SLAC-PUB-9627

January 2003

Single-mode coherent synchrotron radiation instability

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Abstract

The microwave instability driven by the coherent synchrotron radiation (CSR) has been previously studied [1] neglecting effect of the shielding caused by the finite beam pipe aperture. In practice, the unstable mode can be close to the shielding threshold where the spectrum of the radiation in a toroidal beam pipe is discrete. In this paper, the CSR instability is studied in the case when it is driven by a single synchronous mode. A system of equations for the beam-wave interaction is derived and its similarity to the 1D FEL theory is demonstrated. In the linear regime, the growth rate of the instability is obtained and a transition to the case of continuous spectrum is discussed. The nonlinear evolution of the single-mode instability, both with and without synchrotron damping and quantum diffusion, is also studied.

Submitted to Physical Review Special Topics–Accelerators and Beams

*Work supported in part by Department of Energy Contract DE-AC03-76SF00515.

I. INTRODUCTION

A relativistic electron beam moving in a circular orbit in free space can radiate coherently if the wavelength of the synchrotron radiation exceeds the length of the bunch. In accelerators coherent radiation of the bunch is usually suppressed by the screening effect of the conducting walls of the vacuum chamber [2–4]. The screening effect is much less effective for short wavelengths, but if the wavelength is shorter than the length of the bunch (assuming a smooth beam profile), the coherent radiation becomes exponentially small. However, an initial density fluctuation with a characteristic length much shorter than the screening threshold would radiate coherently. If the radiation reaction force is directed so that it drives the growth of the initial fluctuation, one can expect an instability that leads to micro-bunching of the beam and an increased coherent radiation at short wavelengths.

In Ref. [1] the growth rate of the beam instability driven by the coherent synchrotron radiation (CSR) was found using the so called "CSR impedance" [5, 6] that neglects the shielding effect of the walls and assumes a continuous spectrum of radiation. The maximum growth rate was found to correspond to the wavenumber $k = \omega/c$ of the order of $k \sim \Lambda^{2/3} R^{-1}$, where

$$\Lambda = \frac{n_b r_e}{|\eta| \gamma \delta_0^2} \,. \tag{1}$$

Here n_b is the linear bunch density, η is the momentum compaction factor, δ_0 is the rms energy spread in the beam, r_e and γ are classical electron radius and relativistic factor, respectively. For small Λ, the instability is limited to relatively long wavelength where it may be affected by the wall shielding effect [2]. Close to the shielding threshold, one has to take into account that spectrum of the synchronous modes of radiation is discrete, and the instability may be driven by a single synchronous mode rather than a continuous spectrum.

In this paper we study such a single-mode CSR instability. As in Ref. [1], we assume that the bunch is much longer than the wavelength of the modulation and consider a coasting beam model.

The paper is organized as follows. In Section II be briefly review properties of travelling modes in a toroidal waveguide. In Section III we derive linear equations for the beam instability, and in Section IV we solve the dispersion relation and find the growth rate of the instability. In Section V we discuss the transition from a single-mode to multi-mode regime of the instability. In Section VI we derive a system of equations for the evolution of the instability in nonlinear regime, and in Section VII we add to this system terms responsible for synchrotron damping and quantum diffusion. A numerical solution to this system is then obtained that demonstrates continuous growth of the wave amplitude on a long time scale. We also find an approximate asymptotic solution to the system and show a good agreement the numerical one. We summarize the main results of the paper in the last Section VIII.

II. SYNCHRONOUS MODES IN TOROIDAL BEAM PIPE CLOSE TO SHIELDING THRESHOLD

A relativistic beam moving in a toroidal beam pipe interacts with synchronous modes that have phase velocity equal to the speed of light. Such modes in a toroidal pipe have been extensively studied in the past [4, 7, 8]. Recently, a new approach to the problem [9] extended the previous analysis and allowed to treat arbitrary cross section of the toroid.

Following Ref. [9], we assume that the characteristic size of the pipe cross section a is much smaller than the toroid radius R, so that the ratio $\sqrt{a/R}$ is a small parameter. For a given toroid, the synchronous modes have wavenumbers k greater than a minimal value $k_{\mathrm{min}} = \omega_{\mathrm{min}}/c$:

$$k \geq \frac{\omega_{\mathrm{min}}}{c} \sim \frac{R^{1/2}}{a^{3/2}} \gg a^{-1} .$$

The lowest synchronous mode wavenumber is of order of k_0, where

$$k_0 = \frac{\pi}{a}\sqrt{\frac{R}{a}} .$$

For example, for a beam pipe of a square cross-section with the side a, $k_{\mathrm{min}} = 1.52\, k_0$. The loss factor per unit length χ and the group velocity v_g for this mode are

$$\chi = \frac{4.94}{a^2}, \quad 1 - \frac{v_g}{c} = 0.62\, \frac{a}{R} . \tag{2}$$

Note that such modes propagate with the group velocity close to the speed of light. The next mode with a nonzero loss factor has a frequency $\omega_2 = 2.79\, ck_0$ and the loss factor

$\chi = 3.01/a^2$. We emphasize here that the distance between the synchronous modes in the vicinity of $\omega_{\rm min}$ is of the order of their frequency, and in that sense the modes are well separated on the frequency scale. Similar results hold for the round toroidal pipe [9].

III. INTERACTION OF THE BEAM WITH A SINGLE SYNCHRONOUS MODE IN LINEAR APPROXIMATION

The interaction of the beam with electromagnetic waves is usually described in terms of the beam impedance (see, *e.g.*, [10]). For discrete synchronous modes, the beam impedance has singularities centered at the mode frequencies. In this case, a direct application of the standard approach, as we show in Appendix A, may give an incorrect result. In this section, we derive the governing equations describing this interaction starting from the Maxwell-Vlasov system of equations without using the concept of the impedance.

We use a one dimensional model for the beam, neglecting the effect of the finite transverse emittance and considering a distribution function $f(z, \delta, t)$, where z is the longitudinal coordinate measured from a reference particle moving with the speed of light, and δ is the energy offset relative to the nominal energy E_0, $\delta = (E - E_0)/E_0$. We also assume that the modulation wavelength is small compared to the bunch length and consider a coasting beam with the linear density n_b equal to the local linear density of the bunch.

In the linear approximation, the perturbation due to the electromagnetic field can be considered as small:

$$f = f_0(\delta) + f_1(z, \delta, t)\,,$$

with $f_1 \ll f_0$. The linearized Vlasov equation for f_1 is

$$\frac{\partial f_1}{\partial t} - \eta c \delta \frac{\partial f_1}{\partial z} + \frac{e}{\gamma m c} \mathcal{E}(z, t) \frac{\partial f_0}{\partial \delta} = 0\,, \qquad (3)$$

where $\gamma m c^2$ is the nominal beam energy and $\mathcal{E}(z, t)$ is the longitudinal component of the electric field. The function f is normalized so that $\int f dz d\delta$ gives the number of particles in the beam. For what follows, it is convenient to introduce the Fourier transform g_1 of the perturbation of the distribution function

$$g_1(\omega, q, \delta) = \int dt dz\, e^{-i(qz-\omega t)} f_1(z, \delta, t)\,. \tag{4}$$

The electromagnetic field is excited by the beam current. Let us consider a Fourier component of the field with the frequency ω:

$$\boldsymbol{E}_\omega(\boldsymbol{r}, s) = \int dt\, e^{i\omega t} \boldsymbol{E}(\boldsymbol{r}, s, t)\,. \tag{5}$$

where s is the arc length along the beam path and $\boldsymbol{r} = (x, y)$ is the two-dimensional vector in the transverse plane perpendicular to the orbit. This field can be represented as a sum of toroidal modes in an empty waveguide [11]. Assuming that the electric and magnetic fields of the n-th mode of frequency ω are given by

$$\boldsymbol{E}_n(\boldsymbol{r}, s) = \boldsymbol{e}_n(\boldsymbol{r}) e^{-i\omega t + iq(n,\omega)s}\,, \quad \boldsymbol{H}_n(\boldsymbol{r}, s) = \boldsymbol{h}_n(\boldsymbol{r}) e^{-i\omega t + iq(n,\omega)s}\,, \tag{6}$$

where $q(n, \omega)$ is the wavenumber of the n-th mode, we have

$$\boldsymbol{E}_\omega(\boldsymbol{r}, s) = \sum_n C_n(s) \boldsymbol{e}_n(\boldsymbol{r})\,, \quad \boldsymbol{H}_\omega(\boldsymbol{r}, s) = \sum_n C_n(s) \boldsymbol{h}_n(\boldsymbol{r})\,. \tag{7}$$

In these equations, $\boldsymbol{e}_n$ and $\boldsymbol{h}_n$ describe the transverse distribution of the electric and magnetic fields in the mode, respectively, and $C_n(s)$ is the varying in space complex amplitude of the mode. Note that the quantity $\mathcal{E}(z, t)$ is equal to the longitudinal component of $\boldsymbol{E}(\boldsymbol{r}, s, t)$ taken at the location $s = ct + z$ on the axis $\boldsymbol{r} = 0$, $\mathcal{E}(z, t) = E_s(0, s = ct + z, t)$. The Fourier coefficient $C_n(q, \omega)$ of the amplitude is defined by the following equation:

$$C_n(q, \omega) = \int_{-\infty}^{\infty} ds\, e^{-iqs}\, C_n(s)\,. \tag{8}$$

The coefficients $C_n(s)$ can be related to the function g_1 by means of the Lorentz reciprocity theorem [12]:

$$\int d\boldsymbol{S}(\boldsymbol{E}_\omega \times \boldsymbol{H}_n^* + \boldsymbol{E}_n^* \times \boldsymbol{H}_\omega) = -Z_0 \int dV \boldsymbol{j}_\omega \cdot \boldsymbol{E}_n^*\,, \tag{9}$$

where $Z_0 = 4\pi/c$ is the free space impedance. This equation allows us to find the coefficients $C_n(q, \omega)$ in terms of the Fourier component $\boldsymbol{j}_\omega$ of the beam current density. For a filament beam current moving along the axis s, $\boldsymbol{j}_\omega = \boldsymbol{s}\, I_\omega\, \delta(x)\delta(y)$, where I_ω is the frequency component of the current and $\boldsymbol{s}$ is the unit vector in the direction of the beam motion. For ideal conductivity of the wall, integration in Eq. (9) over the volume of the beam pipe between two cross sections $s = s_1$ and $s = s_2$ gives

$$\int \frac{dq}{2\pi} C_n(q,\omega) \left(e^{i[q-q(n,\omega)]s_2} - e^{i[q-q(n,\omega)]s_1}\right)$$
$$= -\frac{1}{N_n} (\boldsymbol{e}_n^*(0) \cdot \boldsymbol{s}) \int_{s_1}^{s_2} ds' I_\omega(s') e^{-iq(n,\omega)s'} . \tag{10}$$

Here N_n is the norm of the n-th mode

$$N_n = \frac{1}{Z_0} \int d\boldsymbol{S}(\boldsymbol{e}_n \times \boldsymbol{h}_n^* + \text{c.c.}) , \tag{11}$$

where the integration goes over the cross section of the pipe and "c.c" stands for the complex conjugate term. The norm N_n does not depend on the location of the cross section in the integral of Eq. (11) and is equal to four times the energy flow in the mode [11].

The beam current $I(s,t)$ can be obtained by integrating the distribution function:

$$I(s,t) = ec \int d\delta f_1(s - ct, \delta, t) ,$$

where we used the relation $z = s - ct$. For the Fourier component $I_\omega(s)$ one finds

$$I_\omega(s) = ec \int \frac{dq}{2\pi} e^{iqs} \int d\delta g_1(\omega - qc, q, \delta) . \tag{12}$$

The amplitudes C_n can be found from Eqs. (10) and (12)

$$C_n(q,\omega) = \frac{i}{N_n} \frac{ec\,(\boldsymbol{e}_n^*(0) \cdot \boldsymbol{s})}{q - q(n,\omega) - i\epsilon} \int d\delta g_1(\omega - qc, q, \delta) . \tag{13}$$

The infinitely small $\epsilon > 0$ in this equation takes into account casuality. Making Fourier transform of Eq. (3) and using Eqs. (5), (7) and (8) yields

$$(\omega + \eta c\delta q)\, g_1(\omega, q, \delta) = -i \frac{e}{\gamma mc} \frac{\partial f_0}{\partial \delta} \sum_n (\boldsymbol{s} \cdot \boldsymbol{e}_n(0))\, C_n(q, \omega + qc) . \tag{14}$$

Substituting then Eq. (13) into Eq. (14) gives

$$(\omega + \eta c\delta q)\, g_1(\omega, q, \delta) = \frac{r_e c^2}{\gamma} \frac{\partial f_0}{\partial \delta} \sum_n \frac{(1-\beta_g)\chi}{v_g(q - q(n, \omega + qc) - i\epsilon)} \int d\delta' g_1(\omega, q, \delta') , \tag{15}$$

where $\beta_g = v_g/c$, $r_e = e^2/mc^2$, v_g is the group velocity , and χ is the loss factor associated with the n-th mode [9]:

$$\chi = \frac{v_g}{1-\beta_g} \frac{|(\boldsymbol{s} \cdot \boldsymbol{e}_n(0))|^2}{N_n} .$$

It follows from this equation, that the dependence of g_1 on δ can be factored out

$$g_1(\omega, q, \delta) = \frac{G(\omega, q)}{\omega + \eta c \delta q} \frac{\partial f_0}{\partial \delta} ,$$

where the function G does not depend on δ. Putting this equation into Eq. (15) yields the dispersion equation

$$1 = \sum_n \frac{\lambda}{q - q(n, \omega + qc) + i\epsilon} \int d\delta \frac{\partial f_0 / \partial \delta}{\omega + \eta c q \delta} , \tag{16}$$

where

$$\lambda = \frac{r_e c^2}{\gamma v_g} (1 - \beta_g) \chi .$$

As always in stability theory, the integration in Eq. (16) goes in the complex plane above the pole $\delta = -\omega / \eta c q$. For a real value of q, Eq. (16) defines a complex frequency ω the imaginary part of which gives the growth rate of the instability. Alternatively, we can consider real ω and find a complex wavenumber q describing a periodic perturbation growing or decaying along the beam pipe.

Note that the frequency of the mode Ω observed in the laboratory frame, where it has a dependence $e^{i(qs - \Omega t)}$, is equal to $\Omega = \omega + qc$.

IV. DISPERSION RELATION FOR A SINGLE MODE

Let us assume that the distribution function $f_0(\delta)$ is Gaussian with the rms energy spread δ_0, $f_0 = (n_b/\delta_0) \rho_0(\delta/\delta_0)$ with $\rho_0(\xi) = e^{-\xi^2/2}/\sqrt{2\pi}$. In the single-mode approximation, we leave only one term in the dispersion equation Eq. (16) which takes the form

$$q - q(n, \Omega) = \frac{n_b \lambda}{\delta_0} \int d\xi \frac{d\rho_0(\xi)/d\xi}{\Omega - qc + \eta c \delta_0 q \xi + i\epsilon} . \tag{17}$$

We expect that the instability develops close to the mode frequency ω_0,

$$\Omega = \omega_0 + \Delta\Omega, \quad \Delta\Omega \ll \omega_0 . \tag{18}$$

The function $q(n, \Omega)$ can be expanded in the vicinity of ω_0,

$$q(n, \Omega) \approx q_0 + \frac{\Delta\Omega}{v_g} . \tag{19}$$

The frequency ω_0 is defined as the frequency of the synchronous mode n under consideration in an evacuated waveguide, $\omega_0 = cq(n, \omega_0)$, and $q_0 = q(n, \omega_0)$ is the wavenumber of this mode. The denominator in the integral of Eq. (17) is $\Omega - qc + \eta c\delta_0 q\xi \approx \Delta\Omega - (q - q_0)c + \eta\omega_0\delta_0\xi$. Eq. (17) then takes the following form

$$\Delta\Omega - v_g\Delta q = -\frac{n_b\lambda v_g}{\eta\omega_0\delta_0^2}\int d\xi \frac{d\rho_0}{d\xi}\left(\frac{\Delta\Omega - c\Delta q}{\eta\omega_0\delta_0} + \xi + i\epsilon\right)^{-1}, \tag{20}$$

where $\Delta q = q - q_0$.

Depending on the ratio $\Delta\Omega/\eta\omega_0\delta_0$, there are two possible regimes for the instability: a large energy spread regime, when $|\Delta\Omega| \ll |\eta\omega_0\delta_0|$, and a "cold beam" approximation when the opposite inequality holds. We consider here the latter case only, as a more relevant to the parameters of the existing accelerators (see below). In this case, we can evaluate the integrand in Eq. (20) asymptotically in the limit $|(\Delta\Omega - c\Delta q)/\eta\omega_0\delta_0| \gg 1$, which results in the cubic dispersion equation:

$$(\Delta\Omega - \Delta q v_g)\left(\Delta\Omega - \Delta q c\right)^2 = -n_b\lambda v_g\eta\omega_0\,. \tag{21}$$

For $\Delta q = 0$, one of the roots has a positive imaginary part:

$$\Delta\Omega = \mu\, e^{i\pi/3}\,, \tag{22}$$

where we introduced the parameter μ

$$\mu = (n_b\lambda v_g\eta\omega_0)^{1/3} = c\left[\frac{r_e n_b\omega_0\eta\chi}{c\gamma}(1 - \beta_g)\right]^{1/3}. \tag{23}$$

Note that for a cold beam there is no threshold for the instability. The estimate of the integral term in the dispersion equation used above neglects the Landau damping and is valid provided $|\mu| \gg \eta\omega_0\delta_0$.

For a general case of arbitrary detuning Δq, Eq. (21) can be written in the dimensionless form as

$$x^2(x + y) + 1 = 0\,, \tag{24}$$

by introducing

$$x = \frac{\Delta\Omega - c\Delta q}{\mu}, \quad y = \frac{c\Delta q(1 - \beta_g)}{\mu}\,. \tag{25}$$

Eq. (24) can be easily solved numerically—it has three roots one of which corresponds to the instability. The dimensionless growth rate $\mathrm{Im}\, x$ as a function of the variable y is plotted in Fig. 1. We see that the maximum growth rate is achieved at zero detuning, $\Delta q = 0$.

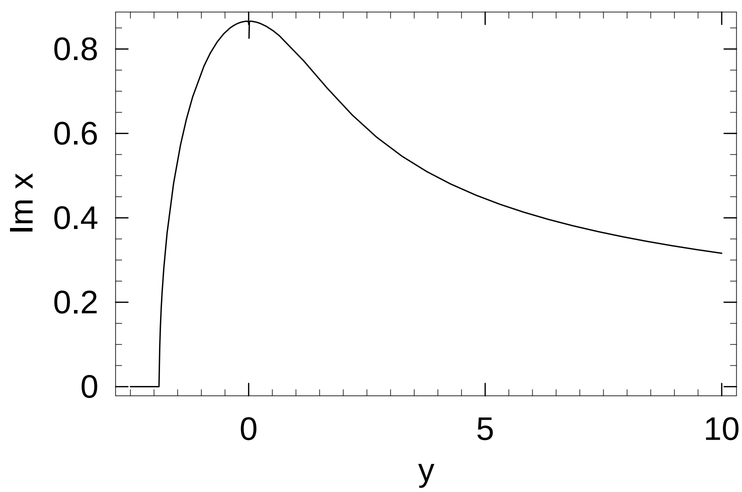

FIG. 1: Dimensionless growth rate $\mathrm{Im}\, x$ as a function of the dimensional detuning y. The maximum of $\mathrm{Im}\, x$ is reached at $y = 0$, and approaches to zero at $y = -1.89$; $\mathrm{Im}\, x = 0$ for $y < -1.89$.

Table 1 gives parameters and compares the growth rate for four accelerators: the Low Energy Ring (LER) and the High Energy Ring (HER) of PEP-II accelerator at SLAC, Advanced Light Source at the Berkeley National Laboratory, and the VUV ring at the National Synchrotron Light Source at BNL. For the ALS, we used beam parameters for the regime in which bursts of infrared radiation were observed [13]. Calculations were made for the lowest synchronous mode assuming a square cross section of the vacuum chamber with the size a equal to the vertical full gap of the beam pipe. Since the real shape of the cross section usually differs from the square, the results in the table should be considered as a rough estimate of the instability parameters. For the linear density of the beam n_b, we used the quantity $N_p/\sqrt{2\pi}\sigma_z$, which gives the maximum linear density in a gaussian bunch (N_p is the number of particles in the bunch, σ_z is the rms bunch length). Note the ratio $\mu/\eta\omega_0\delta_0$ in the last line of the table related the cold beam approximation—it is large in all cases except for the HER PEP-II where it is close to one.

TABLE I: **Parameters relevant to the instability for PEP-II low energy (LER) and high energy (HER) rings, ALS, and VUV NSLS ring.**

Parameter	units	LER	HER	ALS	VUV NSLS
Energy	GeV	3.1	9.0	1.5	0.81
η	10^{-3}	1.3	2.1	1.4	2.4
δ_0	10^{-4}	8.1	6.1	7.1	5.0
n_b	10^{10} cm^{-1}	3.7	0.82	7	3.6
a	cm	5	5	4	4.2
R	m	13.7	165.0	4.0	1.9
$\omega_0/2\pi$	GHz	75.5	260	57	36.6
χ	V/pC/m	18	18	28	25
μ	10^6 s^{-1}	7.5	2.5	18	22
$n_{\rm cr}$	10^{10} cm^{-1}	13	140	3	0.8
$\mu/(\eta\omega_0\delta_0)$		15	1.2	84	50

V. TRANSITION TO CONTINUOUS SPECTRUM

In the previous sections, we focused on the interaction of the beam with a single mode of frequency $\omega_0 \simeq \omega_{\rm min}$ near the shielding threshold of the instability. As was pointed out before, interaction with high-frequency modes at $\omega \gg \omega_{\rm min}$ can be treated in terms of the CSR impedance [1]. In this section, we consider the transition from the single-mode regime to the continuous spectrum interaction, and find a criterion on the electron beam density which determines such a transition.

The spectrum of synchronous modes in a toroidal waveguide with perfectly conducting walls consists of discrete modes. The width of the spectral lines in an evacuated waveguide is infinitely thin, corresponding to delta functions at the mode frequencies. Excitation of those modes by the beam can be considered as broadening of those infinitely thin lines, so that they can be characterized by some width $\Delta\omega_{\rm mode}$. When this width becomes comparable or exceeds the distance between the modes $\Delta\omega$, the spectrum can be considered as continuous.

The average distance between the modes $\Delta\omega$ can be estimated as $\Delta\omega \sim (dN_{\rm mode}/d\omega)^{-1}$, where $dN_{\rm mode}/d\omega$ is the mode density in the frequency space. The latter can be found from the equation for synchrotron power radiation spectrum $dP/d\omega$ of a point charge e (recall that we assume frequencies well below the critical frequency):

$$\frac{dP}{d\omega} = e^2\chi(\omega)\frac{dN_{\rm mode}}{d\omega}\,,$$

where $\chi(\omega)$ is the loss factor per unit length as a function of frequency of the mode. The spectrum of the synchrotron radiation below the critical frequency is $dP/d\omega = 0.52(e^2/R)(kR)^{1/3}$, and the loss factor $\chi(\omega)$ is estimated in Ref. [9] as $\chi \sim a^{-2}\,(ck_0/\omega)^{1/3}$, which gives for $\Delta\omega$

$$\Delta\omega \sim ck_0\left(\frac{ck_0}{\omega}\right)^{2/3}\,.$$

The width of the mode $\Delta\omega_{\rm mode}$ can be estimates as $c\Delta q$ where Δq can be found from the second of Eq. (25) as $\Delta q \sim \Delta y\mu/(1-\beta_g)$. Observing from Fig. 1 that $\Delta y \sim 1$, we conclude that $\Delta q \sim \mu/(1-\beta_g)$. It is interesting to note that $\Delta\omega_{\rm mode} \gg \Delta\Omega$. The overlapping takes place when $c\mu/(1-\beta_g) \gtrsim ck_0(ck_0/\omega)^{2/3}$, or

$$\Lambda\,(\eta\delta_0)^2\left(\frac{k}{k_0}\right)^{2/3}(ka)^2 \gtrsim 1\,, \tag{26}$$

where the parameter Λ is defined by Eq. (1) and $k = \omega/c$.

The growth rate of the instability Γ for a cold beam in the continuous spectrum modes [1] can be estimated as

$$\left(\frac{\Gamma}{c}\right)_{\rm cont} = \Lambda^{1/2}\left(\frac{\eta\delta_0}{a}\right)\left(\frac{k}{k_0}\right)^{2/3}\,.$$

It is easy to check that at Λ given by Eq. (26), $\Gamma_{\rm cont} \sim \mu$, which means that the growth rates in both theories match at the boundary of their validity regions.

Eq. (26) shows that the mode overlapping occurs easier for high frequency modes. In the continuous spectrum model, the maximum growth rate is achieved for $kR \simeq \Lambda^{3/2}$ [1]. Eq.

(26) gives the critical linear bunch density $n_{\rm cr}$ at which overlapping occurs for this frequency,

$$n_{\rm cr} \sim \frac{\gamma\delta_0}{r_e}\,(\eta\delta_0)^{3/5}\left(\frac{R}{a}\right)^{3/5}.$$

The model of Ref. [1] is valid if the beam linear density n_b is larger than $n_{\rm cr}$. It describes the instability of higher modes where the shielding effect of the walls can be neglected. As the same time, the lowest toroidal modes are described by the single mode model developed in this paper. As shown in Table I, typically $n_b \gg n_{cr}$ except for the NSLS VUV ring.

VI. NONLINEAR REGIME OF THE INSTABILITY

When the amplitude of the unstable mode becomes large, the linear theory is not valid any more and one has to use the full Vlasov equation for the distribution function $f(z,\delta,t)$:

$$\frac{\partial f}{\partial t} - \eta c\delta\frac{\partial f}{\partial z} + \frac{e}{\gamma mc}\mathcal{E}(z,t)\frac{\partial f}{\partial \delta} = 0\,. \tag{27}$$

An important approximation that we make in the nonlinear regime is that the evolution of the instability is governed by a single mode with a wavenumber q_w. One would expect that this wavenumber is equal to q_0—the mode that has the maximum growth rate in the linear regime—however, for the sake of generality, we treat q_w as arbitrary (but close to q_0). The derivation of the equation for $\mathcal{E}(z,t)$ describing the interaction of the beam with the mode is presented in Appendix B. The result is given by Eq. (B7) which we reproduce here:

$$\begin{aligned}\mathcal{E}(z,t) &= -ec\chi(1-\beta_g)\frac{q_w}{2\pi}\\ &\times\int_{-\infty}^{\infty}d\delta\int_{-\infty}^{t}dt'\int_0^{2\pi/q_w}dz'\,e^{iq_w(z-z')+ic(q_w-q_0)(1-\beta_g)(t-t')}f(z',\delta,t') + \text{c.c.}\,.\end{aligned}$$

It is convenient to introduce dimensionless variables τ, ζ, and p instead of t, z and δ, respectively, where

$$\tau = \mu t, \quad \zeta = q_w z, \quad p = -\frac{\eta\omega_0}{\mu}\delta\,,$$

and μ is given by Eq. (23). We also introduce the amplitude $A(\tau)$ such that,

$$\mathcal{E} = -\frac{\gamma mc\mu}{e\eta\omega_0}\left[A(\tau)e^{iq_wz} + \text{c.c.}\right]\,,$$

where

$$A(\tau) = \frac{r_e c^2 \chi \eta \omega_0 (1-\beta_g)}{\gamma \mu^2} \frac{q_w}{2\pi}$$
$$\times \int_{-\infty}^{\infty} d\delta \int_{-\infty}^{t} dt' \int_0^{2\pi/q_w} dz' \, e^{-iq_w z' + ic(q_w - q_0)(1-\beta_g)(t-t')} f(z', \delta, t') \,, \tag{28}$$

and the dimensionless distribution function

$$F(\zeta, p, \tau) = \frac{1}{2\pi n_b} \frac{\mu}{\eta \omega_0} f \,,$$

normalized by the condition $\int_{-\infty}^{\infty} dp \int_0^{2\pi} d\zeta F(\zeta, p, \tau) = 1$. Note that we use an approximation $cq_w \approx \omega_0$. In these variables, the beam dynamics is described by the following equation,

$$\frac{\partial F}{\partial \tau} + p\frac{\partial F}{\partial \zeta} + \left[A(\tau)e^{i\zeta} + \text{c.c.}\right] \frac{\partial F}{\partial p} = 0 \,, \tag{29}$$

and the amplitude $A(\tau)$, as if follows from Eq. (28), satisfies the equation

$$\frac{\partial A(\tau)}{\partial \tau} = \langle e^{-i\zeta} \rangle + iuA \,, \tag{30}$$

with

$$\langle e^{-i\zeta} \rangle = \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\zeta F(\zeta, p, \tau) \, e^{-i\zeta}, \quad u = \frac{c}{\mu}(q_w - q_0)(1-\beta_g) \,. \tag{31}$$

Note that characteristics of Eq. (29) are equations of motion for a single particle:

$$\frac{d\zeta}{d\tau} = p, \quad \frac{dp}{d\tau} = \left[A(\tau)e^{i\zeta} + \text{c.c.}\right] . \tag{32}$$

Eq. (29), (30) and Eq. (31) constitute a full system of equations. It has a universal form of beam-wave interaction describing particles moving in the external potential of the unstable mode which amplitude and frequency has to be defined in a self-consistent way. These equations have an integral of motion:

$$C = |A|^2 - \langle p \rangle \,, \tag{33}$$

which reflects conservation of energy—the sum of the wave energy and the beam energy is constant during the interaction.

The system of equations (29), (30) and Eq. (31) is encountered in other problems of nonlinear beam-wave interaction, *e.g.*, in the one-dimensional FEL theory [14, 15], with the parameter μ being equivalent to the Pierce parameter ρ. The solution of the system on a limited time interval can be obtained by numerical methods. In the numerical approach, the beam is represented by a finite number M of macroparticles, and the average $\langle e^{i\zeta} \rangle$ is approximated by the sum $\sum_1^M e^{-i\zeta_k}$ over all particles' coordinates ζ_k. The result of such a solution—the absolute value $|A|$ of the amplitude of the wave—is shown in Fig. (2). It shows that the amplitude of an initial small perturbation saturates after an initial exponential growth and exhibits oscillations at frequency of the order of the bounce frequency of particles in the bucket of the excited wave. Fig. 2 agrees with a similar solution obtained earlier in Ref. [14].

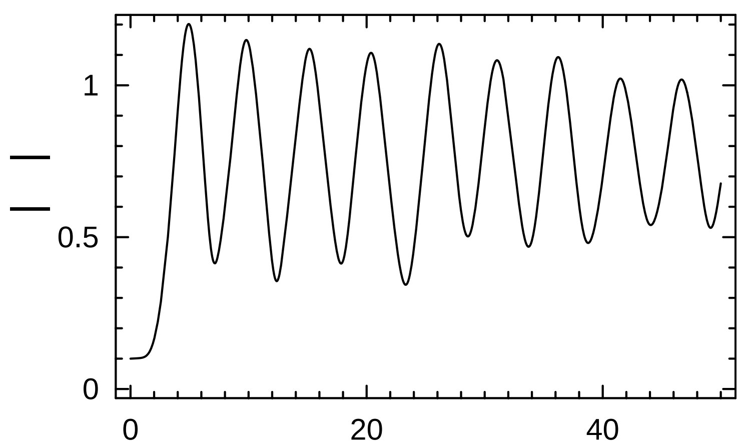

FIG. 2: The dependence of the amplitude $|A|$ versus τ in the nonlinear regime of the instability. After $\tau \simeq 1$, the exponential growth of the linear regime changes to oscillations with the average amplitude $|A| \simeq 1$ and the frequency of the oscillations $\simeq 1$.

VII. SYNCHROTRON DAMPING AND QUANTUM DIFFUSION

Contrary to the FEL theory, where it usually suffices to track the solution on several gain lengths only, for a beam in the storage ring we may be interested in time comparable to the synchrotron damping time. The analysis in this case has to include the synchrotron damping and diffusion due to quantum fluctuations effects. One of the difficulties of such analysis is that the damping time typically is larger than the synchrotron oscillation period in the damping right so that one has also take into account synchrotron oscillations of particle in the bunch. In this section, however, we will consider an idealized formulation which neglects synchrotron oscillations, but includes synchrotron damping and diffusion due to quantum fluctuation in synchrotron radiation. A more detailed study, with account of synchrotron motion, can be found in Ref. [16].

To include the effects of synchrotron damping and quantum diffusion into the interaction of the wave with the beam, we need to use the Vlasov-Fokker-Planck equation [17]. In our dimensionless variables it has the following form

$$\frac{\partial F}{\partial \tau} + p\frac{\partial F}{\partial \zeta} + \left[A(\tau)e^{i\zeta} + \text{c.c.}\right]\frac{\partial F}{\partial p} = \Gamma\frac{\partial}{\partial p}\left(\Delta^2\frac{\partial F}{\partial p} + pF\right), \tag{34}$$

where Γ and Δ are related to the synchrotron radiation damping γ_{SR} and the rms energy spread δ_{SR} due to the quantum fluctuations in the synchrotron radiation:

$$\Gamma = \frac{\gamma_{\text{SR}}}{\mu}, \quad \Delta = \frac{\eta\omega_0\delta_{\text{SR}}}{\mu}.$$

Note that with damping the integral C in Eq. (33) is not conserved any more; we have

$$\frac{d}{d\tau}\left(|A|^2 - \langle p\rangle\right) = \Gamma\langle p\rangle$$

instead of Eq. (33).

First, we will show that Eqs. (30), (32) do not have a steady state solution corresponding to a constant amplitude A. For simplicity, we consider only the synchronous wave with $q_w = q_0$. Indeed, assume that $A(\tau)$ does not depend on time, $A(\tau) = (iA_0/2)e^{i\alpha_0}$, where α_0 is an arbitrary phase. We then have the following equations of motion for particles,

$$\frac{d\zeta}{d\tau} = p, \quad \frac{dp}{d\tau} = -A_0\sin(\zeta + \alpha_0).$$

It is easy to see that these are the pendulum equations with the Hamiltonian H,

$$H(p,\zeta) = \frac{p^2}{2} + A_0[1 - \cos(\zeta + \alpha_0)].$$

The Fokker-Plank equation Eq. (34) in this case has a steady-state solution

$$F(p,\zeta) = Z^{-1}e^{-H(p,\zeta)/\Delta^2},$$

where Z is the normalization constant. The amplitude A_0 and the phase α_0 have to satisfy the condition that follows from Eq. (30) with $dA/d\tau = 0$ (recall that $u = 0$),

$$\langle e^{-i\zeta} \rangle = Z^{-1} \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\zeta e^{-i\zeta}\, e^{-H(p,\zeta)/\Delta^2} = 0\,.$$

Due to periodicity in ζ, this condition is reduced to

$$\int_0^{2\pi} d\zeta e^{-2(A_0/\Delta^2)\sin^2\zeta} \cos\zeta = 0\,.$$

It is easy to see that this equation does not have a solution with $A_0 \neq 0$.

In order to carry out numerical simulation of the Vlasov-Fokker-Plank equation, we note that this equation is equivalent to the set of single-particle equations of motion with damping and an external force $\kappa(\tau)$:

$$\frac{d\zeta}{d\tau} = p, \quad \frac{dp}{d\tau} = [A(\tau)e^{i\zeta} + \text{c.c.}] - \Gamma p + \kappa(\tau)\,.$$

where $\kappa(\tau)$ is a random function of time τ with zero average value $\langle\kappa\rangle = 0$ and the correlation function

$$\langle \kappa(\tau)\kappa(\tau') \rangle = 2\Gamma\Delta^2\, \delta(\tau - \tau')\,.$$

In our simulation, we used a discrete time mesh τ_i with the time step $\tau_{i+1} - \tau_i = \tau_s$ and a finite number of particles M. On each interval, we first solved the system of the differential equations Eqs. (32) and (30) without damping and fluctuations. The damping and fluctuations were taken into account at the end of each step by changing variables p of each particle:

$$p_k \to p_k - \Gamma\tau_s p_k + \sqrt{24\tau_s\Gamma\Delta^2}\,\xi\,,$$

where ξ is a random number uniformly distributed in the range $[-1/2, 1/2]$. This algorithm was tested on the case without the wave, $A = 0$, and also for the case of an external wave with constant amplitude $A = \text{const}$, when the Vlasov-Fokker-Planck equation has analytical solutions. In both cases we found a good agreement between the numerical and analytical solutions.

The simulations were carried out for the parameters close to that of ALS: $\mu = 3.2 \cdot 10^7$ s^{-1}, $\omega_0 = 1.0 \cdot 10^{12}$ s^{-1}, $\Delta = 0.032$. However, to speed up the tracking, we increased the parameter Γ from the ALS value $2.0\,10^{-6}$ to $2.0\,10^{-2}$. We expect that such a rescaling of Γ accelerates the manifestation of the synchrotron damping effects without qualitatively changing the solution. Typically we used from 200 to 800 particles in the simulation.

The results of the tracking for $\tau \approx 1000$ (corresponding to approximately 20 damping times) are shown in Fig. 3 and Fig. 4. Fig. 3 shows the amplitude $A_0(\tau)$, and Fig. 4

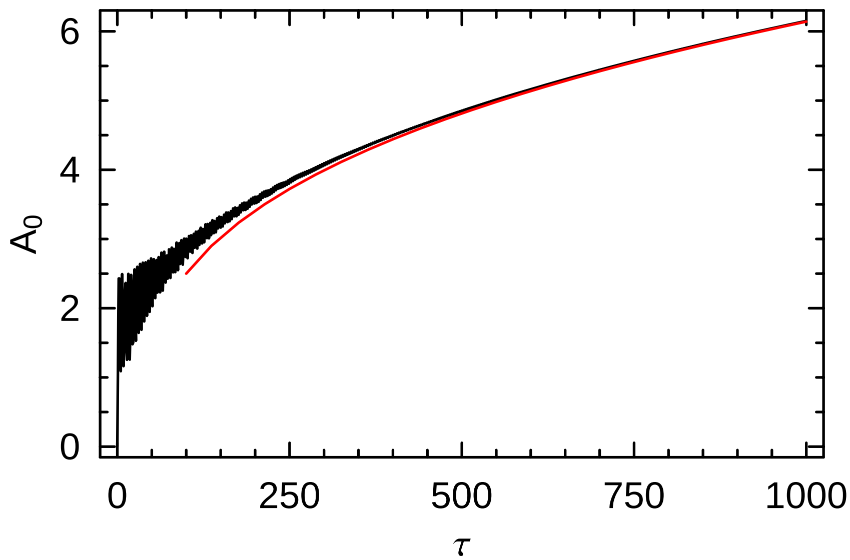

FIG. 3: The absolute value of the amplitude $|A(\tau)|$ as a function of τ. Black curve–result of simulation, red curve–analytical solution of Eq. (42).

shows the average over distribution function momentum $\langle p \rangle$ and the rms spread in p, Δp_{rms}, as functions of time. For the time interval small compared with the damping time $\tau \lesssim 50$, results of tracking reproduce Fig. 2. For larger time intervals, $\tau \gg 50$, we see that the amplitude A_0 keeps growing, and the beam comes to a quasi equilibrium, with a slowly changing values of $\langle p \rangle$ and Δp_{rms}. Note also a relatively small value of Δp_{rms}, which means that particles of the beam are well localized in the p-space.

The numerical results shown in Figs. 3 and 4 give us an indication of an analytical solution to the problem in the limit of large τ. In this solution we assume that

$$A(\tau) = \frac{1}{2} i A_0(\tau) e^{-i\nu(\tau)\tau}\,, \tag{35}$$

where the function $A_0(\tau)$ and frequency $\nu(\tau)$ are slow functions of time. Without loosing generality, we can assume that both A_0 and ν are real. Substituting Eq. (35) into Eq. (32)

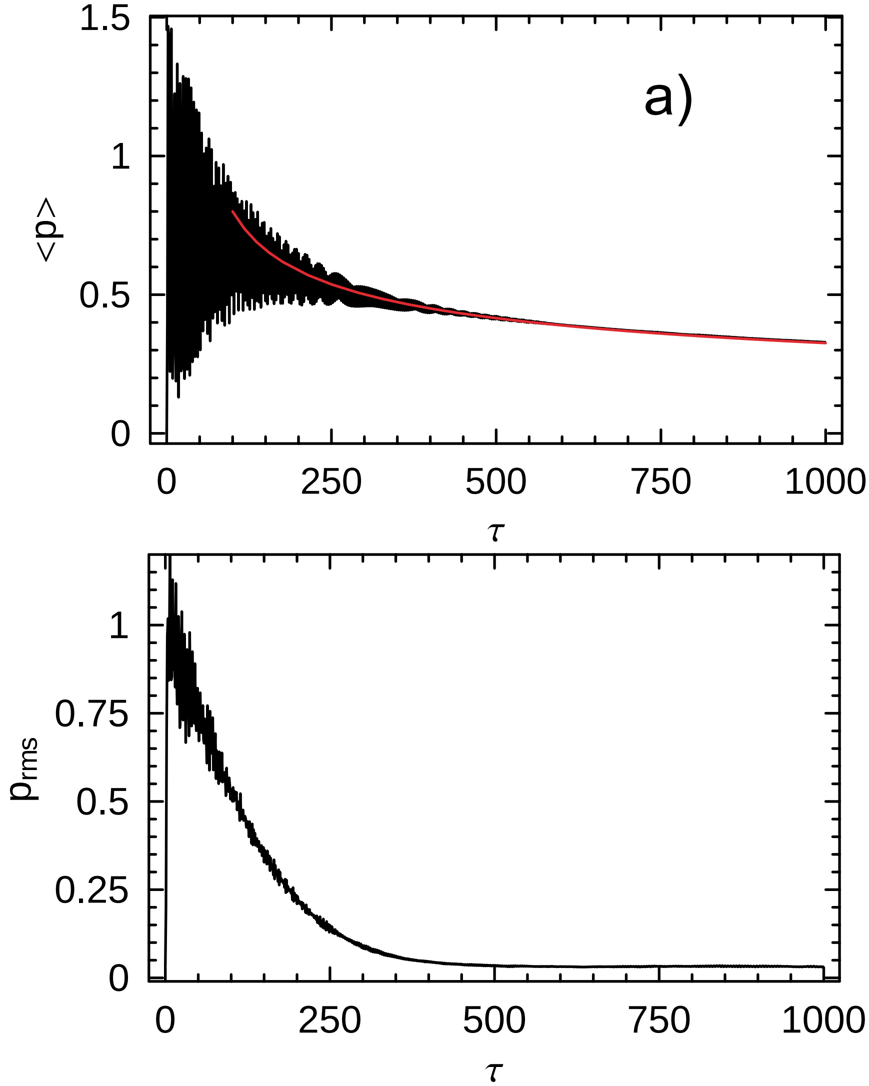

FIG. 4: Numerical simulation of nonlinear regime of the instability: a)—the average momentum $\langle p\rangle$, b)—the rms momentum spread $\Delta p_{\rm rms}$. The red line shows the the result of the analytical model.

yields:

$$\frac{d\zeta}{d\tau} = p, \quad \frac{dp}{d\tau} = -A_0 \sin(\zeta - \nu\tau)\,, \tag{36}$$

where the amplitude A_0 has to be determined in a self-consistent way from Eq. (30).

Let us make a canonical transformation of variables from ζ and p to ξ and r, respectively,

$$\xi = \zeta - \nu\tau, \quad r = p - \nu\,. \tag{37}$$

The Fokker-Plank equation Eq. (34) in new variables takes the form

$$\frac{\partial F}{\partial \tau} + r\frac{\partial F}{\partial \xi} - A_0 \sin\xi \frac{\partial F}{\partial r} = \Gamma \frac{\partial}{\partial r}\left[\Delta^2 \frac{\partial F}{\partial r} + (r+\nu)F\right] . \tag{38}$$

Because A_0 and ν are assumed to vary slowly, in this equation we can neglect their time dependance and consider them as constant. We then find a steady-state solution of Eq. (38) as

$$F(r,\xi) = Z^{-1} \exp\left(\frac{-r^2 - 2A_0 U(\xi)}{2\Delta^2}\right) , \tag{39}$$

where Z is a normalization constant and the potential U is

$$U(\xi) = 1 - \cos\xi + \Upsilon\xi ,$$

with $\Upsilon = \Gamma\nu/A_0$. Note that U depends on time only through adiabatic dependence of A_0 and ν. For the purpose of illustration, Fig. 5 shows the plot of the potential U for $\Upsilon = 0.25$.

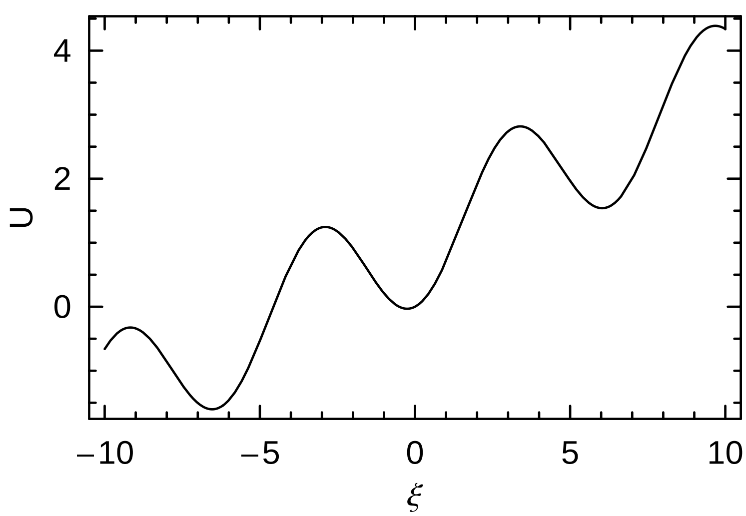

FIG. 5: Potential $U = \Upsilon\xi + (1-\cos\xi)$ for $\Upsilon = 0.25$.

The condition of self-consistency defines time-dependence of the parameters A_0 and ν. Substituting Eq. (35) into Eq. (30) yields:

$$\frac{dA_0}{d\tau} = i\nu A_0 - 2i\langle e^{-i\xi}\rangle . \tag{40}$$

Separating the real and imaginary parts in this equation we get two equations:

$$\frac{dA_0}{d\tau} = -2\langle \sin \xi \rangle, \quad \nu = \frac{2}{A_0} \langle \cos \xi \rangle \,. \tag{41}$$

Note that the non-zero detuning u would shift $\nu \to \nu + u/2$.

Our numerical simulations show that at large τ particles tend to accumulate at the bottom of local wells of the potential U (see Fig. 5) with a small momentum $r \approx 0$. The adiabatic approximation can be simplified even further if the averaging in Eq. (40) is replaced by the value of the function $e^{i\xi}$ taken at the location of the minimum ξ_0 of the potential $U(\xi)$:

$$\langle e^{-i\xi} \rangle \approx e^{-i\xi_0} \,.$$

It is straightforward to show that $\sin \xi_0 = -\Upsilon$. Using Eq. (41), we find $\Upsilon = [1 + A_0^4/(2\Gamma)^2]^{-1/2}$ and equation for $A_0(\tau)$,

$$\frac{dA_0}{d\tau} = \frac{2}{\sqrt{1 + A_0^4/(2\Gamma)^2}} \,. \tag{42}$$

Since this equation determines asymptotic behavior of A in the limit $\tau \to \infty$, the initial condition for it is not well defined. For the purpose of comparison with the numerical solution, we considered an initial condition $A(\tau_0) = A_*$, with A_* as a fitting parameter. The result of integration of Eq. (42) with $A(100) = 2.5$ is shown in Fig. 3 in red color, in good agreement with the numerical solution. It is straightforward to show that for large τ it follows from Eq. (42): $A_0 \propto \tau^{1/3}$. The averaged momentum of the particles $\langle p \rangle$ in this model can be found from equation $r \approx 0$ which gives $\langle p \rangle \approx \nu = 2 \cos \xi_0 / A_0$. This curve is shown as a red line in Fig. 4a.

VIII. CONCLUSION

In this paper, we studied stability of the beam interacting self-consistently with a single synchronous mode in a toroidal wave guide. We first derived the beam-mode interaction equations in linear approximation, obtained the growth rate of the instability, and compared the result with the model of the CSR instability with continuous spectrum. We also showed that the latter follows from the singe-mode model in the limit when the resonances overlap, and obtained the criterion for transition from one regime to the other.

We then derived equations for nonlinear beam-mode interaction, assuming that the interaction is dominated by a single synchronous mode. For relatively small time intervals, the interaction can be described by a system of equations which, after proper scaling, has a universal form that does not depend on parameters of the system. We note that this system of equations is analogous to a one-dimensional free electron laser theory, and leads to the same beam dynamics.

For a storage ring, one is interested in long-term evolution of the instability, when effects of synchrotron damping and quantum diffusion become important. Using the Vlasov-Fokker-Planck equation to describe these effects, we showed, both numerically and analytically, that in the long term there is no saturation of the instability: the amplitude of the mode keeps growing although at a slow rate. Our analytic approximation to the solution describes such an asymptotic behavior and shows a good agreement with the numeric one.

IX. ACKNOWLEDGMENTS

We thank S. Krinsky and Z. Huang for usefull discussions. This work was supported by the Department of Energy, contract DE–AC03–76SF00515.

REFERENCES

[1] S. Heifets and Stupakov G. V., Physical Review Special Topics-Accelerators and Beams **5**, 054402 (2002).

[2] J. Schwinger, *On radiation by electrons in a betatron* (1945), unpublished. Also Report LBL-39088, ed. by M. Furman (1996).

[3] L. Schiff, Rev. Sci. Instr. **17**, 6 (1946).

[4] R. L. Warnock and P. Morton, Part. Accel. **25**, 113 (1990).

[5] J. B. Murphy, S. Krinsky, and R. L. Gluckstern, in *Proc. IEEE Particle Accelerator*

Conference and International Conference on High-Energy Accelerators, Dallas, 1995 (IEEE, Piscataway, NJ, 1996), (IEEE Conference Record 95CH35843).

[6] Y. S. Derbenev, J. Rossbach, E. L. Saldin, and V. D. Shiltsev, *Microbunch Radiative Tail-Head Interaction*, DESY FEL Report TESLA-FEL 95-05, Deutsches Elektronen-Synchrotron, Hamburg, Germany (September 1995).

[7] K.-Y. Ng, Part. Accel **25**, 153 (1990).

[8] M. M. Karliner, N. V. Mityanina, and V. P. Yakovlev, *The impedance of a toroidal chamber with walls of finite conductivity. Waveguide model.*, Tech. Rep. BUDKERINP 93-90, Budker Institute of Nuclear Physics, Novosibirsk, Russia (1993).

[9] G. V. Stupakov and I. A. Kotelnikov, *Shielding and synchrotron radiation in toroidal waveguide*, Report SLAC-PUB-9553, SLAC (2002).

[10] A. W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators* (Wiley, New York, 1993).

[11] L. A. Vainshtein, *Electromagnetic Waves* (Radio i svyaz', Moscow, 1988), in Russian.

[12] R. E. Collin, *Field Theory of Guided Waves* (IEEE Press, New York, 1991), 2nd ed.

[13] J. Byrd, W. P. Leemans, A. Loftsdottir, B. Marcelis, M. C. Martin, W. R. McKinney, F. Sannibale, T. Scarvie, and C. Steier, Phys. Rev. Lett. **89**, 224801 (2002).

[14] R. Gluckstern, S. Krinsky, and H. Okamoto, Phys. Rev. **E47**, 4412 (1993).

[15] N. A. Vinokurov, Z. Huang, O. A. Shevchenko, and K.-J. Kim, Nuclear Instruments and Methods in Physics Research **A 475**, 74 (2001).

[16] S. Heifets, *The single-mode CSR instability for a bunched beam*, Report SLAC-PUB-9626, SLAC (2003).

[17] A. Piwinski, in P. Bryant and S. Turner, eds., *Proc. CERN Accelerator School, Gif-sur-Yvette, 1984* (European Lab. for Particle Physics, Geneva, Switzerland, 1985), no. 85-19 in CERN Yellow Report, pp. 432–462.

APPENDIX A: DERIVATION OF THE DISPERSION RELATION BASED ON SINGLE MODE IMPEDANCE

The dispersion relation in the standard theory of the stability of a coasting beamis [10]

$$1 = i\frac{n_b r_e c^2}{\gamma\delta_0} Z(\Omega) \int \frac{dp\, d\rho_0(p)/dp}{(\Omega - kc) + \eta c\delta_0 kp},$$

where $Z(\Omega)$ is the beam impedance, and k is the wavenumber. The resonant impedance for a mode of frequency ω_n is

$$Z(\Omega) = \frac{R_n}{1 + iQ_n(\omega_n/\Omega - \Omega/\omega_n)}, \tag{A1}$$

where R_n and Q_n are the shunt impedance, and Q-factor of the mode, respectively. If the Q-factor is large, $Q_n \ll 1$, in the vicinity of the resonant frequency $\Omega \approx \omega_n$, the impedance is simplified:

$$Z(\Omega) = \frac{i}{2}\frac{\omega_n}{\Omega - \omega_n}\frac{R_n}{Q_n} = i\frac{\chi_n}{\Delta\Omega}\,. \tag{A2}$$

where the loss factor $\chi_n = \omega_n R_n/2Q_n$.

Eq. (25) now takes the form

$$1 = -\frac{n_b r_e c^2 \chi_n}{\gamma\delta_0\Omega} \int \frac{dp d\rho(p)/dp}{(\Omega - kc) + \eta c\delta_0 kp}. \tag{A3}$$

The same results follows from Eqs. (17-19) for $q = q_n = k$ except for the additional factor $1 - \beta_g$. Eq. (25) is, strictly speaking, implies a localized impedance and has to be corrected for the impedance due to propagating modes with large group velocity.

APPENDIX B: EQUATIONS DESCRIBING INTERACTION OF THE BEAM WITH A SINGLE MODE

The equation for the electric field can be obtained analogously to the derivation of Eq. (13). The only difference is that instead of the Fourier component of the perturbation of the distribution function g_1 one has to use the full distribution function g:

$$C_n(q,\omega) = \frac{i}{N_n}\frac{ec\,(\boldsymbol{e}_n^*(0)\cdot\boldsymbol{s})}{q - q(n,\omega) - i\epsilon} \int d\delta g(\omega - qc, q, \delta)\,, \tag{B1}$$

where

$$g(\omega, q, \delta) = \int dt dz\, e^{-i(qz-\omega t)} f(z, \delta, t)\,. \tag{B2}$$

In a single-mode approximation, we assume that in the nonlinear regime the electromagnetic field is dominated by a mode with the wavenumber q_w close to the synchronous wavenumber q_0. In this case, the distribution function f is a periodic function of z with the period equal to $2\pi/q_w$. This means that the function g given by Eq. (B2) can be represented as

$$g(\omega, q, \delta) = \sum_k g^{(k)}(\omega, \delta)\delta(q - kq_w)\,,$$

where

$$g^{(k)}(\omega, \delta) = q_w \int_0^{2\pi/q_w} dz \int dt f(z, \delta, t) e^{-i(kq_w z - \omega t)} dz\,.$$

The dominant part of the interaction of the beam with the wave is determined by the first harmonic of the distribution function. For this reason, we will keep only harmonics $g^{(\pm 1)}$ in Eq. (B1).

Note that the quantity $\mathcal{E}(z, t)$ in Eq. (27) is equal to the longitudinal component of $\boldsymbol{E}(\boldsymbol{r}, s, t)$ taken at the location $s = ct + z$ on the axis $\boldsymbol{r} = 0$, $\mathcal{E}(z, t) = E_s(0, s = ct + z, t)$. Using Eqs. (B1), and (B2), we find for $\mathcal{E}(z, t)$:

$$\begin{aligned}
\mathcal{E}(z, t) &= \int \frac{d\omega dq}{(2\pi)^2} C_n(q, \omega)\, (\boldsymbol{e}_n(0) \cdot \boldsymbol{s})\, e^{-i\omega t + iq(z+ct)} \\
&= \frac{iec}{N_n} \int \frac{d\omega dq}{(2\pi)^2} \frac{|\boldsymbol{e}_n(0) \cdot \boldsymbol{s}|^2}{q - q(n, \omega) - i\epsilon} e^{-i\omega t + iq(z+ct)} \delta(q - q_w) \int d\delta g_1(\omega - qc, \delta) + \text{c.c.} \\
&= \frac{iec}{N_n} \int \frac{d\omega dq}{(2\pi)^2} \frac{|\boldsymbol{e}_n(0) \cdot \boldsymbol{s}|^2}{q - q(n, \omega) - i\epsilon} e^{-i\omega t + iq(z+ct)} q_w \delta(q - q_w) \\
&\times \int d\delta dt' dz'\, e^{-i(qz' - (\omega - qc)t')} f(z', \delta, t') + \text{c.c.}\,.
\end{aligned} \tag{B3}$$

The integration over the frequency can be carried out using the following relation

$$\int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{q - q(n, \omega) - i\epsilon} = iv_g(q)\Theta(t - t') \left(e^{-i\omega_n(q)(t-t')} + e^{i\omega_n(q)(t-t')} \right)\,, \tag{B4}$$

where $\Theta(t)$ is the step function, $\omega_n(q)$ is the solution of the dispersion relation $q(n, \omega_n) = q$, and $v_g(q) = d\omega_n(q)/dq$ is the group velocity. In Eq. (B4) we took into account that for each q there are two values of ω_n with opposite signs corresponding to waves propagating in opposite directions (we will assume $\omega_n(q) > 0$ below). Substituting Eq. (B4) into Eq. (B3) gives

$$
\begin{aligned}
&\mathcal{E}(z,t) = -\frac{ec}{N_n}\int\frac{dq}{2\pi}\, v_g(q)|\boldsymbol{e}_n(0)\cdot\boldsymbol{s}|^2 q_w\delta(q-q_w)\\
&\times\int d\delta dt' dz' \Theta(t-t')\, e^{iq(z-z')}\left(e^{i(cq-\omega_n(q))(t-t')} + e^{i(cq+\omega_n(q))(t-t')}\right) f(z',\delta,t') + \text{c.c.}\\
&\approx -\frac{ec}{\gamma N_n}\frac{q_w}{2\pi}\, v_g(q_w)|\boldsymbol{e}_n(0)\cdot\boldsymbol{s}|^2\\
&\times\int d\delta dt' dz' \Theta(t-t')\, e^{iq_w(z-z')+i(cq_w-\omega_n(q_w))(t-t')} f(z',\delta,t') + \text{c.c.}\,,
\end{aligned}
\tag{B5}
$$

where we kept only the resonant terms in the equation. One can expand $\omega_n(q_w) \approx \omega_n(q_0) + (q_w - q_0)v_g(q_0)$ and also assume $v_g(q_w) \approx v_g(q_0)$. With this expansion, integration over q in Eq. (B5) with account of both contributions from $+q_0$ and $-q_0$ yields

$$
\begin{aligned}
\mathcal{E}(z,t) = -&\frac{ec}{N_n}\frac{q_w}{2\pi}\, v_g|\boldsymbol{e}_n(0)\cdot\boldsymbol{s}|^2\\
&\times\int d\delta dt' dz' \Theta(t-t')\, e^{iq_w(z-z')+ic(q_w-q_n)(1-\beta_{gn})(t-t')} f(z',\delta,t') + \text{c.c.}\,,
\end{aligned}
\tag{B6}
$$

where v_g, $\beta_g = v_g/c$ and $\boldsymbol{e}_n(0)$ now refer to the synchronous mode. It is worth noting that the retardation time in the distribution function in Eq. (B6) depends on the relative speed of the particles (assumed here close to c) and the group velocity of the wave v_g.

The coefficient in Eq.(B6) is related to the loss factor χ per unit length of the synchronous mode [9]:

$$
\chi = \frac{v_g}{1-\beta_g}\frac{|(\boldsymbol{s}\cdot\boldsymbol{e}_n(0))|^2}{N_n}\,,
$$

so that Eq. (B6) can be also written as

$$
\begin{aligned}
&\mathcal{E}(z,t) = -ec\chi(1-\beta_g)\frac{q_w}{2\pi}\\
&\times\int d\delta dt' dz' \Theta(t-t')\, e^{iq_w(z-z')+ic(q_w-q_0)(1-\beta_g)(t-t')} f(z',\delta,t') + \text{c.c.}\,.
\end{aligned}
\tag{B7}
$$

Eqs. (27) and (B7) describe nonlinear interaction of a synchronous mode with the coasting beam.