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Integral Equation for the Equilibrium State of Colliding Electron Beams *

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Abstract

We study a nonlinear integral equation for the equilbrium phase distribution of stored colliding electron beams. It is analogous to the Haïssinski equation, being derived from Vlasov-Fokker-Planck theory, but is quite different in form. We prove existence of a unique solution, thus the existence of a unique equilibrium state, for sufficiently small current. This is done for the Chao-Ruth model of the beam-beam interaction in one degree of freedom. We expect no difficulty in generalizing the argument to more realistic models.

1 INTRODUCTION

In the theory of stability of stored beams a primary step should be the study of equilibrium states, expected at low current. An equilibrium state should become unstable at some threshold in current, but in order to compute the threshold we must linearize the kinetic equation (Vlasov or Vlasov-Fokker-Planck) about the equilibrium phase space distribution. Historically, investigators have often been lazy about this point, linearizing the Vlasov equation about some state that might be at best a rough approximation to an equilibrium. This may be excused by the fact that determination of the equilibrium is a nonlinear problem, in general rather difficult.

There is one case in which there is a widely known theory of equilibrium that makes some contact with experiment; namely, the case of longitudinal motion of a single stored electron beam subject to a wake field [1, 2]. The theory is based on a model in which the exact longitudinal wake field is replaced by its average over one turn. The averaged wake of course depends only on the distance between source and test particles, not on the position in the ring. With such a wake one may seek a time-independent, factorized solution of the Vlasov-Fokker-Planck (VFP) equation; namely, a product of a Gaussian in the canonical momentum p (proportional to the energy deviation) and the charge density $\rho(q)$, where q is the canonical coordinate (proportional to the distance from the synchronous particle). The equation is satisfied by such a factorized form, provided that the charge density satisfies the Haïssinski equation [1], a nonlinear integral equation. If the wake field satisfies a mild restriction, it is not difficult to prove that the equation has a unique solution in a large function space $\mathcal{S}$, at sufficiently small current. The corresponding solution of the VFP equation is the unique, small-current solution satisfying the principle of detailed balance (with $\rho \in \mathcal{S}$).

There are many ways in which this prototype theory of equilibrium might be extended. For instance, one might include multi-bunch beams, long-range wakes from cavity resonators or resistive walls, nonlinear r.f., proton beams with non-Gaussian distribution in p, localized wakes not averaged over azimuth. Here we are interested in two counter-rotating beams in collision. In mathematical aspects the problem has similarities to the case of a single beam with localized wake contributions.

The beam-beam collision gives a large transverse force that substantially modifies the beams at every collision. Consequently, the equilibrium state, if any, cannot be time-independent. Rather, it must be defined as a phase space distribution that is periodic in azimuthal position s. As a zeroth approximation, one could smear out the localized beam-beam kick, distributing it over a full turn. This has been done in linear stability studies [3]. Here we wish to avoid such a step, accounting fully for the localization. It then follows that we cannot deal with a factorized distribution. We must expect the equilibrium equation to be an integral equation for functions on phase space, not just on coordinate space as in the Haïssinski case. We derive and analyze the simplest instance of such an equation, retaining the full nonlinearity of the beam-beam force.

Some background to the present study is found in a recent paper [4]. There we made an analytic study of equilibria by linearizing the beam-beam force, but retaining the quadratic nonlinearity of the Vlasov equation. We also carried out a numerical integration of the nonlinear VFP system. Here we adopt the notation and equations of motion as given in Ref. [4].

2 FORMULATION OF THE PROBLEM

We treat vertical transverse motion with normalized phase-space variables (q, p) defined in terms of the lattice function $\beta(s)$ and emittance ϵ as

$$q = y(\beta\epsilon)^{-1/2} , \quad p = (\beta y' - \beta' y/2)(\beta\epsilon)^{-1/2} , \qquad (1)$$

where y is the vertical displacement and the prime denotes d/ds. The Hamiltonian of motion unperturbed by the beam-beam interaction is $H = (p^2+q^2)/2$ and the independent "time" variable of Hamilton's equations is the phase advance $\theta = \int_0^s du/\beta(u)$. We distinguish the two beams by superscripts $(1), (2)$.

The Chao-Ruth model [5] is intended to represent flat beams, with large $x : y$ aspect ratio. The force on a particle

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in beam (1) is approximated as though it came from infinite uniform planes of charge perpendicular to the y-axis, distributed with a density $\rho^{(2)}(y)$. This force is concentrated in time, however, at the instant of collision. The resulting kernel function for the beam-beam force is proportional to $\mathrm{sgn}(q-q')$, where $\mathrm{sgn}(x)$ is the signum function, equal to 1 for $x > 0$ and -1 for $x < 0$. For simplicity in notation we take the two beams to have equal properties (tune, energy, bunch height and width, β^*, damping time). The mathematical argument would be the essentially the same with unequal beam properties. The formal Vlasov-Fokker-Planck system is

$$\frac{\partial f^{(1)}}{\partial\theta}+p\frac{\partial f^{(1)}}{\partial q}-\Bigg[q+(2\pi)^{3/2}\xi\sum_n\delta(\theta-2\pi\nu n)\cdot\int_{-\infty}^{\infty}\mathrm{sgn}(q-q')\int_{-\infty}^{\infty}f^{(2)}(q',p',\theta)dq'dp'\Bigg]\frac{\partial f^{(1)}}{\partial p}=2\alpha\frac{\partial}{\partial p}\Bigg[pf^{(1)}+\frac{\partial f^{(1)}}{\partial p}\Bigg]\ ,\quad(\text{and }1\leftrightarrow 2)\ ,\qquad(2)$$

where the distribution function for beam (i) is $f^{(i)}(q,p,\theta)$, the vertical betatron tune is ν, and the beam-beam parameter is $\xi = N\beta^* r_e/((2\pi)^{1/2}\gamma\sigma_y L_x)$. Here β^* is the beta function at the IP, $r_e = e^2/(4\pi\epsilon_0 mc^2)$ is the classical electron radius, γ is the Lorentz factor, L_x is the bunch width, and $\sigma_y = (\beta^*\epsilon)^{1/2}$ is the bunch height. The right hand side of (2) is the Fokker-Planck contribution, with damping constant $\alpha = 1/(2\pi\nu n_d)$, where n_d is the number of turns in a damping time. Our phase space coordinates have been defined so that the damping and diffusion constants are equal.

Equation (2) has only a formal significance, since the θ- dependent factors multiplying the delta function actually change discontinuously at the IP where the delta function acts. Consequently, we cannot say how to evaluate those factors without further analysis. Actually, the correct change of the distribution function at the IP is easy to see. Let $f^{(1)}(q,p,0-)$ and $f^{(1)}(q,p,0+)$ represent the distributions just before and just after $\theta = 0 \pmod{2\pi\nu}$. Then by the usual argument from probability conservation [2] the distribution is changed by the inverse of the kick map; i.e., by the Perron-Frobenius operator for that map:

$$f^{(1)}(q,p,0+) = f^{(1)}(q,p-F(q,0-),0-)\ ,\qquad(3)$$

where

$$F(q,0-) = -(2\pi)^{3/2}\xi\int\mathrm{sgn}(q-q')f^{(2)}(q',p',0-)dq'dp'\ .\qquad(4)$$

For propagation of the distribution function between IP kicks, we have in (2) a linear Fokker-Planck equation with harmonic force. The propagator or fundamental solution of that equation is known [6], namely a function $\Phi(z,z',\theta)$, $z = (z_1,z_2) = (q,p)$ such that for any initial distribution $f(z,0)$ the solution at time θ is

$$f(z,\theta) = \int\Phi(z,z',\theta)f(z',0)dz'\ .\qquad(5)$$

There are several equivalent representations of Φ. The following form, which was derived from a probabilistic argument, is especially convenient for our work:

$$\Phi(z,z',\theta) = \frac{1}{2\pi(\det\Sigma)^{1/2}}\exp\big[-\big(z-e^{A\theta}z'\big)^T\Sigma^{-1}\big(z-e^{A\theta}z'\big)/2\big]\ ,\quad \Sigma = I - e^{A\theta}e^{A^T\theta}\ .\qquad(6)$$

Here T denotes transposition and $e^{A\theta}$ is the transfer matrix for the single-particle harmonic motion with damping. With damping constant α we have

$$e^{A\theta} = e^{-\alpha\theta}\begin{pmatrix} a_+ & b \\ -b & a_- \end{pmatrix}\ ,\quad a_\pm = \cos\Omega\theta \pm (\alpha/\Omega)\sin\Omega\theta\ ,\quad b = (1/\Omega)\sin\Omega\theta\ ,\quad \Omega = (1-\alpha^2)^{1/2}\ ,\quad \det(e^{A\theta}) = e^{-2\alpha\theta}\ .\qquad(7)$$

Let $\mathbf{\Phi}$ denote the operator corresponding to the kernel $\Phi(z,z',\theta)$ in (5). The action of $\mathbf{\Phi}$ has a simple expression in Fourier space. Writing $\hat{h}$ for the Fourier transform of h, we have

$$\widehat{\mathbf{\Phi}h}(v) = \exp\big[-v^Te^{A\theta}\Sigma e^{A^T\theta}v/2\big]\hat{h}(e^{A^T\theta}v)\ .\qquad(8)$$

We can now set down a system of integral equations for the equilibrium distribution. The equations are for the distributions evaluated just *after* the IP, $f^{(i)}(z,0+)$. Henceforth we suppress the time specification $0+$. Starting with $f = (f^{(1)},f^{(2)})$, we propagate one turn by (5) with $\theta = 2\pi\nu$, and then apply the beam-beam kicks according to (3). For equilibrium (periodicity), the result must be the starting f. To state this in equations we first define the linear operator $\mathbf{L}$ by

$$\mathbf{L}f(q) = (2\pi)^{3/2}\int\int\mathrm{sgn}(q-q')K(z'|z'')f(z'')dz'dz''\ ,\qquad(9)$$

where K is the Fokker-Planck propagator for one turn,

$$K(z|z') = \Phi(z,z',2\pi\nu)\ .\qquad(10)$$

The integral equations take the form

$$f^{(i)}(z) = \int K(\ q,\ p+\xi\mathbf{L}f^{(j)}(q)\ |\ z'\ )f^{(i)}(z')dz'\ ,\quad i\neq j\ ,\quad i,j=1,2\ ,\qquad(11)$$

with

$$\int f^{(i)}(z)dz = 1\ .\qquad(12)$$

It is essential that the normalization constraint (12) be regarded as part of the definition of the mathematical system; otherwise in Eqs.(11) there is nothing to set the scale of the beam-beam force. We choose to build in the constraint by redefining the integral equations, dividing the right hand side of (11) by $\int f^{(i)}(z)dz$. Then, since $\int K(z|z')dz = 1$, any solution of the modified equation will automatically satisfy (12). Finally we multiply by $\int f^{(i)}(z)dz$ and rearrange to state the pair of equations as

$$G(f,\xi)=0\;, \tag{13}$$

where $G=(G^{(1)},G^{(2)})$ with

$$G^{(i)}(f,\xi)(z)=f^{(i)}(z)\int f^{(i)}(z')dz' \\ -\int K(\,q,p+\xi\mathbf{L}f^{(j)}(q)\mid z'\,)f^{(i)}(z')dz'\;,\quad i\neq j\;. \tag{14}$$

We know the solution of (13) at $\xi=0$; it is the Gaussian equilibrium in the absence of beam-beam force,

$$G(f_0,0)=0\;,\quad f_0=(f_0^{(1)},f_0^{(2)})\;, \\ f_0^{(i)}=\frac{1}{2\pi}\exp\big(-\frac{1}{2}(q^2+p^2)\big)\;. \tag{15}$$

We apply the implicit function theorem to prove that this solution can be continued in a unique way to a solution $f(\xi)$ of (13) for small $\xi\neq 0$. This requires an implicit function theorem in an infinite-dimensional function space. Let us first recall the intuitive basis of the theorem in finitely many dimensions, so that (13) represents n real (generally nonlinear) equations in n unknowns $f_j\;,\;j=1,\cdots,n$. We wish to solve for the f_j as a function of the parameter ξ, supposing that a solution f_{0j} for $\xi=0$ is known. Supposing that G is smooth, we can expand it by Taylor's formula with remainder R about the point $(f_0,0)$:

$$G(f,\xi)=G_f(f_0,0)(f-f_0)+G_\xi(f_0,0)\xi+R(f,\xi)=0\;. \tag{16}$$

If the Jacobian matrix $G_f=\{\partial G_i/\partial f_j\}$ is non-singular at the expansion point, and the nonlinear remainder R is small, an approximate solution of our problem is

$$f(\xi)\approx f_0-G_f(f_0,0)^{-1}G_\xi(f_0,0)\xi\;. \tag{17}$$

The implicit function theorem takes into account the nonlinear term, and assures us that for sufficiently small ξ there will be a unique exact solution of (13) close to the approximation (17). The Jacobian is required to be nonsingular *only* at the single point $(f_0,0)$.

In the infinite-dimensional case we must first decide on the arena of the discussion: in what space of functions do we seek a solution of (13)? Physicists are usually familiar with Hilbert space, but here we can get by with a simpler notion, a Banach space. Like the Hilbert space, it is a complete linear space with a norm, but is not required to have a scalar product. For instance, the set of all continuous functions $f(x)$ on the unit interval $[0,1]$ is a Banach space if the norm is defined as $\|f\|=\max_{[0,1]}|f(x)|$. Secondly, we must give a meaning to the Jacobian G_f when f is a function rather than a finite-dimensional vector. A simple possibility is the Fréchet derivative, which for a function $G(f)$ on a Banach space is defined at f_0 as a linear operator $G_f(f_0)$ such that

$$\lim_{\|h\|\to 0}\frac{1}{\|h\|}\|G(f_0+h)-G(f_0)-G_f(f_0)h\|=0\;. \tag{18}$$

We are now ready to state the implicit function theorem in Banach space, in a form sufficiently general for our purposes (but hardly the most general).

> **Theorem**: Let B be a Banach space, and suppose that G is a continuously differentiable mapping (operator) of $B\times I$ into B, where $I=(-\Delta\xi,\Delta\xi)$ is an open interval, the domain of ξ. The continuous differentiability implies that the partial (Fréchet) derivatives $G_f(f,\xi)\;,\;G_\xi(f,\xi)$ exist and are continuous in $B\times I$. Let $f_0\in B$ be a solution of $G(f_0,0)=0$, and suppose that $G_f(f_0,0)$ is a continuous linear map of B onto B with a continuous inverse. Then there exists a unique solution $f(\xi)$ of $G(f,\xi)=0$ such that $f(0)=f_0$, for ξ in some interval $I_0=(-\delta\xi\;,\;\delta\xi)\subset I\;,\;\delta\xi\neq 0$. Moreover, for $\xi\in I_0$ this solution has a continuous derivative with respect to ξ and $(G_f(f(\xi),\xi))^{-1}$ exists. The derivative is given by $f'(\xi)=-(G_f(f(\xi),\xi))^{-1}G_\xi(f(\xi),\xi)$.

The theorem alone does not give us an estimate of the size of the interval I_0 in which the solution exists. In specific cases analytic estimates can be made, but they may be pessimistic. In our problem, we mainly seek assurance that an equilibrium exists for sufficiently small current. We shall have to rely on numerical calculations to determine a maximum interval of existence. Calling on experience with the Haïssinski equilibrium, we expect that as the current is increased the equilibrium will become unstable long before it ceases to exist.

To apply the implicit function theorem to (13), a crucial matter is to find a suitable space B. As is usual in applications of functional analysis, this requires some experimentation. The space has to be tailored to fit the properties of the operator. A primary requirement is that $B\times I$ be mapped into B, and that is relatively easy to check for some candidates for B. Further requirements such as invertibility of $G_f(f_0,0)$ may be harder to verify, and lead us to refine the choice, perhaps taking a subspace of an initial candidate for B.

After various estimates of integrals we find that a suitable B consists of all pairs $f=(f^{(1)},f^{(2)})$ of continuous functions on the phase space $\mathbb{R}^2$ such that the following expression, identified as the norm, is finite:

$$\|f\|=\max_i\;\sup_{z\in\mathbb{R}^2}|\,(1+\|z\|^{2a})f^{(i)}(z)\,|\;,\quad a>2\;, \tag{19}$$

where $\|z\|=(z_1^2+z_2^2)^{1/2}$ and sup (supremum) denotes the least upper bound. This is a "big" space, in the sense that it contains functions with slow, polynomial decrease at infinity, whereas intuition and the results of Ref.[4] indicate that the actual decrease of the solution is close to Gaussian. The advantage of a big space is that our assertion of uniqueness of the solution means uniqueness in a bigger universe. The disadvantage is that our resulting theorem will give no close information on the actual fall-off of the solution, since it merely asserts that the solution is in B. We did not succeed in finding a space with Gaussian fall-off, mapped into itself by G.

3 SOME HIGHLIGHTS OF THE PROOF

Here we give a few main points of the proof, deferring full details to a future report. We have to verify the three main hypotheses of the implicit function theorem, namely

1. $G : B \times I \to B$
2. G_f, G_ξ exist and are continuous in $B \times I$
3. $G_f(f_0, 0)^{-1}$ exists and is continuous

Suppose that $f \in B$. Then its Fourier transform $\hat{f}$ exists and is bounded. By (8) one then sees that every derivative of $\mathbf{K}f(z) = \int K(z|y)f(y)dy$ exists and is bounded, being the Fourier transform of an absolutely integrable function. For estimates of the action of $\mathbf{K}$ on f we can prove the lemma

$$\left| \frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n} \int \frac{K(z|y)dy}{1 + \|y\|^{2a}} \right| \le \frac{M_{mn}}{1 + \|z\|^{2a}} , \qquad (20)$$

for any $m \ge 0, n \ge 0$, where the constant M_{mn} depends on a and the parameters defining K. Using these results one verifies that hypothesis (1) holds.

To check (2) for G_f, we compute the formal variational derivative of G, applied to a variation $h \in B$. That is a linear integral operator $\mathcal{L}$ applied to h. Then some work with the lemma and the mean-value theorem shows that $\mathcal{L}$ is indeed the Fréchet derivative. In fact, the numerator under the limit in (18) is $\mathcal{O}(\|h\|^2)$ if $G_f = \mathcal{L}$.

The hardest part of the proof is verifying item (3). In textbook examples it is usual to suppose that $G_f - 1$ is a compact operator, in which case one can apply Fredholm theory to discuss existence of G_f^{-1}. In the present case this operator appears to be non-compact, and we have to resort to a more subtle method. We get the inverse by proving uniform convergence of an operator power series development, and the convergence is at a slow rate determined by the damping constant. Thus, the proof fails for a proton system with no damping, and it does not seem at all likely that one could get a proof for zero damping by somehow taking a limit.

Since the power series method is interesting and novel, we give a few details. We have to show that the equation

$$G_f(f_0, 0)x = y \qquad (21)$$

has a unique solution $x \in B$ for any $y \in B$. At zero current G_f breaks into two identical and independent blocks for the two beams. Then (21) for one block takes the form

$$x(z) + f_0(z) \int x(z')dz' - \mathbf{K}x(z) = y(z) , \qquad (22)$$

where now x and y are single functions, not pairs. We discuss (22) in the space B_1, which is defined in the same way as B, except that it consists of single functions; i.e., $B = B_1 \times B_1$. For any $x \in B_1$,

$$\int \mathbf{K}x(z)dz = \int x(z)dz , \qquad (23)$$

from which it follows that any solution of (22) must satisfy

$$\int x(z)dz = \int y(z)dz . \qquad (24)$$

Consequently, any solution of (22) must also be a solution of

$$x(z) - \mathbf{K}x(z) = p(z) , \qquad (25)$$
$$p(z) = y(z) - f_0(z) \int y(z')dz' , \quad \int p(z)dz = 0 .$$

We look for solutions of (22) among the solutions of (25). Iterating (25) $n-1$ times we find

$$x = \mathbf{K}^n x + \sum_{m=1}^{n-1} \mathbf{K}^m p + p . \qquad (26)$$

Here the story is different from the familiar case of the Neumann series, since the term $\mathbf{K}^n x$ does not vanish in the limit of large n. By the semigroup property of the linear Fokker-Planck evolution, the kernel of $\mathbf{K}^n$ is given by (6) with $\theta = 2\pi n\nu$. If $x \in B_1$ the integral defining $\mathbf{K}^n x$ converges uniformly in n, since the integrand is majorized by $|x(z')|$ and $\int |x(z)|dz < \infty$. We may then take the limit under the integral to obtain

$$\lim_{n\to\infty} \mathbf{K}^n x(z) = f_0(z) \int x(z)dz . \qquad (27)$$

Thus, from (26) a solution of (25) in B_1 is expected to have the form

$$x = f_0 \int x(z')dz' + \sum_{m=1}^{\infty} \mathbf{K}^m p + y - f_0 \int y(z')dz' . \qquad (28)$$

A candidate for a solution of (22) must satisfy (24), so that from (28) the unique candidate is

$$x(z) = \sum_{m=1}^{\infty} \mathbf{K}^m p(z) + y(z) . \qquad (29)$$

We are now faced with a delicate step of the proof, to show that for any $y \in B_1$ the series in (29) converges and represents an element of B_1. Once that it is done, it is easy to check that (29) represents a solution of (22), unique in B_1.

To estimate the $\mathbf{K}^n p$ we formally subtract $\exp(-z^T z/2) \int p(z')dz'$, which is zero, and then do some analysis with the mean value theorem to get the following bound:

$$
\begin{aligned}
I_n &= \left| 2\pi(\det \Sigma_n)^{1/2} \mathbf{K}^n p(z) \right| \\
&= \left| \int \left[\exp[-(z - e^{n\theta A} z')^T (z - e^{n\theta A} z')/2] \right. \right. \\
&\qquad \left. \left. - \exp[-z^T z/2] \right] p(z') dz' \right| \\
&\leq \frac{M(1 - \exp(-z^T z/2))}{z^T z} (1 + \|z\|) e^{-n\theta\alpha} , \\
&\qquad \theta = 2\pi\nu , \qquad (30)
\end{aligned}
$$

where α is the damping constant. (We write M for a generic constant in majorizations. In any statement M may have a value larger than in any previous statement.) Now (30) is enough to show uniform convergence (in the maximum norm) of the series in (29) over any finite ball $\|z\| < r$, but not enough to show that the sum of the series belongs to B_1. To complete the job we get a bound by a different method which fails at small $\|z\|$ but works for $\|z\| > r$. For that we break the integral in (30) into two parts, one for $\|\exp(n\theta A)z'\| < b\|z\|$ and the other for $\|\exp(n\theta A)z'\| > b\|z\|$, with $0 < b < 1$. In the second region the coefficient of $p(z')$ is not small, and we have to rely wholly on the fall-off of $p(z')$. Using appropriate estimates for the two regions (and supposing $\alpha < 1/2$, which is more than safe for real machines) we find

$$
I_n \leq \frac{M}{1 + \|z\|^{2a}} e^{-n\theta\alpha} , \quad \|z\| > r . \qquad (31)
$$

Combining (30) and (31) we have for all z that

$$
|\mathbf{K}^n p(z)| \leq \frac{M}{1 + \|z\|^{2a}} e^{-n\theta\alpha} , \qquad (32)
$$

from which it follows that x as given in (29) exists and belongs to B_1. Furthermore, this function satisfies the original equation (22):

$$
\begin{aligned}
&\sum_{m=1}^{\infty} \mathbf{K}^m p + y + f_0 \int [\sum_{m=1}^{\infty} \mathbf{K}^m p(z') + y(z')] dz' \\
&- \sum_{m=2}^{\infty} \mathbf{K}^m p - \mathbf{K} y = y \qquad (33)
\end{aligned}
$$

since we know that $\mathbf{K} f_0 = f_0$ and $\int \sum_{m=1}^{\infty} \mathbf{K}^m p(z) dz = \sum_{m=1}^{\infty} \int \mathbf{K}^m p(z) dz = \sum_{m=1}^{\infty} \int p(z) dz = 0$, the reversal of sum and integral in the latter being justified by (32).

To prove that the solution is unique, suppose that there were two solutions x_1 , x_2 in B_1. Then $x = x_1 - x_2$ satisfies (22) with $y = 0$, from which it follows that $\int x(z) dz = 0$, hence $x - \mathbf{K} x = 0$. Iterating the latter equation, we have $x = \mathbf{K}^n x = \lim_{n \to \infty} \mathbf{K}^n x = f_0 \int x(z) dz = 0$. Finally, the continuity of $G_f(f_0, 0)^{-1}$ is clear, since a small change in y evidently produces a small change in x.

4 CONCLUSION

We have sketched the proof that there is a unique solution to the Vlasov-Fokker-Planck system for the Chao-Ruth model of colliding electron beams at sufficiently small current. The details of the various estimates involved will be given in a longer report. We are fairly confident that the proof will go through in almost the same way for other models in one degree of freedom [3] and for the model in two degrees of freedom in which the force is obtained from the two-dimensional Poisson equation. The case of protons, without radiation damping, is an entirely different story. One expects infinitely many approximate equilibria [4, 7], but the question of exact equilibria is open.

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