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# Factorization and Sudakov Resummation in Leptonic Radiative B Decay

S.W. BOSCH<sup>a</sup>, R.J. HILL<sup>b</sup>, B.O. LANGE<sup>a</sup> AND M. NEUBERT<sup>a</sup>

<sup>a</sup>*Newman Laboratory for Elementary-Particle Physics, Cornell University  
Ithaca, NY 14853, U.S.A.*

<sup>b</sup>*Stanford Linear Accelerator Center, Stanford University  
Stanford, CA 94309, U.S.A.*

## Abstract

Soft-collinear effective theory is used to prove factorization of the  $B \rightarrow \gamma l \nu$  decay amplitude at leading power in  $\Lambda/m_b$ , including a demonstration of the absence of non-valence Fock states and of the finiteness of the convolution integral in the factorization formula. Large logarithms entering the hard-scattering kernel are resummed by performing a two-step perturbative matching onto the low-energy effective theory, and by solving evolution equations derived from the renormalization properties of the leading-order  $B$ -meson light-cone distribution amplitude. As a byproduct, the evolution equation for heavy-collinear current operators in soft-collinear effective theory is derived.

# 1 Introduction

The proposal of a “soft-collinear effective theory” (SCET) for the strong interactions of collinear and soft particles has been an important step toward understanding the factorization properties of hard exclusive processes in QCD [1]. In particular, it raises the prospects for rigorously proving QCD factorization theorems for hadronic and radiative  $B$ -meson decays into light particles such as  $B \rightarrow \pi\pi$  [2] and  $B \rightarrow K^*\gamma$  [3, 4], which are of great importance to the physics program at the  $B$  factories. A challenge common to these decays and many others is to understand the interactions of collinear particles with the soft spectator quark inside the  $B$  meson, which give rise to convolutions of hard-scattering kernels with  $B$ -meson light-cone distribution amplitudes (LCDAs). The first systematic analysis of these interactions in the framework of SCET has recently been performed by two of us [5]. In that work we have developed the formulation of SCET appropriate for the discussion of exclusive  $B$  decays into light particles.

The radiative, semileptonic decay  $B \rightarrow \gamma l\nu$  provides a clean environment for the study of soft-collinear interactions [6]. This process is particularly simple in that no hadrons appear in the final state. Yet, there is sensitivity to the light-cone structure of the  $B$  meson, probed by the coupling of the high-energy photon to the soft spectator quark inside the heavy meson. In this paper, we apply the formalism of [5] to prove factorization for this decay and systematically resum large Sudakov logarithms. The arguments we will present apply, with minimal modifications, to more complicated decays such as  $B \rightarrow K^*\gamma$ .

Several other groups have recently studied the decay  $B \rightarrow \gamma l\nu$ . In [7] a QCD factorization formula was established at next-to-leading order (NLO) in  $\alpha_s$ , and it was demonstrated that the leading-order LCDA of the  $B$  meson is sufficient to describe the decay amplitude at leading power in  $\Lambda/m_b$  (with  $\Lambda$  a typical hadronic scale), contrary to the findings of [6]. QCD factorization formulae have also been proposed for the related processes  $B \rightarrow \gamma\gamma$  and  $B \rightarrow \gamma l^+l^-$  [8, 9]. Arguments in favor of factorization in higher orders were given in [10] using a formulation of SCET different from the one adopted here.

In the present work we provide the first complete proof of factorization for a  $B$  decay in which hard-spectator interactions are relevant at leading power. We believe that our approach is simpler and more transparent than that put forward in [10]. The discussion of factorization we will present is more complete in that we prove the absence of non-valence Fock-state contributions and the convergence of the convolution integral to all orders in perturbation theory. We present the first correct result for the perturbative hard-scattering kernel in a scheme which uses the conventional, mass-independent definition of the LCDA. In addition, we discuss in detail the complete renormalization-group (RG) resummation of large logarithms. To this end, we solve evolution equations for the different components of the hard-scattering kernel, which follow from the renormalization properties of the  $B$ -meson LCDA. In this context we clarify the connection between the anomalous dimension of heavy-collinear currents in SCET and the cusp anomalous dimension encountered in the study of Wilson loops with light-like segments [11, 12].

Our main goal is to establish the QCD factorization formula [7]

$$\mathcal{A}(B^- \rightarrow \gamma l^- \bar{\nu}_l) \propto m_B f_B Q_u \int_0^\infty dl_+ \frac{\phi_+^B(l_+, \mu)}{l_+} T(l_+, E_\gamma, m_b, \mu) \quad (1)$$

to all orders in perturbation theory and at leading power in  $\Lambda/m_b$ . Here  $Q_u = \frac{2}{3}$  is the electric charge of the up-quark (in units of  $e$ ),  $f_B$  is the  $B$ -meson decay constant,  $\phi_+^B$  is a leading-order LCDA of the  $B$ -meson, and  $T = 1 + O(\alpha_s)$  is a perturbative hard-scattering kernel. Factorization holds as long as the photon is energetic in the  $B$ -meson rest frame, meaning that  $E_\gamma$  is of the order of the  $b$ -quark mass. The physics underlying the factorization formula is that a high-energy photon coupling to the soft constituents of the  $B$ -meson produces quantum fluctuations far off their mass shell, which can be integrated out in a low-energy effective theory. Specifically, when the photon couples to the  $b$ -quark it takes it off shell by an amount of order  $m_b^2$ , producing a hard quantum fluctuation that can be treated using the methods of heavy-quark effective theory (HQET) [13]. (The resulting contribution to the amplitude is, in fact, power suppressed.) When the photon couples to a soft light parton inside the  $B$  meson it produces a “hard-collinear” mode that is off shell by an amount of order  $m_b \Lambda$ . Once these short-distance modes are integrated out, the decay amplitude factorizes into a soft component (the LCDA) and a hard-scattering kernel.

For the analysis of the  $B \rightarrow \gamma l \nu$  decay amplitude we work in the  $B$ -meson rest frame and choose the photon momentum along the  $z$  direction, such that  $q^\mu = E_\gamma n^\mu$ , where  $n^\mu = (1, 0, 0, 1)$  is a light-like vector. The two transverse polarization states of the photon can be expressed in terms of the basis vectors  $\varepsilon_{\mp}^\mu = \frac{1}{\sqrt{2}}(0, 1, \mp i, 0)$ , which correspond to left- and right-circular polarization, respectively. It is convenient to define a second light-cone vector  $\bar{n}^\mu = (1, 0, 0, -1)$ . Any 4-vector can then be expanded as  $p^\mu = \frac{1}{2}(p_+ \bar{n}^\mu + p_- n^\mu) + p_\perp^\mu$ , where  $p_+ = n \cdot p$  and  $p_- = \bar{n} \cdot p$ .

In order to prove the factorization formula (1) one needs to show that [14]:

1. The decay amplitude can be expanded in powers of transverse momenta and, at leading order, can be expressed in terms of a convolution with the  $B$ -meson LCDA as shown in (1).
2. After subtraction of infrared contributions corresponding to the  $B$ -meson decay constant and LCDA, the leading contributions to the amplitude come from hard internal lines, i.e., the hard-scattering kernel  $T$  is free of infrared singularities to all orders in perturbation theory.
3. The convolution integral of the hard-scattering kernel with the LCDA is convergent.
4. Non-valence Fock states do not give rise to leading contributions.

In [10] the authors have discussed factorization for the decay  $B \rightarrow \gamma l \nu$ , albeit without addressing the last two points in this list. (The same criticism applies to the treatment of factorization for the decay  $B \rightarrow D \pi$  presented in [15].) In the following section we

provide the first complete proof of a QCD factorization theorem including hard-spectator interactions.

Before entering the technical details of the factorization proof, we stress that (1) provides the simplest example of a factorization formula for a decay in which the hard-scattering mechanism involves three different mass scales: a hard scale  $E_\gamma \sim m_b$ , a soft scale  $l_+ \sim \Lambda$ , and an intermediate scale  $\sqrt{2E_\gamma l_+} \sim \sqrt{m_b \Lambda}$ . Indeed, a diagrammatic analysis of one-loop diagrams for the decay amplitude reveals that leading contributions arise from three different regions of loop momenta: hard momenta  $k \sim m_b$ , soft momenta  $k \sim \Lambda$ , and hard-collinear momenta scaling like  $(k_+, k_-, k_\perp) \sim (\Lambda, m_b, \sqrt{m_b \Lambda})$ . This suggests a second stage of “perturbative” factorization [5, 10], which we will establish below. It says that the hard-scattering kernel itself can be factorized as

$$T(l_+, E_\gamma, m_b, \mu) = H\left(\frac{2E_\gamma}{\mu}, \frac{2E_\gamma}{m_b}\right) \cdot J\left(\frac{2E_\gamma l_+}{\mu^2}\right). \quad (2)$$

The hard component  $H$  accounts for the short-distance corrections from quantum fluctuations that are off shell by an amount of order  $m_b^2$ . They arise from the coupling of hard or hard-collinear particles to the heavy quark. The jet function  $J$  accounts for short-distance fluctuations that are off shell by an amount of order  $m_b \Lambda$ , which result from the coupling of hard-collinear particles to the light spectator quark. The factorization formula (2) thus separates the physics on two different short-distance scales. While this is necessary to gain full control over large logarithms arising in the perturbative calculation of the hard-scattering kernel, it is not a necessary step in the proof of the QCD factorization formula (1), which describes the separation of short- and long-distance physics.

## 2 Proof of factorization

We use the formulation of SCET developed in [5], which is applicable to any exclusive  $B$  decay into light particles. In the present case the  $B$  meson is the only hadron in the process, and so the relevant degrees of freedom in the low-energy effective theory are soft partons. The strong-interaction Lagrangian consists of the ordinary QCD Lagrangian for light quarks and gluons (restricted to the subspace of soft Fourier modes) and of the HQET Lagrangian for heavy quarks. In the full theory the hadronic part of the decay amplitude is given by the  $B$ -meson matrix element of the time-ordered product of a weak, flavor-changing current and the electromagnetic current. At leading order in the effective theory this object is matched onto flavor-changing bilocal operators of the form  $A^{(\text{em})}(z) \bar{q}_s(z) \dots h(0)$ , where  $A^{(\text{em})}$  is the photon field,  $h$  is the effective-theory field for the heavy  $b$ -quark, and  $q_s$  is the field for a soft  $u$ -quark. The separation  $z$  between the fields is nearly light-like,  $z^2 \sim 1/(m_b \Lambda) \approx 0$ . A space-time picture of the decay process is as follows: The virtual light quark produced at the weak vertex is off shell by an amount of order  $m_b \Lambda$  and almost collinear with the photon direction. The electromagnetic vertex is thus located at a distance  $z^\mu = t n^\mu + s \bar{n}^\mu + z_\perp^\mu$  relative to the weak vertex, where  $t \sim 1/\Lambda$

is the parametrically largest component. Because of the scaling properties of the photon and soft spectator momenta one can replace  $\bar{q}_s(z) \simeq \bar{q}_s(tn)$  and  $A^{(\text{em})}(z) \simeq A^{(\text{em})}(s\bar{n})$  to leading power. From the discussion in [5], it follows that at leading order in  $\Lambda/m_b$  there exists a unique type of SCET operators that mediate this decay and are allowed by gauge and reparameterization invariance. They are

$$\begin{aligned}
& \sum_{q=u,b} ie Q_q \int d^4x T \left\{ [\bar{u}\gamma^\mu(1-\gamma_5)b](0), [\bar{q} \mathcal{A}^{(\text{em})} q](x) \right\} \\
& \rightarrow \sum_i \int ds dt \widehat{C}_i(t, s, v \cdot q, m_b, \mu) \bar{Q}_s(tn) \mathcal{A}_{c\perp}^{(\text{em})}(s\bar{n}) \frac{\not{n}}{2} \Gamma_i \mathcal{H}(0) \\
& = \sum_i \int dt \widetilde{C}_i(t, \bar{n} \cdot q, v \cdot q, m_b, \mu) \bar{q}_s(tn) S(tn, 0) \mathcal{A}_{c\perp}^{(\text{em})}(0) \frac{\not{n}}{2} \Gamma_i h(0), \tag{3}
\end{aligned}$$

where  $\mathcal{H} = S^\dagger h$  and  $\bar{Q}_s = S^\dagger q_s$  with the path-ordered exponential

$$S(x) = P \exp \left( ig \int_{-\infty}^0 dw n \cdot A_s(x + wn) \right) \tag{4}$$

are gauge-invariant combinations of SCET fields and Wilson lines, and  $A_s$  is the soft gluon field. The combination  $S(x, y) \equiv S(x) S^\dagger(y)$  appearing in the last line of (3) represents a soft Wilson line connecting the two points  $x$  and  $y$  on a straight segment. Physically, this string operator arises because the virtual quark propagating between the two vertices can emit multiple soft gluons without power suppression. The quantity  $\mathcal{A}_{c\perp}^{(\text{em})}$  is the electromagnetic analog of the gauge-invariant collinear gluon field defined in [5]. To first order in  $e$  we have

$$\mathcal{A}_{c\perp}^{(\text{em})}(0) = \bar{n}_\alpha \gamma_\mu^\perp \int_{-\infty}^0 dw e F^{\alpha\mu}(w\bar{n}). \tag{5}$$

The Feynman rule for this object is simply  $e \not{\varepsilon}^*$ , where  $\varepsilon$  is the photon polarization vector. Finally, from the fact that the leptonic weak current  $\bar{\nu}\gamma_\mu(1-\gamma_5)l$  is conserved (in the limit where the lepton mass is neglected) it follows that the relevant Dirac structures  $\Gamma_i$  in (3) can be taken as  $\Gamma_1 = \gamma^\mu(1-\gamma_5)$  and  $\Gamma_2 = n^\mu(1+\gamma_5)$ .

In the last step in (3) we have used that the photon momentum  $q$  is an external momentum, so that the integration over  $s$  can be performed and leads to new coefficients  $\widetilde{C}_i(t, \bar{n} \cdot q, v \cdot q, m_b, \mu) = \int ds e^{is\bar{n}\cdot q} \widehat{C}_i(t, s, v \cdot q, m_b, \mu)$ . Note that the scalar products  $\bar{n} \cdot q = 2E_\gamma$  and  $v \cdot q = E_\gamma$  are both determined in terms of the photon energy; however, for a while it will be useful to distinguish between these two variables.

SCET power counting shows that the soft fields  $\mathcal{H}$  and  $\bar{Q}_s$  scale like  $\Lambda^{3/2}$ , the integration variable  $t$  scales like  $1/\Lambda$ , and the Wilson coefficients  $\widetilde{C}_i$  scale like 1. It follows that the hadronic components of the SCET operators on the right-hand side of (3) scale like  $\Lambda^2$ , which is one power of  $\Lambda$  less than a local current containing two soft quark fields. That one can write down such operators is a consequence of the fact that transverse

collinear fields have unsuppressed interactions with soft light quarks [5]. As a result, at leading power the Wilson coefficients  $\tilde{C}_i$  receive contributions only from Feynman diagrams with a photon attached to the light spectator quark in the  $B$  meson.

Following [16], we define the two leading-order LCDAs for the  $B$  meson in position space in terms of the HQET matrix element

$$\begin{aligned} & \frac{1}{\sqrt{m_B}} \langle 0 | \bar{q}_s(z) S(z, 0) \Gamma h(0) | \bar{B}(v) \rangle \\ &= -\frac{iF(\mu)}{2} \text{tr} \left[ \left( \tilde{\phi}_+^B(\tau, \mu) + \frac{\not{z}}{2\tau} \left[ \tilde{\phi}_-^B(\tau, \mu) - \tilde{\phi}_+^B(\tau, \mu) \right] \right) \Gamma \frac{1 + \not{v}}{2} \gamma_5 \right]. \end{aligned} \quad (6)$$

Here  $z^2 = 0$  is a null vector,  $v$  is the  $B$ -meson velocity, and  $\tau = v \cdot z - i0$ . The quantity  $F(\mu)$  corresponds to the asymptotic value of the product  $f_B \sqrt{m_B}$  in the heavy-quark limit [17]. The two distribution amplitudes obey the normalization  $\tilde{\phi}_\pm^B(0, \mu) = 1$  at  $\tau = 0$ . In our case  $z = tn$ , and because of the presence of the factor  $\not{v}$  in (3) it follows that only the function  $\tilde{\phi}_+^B(\tau, \mu)$  contributes, to which we refer as *the* leading-order LCDA. Note that with our choice of the light-cone basis vector  $n$  we have  $\tau = t v \cdot n = t$ . Performing the relevant traces over Dirac matrices, we find that at leading power in  $\Lambda/m_b$  the decay amplitude vanishes if the photon has right-circular polarization, while for a photon with left-circular polarization it is given by

$$\begin{aligned} \mathcal{A}(B^- \rightarrow \gamma_L l^- \bar{\nu}_l) &= \frac{iG_F}{\sqrt{2}} V_{ub} e \bar{u}_l(p_l) \not{\epsilon}^* (1 - \gamma_5) v_\nu(p_\nu) \\ &\times \sqrt{m_B} F(\mu) \int dt \tilde{C}_1(t, \bar{n} \cdot q, v \cdot q, m_b, \mu) \tilde{\phi}_+^B(t, \mu) + \dots, \end{aligned} \quad (7)$$

where the dots represent power-suppressed contributions. Only the SCET operator with Dirac structure  $\Gamma_1 = \gamma^\mu (1 - \gamma_5)$  contributes to the decay amplitude. In order to recast the above result in a form resembling the factorization formula (1) we introduce the Fourier transforms of the Wilson coefficient function and the LCDA as [5, 16]

$$\begin{aligned} C_1(l_+, \bar{n} \cdot q, v \cdot q, m_b, \mu) &= \int dt e^{-i l_+ t} \tilde{C}_1(t, \bar{n} \cdot q, v \cdot q, m_b, \mu), \\ \phi_+^B(\omega, \mu) &= \frac{1}{2\pi} \int dt e^{i \omega t} \tilde{\phi}_+^B(t, \mu). \end{aligned} \quad (8)$$

The analytic properties of the function  $\tilde{\phi}_+^B(t, \mu)$  in the complex  $t$ -plane imply that  $\phi_+^B(\omega, \mu) = 0$  if  $\omega < 0$ . Finally, the HQET parameter  $F(\mu)$  is related to the physical  $B$ -meson decay constant through  $f_B \sqrt{m_B} = K_F(m_b, \mu) F(\mu) [1 + O(\Lambda/m_b)]$ , where at NLO in the  $\overline{\text{MS}}$  scheme [17]

$$K_F(m_b, \mu) = 1 + \frac{C_F \alpha_s(\mu)}{4\pi} \left( 3 \ln \frac{m_b}{\mu} - 2 \right). \quad (9)$$

Combining these results, it follows that the terms shown in the second line of (7) equal those on the right-hand side of the factorization formula (1) if we identify the hard-scattering kernel as

$$\frac{Q_u}{l_+} T(l_+, E_\gamma, m_b, \mu) = K_F^{-1}(m_b, \mu) C_1(l_+, 2E_\gamma, E_\gamma, m_b, \mu). \quad (10)$$

Let us now discuss factorization in the context of our formalism. The fact that the position-space SCET operators appearing on the right-hand side of (3) contain component fields with light-like separation implies that transverse parton momenta can be set to zero at leading power. As we have seen, this naturally leads to the appearance of LCDAs. In SCET the hard-scattering kernel  $T$  is identified with a Wilson coefficient. The absence of infrared singularities then follows from the very existence of a low-energy effective theory, because Wilson coefficients arise from matching and by construction are insensitive to infrared physics. At this point we have achieved as much as [10].

We proceed to prove the convergence of the convolution integral in (1). The key ingredient here is to note that the invariance of SCET operators under reparameterizations of the light-cone basis vectors  $n$  and  $\bar{n}$  [18] can be used to deduce the dependence of Wilson coefficient functions on the separation  $t$  between the component fields of non-local operators [5]. In our case, invariance of the operators in (3) under the rescaling transformation  $n^\mu \rightarrow n^\mu/\alpha$  and  $\bar{n}^\mu \rightarrow \alpha\bar{n}^\mu$  (with fixed  $v$ ) implies that

$$\begin{aligned} \tilde{C}_i(t, \bar{n} \cdot q, v \cdot q, m_b, \mu) &= \tilde{C}_i(\alpha t, \alpha \bar{n} \cdot q, v \cdot q, m_b, \mu) \\ &= \tilde{H}_i(v \cdot q, m_b, \mu) \cdot \tilde{J}(\alpha t, \alpha \bar{n} \cdot q, \mu) \end{aligned} \quad (11)$$

to all orders in perturbation theory. In other words, the variables  $t$  and  $\bar{n} \cdot q$  can only appear in the combination  $\bar{n} \cdot q/t$ , but not individually. (Similarly, the variable  $t$  enters the argument of the LCDA in (6) in the reparameterization-invariant combination  $\tau = t v \cdot n$ .) In the second step we have used the fact that  $v \cdot q$  and  $m_b$  enter only through interactions of hard or hard-collinear gluons with the heavy quark. The corresponding modes can be integrated out in a first matching step and lead to the functions  $\tilde{H}_i$ , which depend on the Dirac structure of the weak current containing the heavy quark. The non-localities of the component fields in (3) result from the coupling of hard-collinear fields to the soft spectator quark in the  $B$  meson. These effects live on scales of order  $m_b \Lambda$  and can be integrated out in a second step, leading to the function  $\tilde{J}$ . Since the coefficients  $\tilde{C}_i$  are dimensionless, it follows that (with a slight abuse of notation)

$$\tilde{C}_i(t, \bar{n} \cdot q, v \cdot q, m_b, \mu) = \tilde{H}_i\left(\frac{2v \cdot q}{\mu}, x_\gamma\right) \cdot \tilde{J}\left(\frac{\bar{n} \cdot q}{\mu^2 t}\right), \quad (12)$$

where  $x_\gamma \equiv 2v \cdot q/m_b = 2E_\gamma/m_b$  is a scaling variable of order 1. Corrections to this perturbative factorization formula are suppressed by a ratio of the intermediate scale  $\bar{n} \cdot q/t \sim m_b \Lambda$  and a hard scale of order  $m_b^2$ . (While the fact that these corrections scale like  $\Lambda/m_b$  is an immediate consequence of our discussion here, it is not obvious in the

context of the approach proposed in [10], which is based on an expansion in powers of  $\sqrt{\Lambda/m_b}$ .) The corresponding result for the hard-scattering kernel obtained after Fourier transformation has the form shown in (2) if we identify

$$\begin{aligned} H\left(\frac{2v \cdot q}{\mu}, x_\gamma\right) &= K_F^{-1}(m_b, \mu) \tilde{H}_1\left(\frac{2v \cdot q}{\mu}, x_\gamma\right), \\ \frac{Q_u}{l_+} J\left(\frac{\bar{n} \cdot q l_+}{\mu^2}\right) &= \int dt e^{-il_+ t} \tilde{J}\left(\frac{\bar{n} \cdot q}{\mu^2 t}\right), \end{aligned} \tag{13}$$

where  $\bar{n} \cdot q l_+ = 2q \cdot l = 2E_\gamma l_+$ . Since the dependence of the coefficient functions on the renormalization scale is logarithmic, it follows that to all orders in perturbation theory the Wilson coefficients in (12) scale like  $\tilde{C}_i \sim 1$  modulo logarithms. (Correspondingly, the kernel scales like  $T \sim 1$  modulo logarithms.) The convergence of the convolution integral in (7) in the infrared region  $t \rightarrow \infty$ , corresponding to the region  $l_+ \rightarrow 0$  in the factorization formula (1), then follows to all orders in perturbation theory as long as the integral converges at tree level. Because the  $B$  meson has a spatial size of order  $1/\Lambda$  due to confinement, the bilocal matrix element must vanish faster than  $1/t$  for  $t \gg 1/\Lambda$ , and so the integral over  $t$  is convergent.

The final step in the factorization proof is to demonstrate the power suppression of more complicated projections onto the  $B$  meson involving higher Fock states or transverse parton momenta. The fact that such projections do not contribute at leading power follows from the rules for constructing SCET operators out of gauge-invariant building blocks, as explained in [5]. Projections sensitive to transverse momentum components contain extra derivatives and so are power suppressed. Projections corresponding to non-valence Fock states contain insertions of the soft (subscript “ $s$ ”) gluon field

$$\mathcal{A}_s^\nu(x) = [S^\dagger(iD_s^\mu S)](x) = \int_{-\infty}^0 dw n_\alpha [S^\dagger g G_s^{\alpha\nu} S](x + wn). \tag{14}$$

Since  $\mathcal{A}_s$  scales like  $\Lambda$ , such insertions lead to power suppression unless this field is integrated over a domain of extension  $1/\Lambda$ . It follows that the only possibility for a leading-power contribution from non-valence Fock states would be to include a factor of  $\int du \mathcal{A}_s^\nu(un)$  with  $u \sim 1/\Lambda$  (modulo logarithmic dependence on  $u$  in the Wilson coefficients). Reparameterization invariance requires that this object must be accompanied by a factor of  $n$  in the numerator.<sup>1</sup> Since  $n \cdot \mathcal{A}_s = 0$  by definition, the only possibility would be to include an insertion of  $\int du \not{n} \mathcal{A}_{s\perp}(un)$  somewhere between the light-quark field  $\bar{Q}_s$  and the Dirac matrix  $\Gamma_i$  in the SCET operators in (3). However, any such insertion vanishes, since  $n^2 = 0$ . It is important for this argument that the heavy quark can be integrated out before one removes the off-shell modes resulting from soft-collinear interactions. The heavy quark therefore decouples from such interactions. This ensures

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<sup>1</sup>The reason is that the integration variable  $u$  scales like  $\bar{n}$ , and that the basis vectors  $n$  and  $\bar{n}$  always come in pairs since they are introduced through the light-cone decomposition of 4-vectors. The only alternative to having a factor of  $n$  in the numerator would be to add a factor of  $\bar{n}/(t\bar{n} \cdot q)$  or  $\bar{n}/(u\bar{n} \cdot q)$ , both of which scale like  $\Lambda/m_b$  and thus would lead to power suppression.



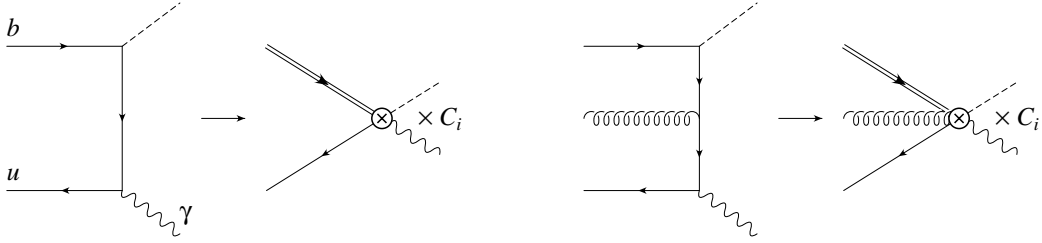


Figure 1: Tree-level matching calculation for the Wilson coefficients  $C_1$  and  $C_2$ , without (left) and with (right) an external soft gluon. The dashed line denotes the flavor-changing weak current. The resulting non-local operators in SCET are denoted by a crossed circle. Double lines represent effective heavy-quark fields in HQET.

that the factor  $\not{n}$  cannot appear to the right of  $\Gamma_i$ , and it excludes the appearance of  $v \cdot n$  instead of  $\not{n}$ .

We have thus completed the proof of the factorization formula (1) to all orders in perturbation theory, and at leading power in  $\Lambda/m_b$ . The remainder of this paper is devoted to the calculation of the hard-scattering kernel at NLO in RG-improved perturbation theory, including a complete resummation of large logarithms.

### 3 Calculation of the hard-scattering kernel

The Wilson coefficients  $\tilde{C}_i$  in (3) are derived by matching perturbative expressions for operator matrix elements in the full theory onto corresponding expressions in the effective theory. Because by construction the Wilson coefficients are insensitive to infrared physics the matching can be done using on-shell external quark states. We thus assign incoming momenta  $m_b v$  to the heavy quark and  $l$  (with  $l^2 = 0$ ) to the soft light quark. At tree-level the relevant amplitude in the full theory is obtained from the first Feynman diagram shown in Figure 1. At leading power only photon emission from the light spectator quark contributes; emission from the  $b$ -quark is suppressed by one power of  $\Lambda/m_b$ . The corresponding amplitude in the effective theory follows from the second diagram. A straightforward matching calculation yields for the Wilson coefficients at this order  $C_1 = Q_u/l_+$  and  $C_2 = 0$ , where  $l_+ = n \cdot l - i0$ . (In general, the variable  $l_+$  is conjugate to the coordinate  $t$  in (3). In the present case,  $l_+$  coincides with the plus component of the spectator momentum because of the particular external state we choose for the matching calculation.) The corresponding results in position space are  $\tilde{C}_1 = iQ_u \theta(t)$  and  $\tilde{C}_2 = 0$ .

While these results are most easily derived by matching amplitudes with two external quarks, the Wilson coefficients are independent of the nature of the external states. Alternatively, therefore, they can be determined by matching amplitudes with external soft gluons. Gluon emissions from the external quark lines cancel in the matching, since those emissions are the same in the full theory and in SCET. However, gluon emissions

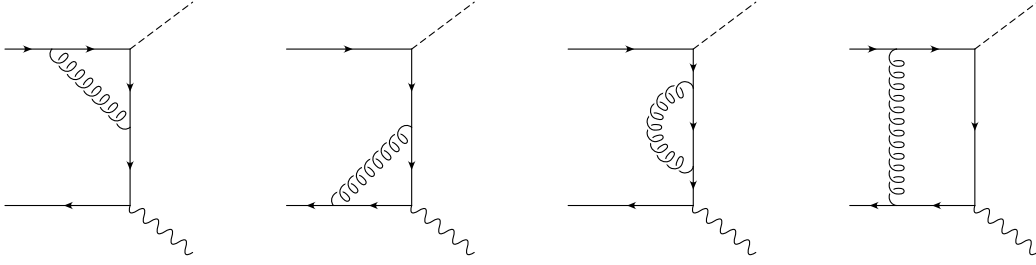


Figure 2: One-loop diagrams in the full theory contributing at leading power to the  $B \rightarrow \gamma l \nu$  decay amplitude.

from the internal quark propagator, which is integrated out in the effective theory, are contained in the Wilson line  $S(tn, 0)$ , which sums up an infinite number of soft gluon insertions. The one-gluon example is illustrated in the last two diagrams in Figure 1. Evaluating these graphs one readily recovers the results given above. In that way one confirms (at tree level) our general result about the absence of operators containing additional insertions of the soft gluon field  $\mathcal{A}_s$ .

Beyond tree level the coefficient functions can be written in the form

$$C_i = \frac{Q_u}{l_+} \left[ \delta_{i1} + \frac{C_F \alpha_s(\mu)}{4\pi} c_i + \dots \right]. \quad (15)$$

To obtain the NLO corrections  $c_i$  we evaluate the one-loop contributions to the decay amplitude in the full theory and in SCET. The relevant diagrams in full QCD are shown in Figure 2. In addition there is a contribution from the wave-function renormalization for the heavy quark. Using the  $\overline{\text{MS}}$  regularization scheme with  $d = 4 - 2\epsilon$  dimensions and anti-commuting  $\gamma_5$ , we find the following expressions for the contributions  $A_i$  corresponding to the coefficients  $c_i$  (before subtraction of the pole terms):

$$\begin{aligned} A_1^{\text{QCD}} &= \left( \frac{m_b}{\mu} \right)^{-2\epsilon} \left[ -\frac{1}{2\epsilon} - \ln^2 \frac{2E_\gamma l_+}{m_b^2} - 2(1 - 2 \log x_\gamma) \ln \frac{2E_\gamma l_+}{m_b^2} - 4 \ln^2 x_\gamma \right. \\ &\quad \left. + \frac{2 - 3x_\gamma}{1 - x_\gamma} \ln x_\gamma - 2L_2(1 - x_\gamma) - 2 - \pi^2 \right] \\ &\quad + \left( \frac{2E_\gamma l_+}{\mu^2} \right)^{-\epsilon} \left( -\frac{2}{\epsilon} - 5 \right) + \left( \frac{-2v \cdot l}{\mu} \right)^{-2\epsilon} \left[ -\frac{1}{\epsilon^2} + \ln^2 \left( \frac{-2v \cdot l}{l_+} \right) + \frac{7\pi^2}{12} \right], \\ A_2^{\text{QCD}} &= \left( \frac{m_b}{\mu} \right)^{-2\epsilon} \frac{x_\gamma \ln x_\gamma}{1 - x_\gamma}. \end{aligned} \quad (16)$$

In the expression for  $A_1^{\text{QCD}}$  the first bracket contains the vertex correction for the weak current and the contribution from wave-function renormalization for the heavy quark, the second term corresponds to the vertex correction for the electromagnetic current and the self-energy insertion on the intermediate quark propagator, and the last term corresponds

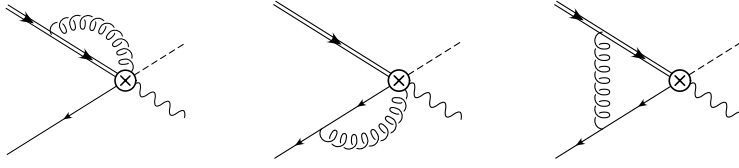


Figure 3: One-loop diagrams in the effective theory whose contribution to the amplitude needs to be subtracted in the calculation of the Wilson coefficients.

to the box diagram. Note that the various terms depend on three different mass scales:  $m_b$  (hard),  $2E_\gamma l_+ \sim m_b \Lambda$  (hard-collinear), and  $l \sim \Lambda$  (soft). Whereas the first two scales are perturbative, the contribution from the box diagram is dominated by soft physics. Indeed, power counting shows that the hard and collinear contributions to the box graph only contribute at subleading power in  $\Lambda/m_b$ , as has been observed previously in [7]. Note that the box contribution involves components of the soft spectator momentum other than  $l_+$  [6]. However, this dependence will cancel in the matching.

In the next step we evaluate the corresponding contributions at one-loop order in the effective theory. Generically, they are of the form  $C_{1\text{-loop}} \otimes \langle O_{\text{SCET}} \rangle_{\text{tree}} + C_{\text{tree}} \otimes \langle O_{\text{SCET}} \rangle_{1\text{-loop}}$ . The second contribution, which involves one-loop matrix elements of SCET operators convoluted with tree-level coefficient functions, must be subtracted from the amplitudes obtained in the full theory. The relevant diagrams are shown in Figure 3. We find

$$\begin{aligned}
 A_1^{\text{SCET}} &= \left(\frac{l_+}{\mu}\right)^{-2\epsilon} \left(-\frac{1}{\epsilon^2} - \frac{3\pi^2}{4}\right) + \left(\frac{-2v \cdot l}{\mu}\right)^{-2\epsilon} \left[-\frac{1}{\epsilon^2} + \ln^2\left(\frac{-2v \cdot l}{l_+}\right) + \frac{7\pi^2}{12}\right], \\
 A_2^{\text{SCET}} &= 0.
 \end{aligned}
 \tag{17}$$

The first term in the expression for  $A_1^{\text{SCET}}$  corresponds to the first two diagrams in the figure, while the second term is obtained from the last graph. Note that this is precisely the same contribution as obtained from the box diagram in the full theory.

The difference of the expressions given in (16) and (17) determines the NLO contributions to the Wilson coefficient functions. Subtracting the pole terms in the  $\overline{\text{MS}}$  scheme, we find

$$\begin{aligned}
 c_1 &= -2 \ln^2 \frac{m_b}{\mu} + (5 - 4 \ln x_\gamma) \ln \frac{m_b}{\mu} + \ln^2 \frac{2E_\gamma l_+}{\mu^2} - 2 \ln^2 x_\gamma \\
 &\quad + \frac{2 - 3x_\gamma}{1 - x_\gamma} \ln x_\gamma - 2L_2(1 - x_\gamma) - 7 - \frac{\pi^2}{4}, \\
 c_2 &= \frac{x_\gamma \ln x_\gamma}{1 - x_\gamma}.
 \end{aligned}
 \tag{18}$$

Our result for the Wilson coefficient  $C_1$  agrees with the expressions for the hard-scattering kernel given in [7] and [10]; however, eq. (10) shows that identifying the kernel with

$C_1$  misses the large logarithms encountered when the HQET matrix element  $F(\mu)$  is related to the physical  $B$ -meson decay constant. (In other words, these papers implicitly assume an  $m_b$ -dependent definition and normalization of the  $B$ -meson LCDA, which is unconventional. In our approach the LCDA is normalized to unity, and all mass dependence is explicit in the hard-scattering kernel.) Including these corrections and simplifying the answer, we obtain for the hard-scattering kernel at NLO the final result

$$T(l_+, E_\gamma, m_b, \mu) = 1 + \frac{C_F \alpha_s(\mu)}{4\pi} \left[ -2 \ln^2 \frac{2E_\gamma}{\mu} + 2 \ln \frac{2E_\gamma}{\mu} + \ln^2 \frac{2E_\gamma l_+}{\mu^2} - \frac{x_\gamma \ln x_\gamma}{1 - x_\gamma} - 2L_2(1 - x_\gamma) - 5 - \frac{\pi^2}{4} \right]. \quad (19)$$

Note that, despite appearance, the scale dependence of the expression in brackets is of the form  $2 \ln^2(\mu/l_+) - 2 \ln(\mu/l_+) + \mu$ -independent terms, indicating that it can be canceled by the scale dependence of the LCDA  $\phi_+^B(l_+, \mu)$  under the convolution integral in (1). This would be impossible if the argument of the  $\ln^2$ -term depended on the hard scales  $E_\gamma$  or  $m_b$ .

## 4 RG evolution and resummation

The hard-scattering kernel  $T$  contains the logarithms  $\ln(2E_\gamma/\mu)$  and  $\ln(2E_\gamma l_+/\mu^2)$ , which cannot be made small simultaneously for any choice of the renormalization scale  $\mu$ . While for realistic values of the photon energy ( $E_\gamma \sim m_b/2 \sim 2.5$  GeV) these logarithms are numerically not too large, it is conceptually interesting to gain control over the perturbative expansion of the kernel by summing the various logarithms to all orders in perturbation theory.

It was proposed in [7] to choose the renormalization scale of order  $\mu^2 \sim m_b \Lambda$ , and to identify the remaining large logarithms in (18) with those arising in the matching of heavy-collinear current operators onto SCET. The precise nature of this identification beyond one-loop order was however left unclear. In the spirit of an effective-theory approach one would rather prefer to take the renormalization scale further down to a value  $\mu = \text{few} \times \Lambda_{\text{QCD}}$  independent of the  $b$ -quark mass and the photon energy, yet large enough for a perturbative treatment. In this way all dependence on the large scales  $m_b$  and  $E_\gamma$  becomes explicit and is contained in Wilson coefficients of the effective theory.

To gain full control over the large logarithms in the perturbative expansion of the kernel we perform the matching onto SCET in two steps [5, 10]. In the first step the off-shell fluctuations of the heavy  $b$  quark are integrated out by matching onto HQET. Hard-collinear modes with momenta scaling like  $(k_+, k_-, k_\perp) \sim (\Lambda, m_b, \sqrt{m_b \Lambda})$  are integrated out in a second step. In contrast with [10], we avoid the explicit construction of the intermediate effective theory. Since this theory is needed only for RG improvement, it suffices to perform the matching diagrammatically using the method of regions [19, 20]. Specifically, because the intermediate mass scale of order  $m_b \Lambda$  always arises from a scalar product of a soft momentum with a collinear momentum, one can perturbatively match

onto the intermediate theory by simply setting the soft momentum  $l = 0$ . In that way only hard fluctuations associated with couplings to the heavy quark are integrated out. This yields precisely the function  $H$  in (2).

Let us illustrate this for the case at hand. For  $l = 0$  the second and third diagrams in Figure 2 involve scaleless integrals that vanish in dimensional regularization, while the box graph can readily be shown to vanish at leading power. The remaining contribution from the weak vertex correction and wave-function renormalization for the heavy quark yields

$$A_1^{\text{QCD}}|_{l=0} = \left(\frac{m_b}{\mu}\right)^{-2\epsilon} \left[ -\frac{1}{\epsilon^2} - \frac{5}{2\epsilon} + \frac{2 \ln x_\gamma}{\epsilon} - 2 \ln^2 x_\gamma + \frac{2 - 3x_\gamma}{1 - x_\gamma} \ln x_\gamma - 2L_2(1 - x_\gamma) - 6 - \frac{\pi^2}{12} \right], \quad (20)$$

while the expression for  $A_2^{\text{QCD}}|_{l=0}$  is the same as that for  $A_2^{\text{QCD}}$  given in (16). All graphs in the effective theory involve tadpole integrals and vanish in dimensional regularization. Hence, after  $\overline{\text{MS}}$  subtractions the above result determines the hard function  $\tilde{H}_1$  defined in (11). Using the first relation in (13), it then follows that

$$H\left(\frac{2E_\gamma}{\mu}, x_\gamma\right) = 1 + \frac{C_F \alpha_s(\mu)}{4\pi} \left[ -2 \ln^2 \frac{2E_\gamma}{\mu} + 2 \ln \frac{2E_\gamma}{\mu} - \frac{x_\gamma \ln x_\gamma}{1 - x_\gamma} - 2L_2(1 - x_\gamma) - 4 - \frac{\pi^2}{12} \right]. \quad (21)$$

According to (2), the difference between (19) and (21) determines the one-loop contribution to the jet function, which is thus given by

$$J\left(\frac{2E_\gamma l_+}{\mu^2}\right) = 1 + \frac{C_F \alpha_s(\mu)}{4\pi} \left( \ln^2 \frac{2E_\gamma l_+}{\mu^2} - 1 - \frac{\pi^2}{6} \right). \quad (22)$$

In order to proceed we need RG equations obeyed by the various coefficient functions. The fact that the decay amplitude in (1) is scale independent links the scale dependence of the hard-scattering kernel to the evolution of the LCDA  $\phi_+^B(\omega, \mu)$  [16]. By analyzing the renormalization properties of this function, one finds that the hard-scattering kernel satisfies the integro-differential equation (for  $l_+ > 0$ ) [21]<sup>2</sup>

$$\frac{d}{d \ln \mu} T(l_+, \mu) = \left[ \Gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu}{l_+} + \gamma(\alpha_s) \right] T(l_+, \mu) + \int_0^\infty d\omega \Gamma(\omega, l_+, \alpha_s) T(\omega, \mu), \quad (23)$$

where  $\Gamma_{\text{cusp}}$  is the universal cusp anomalous dimension familiar from the theory of the renormalization of Wilson loops [11]. The function  $\Gamma$  obeys  $\int d\omega \Gamma(\omega, \omega', \alpha_s) = 0$ . At one-loop order, it is given by

$$\Gamma^{1\text{-loop}}(\omega, \omega', \alpha_s) = -\Gamma_{\text{cusp}}^{1\text{-loop}}(\alpha_s) \left[ \frac{\omega'}{\omega} \frac{\theta(\omega - \omega')}{\omega - \omega'} + \frac{\theta(\omega' - \omega)}{\omega' - \omega} \right]_+, \quad (24)$$

---

<sup>2</sup>For simplicity of notation, we omit the arguments  $E_\gamma$ ,  $m_b$ , and  $x_\gamma$  for the remainder of this section. Also, unless otherwise indicated,  $\alpha_s \equiv \alpha_s(\mu)$ .

where the plus distribution is defined such that, when  $\Gamma$  is integrated with a function  $f(\omega)$ , one must replace  $f(\omega) \rightarrow f(\omega) - f(\omega')$  under the integral. It is straightforward to check that our one-loop result in (19) is a solution to (23) at order  $\alpha_s$ .

From the factorization property of the hard-scattering kernel exhibited in (2) and the functional forms of the hard and jet functions given above, it follows that the hard component and the jet function obey the RG equations

$$\begin{aligned} \frac{d}{d \ln \mu} H(\mu) &= \left[ -\Gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu}{2E_\gamma} + \gamma(\alpha_s) - \gamma'(\alpha_s) \right] H(\mu), \\ \frac{d}{d \ln \mu} J(l_+, \mu) &= \left[ \Gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{2E_\gamma l_+} + \gamma'(\alpha_s) \right] J(l_+, \mu) + \int_0^\infty d\omega \Gamma(\omega, l_+, \alpha_s) J(\omega, \mu). \end{aligned} \quad (25)$$

The reasoning here is analogous to an argument presented by Korchemsky and Sterman in their discussion of the  $B \rightarrow X_s \gamma$  photon spectrum [22]. The anomalous dimensions  $\gamma$  and  $\gamma'$  do not have a simple geometric interpretation and must be determined by explicit calculation. From (21) and (22) we find

$$\gamma(\alpha_s) = -2C_F \frac{\alpha_s}{4\pi} + O(\alpha_s^2), \quad \gamma'(\alpha_s) = O(\alpha_s^2). \quad (26)$$

From the discussion above, it follows that in the intermediate theory the amplitude is represented in terms of the time-ordered product of the electromagnetic current with a heavy-collinear SCET current operator of the type  $\bar{\chi} \Gamma_1 h$ , where  $\chi$  is a hard-collinear quark field. The corresponding matching relation for such currents reads [23]

$$\bar{u} \gamma^\mu (1 - \gamma_5) b \rightarrow C_3^{\text{SCET}}(\mu) \bar{\chi} \gamma^\mu (1 - \gamma_5) h + \dots, \quad (27)$$

where the ellipses represent terms with different Dirac structure. The Wilson coefficient  $C_3^{\text{SCET}}(\mu)$  coincides with our function  $\tilde{H}_1(\mu)$ . The first relation in (25) then determines the exact form of the RG equation obeyed by the Wilson coefficients of heavy-collinear currents in SCET, which we have derived here for the first time. Based on a comparison of one-loop results obtained in SCET with expressions presented in [22] for the photon energy spectrum in inclusive  $B \rightarrow X_s \gamma$  decays, previous authors have conjectured a relation between the coefficient of the  $\ln(\mu/2E_\gamma)$  term in (25) and the cusp anomalous dimension [23, 24]; however, the nature and origin of this connection beyond one-loop order was left unclear. Note, in particular, that the minus sign in front of the logarithm in the RG equation for  $H(\mu)$  does not allow for a simple interpretation in terms of a cusp singularity of a Wilson loop. Our derivation establishes two important facts: First, to all orders in perturbation theory only a single logarithm of the ratio  $\mu/2E_\gamma$  appears in the RG equation. Secondly, the coefficient of the logarithm equals minus the cusp anomalous dimension. The first observation is a crucial one. The fact that no higher powers of logarithms appear in the RG equations cannot be inferred from a fixed-order calculation in SCET. It follows from remarkable properties of Wilson loops discussed long ago by Korchemsky and Radyushkin [11]. Without this insight it would be impossible to integrate the RG equations. The second observation implies that the coefficient of

the logarithmic term is known to two-loop order, which allows for the resummation of Sudakov logarithms at NLO.

We now discuss the general solution of the evolution equations (23) and (25), which is non-trivial due to the convolution integral involving the function  $\Gamma(\omega, \omega', \alpha_s)$ . Note that the kernel  $\Gamma$  is not simply a function of the difference  $(\omega - \omega')$ , and so the convolution cannot be turned into a product using Fourier transformation. Nevertheless, an exact solution can be written down in terms of a new function [21]

$$\mathcal{F}(\alpha_s, a) = \int d\omega \Gamma(\omega, \omega', \alpha_s) \left(\frac{\omega}{\omega'}\right)^a. \quad (28)$$

At one-loop order we find from (24)

$$\mathcal{F}^{1\text{-loop}}(\alpha_s, a) = \Gamma_{\text{cusp}}^{1\text{-loop}}(\alpha_s) \left[ \psi(1+a) + \psi(1-a) + 2\gamma_E \right], \quad (29)$$

where  $\psi(z)$  is the logarithmic derivative of the Euler  $\Gamma$ -function. We start by solving the first equation in (25) with the initial condition for  $H(\mu_h)$  evaluated at a high scale  $\mu_h \sim m_b$ , for which it does not contain large logarithms. We then evolve the function  $H(\mu)$  down to an intermediate scale  $\mu_i \sim \sqrt{m_b \Lambda}$  and multiply it by the result  $J(l_+, \mu_i)$  for the jet function, which at the intermediate scale is free of large logarithms and can be written in the general form  $J(l_+, \mu_i) \equiv \mathcal{J}[\alpha_s(\mu_i), \ln(2E_\gamma l_+/\mu_i^2)]$ . This determines the kernel  $T(l_+, \mu_i)$  at the intermediate scale. Finally, we solve (23) and compute the evolution down to a low-energy scale  $\mu \sim \text{few} \times \Lambda_{\text{QCD}}$ . The exact solution is given by

$$T(l_+, \mu) = H(\mu_h) \mathcal{J}[\alpha_s(\mu_i), \nabla_\eta] \exp U(l_+, \mu, \mu_i, \mu_h, \eta) \Big|_{\eta=0}, \quad (30)$$

where the notation  $\mathcal{J}[\alpha_s(\mu_i), \nabla_\eta]$  means that one must replace each logarithm of the ratio  $2E_\gamma l_+/\mu_i^2$  by a derivative with respect to an auxiliary parameter  $\eta$ . The evolution function  $U$  is given by

$$\begin{aligned} U(l_+, \mu, \mu_i, \mu_h, \eta) = & \int_{\alpha_s(\mu_h)}^{\alpha_s(\mu_i)} d\alpha \frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)} \left[ \ln \frac{2E_\gamma}{\mu_h} - \int_{\alpha_s(\mu_h)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} \right] - \int_{\alpha_s(\mu_h)}^{\alpha_s(\mu_i)} d\alpha \frac{\gamma'(\alpha)}{\beta(\alpha)} \\ & - \int_{\alpha_s(\mu_i)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)} \left[ \ln \frac{l_+}{\mu} + \int_{\alpha}^{\alpha_s(\mu)} \frac{d\alpha'}{\beta(\alpha')} \right] + \int_{\alpha_s(\mu_h)}^{\alpha_s(\mu)} d\alpha \frac{\gamma(\alpha)}{\beta(\alpha)} \\ & + \eta \ln \frac{2E_\gamma l_+}{\mu_i^2} + \int_{\alpha_s(\mu_i)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)} \mathcal{F} \left( \alpha, \eta - \int_{\alpha_s(\mu_i)}^{\alpha} d\alpha' \frac{\Gamma_{\text{cusp}}(\alpha')}{\beta(\alpha')} \right), \quad (31) \end{aligned}$$

where  $\beta(\alpha_s) = d\alpha_s/d \ln \mu$ . Close inspection shows that the result for the hard-scattering kernel is independent of the two matching scales  $\mu_h$  and  $\mu_i$ .

Given this exact result, it is straightforward to derive approximate expressions for the kernel at given orders in RG-improved perturbation theory, by using perturbative expansions of the anomalous dimensions and  $\beta$ -function to the required order. Unfortunately, controlling terms of  $O(\alpha_s)$  in the evolution function  $U$  would require knowledge of the cusp anomalous dimension at three-loop order (as well as knowledge of all other anomalous dimensions at two-loop order), which at present is lacking. We will, however, control the dependence on the variables  $l_+$  and  $E_\gamma$  to  $O(\alpha_s)$ . As usual, we write

$$\beta(\alpha_s) = -2\alpha_s \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1}, \quad \Gamma_{\text{cusp}}(\alpha_s) = \sum_{n=0}^{\infty} \Gamma_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1}, \quad (32)$$

and similarly for the anomalous dimensions  $\gamma$  and  $\gamma'$ . The relevant expansion coefficients are  $\Gamma_0 = 4C_F$ ,  $\Gamma_1 = 4C_F \left[ \left(\frac{67}{9} - \frac{\pi^2}{3}\right) C_A - \frac{20}{9} T_F n_f \right]$ ,  $\gamma_0 = -2C_F$ ,  $\gamma'_0 = 0$ , and  $\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f$ ,  $\beta_1 = \frac{34}{3} C_A^2 - \frac{20}{3} C_A T_F n_f - 4C_F T_F n_f$ . Defining the ratios  $r_1 = \alpha_s(\mu_i)/\alpha_s(\mu_h)$  and  $r_2 = \alpha_s(\mu)/\alpha_s(\mu_i)$ , we obtain our final result

$$\begin{aligned} T(l_+, \mu) &= e^{U_0(\mu, \mu_i, \mu_h)} \left(\frac{l_+}{\mu}\right)^{c \ln r_2} \left(\frac{2E_\gamma}{\mu_h}\right)^{-c \ln r_1} \\ &\times \left\{ 1 + \frac{C_F \alpha_s(\mu_h)}{4\pi} \left[ -2 \ln^2 \frac{2E_\gamma}{\mu_h} + 2 \ln \frac{2E_\gamma}{\mu_h} - \frac{x_\gamma \ln x_\gamma}{1-x_\gamma} - 2L_2(1-x_\gamma) - 4 - \frac{\pi^2}{12} \right] \right. \\ &\quad + \frac{C_F \alpha_s(\mu_i)}{4\pi} \left[ \left( \ln \frac{2E_\gamma l_+}{\mu_i^2} - \psi(1+c \ln r_2) - \psi(1-c \ln r_2) - 2\gamma_E \right)^2 \right. \\ &\quad \left. \left. - \psi'(1+c \ln r_2) + \psi'(1-c \ln r_2) - 1 - \frac{\pi^2}{6} \right] \right. \\ &\quad \left. + \frac{\Gamma_0}{2\beta_0} \left( \frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0} \right) \left[ \frac{\alpha_s(\mu) - \alpha_s(\mu_i)}{4\pi} \ln \frac{l_+}{\mu} - \frac{\alpha_s(\mu_i) - \alpha_s(\mu_h)}{4\pi} \ln \frac{2E_\gamma}{\mu_h} \right] \right\}, \end{aligned} \quad (33)$$

where  $c = \Gamma_0/2\beta_0$ , and

$$\begin{aligned} U_0(\mu, \mu_i, \mu_h) &= \frac{\Gamma_0}{4\beta_0^2} \left\{ (1 - \ln r_1) \frac{4\pi}{\alpha_s(\mu_h)} + (1 + \ln r_2) \frac{4\pi}{\alpha_s(\mu)} - \frac{8\pi}{\alpha_s(\mu_i)} \right. \\ &\quad \left. + \frac{\beta_1}{2\beta_0} (\ln^2 r_1 + \ln^2 r_2) + \left( \frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0} \right) \left( \ln \frac{r_1}{r_2} + 2 - r_1 - \frac{1}{r_2} \right) \right\} \\ &\quad - \frac{\gamma_0}{2\beta_0} \ln(r_1 r_2) - \ln \frac{\Gamma(1+c \ln r_2)}{\Gamma(1-c \ln r_2)} - 2\gamma_E c \ln r_2 + O(\alpha_s) \end{aligned} \quad (34)$$

corresponds to the evolution function  $U$  evaluated with  $\eta = 0$ ,  $l_+ = \mu$ , and  $2E_\gamma = \mu_h$ . The only piece missing for a complete resummation at NNLO is the  $O(\alpha_s)$  contribution to  $U_0$ , which is independent of  $l_+$  and  $E_\gamma$ .



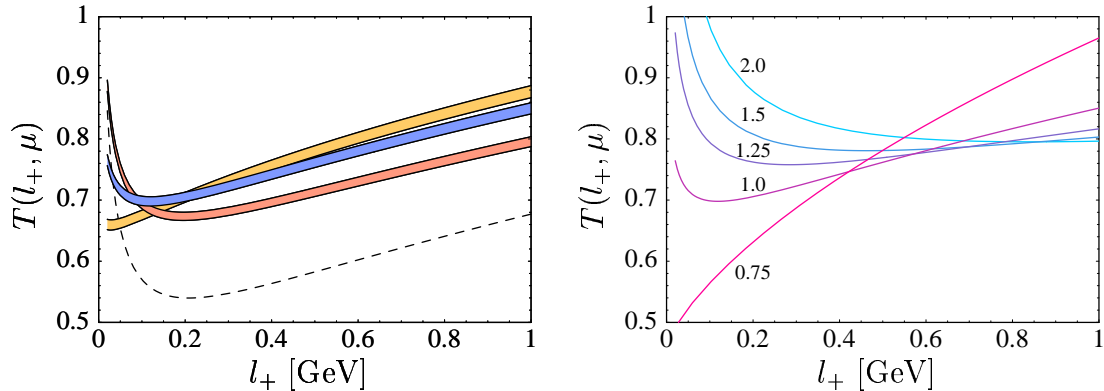


Figure 4: RG-improved predictions for the hard-scattering kernel at maximum photon energy. *Left:* Results at  $\mu = 1$  GeV. The bands refer to different values of the intermediate matching scale:  $\mu_i^2 = \Lambda_h m_b$  (center),  $2\Lambda_h m_b$  (top),  $0.5\Lambda_h m_b$  (bottom). Their width reflects the sensitivity to the high-energy matching scale  $\mu_h^2$ , varied between  $2m_b^2$  and  $0.5m_b^2$ . The dashed line shows the result obtained at one-loop order. *Right:* Dependence of the kernel on the renormalization scale  $\mu$ , varied between 0.75 GeV and 2.0 GeV as indicated on the curves.

In order to study the importance of RG improvement and Sudakov resummation, we compare in the left-hand plot in Figure 4 the result for the resummed hard-scattering kernel in (33) with the one-loop approximation in (19). We choose the highest possible value of the photon energy ( $E_\gamma = m_b/2$  with  $m_b = 4.8$  GeV) so as to maximize the values of the large logarithms, and plot the function  $T(l_+, \mu)$  for  $\mu = 1$  GeV and different choices of the matching scales  $\mu_h$  and  $\mu_i$ . Here and below, the “natural” choices  $\mu_h = 2E_\gamma$  and  $\mu_i = \sqrt{2E_\gamma\Lambda_h}$ , where  $\Lambda_h = 0.5$  GeV serves as a typical hadronic scale, are taken as default values. We use the two-loop running coupling normalized at  $\alpha_s(m_b) = 0.22$  and set  $n_f = 4$  for the number of light quark flavors. (For simplicity, we do not match onto a three-flavor theory even for low renormalization scales.) We find that resummation effects decrease the magnitude of the radiative corrections, i.e., the resummed kernel is closer to the tree-level value ( $T = 1$ ) than the one-loop result. The fact that after Sudakov resummation the radiative corrections are moderate in magnitude persists even for asymptotically large  $b$ -quark masses. For instance, setting  $m_b = 50$  GeV we find after resummation  $T(l_+, \mu) = 0.74$  at  $l_+ = \mu = 1$  GeV. (Fixed-order perturbation theory breaks down for such large values of the quark mass. From (19) we would obtain  $T(l_+, \mu) = 0.08$  with these parameter values.) The figure also exhibits that our results are stable under variation of the two matching scales. Varying  $\mu_i^2$  and  $\mu_h^2$  by factors of 2 changes the result for the kernel by less than 10%. This suggests that the unknown NNLO corrections to the function  $U_0$  in (34) are perhaps not very important.

The scale dependence of the resummed expression for the kernel is illustrated in the right-hand plot in Figure 4, which shows the functional dependence of  $T(l_+, \mu)$  for maximal photon energy and several values of  $\mu$ . The matching scales are set to their

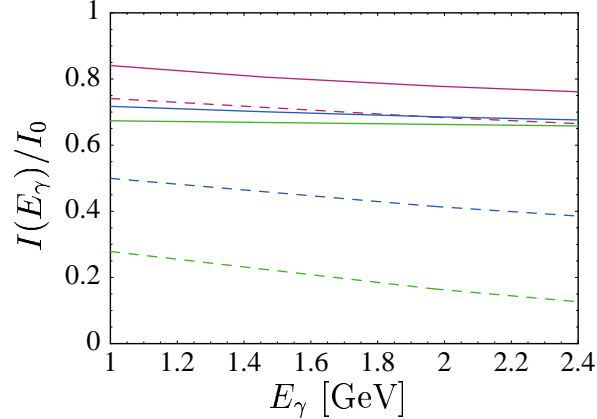


Figure 5: Energy dependence of the convolution integral  $I(E_\gamma)$  normalized to its tree-level value, assuming the model (36) for the LCDA at a low scale  $\mu_0$  such that  $\alpha_s(\mu_0) = 0.5$  (top),  $0.75$  (center), and  $1.0$  (bottom). The matching scales are set to their default values. The solid curves correspond to the resummed kernel, while the dashed ones are obtained at one-loop order.

default values  $\mu_h = m_b = 4.8 \text{ GeV}$  and  $\mu_i = \sqrt{\Lambda_h m_b} \simeq 1.55 \text{ GeV}$ . We observe a significant scale dependence of the kernel, especially as one lowers  $\mu$  below the intermediate scale  $\mu_i$ . In other words, the second stage of running (for  $\mu < \mu_i$ ), which we have computed for the first time in the present paper, is numerically significant.

Finally, it is interesting to study resummation effects for the convolution integral

$$I(E_\gamma) = \int_0^\infty dl_+ \frac{\phi_+^B(l_+, \mu)}{l_+} T(l_+, E_\gamma, m_b, \mu). \quad (35)$$

At tree-level, this integral has been denoted by  $I_0 = 1/\lambda_B$ , where  $\lambda_B$  is a low-energy hadronic parameter [2]. To evaluate the integral beyond tree-level one needs to assume a particular form of the LCDA. In [16], a model function was derived from a QCD sum-rule analysis of the matrix element of the bilocal HQET operator in (6). This study motivated the ansatz

$$\phi_+^B(l_+, \mu_0) = \frac{l_+}{\lambda_B^2} e^{-l_+/\lambda_B}, \quad \text{with} \quad \lambda_B = \frac{2}{3} (m_B - m_b) \approx 0.32 \text{ GeV}. \quad (36)$$

Such a nonperturbative calculation, which does not control the scale dependence of the LCDA, is reasonable only at a low hadronic scale  $\mu_0$ . For larger values of  $\mu$  evolution effects introduce a radiative tail in the LCDA. We thus expect significant modifications of the convolution integral due to radiative corrections. While the function  $I(E_\gamma)$  is formally  $\mu$ -independent, our model estimate will depend on the value of the scale  $\mu_0$  at which the functional form given in (36) is assumed to be correct. In Figure 5, we show results for the function  $I(E_\gamma)$  in units of  $I_0$  for three different values of  $\mu_0$ . After RG resummation we observe a modest reduction of  $I(E_\gamma)$  with respect to its tree-level value, which is fairly insensitive to the precise value of  $\mu_0$  and only shows a mild energy

dependence. In a rough approximation we have  $I(E_\gamma) \approx 0.75I_0$ . In contrast, the results obtained at one-loop order are strongly sensitive to the choice of  $\mu_0$  and exhibit a more pronounced dependence on the photon energy.

## 5 Conclusions

We have applied soft-collinear effective theory to prove a QCD factorization formula for the radiative semileptonic decay  $B \rightarrow \gamma l \nu$ , stating that at leading power in  $\Lambda/m_b$  the decay amplitude can be written as a convolution of a perturbative hard-scattering kernel with the leading-order  $B$ -meson light-cone distribution amplitude. Besides deriving the precise form of this convolution and establishing that the kernel is infrared finite to all orders in perturbation theory, we have shown that the convolution integral is free of endpoint singularities, and that non-valence Fock states of the  $B$  meson do not contribute at leading power.

We believe that the analysis of the factorization properties of the decay amplitude is most transparent in the framework of the formulation of soft-collinear effective theory developed in [5], where full QCD is matched onto a low-energy effective theory in which only long-distance modes are kept as dynamical degrees of freedom. In the present case these are the soft constituents of the  $B$  meson and the collinear photon field. For more complicated processes with energetic, light final-state mesons one would also have to introduce collinear quark and gluon fields. Many of the techniques used in our analysis (such as soft-collinear gauge invariance, reparameterization invariance, and renormalization-group improvement) are equally relevant in such a more general context.

The second part of our analysis was devoted to the calculation of the hard-scattering kernel in the factorization formula using renormalization-group improved perturbation theory. We have established a second, perturbative factorization formula, according to which the different short-distance scales entering in the calculation of the kernel (hard scales of order  $m_b$  and hard-collinear scales of order  $\sqrt{m_b \Lambda}$ ) can be separated into a hard function and a jet function. The corresponding two classes of large logarithms can be systematically resummed by solving evolution equations derived from the renormalization properties of the leading-order  $B$ -meson light-cone distribution amplitude. As a byproduct, we have elucidated the relation between the anomalous dimension of heavy-collinear currents in the effective theory and the universal cusp anomalous dimension encountered in the study of Wilson loops with light-like segments. In contrast to previous analyses, we have performed a complete resummation of Sudakov logarithms down to a low-energy scale  $\mu \sim \text{few} \times \Lambda_{\text{QCD}}$  independent of the heavy-quark mass. Only in that way all dependence on  $m_b$  and the photon energy  $E_\gamma$  is explicitly contained in the hard-scattering kernel. Our results (30) and (31) give the exact analytic solution for the kernel, valid to all orders in perturbation theory.

The discussion of the decay  $B \rightarrow \gamma l \nu$  presented here can be taken over almost verbatim to analyze related processes such as  $B \rightarrow \gamma \gamma$  and  $B \rightarrow \gamma l^+ l^-$ . The interesting observation that at leading power the long-distance effects in these processes are univer-

sal (for fixed photon energy) [9] can be understood in our formalism as follows: Once short-distance modes related to hard and collinear interactions with the heavy quark are integrated out at a scale  $\mu_h \sim m_b$ , the evolution of the resulting operators in the effective theory is independent of the Dirac structure of the relevant current operators. This is true for the evolution of heavy-collinear currents (running from  $\mu_h$  down to an intermediate scale  $\mu_i \sim \sqrt{m_b \Lambda}$ ), and for the bilocal heavy-light currents whose matrix elements are expressed in terms of the leading-order  $B$ -meson light-cone distribution amplitude (evolution from  $\mu_i$  down to a low-energy hadronic scale). Process-dependent corrections arise only in the initial matching at the high-energy scale and thus are calculable in an expansion in powers of  $\alpha_s(m_b)$ .

From a phenomenological point of view, an important finding of our analysis is that Sudakov resummation does not lead to a strong suppression of the decay amplitudes. After renormalization-group improvement we find moderate corrections to the kernel which are smaller than at one-loop order, and which stay at the level of 20–30% even in the hypothetical case of asymptotically large heavy-quark mass.

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