\mu 1} = B_{\mu 1}. \end{aligned} \tag{5.7}$$

As we see, the g and B fields are exactly the same before and after T-duality; in other words parallelizable pp-waves are invariant under T-duality. This invariance is a direct consequence of the specific form of our metric and B-field, namely, $B_{\mu 1} = g_{\mu 1}$, which is dictated by parallelizability.

So far, we have shown that any parallelizable pp-wave solution of type IIB is also a solution of IIA, related by T-duality. In the above arguments we have only considered the bosonic fields g and B. One should also consider fermions and check if the above T-duality invariance also holds in the fermionic sector. First, we note that the two conserved supersymmetries of the IIB background have the same chirality, while those of IIA have different chiralities. Compactification imposes a boundary condition on fermions, which is not necessarily compatible with their chirality and as a result, the number of supersymmetries may change under T-duality. While individual supersymmetries may be affected by compactification, the total number of kinematical supersymmetries (16) is not, and hence 16 such supercharges survive compactification and T-duality. The difference between IIB and IIA only arises in the non-kinematical supersymmetries. If the number of supercharges of a type IIA parallelizable pp-wave is $N_A = 4k + 2$ $(k = 4, 5)$, then the corresponding T-dual IIB solution has $N_B = N_A \pm (N_A - 16)$; the $+/-$ depends on the "orientation" with respect to the H-field. Because of the existence of a non-trivial H-field, the two different orientations on the compactification circle lead to two different solutions, differing by the number of supersymmetries. For the $N_A = 4k$ $(k = 4, 5, 6)$ cases, after T-duality we have $N_B = N_A$ or $N_B = N_A \pm (N_A - 16)$. Which of these cases we obtain depends on the details of the supergravity solution. Therefore, invariance under T-duality can only be exact (in the sense that it holds in both bosonic and fermionic sectors) for $N_A = N_B = 16$, $N_A = N_B = 20$ and $N_A = N_B = 24$. As examples of these cases we mention the plane-wave coming as the Penrose limit of $AdS_{3,7} \times S^{7,3}$ $(N_A = N_B = 16)$ and $AdS_3 \times S^3 \times T^4$ $(N_A = N_B = 24)$ solutions.

[17] Given $H = dB$ there is a $U(1)$ gauge freedom in the definition of B and we choose the gauge which is compatible with the translational symmetry along X^1.

6 String theory on parallelizable pp-waves

As noted in [7], the string theory (sigma model) on a generic homogeneous plane-wave is solvable in the light-cone gauge. In [7] the interest was mainly in backgrounds with non-zero RR flux. In this section we formulate string theory on parallelizable pp-waves and show that, in this case, string theory is simpler than the generic plane-wave with RR flux. The formulation of string theory on some special parallelizable pp-waves, namely those coming from the Penrose limit of $AdS_3 \times S^3$ and its variants, have been previously considered in [30, 32, 35], although the connection to parallelizability was not made. We first focus on the bosonic sector of strings and in the next section study fermions in the Green-Schwarz (GS) formulation. However, since our backgrounds are only in the NS-NS sector, the RNS formulation can also be used.

6.1 Bosonic sector

The non-linear sigma model in the NS-NS background of $G_{\mu\nu}$ and $B_{\mu\nu}$ fields is given by

$$S = \frac{-1}{4\pi\alpha'} \int d^2\sigma \ \sqrt{-g} \left(g^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right) \ , \tag{6.1}$$

where $a, b = 1, 2$ and g_{ab} is the worldsheet metric. For the background defined through eqs. (4.9,4.10,4.11), we have

$$\begin{aligned} S = \frac{-1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} \Big[g^{ab} \big(-2\partial_a U \partial_b V - \frac{1}{4} h_{ik} h_{jk} X^i X^j \partial_a U \partial_b U + \partial_a X^i \partial_b X^i \big) \\ + \epsilon^{ab} h_{ij} X^j \partial_a U \partial_b X^i \Big] \ . \end{aligned} \tag{6.2}$$

We next fix the conformal symmetry in the gauge $g_{\tau\sigma} = 0$ and $-g_{\tau\tau} = g_{\sigma\sigma} = 1$, and to avoid ghosts we choose light-cone gauge, in which

$$\partial_\tau U = p^+ = const. \tag{6.3}$$

Then V is constrained to satisfy

$$\begin{aligned} p^+ \partial_\sigma V &= \partial_\tau X^i \partial_\sigma X^i \ , \\ 2p^+ \partial_\tau V &= \partial_\tau X^i \partial_\tau X^i + \partial_\sigma X^i \partial_\sigma X^i - \frac{1}{4} (p^+)^2 h_{ik} h_{jk} X^i X^j \ . \end{aligned} \tag{6.4}$$

Plugging the above into the action (6.2), the light-cone action for the transverse modes X^i is seen to be

$$S_{LC} = \frac{1}{4\pi\alpha'} \int d^2\sigma \left[\partial_\tau X^i \partial_\tau X^i - \partial_\sigma X^i \partial_\sigma X^i - \frac{p^{+2}}{4} h_{ik} h_{jk} X^i X^j - p^+ h_{ij} X^j \partial_\sigma X^i \right]$$

$$= \frac{1}{4\pi\alpha'} \int d^2\sigma \left[(\partial_\tau X^i)^2 - (\partial_\sigma X^i + \frac{p^+}{2} h_{ij} X^j)^2 \right] . \tag{6.5}$$

Without loss of generality we can take the h_{ij}'s as in equation (4.11). Since the analysis for the different modes is quite similar, here we only focus on the X^1, X^2 components, whose equations of motion are

$$(\partial_\tau^2 - \partial_\sigma^2) X^i + \left(a_1 p^+\right)^2 X^i - 2 a_1 p^+ \epsilon_{ij} \partial_\sigma X^j = 0 \ , \quad i,j = 1,2 \ . \tag{6.6}$$

The above equation is solved by $x_i e^{i(\omega_n^\pm \tau - 2n\sigma)}$ where

$$\omega_n^\pm = 2n \pm a_1 p^+ \ , \tag{6.7}$$

and the closed string boundary conditions

$$X^i(\sigma) = X^i(\sigma + \pi) \ , \tag{6.8}$$

fixes n to be integer. Therefore, the most generic solution to equation (6.6) can be written as

$$X_R^1 = \frac{1}{2} \sum_{n\in\mathbb{Z}} \frac{1}{\omega_n^+} \alpha_n^R e^{i(\omega_n^+ \tau - 2n\sigma)} + \frac{1}{\omega_n^-} \beta_n^R e^{i(\omega_n^- \tau - 2n\sigma)} \ , \tag{6.9a}$$

$$X_R^2 = \frac{1}{2} \sum_{n\in\mathbb{Z}} \frac{+i}{\omega_n^+} \alpha_n^R e^{i(\omega_n^+ \tau - 2n\sigma)} + \frac{-i}{\omega_n^-} \beta_n^R e^{i(\omega_n^- \tau - 2n\sigma)} \ , \tag{6.9b}$$

for the right-movers, and for the left-moves

$$X_L^1 = \frac{1}{2} \sum_{n\in\mathbb{Z}} \frac{1}{\omega_n^+} \alpha_n^L e^{i(\omega_n^+ \tau + 2n\sigma)} + \frac{1}{\omega_n^-} \beta_n^L e^{i(\omega_n^- \tau + 2n\sigma)} \ , \tag{6.10a}$$

$$X_L^2 = \frac{1}{2} \sum_{n\in\mathbb{Z}} \frac{i}{\omega_n^+} \alpha_n^L e^{i(\omega_n^+ \tau + 2n\sigma)} + \frac{-i}{\omega_n^-} \beta_n^L e^{i(\omega_n^- \tau + 2n\sigma)} \ . \tag{6.10b}$$

The reality of X's implies that

$$(\alpha_n^R)^\dagger = \beta_{-n}^R \ , \quad (\alpha_n^L)^\dagger = \beta_{-n}^L \ , \tag{6.11}$$

and $X^i = X_R^i + X_L^i$.

Note that for a generic $a_1 p^+$, there is no zero-mode and the mode expansions of (6.9,6.10) can be used without ambiguity. For $\frac{1}{2} a_1 p^+ \in \mathbb{Z}$, however, we have zero frequency modes; for such cases one may extract the zero-modes:

$$X^1 = (x^1 + p^1\tau) \cos a_1 p^+ \sigma + (x^2 + p^2\tau) \sin a_1 p^+ \sigma + \sum_{n \neq a_1 p^+/2} \ Oscil. \tag{6.12a}$$

$$X^2 = (x^2 + p^2\tau) \cos a_1 p^+ \sigma - (x^1 + p^1\tau) \sin a_1 p^+ \sigma + \sum_{n \neq a_1 p^+/2} \ Oscil. \tag{6.12b}$$

In these cases, for a constant $\tau = \tau_0$ slice in the X^1, X^2 plane, the string forms a circle of radius $X_1^2 + X_2^2 = (x^1 + p^1\tau_0)^2 + (x^2 + p^2\tau_0)^2$, with some wiggles superposed on it. To understand the physics of such strings it is helpful to work out the angular momentum of the center of mass of the string:

$$L = \frac{1}{\pi}\int_0^\pi (X^1\partial_\tau X^2 - X^2\partial_\tau X^1)d\sigma = x^1p^2 - x^2p^1 \ , \tag{6.13}$$

which is constant. In this case strings are circles whose radius grows with time, while carrying constant angular momentum. This is as expected if we note that the NS-NS field acts like a magnetic field. However, the effect of du^2 terms in the background metric appears in the fact that the $p^i\tau$ terms in (6.12) are multiplied by $\sin a_1p^+\sigma$ and $\cos a_1p^+\sigma$ factors, i.e., the center of mass of a string on the average has zero momentum and is confined to stay around $X = 0$. Similar behaviour have been observed for other plane-waves (e.g. see [7]).

One of the remarkable differences between our case and the other plane-waves (which involve RR fluxes), is that the spacing between the energy levels in our case is just given by integers (in other words $\omega_n^\pm$ is a linear function of n).

We would like to point out that due to the presence of background fields, the light-cone Hamiltonian differs from $\int \partial_\tau V$, explicitly

$$\begin{aligned}
H_{LC} = \Pi_+ \ &= \int d\sigma \ (\partial_\tau V + p^+a_1^2X_i^2 + a_1\epsilon_{ij}X^i\partial_\sigma X^j) \\
&= \frac{1}{2p^+}\int d\sigma \ \Big[(\partial_\tau X^i)^2 + (\partial_\sigma X^i + p^+a_1\epsilon_{ij}X^j)^2\Big] \\
&= \frac{1}{2p^+}\sum_{n\in\mathbb{Z}} \alpha_n^R\beta_{-n}^R + \beta_n^R\alpha_{-n}^R + \alpha_n^L\beta_{-n}^L + \beta_n^L\alpha_{-n}^L \ .
\end{aligned} \tag{6.14}$$

Note that the last line of the above equation is written for $\frac{1}{2}a_1p^+ \notin \mathbb{Z}$. For integer values of $\frac{1}{2}a_1p^+$, one should add $(p^1)^2 + (p^2)^2$ to the sum over oscillators.

Given the mode expansions, we can proceed with the quantization of strings by imposing

$$[X^i(\sigma), X^j(\sigma')] = 0 \ , \quad [P^i(\sigma), P^j(\sigma')] = 0 \ , \quad [X^i(\sigma), P^j(\sigma')] = i\delta^{ij}\delta(\sigma - \sigma') \ , \tag{6.15}$$

with $P^i = \partial_\tau X^i$, leading to

$$[\alpha_n^R, \beta_m^R] = [\alpha_n^L, \beta_m^L] = \omega_n^-\delta_{m+n} \ , \tag{6.16}$$

and all the other combinations commuting. For the cases where we have "zero-modes" ($\frac{1}{2}p^+a_1 \in \mathbb{Z}$), $[x^i, p^j] = i\delta^{ij}$. The above mode expansion and commutators is similar to the twisted sector of strings on orbifolds. Using the commutation relations (6.16) we can write the Hamiltonian in a normal ordered form

$$H_{LC} = \frac{1}{2p^+} \sum_{i=1}^{4} \sum_{n\geq 0} \alpha^R_{-n\ i} \beta^R_{n\ i} + \beta^R_{-n\ i} \alpha^R_{n\ i} + \alpha^L_{-n\ i} \beta^L_{n\ i} + \beta^L_{-n\ i} \alpha^L_{n\ i} + E_0 \ , \tag{6.17}$$

where E_0 is the zero point energy coming from regularizing sums like $\sum(n-\phi)$, and has value

$$E_0 = -\frac{2}{3} - \sum_{i=1}^{4} \phi_i^2 \ , \tag{6.18}$$

where ϕ_i are the non-integer part of $\frac{1}{2}p^+ a_i$, i.e. $\phi_i = \frac{1}{2}p^+ a_i - [\frac{1}{2}p^+ a_i]$.

6.2 Fermionic sector

The fermionic NS-NS sector of the Green-Schwarz action, expanded to second order[18] in the fermions, for type IIA string theory is

$$S_F \ = \ \frac{i}{\pi\alpha'} \int d^2\sigma \bar{\theta} \beta^{ab} \partial_a X^\mu \Gamma_\mu \hat{D}_b \theta \ , \tag{6.19}$$

with $\beta^{ab} = \sqrt{-h} h^{ab} \sigma_0 - \epsilon^{ab} \sigma_3$, $\hat{D}_b$ the pull-back of the superspace covariant derivative with torsion

$$\hat{D}_a \ = \ D_a \ + \ \frac{1}{8} \partial_a X^\mu \Gamma^{\rho\sigma} \sigma_3 H_{\mu\rho\sigma} \ , \tag{6.20}$$

and the normal covariant derivative

$$D_a \ = \ \partial_a \ + \ \frac{1}{4} (\partial_a X^\mu) \omega^{ab}_\mu \Gamma_{ab} \tag{6.21}$$

contains the spin connection ω^{ab}_μ. The NS-NS sector of the IIB Green-Schwarz action is the same as that presented above, with the difference in the solutions arising from the chirality of the two spinors, as in the discussion of supersymmetry counting in section 4.2.

We work in light-cone gauge, where we impose the conditions

$$X^+ = p^+ \tau \ , \qquad\qquad \sqrt{-h} h^{ab} = \eta^{ab} \ , \tag{6.22}$$

and fix the κ symmetry by imposing the additional constraint $\Gamma^+ \theta^\alpha = 0$, giving

$$S_F \ = \ \frac{-ip^+}{\pi\alpha'} \int d^2\sigma \bar{\theta} \Gamma^- \left(\hat{D}_\tau \sigma_0 \ + \ \hat{D}_\sigma \sigma_3 \right) \theta \ , \tag{6.23}$$

[18] Terms of higher order do not contribute in light-cone gauge.

with

$$\hat{D}_\sigma = \partial_\sigma \ , \tag{6.24a}$$

$$\hat{D}_\tau = \partial_\tau + \frac{p^+}{4}\left(\omega_{+\mu\nu} + \frac{1}{2}H_{+\mu\nu}\right)\Gamma^{\mu\nu} \ , \tag{6.24b}$$

yielding the following form for the fermionic action

$$S_F = \frac{-ip^+}{\pi\alpha'}\int d^2\sigma\sqrt{2}\left(\bar{\theta}^1\Gamma^-\partial_+\theta^1 + \bar{\theta}^2\Gamma^-\partial_-\theta^-\right) - \frac{p^+}{8}\left(\bar{\theta}^1\Gamma^-\Gamma^{IJ}\theta^1 - \bar{\theta}^2\Gamma^-\Gamma^{IJ}\theta^2\right)H_{+IJ} \ , \tag{6.25}$$

where I, J range over the transverse directions, and $\partial_\pm = \frac{1}{\sqrt{2}}(\partial_\tau \pm \partial_\sigma)$. Precisely half the components of the fermions have been projected out by the gauge fixing to leave only the physical degrees of freedom. The equations of motion are

$$\left(\partial_+ - \frac{p^+}{8\sqrt{2}}\Gamma^{IJ}H_{+IJ}\right)\theta^1 = 0, \tag{6.26a}$$

$$\left(\partial_- + \frac{p^+}{8\sqrt{2}}\Gamma^{IJ}H_{+IJ}\right)\theta^2 = 0, \tag{6.26b}$$

which do not couple θ^1 to θ^2, but do mix components within the individual fermions. The closed string boundary conditions are

$$\theta^1, \theta^2|_{\sigma=0} = \theta^1, \theta^2|_{\sigma=\pi}. \tag{6.27}$$

We solve the equations of motion (6.26) by separating the solution in light-cone coordinates, then expanding the solution into a complete set of functions. The ansatz is

$$\theta^1(\tau,\sigma) = e^{in_1(\tau+\sigma)}\ e^{im_1(\tau-\sigma)}\ \psi^1_{m_1,n_1} \ , \ \theta^2(\tau,\sigma) = e^{in_2(\tau-\sigma)}\ e^{im_2(\tau+\sigma)}\ \psi^2_{m_2,n_2} \ ,$$

with ψ^1 and ψ^2 constant spinors which are subject to the gauge fixing condition. Introducing the mass matrix $M = \frac{-ip^+}{16}\not{H}_+$, with the matrix $\not{H}_+$ written out in equation (A-10), we have

$$(n_1 - M)\,\psi^1_{m_1,n_1} = 0 \ , \tag{6.28a}$$

$$(n_2 + M)\,\psi^2_{m_2,n_2} = 0 \ , \tag{6.28b}$$

which are a pair of eigenvalue equations for n_1 and n_2. Imposing the boundary condition (6.27), we see that

$$m_1 \in 2\mathbb{Z} + n_1 \ , \qquad m_2 \in 2\mathbb{Z} + n_2 \ . \tag{6.29}$$

The final mode expansions are

$$\theta^1(\tau,\sigma) \;=\; \sum_{m_1} e^{in_1(\tau+\sigma)}\; e^{im_1(\tau-\sigma)}\; \psi^1_{m_1,n_1}\ , \tag{6.30a}$$

$$\theta^2(\tau,\sigma) \;=\; \sum_{m_2} e^{in_2(\tau-\sigma)}\; e^{im_2(\tau+\sigma)}\; \psi^2_{m_2,n_2}\ . \tag{6.30b}$$

The reality condition implies that

$$\left(\psi^\alpha_{m_\alpha,n_\alpha}\right)^\dagger \;=\; \psi^\alpha_{-m_\alpha,-n_\alpha}\ . \tag{6.31}$$

Upon quantization

$$\{(\psi^\alpha_{m_\alpha,n_\alpha})^\dagger,\ \psi^\beta_{m'_\beta,n'_\beta}\} = \delta_{\alpha\beta}\delta_{mm'}\delta_{nn'}\ .$$

It is straightforward to work out the light-cone Hamiltonian for the fermionic modes

$$H_{LC} = \frac{1}{2p^+}\sum_{m_\alpha\geq 0}(\psi^\alpha)^\dagger m_\alpha \psi^\alpha - E_0\ , \tag{6.32}$$

where E_0 is the zero point energy and is defined in eq. (6.18). As we expect, the bosonic and fermionic zero point energies are the same up to a sign and hence there is no total zero point energy in the spectrum, a confirmation of the supersymmetry of the background.

The parallelizable solutions we are considering are parameterized by four constants, as in (4.11). As an example, consider the background with $-a_1 = a_2 = a_3 = a_4$. This background preserves 22 supercharges in type IIA and either 16 or 28 supercharges in type IIB, depending on whether the spinors are chosen to be both left-handed or right-handed, respectively. The mass matrix for this configuration is

$$M = \frac{a_1\, p^+}{4}\, diag(1,1,-1,-1,-1,1,-1,1,0,2,-2,0,0,0,0,0), \tag{6.33}$$

where we only need to consider the half of the mass matrix acting on the subspace surviving the gauge fixing. Now, for the modes for which the mass vanishes, equation (6.28) implies the existence of zero-modes. After fixing the κ symmetry by imposing the gauge $\Gamma^+\theta^\alpha = 0$, the counting of the zero-modes for the physical degrees of freedom parallels the counting of the non-kinematical supercharges presented in section 4.2. The number of zero-modes depends on whether we are working with type IIA or IIB, and in type type IIB, whether we choose both spinors to be left-handed or right-handed. The number of zero-modes can be ascertained by looking at tables 1 and 2, subtracting 16 for the standard kinematical supersymmetries, which in the present discussion correspond to the modes orthogonal to those removed by the gauge fixing. This leads to a subtlety, which is that the number of zero-modes coming from left handed spinors θ here are associated with the number of non-kinematical supercharges (for counting purposes only) for right handed spinors in section 4.2, and vice-verse. The total counting in type IIA, of course, is the same. Therefore, for this example, there are 6 zero-modes in type IIA and zero (both spinors right-handed) or 12 (both left-handed) zero-modes in type IIB.

The example $-a_1 = 1, a_2 = a_3 = a_4 = 1/r$, with r defined in (4.19), arising as the Penrose limit of $AdS_3 \times S^7$, is half supersymmetric, preserving 16 supercharges in both type IIA and IIB, and as a result, the fermionic solutions exhibit no zero-modes. The mass matrix is

$$M = \frac{p^+}{8r^2}\, diag(r^2-1, r^2-1, r^2+3, r^2-1, r^2+1, r^2-3, r^2+1, r^2+1). \tag{6.34}$$

This analysis can be extended to other configurations, with the general result that the number of zero-modes, in the solutions to the fermionic sector, is equal to the number of non-kinematical supercharges preserved by the background, taking account of the type IIB reversal discussed above.

7 Compactification and T-duality

In section 5 we studied T-duality of parallelizable pp-waves at the supergravity level. In this section we extend the T-duality analysis to string theory. As we will see, because of the nature of the background, only right-movers contribute to the center of mass modes. This, and other peculiar features of strings under T-duality will be examined in this section.

We begin by writing the sigma model action in the parallelizable pp-wave after a Michelson rotation, i.e., the background given in eqs. (5.4),(5.6). After fixing the light-cone gauge:

$$\tilde{S}_{LC} = \frac{1}{4\pi\alpha'} \int d^2\sigma \left[\partial_\tau \tilde{X}^i \partial_\tau \tilde{X}^i - \partial_\sigma \tilde{X}^i \partial_\sigma \tilde{X}^i - 4p^+ a_1 \tilde{X}^2 (\partial_\tau \tilde{X}^1 + \partial_\sigma \tilde{X}^1) \right] \ , \tag{7.1}$$

where we have only presented the action for the $\tilde{X}^1, \tilde{X}^2$ components, the other X's are similar and here we skip them. The equations of motion are

$$(\partial_\tau^2 - \partial_\sigma^2)\tilde{X}^i - 2a_1 p^+ \epsilon_{ij} (\partial_\tau \tilde{X}^j + \partial_\sigma \tilde{X}^j) = 0 \ , \quad i,j = 1,2 \ . \tag{7.2}$$

Inserting $x_i e^{i(\omega\tau - 2n\sigma)}$ we find

$$\omega = 2n \quad \text{or} \quad \tilde{\omega}_n^\pm = 2n \pm 2a_1 p^+ \ . \tag{7.3}$$

Imposing the closed string boundary conditions restricts n to the integers. Then the mode expansions are

$$\tilde{X}^1_R = \tilde{x}^1 + \tilde{p}^1(\tau - \sigma) + \sum_{n\in\mathbb{Z}} \frac{i}{2n}\tilde{\alpha}^R_n e^{2in(\tau-\sigma)} \ , \tag{7.4a}$$

$$\tilde{X}^2_R = \tilde{x}^2 + \tilde{p}^2(\tau - \sigma) + \sum_{n\in\mathbb{Z}} \frac{i}{2n}\tilde{\beta}^R_n e^{2in(\tau-\sigma)} \ , \tag{7.4b}$$

for the right-movers and[19]

$$\tilde{X}^1_L = \frac{1}{2}\sum_{n\in\mathbb{Z}} \frac{1}{\tilde{\omega}^+_n}\tilde{\alpha}^L_n e^{i(\tilde{\omega}^+_n\tau+2n\sigma)} + \frac{1}{\tilde{\omega}^-_n}\tilde{\beta}^L_n e^{i(\tilde{\omega}^-_n\tau+2n\sigma)} \ , \tag{7.5a}$$

$$\tilde{X}^2_L = \frac{1}{2}\sum_{n\in\mathbb{Z}} \frac{i}{\tilde{\omega}^+_n}\tilde{\alpha}^L_n e^{i(\tilde{\omega}^+_n\tau+2n\sigma)} + \frac{-i}{\tilde{\omega}^-_n}\tilde{\beta}^L_n e^{i(\tilde{\omega}^-_n\tau+2n\sigma)} \ . \tag{7.5b}$$

for the left-movers. We see that only the right movers have the usual momentum mode. Note also that for the non-compact case, $\tilde{p}^1 = \tilde{p}^2 = 0$, resulting from the boundary conditions,

Next, we quantize the strings by imposing the usual commutation relations (6.15), where now the conjugate momenta to $\tilde{X}^i$ are

$$\tilde{P}^1 = \partial_\tau\tilde{X}^1 - 2p^+a_1\tilde{X}^2 \ , \tag{7.6a}$$

$$\tilde{P}^2 = \partial_\tau\tilde{X}^2 \ . \tag{7.6b}$$

The canonical quantization conditions lead to

$$\begin{aligned}
[\tilde{x}^1,\tilde{x}^2] = -\frac{i}{2p^+a_1} \quad &, \quad [\tilde{x}^i,\tilde{p}^j] = 0 \ , \quad [\tilde{p}^1,\tilde{p}^2] = 0 \ , \\
[\tilde{\alpha}^R_n,\tilde{\alpha}^R_m] = [\tilde{\beta}^R_n,\tilde{\beta}^R_m] = \frac{-4n^3}{4n^2-(p^+a_1)^2}\delta_{m+n} \quad &, \quad [\tilde{\alpha}^R_n,\tilde{\beta}^R_m] = ip^+a_1\frac{2n^2}{4n^2-(p^+a_1)^2}\delta_{m+n} \ , \\
[\tilde{\alpha}^L_n,\tilde{\alpha}^L_m] = [\tilde{\beta}^L_n,\tilde{\beta}^L_m] = 0 \quad &, \quad [\tilde{\alpha}^L_n,\tilde{\beta}^L_m] = \frac{4(n+p^+a_1)^2}{2n+p^+a_1}\delta_{m+n} \ .
\end{aligned} \tag{7.7}$$

We note that the "zero-modes", $\tilde{x}^i, \tilde{p}^i$, have unusual commutation relations, e.g. $[\tilde{x}^i,\tilde{p}^j] = 0$, and also the zero-modes of $\tilde{X}^1$ and $\tilde{X}^2$ are non-commuting.

We would like to stress that in the rotating frame coordinates, the light-cone constraints are different from the usual ones

$$p^+\partial_\sigma\tilde{V} = \partial_\tau\tilde{X}^i\partial_\sigma\tilde{X}^i + 2p^+a_1\tilde{X}^2\partial_\sigma\tilde{X}^1 \ , \tag{7.8a}$$

$$2p^+\partial_\tau\tilde{V} = \partial_\tau\tilde{X}^i\partial_\tau\tilde{X}^i + \partial_\sigma\tilde{X}^i\partial_\sigma\tilde{X}^i + 4p^+a_1\tilde{X}^2\partial_\tau\tilde{X}^1 \ , \tag{7.8b}$$

[19] For $p^+a_1 \in \mathbb{Z}$, there is a "zero-mode" for left-movers, which should be extracted in a manner similar to eq. (6.12).

and the light-cone Hamiltonian is

$$\tilde{H}_{LC} = \frac{1}{2p^+} \int d\sigma \left[(\partial_\tau \tilde{X}^i)^2 + (\partial_\sigma \tilde{X}^i)^2 + 4p^+ a_1 \tilde{X}^2 \partial_\sigma \tilde{X}^1 \right] . \tag{7.9}$$

Now we proceed with compactification. Although from the solutions (7.4) and (7.5), it may seem that both the $\tilde{X}^1$ and $\tilde{X}^2$ directions can have zero-modes (i.e., we have translational symmetry along $\tilde{X}^1$ and $\tilde{X}^2$ at the level of the equations of motion(7.2)), it is only possible to compactify the system along the $\tilde{X}^1$ direction. This can be seen from equations (7.6) or (7.8). Putting $\tilde{X}^1$ on a circle of radius R_1, $\tilde{p}^1$ can become non-zero as a winding mode

$$\tilde{p}^1 = w_1 R_1 \ , \quad w_1 \in \mathbb{Z} \ , \tag{7.10}$$

while $\tilde{p}^2$ is still zero as a result of the closed strings boundary conditions.

Since $[\tilde{x}^1, \tilde{p}^1] = 0$, there is no extra quantization condition on $\tilde{p}^1$ due to the momentum modes. In fact the momentum conjugate to $\tilde{x}^1$ is $\tilde{x}^2$ and it should have a discrete spectrum. Noting that $\tilde{x}^1, \tilde{x}^2$ are the center of mass coordinates of strings along $\tilde{X}^1, \tilde{X}^2$, $[\tilde{x}^1, \tilde{x}^2] \neq 0$ means that the $\tilde{x}^1, \tilde{x}^2$ space is a non-commutative cylinder where $\tilde{x}^2$ is along the axis [29], its radius is R_1 and the fuzziness is proportional to $\frac{1}{2p^+ a_1}$. As discussed in [29], $\tilde{x}^2$ has a discrete spectrum, $\tilde{x}^2 \sim \frac{m_1}{2R_1 p^+ a_1}$ with $m_1 \in \mathbb{Z}$. Upon T-dualizing along X^1 and exchanging R_1 with $\frac{1}{R_1}$, it is easy to show that the Hamiltonian remains unchanged if we also exchange m_1 with the winding w_1. In the fermionic sector as usual, duality acts by changing the chirality of one of the fermions. As we have argued in previous sections, this may change the number of supersymmetries.

8 D-Branes on parallelizable plane-waves

In this section we study the existence of BPS Dp-branes in parallelizable pp-waves backgrounds. Dp-branes have been studied in various plane-wave backgrounds (e.g. see [36, 37, 31, 32, 38]). In general there are two ways of addressing the question of D-branes, one is through open strings with Dirichlet boundary conditions, as it was first introduced in [39], or through closed strings and the boundary state formulation [40]. Here we follow the construction via open strings. As we will show, due to presence of the B-field, three different situations can arise.

To start with, let us focus on the bosonic modes. First we note that, as is evident from eq. (6.3), the light-cone gauge condition fixes the boundary condition the or U component to be Neumann. Therefore, U lies inside the worldvolume of all Dp-branes we are going to study. As for the other components, we start with the light-cone action (6.5). This leads to the equations of motion eq. (6.6) and the boundary conditions

$$\int d\tau \delta X^i (\partial_\sigma X^i + p^+ a_1 \epsilon^{ij} X_j)|_{\sigma=0}^{\pi} \ . \tag{8.1}$$

Since the argument for X's along the directions where there is an H-field present is similar, here we consider H_{+12} and the $(1,2)$ plane only. If some of the a_i's in (4.11) are zero, the situation is of course the same as for flat space.

One can recognize three possibilities:

i) X^1 and X^2 are both transverse to brane.

ii) X^1 and X^2 are both inside the brane.

iii) Only X^1 or X^2 is along the brane brane.

Since these cases have been mentioned in [31], we will be very brief with them.

i) This case is realized by imposing Dirichlet boundary conditions on both X^1 and X^2, i.e., $\delta X^i|_{\sigma=0,\pi} = 0$. The boundary conditions force the frequency of the string modes to be integer valued and hence

$$X^1 = (x_1 \cos p^+ a_1 \sigma - x_2 \sin p^+ a_1 \sigma)\sigma + \sum_{n\neq 0} \frac{\sin n\sigma}{n} \big(\alpha_n \cos(n\tau + p^+ a_1 \sigma) - \beta_n \sin(n\tau + p^+ a_1 \sigma)\big)$$

$$X^2 = (x_2 \cos p^+ a_1 \sigma + x_1 \sin p^+ a_1 \sigma)\sigma + \sum_{n\neq 0} \frac{\sin n\sigma}{n} \big(\alpha_n \sin(n\tau + p^+ a_1 \sigma) - \beta_n \cos(n\tau + p^+ a_1 \sigma)\big)$$

(Once $p^+ a_1 \in \mathbb{Z}$ we have extra zero-modes.) Upon quantization we find

$$[\alpha_n, \alpha_m] = [\beta_m, \beta_n] = n\delta_{m+n} \ . \tag{8.2}$$

In this case the V direction also satisfies a Neumann boundary condition and hence lies inside the brane. The brane is located in such a way that H_{+12} only has one leg along the brane. As discussed in [32, 33], the theory living on such branes is a "dipole" gauge theory. As seen from the above mode expansions, these branes are stuck at $X^1 = X^2 = 0$. In general, following [36, 31], it is straightforward to show that these branes preserve half of the kinematical and half of the non-kinematical supersymmetries.

ii) In this case, to satisfy the boundary conditions we demand that

$$\partial_\sigma X^i + p^+ a_1 \epsilon^{ij} X_j|_{\sigma=0,\pi} = 0 \ , \tag{8.3}$$

which is a modified Neumann boundary condition. It is well known that the existence of a background B-field will change the boundary conditions. The V direction should also satisfy a modified boundary condition

$$\partial_\sigma V + p^+ a_1 \epsilon^{ij} X_j \partial_\tau X^j|_{\sigma=0,\pi} = 0 \ . \tag{8.4}$$

The modified boundary conditions (8.3) and (8.4) may be understood from the fact that the brane now contains U, V, X^1, X^2, and the H_{+12} field (which lies completely inside the brane) can be treated as a background electric field equal to $p^+ a_1 x^2$ on the brane. For the low energy theory on such branes we expect to find just a gauge theory with a non-constant background electric field.

The frequencies of the normal modes satisfy the boundary conditions (8.3), similar to the case *i)*, and are found to be integer valued and hence the mode expansions are

$$\begin{aligned}
X^1 &= x_1 \cos p^+ a_1 \sigma - x_2 \sin p^+ a_1 \sigma + \sum_{n\neq 0} \frac{\cos n\sigma}{n} \left(\alpha_n \cos(n\tau + p^+ a_1 \sigma) - \beta_n \sin(n\tau + p^+ a_1 \sigma)\right) \\
X^2 &= x_2 \cos p^+ a_1 \sigma + x_1 \sin p^+ a_1 \sigma + \sum_{n\neq 0} \frac{\cos n\sigma}{n} \left(\alpha_n \sin(n\tau + p^+ a_1 \sigma) - \beta_n \cos(n\tau + p^+ a_1 \sigma)\right)
\end{aligned}$$

iii) In this case, we demand that X^1 satisfy Neumann b.c.'s, $\partial_\sigma X^1|_{\sigma=0,\pi} = 0$ and X^2 to satisfy Dirichelet b.c.'s, $\delta X^2 = 0$. It is not hard to check that, as a result, V must satisfy the usual Neumann boundary condition, $\partial_\sigma V|_{\sigma=0,\pi} = 0$. One of the differences between this case and the previous two cases is that the frequencies of the string modes are now non-integer

$$\omega_n^\pm = n \pm p^+ a_1 \ , \tag{8.5}$$

with the mode expansions

$$X^1 = \frac{1}{2} \sum_{n\in\mathbb{Z}} \left(\frac{1}{\omega_n^+} \alpha_n e^{i\omega_n^+ \tau} - \frac{1}{\omega_n^-} \beta_n e^{i\omega_n^- \tau} \right) \cos n\sigma \ , \tag{8.6a}$$

$$X^2 = \frac{1}{2} \sum_{n\in\mathbb{Z}} \left(\frac{1}{\omega_n^+} \alpha_n e^{i\omega_n^+ \tau} + \frac{1}{\omega_n^-} \beta_n e^{i\omega_n^- \tau} \right) \sin n\sigma \ , \tag{8.6b}$$

with the reality condition $\beta_n^\dagger = \alpha_{-n}$. Imposing the quantization conditions (6.15) leads to

$$[\alpha_n, \beta_m] = \omega_n^+ \delta_{m+n} \ . \tag{8.7}$$

We should note that the above mode expansion is for $p^+a_1 \notin \mathbb{Z}$. For integer values of p^+a_1, however, we have a zero-mode and the expansion is

$$X^1 = (x + p\tau)\cos p^+ a_1 \sigma + \sum_{n \neq p^+ a_1} Oscil. \tag{8.8a}$$

$$X^2 = (x + p\tau)\sin p^+ a_1 \sigma + \sum_{n \neq p^+ a_1} Oscil. \tag{8.8b}$$

where upon quantization we obtain $[x, p] = i$.

In this case the brane is located so that the H_{+12} field has two legs along the brane and one transverse to it. In the gauge $B_{+1} = p^+ a_1 X^2$, the B-field resides completely inside the brane, however, the value of the B-field is zero exactly on the brane which is necessarily at $X^2 = 0$. Again the B_{+1}-field can be understood as a background electric field on the brane, which is now proportional to one of the scalar fields (the "transverse" directions to the brane). This in particular gives a mass to that scalar field, so that its lowest excitation is no longer massless. This can also be observed from the mode expansion (8.6). A more detailed analysis of the theory living on these branes is postponed to future works.

In principle, one can also work out the fermionic modes, but since the computations are very similar to those which appear in [31], we do not present them here.

Finally, we would like to note that it is possible to have a combination of the three cases discussed above. For example, in the the background $a_1 = a_2 = a_3 = a_4 \neq 0$, with 28 supersymmetries, we might have a D5-brane along the $UV1345$ directions, a mixture of all three cases.

9 Discussion

In this paper we classified and studied, to some extent, the parallelizable pp-waves. We first briefly studied implications of parallelizability for a general supergravity solution and proved that the vanishing of the gravitino variation for parallelizable backgrounds can be solved with 32 independent solution. One should note that the converse is not true, that is, not all the cases for which the gravitino variation has 32 solutions are parallelizable, for example, the famous $AdS_5 \times S^5$ background.

Strings on AdS_3 have been studied in detail using the SL(2,$\mathbb{R}$) group manifold and the corresponding Kac-Moodi algebra [41]. It would be very interesting to extend the definition of the WZW models and the Maldacena-Ooguri setup to the S^7 case; although it is not a group manifold, it has a nice (non-associative) algebraic (octonionic) structure. This may help clarify the issue of what the (gauge) field theory dual to strings on $AdS_{3,7} \times S^{7,3}$ might be.

We then turned to parallelizable pp-waves and proved that all parallelizable pp-waves are necessarily homogeneous plane-waves and the converse is true if the μ_{ij} in the du^2 term ($\mu_{ij} x^i x^j$) has doubly degenerate eigenvalues. The parallelizable pp-waves may be classified by their supersymmetry, where for type IIB the maximal supersymmetry is 28 and the others differ by steps of four (down to 16), while type IIA supersymmetry may have 24, 22, 20, 18 and 16 supercharges. We also discussed the invariance of the bosonic sector of parallelizable pp-waves under T-duality and discussed how the fermions and number of supercharges may change under T-duality.

We studied string theory on the parallelizable pp-wave backgrounds and showed that the sigma model is simpler than for other plane-wave backgrounds. This simplicity might help in working out the vertex operators, making it possible to study string scattering processes and to evaluate the S-matrix elements, whose existence for plane-wave backgrounds has been argued in [42]. Working out the proper vertex operators and string scattering amplitudes is another interesting open question we postpone to future works.

We also very briefly discussed the half BPS Dp-branes and the restrictions on the possible Dirichelet or Neumann boundary conditions arising from parallelizable pp-wave backgrounds. The classification of possible Dp-branes, branes at angles, intersecting branes and most importantly, the theory residing on branes in the parallelizable pp-wave backgrounds and the corresponding supergravity solutions, along the lines of [43, 44], deserve more detailed analysis.

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APPENDIX - Conventions

We briefly review our conventions in this appendix. We use the mostly plus metric. Greek indices $\mu, \nu, ...$ range over the curved (world) indices, while Latin indices $a, b, ...$ denote tangent space indices and i, j label coordinates on the space transverse to the light-cone directions. The curved space Gamma matrices are defined via contraction with vierbeins as usual, $\Gamma^\mu = e^\mu_a \Gamma^a$.

We may rewrite the two Majorana-Weyl spinors in ten dimensional type IIA and IIB theories as a pair of Majorana spinors $\chi^\alpha, \alpha = 1, 2$, subject to the chirality conditions appropriate to the theory,

$$\Gamma^{11}\,\chi^1 \;=\; +\,\chi^1 \;, \qquad\qquad \Gamma^{11}\,\chi^2 \;=\; \pm\,\chi^2 \;, \tag{A-1}$$

where for the second spinor we choose $-$ for non-chiral type IIA and $+$ for chiral type IIB theories, and treat the index α labeling the spinor as an SL(2,$\mathbb{R}$) index. Where Pauli matrices appear, they act on this auxiliary index, with σ_3 acting analogously to the chirality operator in type IIA. In what follows, we suppress the spinor index as well as the auxiliary index.

Type II string theories contain two Majorana-Weyl gravitinos ψ^α_μ, $\alpha = 1, 2$, which are of the same (opposite) chirality in IIB (IIA). The supersymmetry variation of these gravitinos in string frame is

$$\delta\psi_\mu \;=\; \hat{\mathfrak{D}}_\mu \epsilon, \tag{A-2}$$

where the supercovariant derivative is defined as [27, 45]

$$\hat{\mathfrak{D}}_\mu = \nabla_\mu + \frac{1}{8}\sigma_3 \Gamma^{ab} H_{\mu ab} - \frac{1}{16} e^\phi \left(\sigma_3 \Gamma^{ab} F_{ab} - \frac{1}{12} \Gamma^{abcd} F_{abcd} \right) \Gamma_\mu \;, \tag{A-3}$$

for type IIA theory and

$$\hat{\mathfrak{D}}_\mu = \nabla_\mu + \frac{1}{8}\sigma_3 \Gamma^{ab} H_{\mu ab} + \frac{i}{8} e^\phi \left(\sigma_2 \Gamma^a \partial_a \chi - \frac{i}{6} \sigma_1 \Gamma^{abc} F_{abc} + \frac{1}{240} \sigma_2 \Gamma^{abcde} F_{abcde} \right) \Gamma_\mu \;, \tag{A-4}$$

for type IIB, with the spin connection ω^{ab}_μ appearing in the covariant derivative $\nabla_\mu = (\partial_\mu + \frac{1}{4}\omega^{ab}_\mu \Gamma_{ab})$. In these expressions, ϕ is the dilaton, χ is the axion, H the three-form field strength from the NS-NS sector, and the F's represent the appropriate RR field strengths.

For pure NS-NS backgrounds relevant to this paper, the dilatino variation for both types IIA and IIB, is

$$\delta\lambda \;=\; \left(\frac{1}{2} \Gamma^a (\partial_a \phi) \;-\; \frac{1}{4} \Gamma^{abc} H_{abc} \sigma_3 \right) \epsilon \;. \tag{A-5}$$

A convenient choice of basis for 32×32 Dirac matrices, which we denote by Γ^μ, can be written in terms of 16×16 matrices γ^μ such that

$$\Gamma^+ = i\begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad \Gamma^- = i\begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} \gamma^i & 0 \\ 0 & -\gamma^i \end{pmatrix}, \quad \Gamma^{11} = \begin{pmatrix} \gamma^{(8)} & 0 \\ 0 & -\gamma^{(8)} \end{pmatrix}, \tag{A-6}$$

and the γ^i satisfying $\{\gamma^i, \gamma^j\} = 2\delta_{ij}$ with δ_{ij} the metric on the transverse space. Choosing a chiral basis for the γ's, we have $\gamma^{(8)} = diag(1_8, -1_8)$. The above matrices satisfy

$$\begin{gathered} (\Gamma^+)^\dagger \;=\; -\Gamma^-, \qquad (\Gamma^-)^\dagger \;=\; -\Gamma^+, \qquad (\Gamma^+)^2 = (\Gamma^-)^2 = 0, \\ \left[\Gamma^+, \not{H}_+\right] = 0, \qquad \{\Gamma^{11}, \Gamma^\pm\} = 0, \qquad \{\Gamma^{11}, \Gamma^i\} \;=\; 0, \qquad \left[\Gamma^\pm, \Gamma^{ij}\right] \;=\; 0 \;, \end{gathered} \tag{A-7}$$

and $\Gamma^\pm \Gamma^i ... \Gamma^j \Gamma^\pm = 0$ if the same signs appearing on both sides.

We define light-cone coordinates $x^\pm = (x^0 \pm x^9)/\sqrt{2}$ and likewise for the light-like Gamma matrices $\Gamma^\pm = (\Gamma^0 \pm \Gamma^9)/\sqrt{2}$, and also define antisymmetric products of γ matrices with weight one, $\gamma^{ab...cd} \equiv \gamma^{[a}\gamma^b ... \gamma^c\gamma^{d]}$, choosing for the γ matrices a representation such that

$$\gamma^{12} \;=\; \;\; i\, diag(++--,++--,++--,++--) \;, \tag{A-8a}$$

$$\gamma^{34} \;=\; -i\, diag(++--,--++,++--,--++) \;, \tag{A-8b}$$

$$\gamma^{56} \;=\; \;\; i\, diag(+-+-,+-+-,+-+-,+-+-) \;, \tag{A-8c}$$

$$\gamma^{78} \;=\; -i\, diag(+-+-,-+-+,-+-+,+-+-) \;, \tag{A-8d}$$

for which

$$\gamma^{12345678} \;\equiv\; \gamma^{(8)} \;=\; diag(++++,++++,----,----) \;. \tag{A-9}$$

The following combination appears in the paper, which we write using the results of section 4.1 (in particular (4.11)),

$$\not{H}_+ \;\equiv\; \Gamma^{ij} H_{+ij} \;=\; \begin{pmatrix} \gamma^{ij} H_{+ij} & 0 \\ 0 & \gamma^{ij} H_{+ij} \end{pmatrix} \;, \tag{A-10}$$

and

$$\begin{aligned} \gamma^{ij} H_{+ij} \;&=\; 2\left(a_1\gamma^{12} + a_2\gamma^{34} + a_3\gamma^{56} + a_4\gamma^{78}\right) \\ &=\; 2i\, diag(c_1, c_2, -c_2, -c_1, c_3, c_4, -c_4, -c_3, c_5, c_6, -c_6, -c_5, c_7, c_8, -c_8, -c_7) \;, \end{aligned} \tag{A-11}$$

where we also define the constants

$$c_1 = a_1 - a_2 + a_3 - a_4 \ , \tag{A-12a}$$

$$c_2 = a_1 - a_2 - a_3 + a_4 \ , \tag{A-12b}$$

$$c_3 = a_1 + a_2 + a_3 + a_4 \ , \tag{A-12c}$$

$$c_4 = a_1 + a_2 - a_3 - a_4 \ , \tag{A-12d}$$

$$c_5 = a_1 - a_2 + a_3 + a_4 \ , \tag{A-12e}$$

$$c_6 = a_1 - a_2 - a_3 - a_4 \ , \tag{A-12f}$$

$$c_7 = a_1 + a_2 + a_3 - a_4 \ , \tag{A-12g}$$

$$c_8 = a_1 + a_2 - a_3 + a_4 \ . \tag{A-12h}$$

References

[1] G. T. Horowitz and A. R. Steif, "Space-Time Singularities In String Theory," *Phys. Rev. Lett.* **64**, 260 (1990).

[2] R. Penrose, "Any Spacetime Has a Plane Wave Limit," Differential Geometry and Relativity, Reidel, Dordrecht, 1976, pp. 271.

[3] V. Hubeny, M. Rangamani, E. Verlinde, "Penrose Limits and Non-local theories," *JHEP* **0210** (2002) 020, hep-th/0205258.

[4] R. Gueven, "Plane wave limits and T-duality," *Phys. Lett.* **B482**, 255 (2000), hep-th/0005061.

[5] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "Penrose limits and maximal supersymmetry," *Class. Quant. Grav.* **19** (2002) L87, hep-th/0201081;

M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, "Penrose limits, supergravity and brane dynamics," *Class. Quant. Grav.* **19**, 4753 (2002), hep-th/0202111.

[6] J. Figueroa-O'Farrill and G. Papadopoulos, "Maximally supersymmetric solutions of ten-dimensional and eleven-dimensional supergravities," *JHEP* **0303**, 048 (2003), hep-th/0211089.

[7] O. Jofre, C. Nunez, "Strings in Plane Wave Backgrounds Revisited," *Phys.Rev.* **D50** (1994) 5232, hep-th/9311187;

R.R.Metsaev, "Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background," *Nucl.Phys.* **B625** (2002) 70;

R.R. Metsaev, A.A. Tseytlin, "Exactly solvable model of superstring in Ramond-Ramond plane wave background," *Phys.Rev.* **D65** (2002) 126004, hep-th/0202109.

[8] D. Berenstein, J. Maldacena, H. Nastase, "Strings in flat space and pp waves from $\mathcal{N} = 4$ Super Yang Mills," *JHEP* **0204** (2002) 013, hep-th/0202021.

[9] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov, W. Skiba, "PP-wave string interactions from perturbative Yang-Mills theory," *JHEP* **0207** (2002) 017, hep-th/0205089;

A. Santambrogio, D. Zanon, "Exact anomalous dimensions of N=4 Yang-Mills operators with large R charge," *Phys. Lett.* **B545** (2002) 425, hep-th/0206079.

[10] For example see:

N. Itzhaki, Igor R. Klebanov, Sunil Mukhi, "PP Wave Limit and Enhanced Supersymmetry in Gauge Theories," *JHEP* **0203** (2002) 048, hep-th/0202153;

L. A. Pando Zayas, J. Sonnenschein, "On Penrose Limits and Gauge Theories," *JHEP* **0205** (2002) 010, hep-th/0202186;

M. Alishahiha, M. M. Sheikh-Jabbari, "The PP-Wave Limits of Orbifolded $AdS_5 \times S^5$," *Phys.Lett.* **B535** (2002) 328, hep-th/0203018.

[11] R. R. Metsaev and A. A. Tseytlin, "Order Alpha-Prime (Two Loop) Equivalence Of The String Equations Of Motion And The Sigma Model Weyl Invariance Conditions: Dependence On The Dilaton And The Antisymmetric Tensor," *Nucl. Phys.* **B293**, 385 (1987).

[12] L. Pando-Zayas, "Ricci-Parallelizable Spaces in the NS-NS Sector: $AdS_3 \times S^7$," hep-th/0007134.

[13] I. Bena and R. Roiban, "Supergravity pp-wave solutions with 28 and 24 supercharges," hep-th/0206195.

[14] J. Michelson, "(Twisted) Toroidal Compactification of pp-Waves," *Phys. Rev.* **D66** (2002) 066002, hep-th/0203140.

[15] V. Hubeny, M. Rangamani, "No horizons in pp-waves," *JHEP* **0211** (2002) 021, hep-th/0210234; "Causal structures of pp-waves," *JHEP* **0212** (2002) 043, hep-th/0211195.

[16] M. J. Duff, B. E. Nilsson and C. N. Pope, "Kaluza-Klein Supergravity," *Phys. Rept.* **130**, 1 (1986).

E. Braaten, T. L. Curtright and C. K. Zachos, "Torsion And Geometrostasis In Nonlinear Sigma Models," *Nucl. Phys.* **B260**, 630 (1985).

[17] M. Fabinger, S. Hellerman, "Stringy Resolutions of Null Singularities," hep-th/0212223.

[18] M. Blau and M. O'Loughlin, "Homogeneous plane waves," *Nucl. Phys.* **B 654**, 135 (2003), hep-th/0212135.

[19] J. F. Adams, "On the non-existence of elements of Hopf invariant one," *Ann. Math.* **72**, 20 (1960).

[20] S. Okubo, "Introduction to Octonion and Other Non-Associative Algebras in Physics," Cambridge University Press, 1995.

[21] E. Cartan and J. Schouten, Proc. K. Ned. Akad. Wet. Amsterdam 29 (1926) 803, 933.

[22] Charles W. Misner, Kip S. Thorne and John Archibald Wheeler, "Gravitation," W. H. Freeman, San Francisco, 1973.

[23] G. T. Horowitz, J. M. Maldacena and A. Strominger, "Nonextremal Black Hole Microstates and U-duality," *Phys. Lett.* **B383**, 151 (1996), hep-th/9603109.

[24] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)], hep-th/9711200.

[25] F. Gursey and C. H. Tze, "Octonionic Torsion On S(7) And Englert's Compactification Of D = 11 Supergravity," *Phys. Lett.* **B127**, 191 (1983).

[26] M. Gunaydin and F. Gursey, "Quark Structure And Octonions," J. Math. Phys. **14**, 1651 (1973).

[27] M. Cvetic, H. Lu, C. N. Pope and K. S. Stelle, "Linearly-realised worldsheet supersymmetry in pp-wave background," hep-th/0209193.

[28] C. Duval and G. Gibbons, "Celestial mechanics, conformal structures, and gravitational waves," *Phys. Rev.* **D43** (1991) 3907.

[29] M. Chaichian, A. Demichev, P. Presnajder, M.M. Sheikh-Jabbari, A. Tureanu, "Quantum Theories on Noncommutative Spaces with Non-trivial Topology: Aharonov-Bohm and Casimir Effects," *Nucl. Phys.* **B611** (2001) 383, hep-th/0101209.

[30] J.G. Russo, A.A. Tseytlin, "On solvable models of type IIB superstring in NS-NS and R-R plane wave backgrounds," *JHEP* **0204** (2002) 021, hep-th/0202179.

S. Stanciu and J. Figueroa-O'Farrill, "Penrose limits of Lie branes and a Nappi-Witten braneworld," hep-th/0303212.

[31] Y. Michishita, "D-branes in NSNS and RR pp-wave backgrounds and S-duality," *JHEP* **0210** (2002) 048, hep-th/0206131.

[32] O. J. Ganor, U. Varadarajan, "Nonlocal Effects on D-branes in Plane-Wave Backgrounds," *JHEP* **0211** (2002) 051, hep-th/0210035.

[33] M. Alishahiha, O. J. Ganor, "Twisted Backgrounds, PP-Waves and Nonlocal Field Theories," hep-th/0301080.

[34] E. Bergshoeff, C.M. Hull, T. Ortin, "Duality in the Type–II Superstring Effective Action," *Nucl. Phys.* **B451** (1995) 547, hep-th/9504081.

[35] Y. Hikida, "Superstrings on NSNS PP-Waves and Their CFT Duals," hep-th/0303222.

[36] A. Dabholkar, S. Parvizi, "Branes in PP-wave Background," *Nucl.Phys.* **B641** (2002) 223, hep-th/0203231.

[37] O. Bergman, M. R. Gaberdiel, M. B. Green, "D-brane interactions in type IIB plane-wave background," hep-th/0205183.

[38] J. Kim, B-H. Lee, H. S. Yang, "Superstrings and D-branes in Plane Wave," hep-th/0302060.

[39] Joseph Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges," *Phys. Rev. Lett.* **75** (1995) 4724, hep-th/9510017.

[40] C. G. Callan, Jr., I. R. Klebanov, " D-Brane Boundary State Dynamics," *Nucl.Phys.* **B465** (1996) 473, hep-th/9511173.

[41] J. Maldacena, H. Ooguri, "Strings in AdS_3 and the $SL(2,R)$ WZW Model. Part 1: The Spectrum," *J. Math. Phys.* **42** (2001) 2929, hep-th/0001053; "Strings in AdS_3 and the $SL(2,R)$ WZW Model. Part 2: Euclidean Black Hole," *J. Math. Phys.* **42** (2001) 2961,

hep-th/0005183; "Strings in AdS_3 and the $SL(2, R)$ WZW Model. Part 3: Correlation Functions," *Phys. Rev.***D65** (2002) 106006, hep-th/0111180.

[42] D. Bak, M. M. Sheikh-Jabbari, "Strong Evidence In Favor OF The Existence Of S-Matrix For Strings In Plane Waves," *JHEP* **0302** (2003) 019, hep-th/0211073;

K. Bobkov,"Graviton-Scalar Interaction in the PP-Wave Background," hep-th/0303007.

[43] A. Kumar, R. R. Nayak and Sanjay, "D-brane solutions in pp-wave background," *Phys. Lett.* **B541**, 183 (2002), hep-th/0204025.

[44] M. Alishahiha and A. Kumar, "D-brane solutions from new isometries of pp-waves," *Phys. Lett.* **B542**, 130 (2002), hep-th/0205134.

[45] M. Alishahiha, M. A. Ganjali, A. Ghodsi and S. Parvizi, "On type IIA string theory on the PP-wave background," hep-th/0207037.