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Continuum methods in lattice perturbation theory

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We show how methods of continuum perturbation theory can be used to simplify perturbative lattice calculations. We use the technique of asymptotic expansions to expand lattice loop integrals around the continuum limit. After the expansion, all nontrivial dependence on momenta and masses is encoded in continuum loop integrals and the only genuine lattice integrals left are tadpole integrals. Using integration-by-parts relations all of these can be expressed in terms of a small number of master integrals. Four master integrals are needed for bosonic one loop integrals, sixteen in QCD with Wilson or staggered fermions.

1. Introduction

In many cases, the results of numerical simulations of QCD on the lattice need to be matched by equally precise *perturbative* calculations in lattice regularization to become phenomenologically relevant. In the past, the need (or the motivation) for precise matching calculations has been limited by the large systematic errors induced by the quenched approximation. By now, simulations with fairly light dynamical quarks are feasible and precise calculations in lattice perturbation theory are becoming increasingly important. The dominant uncertainty in the first determinations of the strong coupling constant from lattice simulations with three dynamical quarks, for example, are unknown higher order perturbative corrections to the relation between the coupling measured on the lattice and the $\overline{\text{MS}}$ coupling [1,2].

The introduction of a space-time lattice to regulate the theory considerably complicates perturbative calculations. The propagators and vertices become functions of sines and cosines of the momenta and not a single loop integral in four dimensions can be evaluated analytically. In the standard approach to lattice perturbation theory the relevant diagrams are therefore evaluated numerically, which has several drawbacks: (a) the amount of numerical computations necessary for realistic calculations is huge; (b) cancellations be-

tween individual diagrams can render numerical results unstable; (c) the continuum limit, i.e. the limit in which the inverse lattice spacing becomes much larger than external momenta and masses, has to be taken numerically as well. There are a number of techniques to both reduce the number and increase the precision of the numerical integrations involved [3–5], but we feel that the computational tools for perturbative calculations on the lattice are not as highly developed as the methods used in the continuum.

In a recent paper [6], we have proposed to use continuum methods to simplify lattice calculations. In particular, two tools originally developed for the evaluation of continuum loop integrals turn out to be useful: the technique of asymptotic expansions [7] and the use of integration-by-parts relations [8]. The method of asymptotic expansions can be used to expand lattice loop integrals around the continuum limit. This expansion splits the loop integrals into two parts: the “hard” part is a sum of lattice tadpole integrals, while the “soft” part consists of ordinary continuum integrals. In a second step, we use integration-by-parts relations to express all lattice tadpole integrals through a few master integrals. In a theory involving only bosonic fields, there are four such master integrals at one loop. In QCD with Wilson or staggered fermions, the number of master integrals increases to sixteen. The master integrals can be chosen to be

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convergent and are evaluated numerically.

We begin by quickly reviewing perturbation theory in lattice regularization and some of the complications that arise with this particular regulator. After illustrating the above method with a simple example, we discuss its application to the case of staggered fermions.

2. Lattice perturbation theory

When calculable at all, phenomenological results in QCD are often obtained in a factorized form: a physical quantity is given as a product of a perturbative short-distance part times a low-energy contribution. The splitting into the two parts is regularization dependent. If one manages to evaluate the low energy part with a lattice simulation, one also needs the high energy part in the same regularization. This requires a perturbative calculation in lattice regularization. Such calculations are, however, tedious. First of all, the propagators are much more complicated than in the continuum. The simplest discretization of the bosonic propagator in four dimensions is

$$G_B(k) = \frac{1}{(\widehat{k}^2 + m^2)}, \quad (1)$$

where

$$\widehat{k}^2 = \sum_{\mu=1}^4 \widehat{k}_\mu^2 \quad \text{and} \quad \widehat{k}_\mu = 2 \sin \frac{k_\mu}{2}. \quad (2)$$

Loop integrals involve products of these propagators, integrated over the Brillouin zone; a typical loop integral has the form

$$\int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{1}{(\widehat{k}^2 + m^2)} \frac{1}{((p + k)^2 + m^2)}, \quad (3)$$

where p is the external momentum. Note that we have expressed all quantities in units of the lattice spacing a : $k_\mu = k_\mu^{\text{phys}} a$, $m = m_{\text{phys}} a$, etc.

In lattice gauge theories not only the propagators, but also the vertices take a rather complicated form. The gluonic action is given in terms of Wilson loops, whose expansion in terms of the gauge field yields interactions among any number

of gluons. The four-gluon vertex arising from the simplest Wilson loop, for example, involves one hundred terms with up to four powers of sines and cosines of the in- and outgoing momenta (see e.g. [9]).

3. Asymptotic expansion around the continuum limit

We now discuss the expansion of the massive tadpole integral

$$G(m) = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{1}{(\widehat{k}^2 + m^2)}, \quad (4)$$

around the continuum limit, corresponding to the expansion around $m = m_{\text{phys}} a = 0$. Although this is the simplest possible example, it illustrates all the important features of our approach.

We begin by mapping the integration region in Eq.(3) to an infinite volume and defining new integration variables η_μ ,

$$\eta_\mu = \tan(k_\mu/2). \quad (5)$$

In terms of the new variables, the loop integrations in Eq.(4) range from $-\infty$ to $+\infty$:

$$G(m) = \frac{1}{4\pi^4} \int_{-\infty}^{\infty} \prod_{i=1}^4 \frac{d\eta_i}{(1 + \eta_i^2)} \left[\frac{m^2}{4} + D_B(\eta) \right]^{-1-\delta}, \quad (6)$$

where

$$D_B(\eta) = \sum_{i=1}^4 \frac{\eta_i^2}{(1 + \eta_i^2)}. \quad (7)$$

After the variable transformation, the form of the integrand becomes more similar to a continuum integral. In addition to the change of variables, we have also introduced an *intermediate regulator* by raising the propagator to the power $1 + \delta$ in Eq. (6).

The expansion of $G(m)$ is obtained as a sum of two parts:

$$G(m) = G_{\text{soft}}(m) + G_{\text{hard}}(m). \quad (8)$$

The soft and hard contributions are calculated by applying the following procedure to the integrand in Eq.(6):

- Soft: assume that all the components of the loop momentum η are small, $\eta_i \sim m \ll 1$. Perform the Taylor expansion of the integrand in Eq.(6) in the small quantities η_i and m . The expansion coefficients in this region are standard continuum one loop integrals, regularized analytically. Note that *no* restriction on the integration region is introduced.
- Hard: assume that all the components of the loop momentum are large, $\eta_i \sim 1 \gg m$ and Taylor expand the integrand in m . The expansion coefficients are massless lattice tadpole integrals.

As in dimensional regularization, scaleless integrals are set to zero. Higher orders of the soft part contain more and more ultraviolet divergent integrals. The ultraviolet divergences manifest themselves as poles in δ :

$$G_{\text{soft}}(m) = \frac{m^2}{16\pi^2} \left(-1 - \frac{1}{\delta} + 2 \ln \frac{m}{2} \right) + \frac{m^4}{256\pi^2} \left(3 + \frac{2}{\delta} - 4 \ln \frac{m}{2} \right) + O(m^6). \quad (9)$$

In the final result, these poles will cancel against infrared divergences appearing in the expansion of the hard part, whose evaluation we discuss in the next section.

4. Integration-by-parts relations between lattice integrals

While the soft part is given in terms of continuum integrals, the hard part involves the functions

$$H(\{a_i\}; n) = \int_{-\infty}^{\infty} \prod_{i=1}^4 \frac{d\eta_i}{(1 + \eta_i^2)^{a_i}} [D_B(\eta)]^{-n-\delta}. \quad (10)$$

The functions $H(\{a_i\}; n)$, with a_i and n integers, provide the full set of genuine lattice integrals needed for any one-loop calculation. The integrals occurring in the hard part are process independent and can therefore be calculated once

and for all for a given lattice action (the hard integrals occurring in the case of staggered fermions are given in section 5).

The hard part of Eq. (6) is obtained by expanding the integrand in m

$$4\pi^4 G(m)_{\text{hard}} = H(\vec{1}, 1) - \frac{m^2}{4} (1 + \delta) H(\vec{1}, 2) + \frac{m^4}{32} (1 + \delta) (2 + \delta) H(\vec{1}, 3) + \dots,$$

with $\vec{1} = \{1, 1, 1, 1\}$. The hard part of integrals with external momenta or tensor structure also involves functions H with index values $a_i \neq 1$.

We now use integration-by-parts identities to fully exploit the algebraic relations between the various integrals $H(\{a_i\}, n)$. These relations are derived using the fact that the integral of a total derivative vanishes in analytic regularization:

$$0 = \int_{-\infty}^{\infty} d^d \eta \frac{\partial}{\partial \eta_\mu} \left\{ \eta_\mu \prod_{i=1}^4 \frac{1}{(1 + \eta_i^2)^{a_i}} [D_B(\eta)]^{-n-\delta} \right\}, \quad (11)$$

for each value of $\mu = 1, 2, 3, 4$. Computing the derivative in Eq.(11) and rewriting the resulting expression in terms of the functions $H(\{a_i\}, n)$, we obtain an algebraic relation between integrals with different values of $\{a_i\}$ and n . Another equation can be obtained by partial fractioning, i.e. by using the linear dependence of the five ‘‘propagators’’ in the function $H(\{a_i\}, n)$. The complete set of algebraic relations is therefore

$$0 = \left\{ \mathbf{n}^- + \sum_{i=1}^d (\mathbf{a}_i^+ - 1) \right\} H, \quad (12)$$

$$0 = \left\{ 1 + 2 a_i (\mathbf{a}_i^+ - 1) + 2(n + \delta) \mathbf{n}^+ \mathbf{a}_i^+ (\mathbf{a}_i^+ - 1) \right\} H.$$

The conventions are such that the operator \mathbf{a}_i^\pm increases (decreases) the index a_i by one.

Similar integration-by-parts relations for lattice integrals were first studied in [3,4], where it was shown that the entire class of integrals $H(\{a_k\}; n)$ can be reduced to d master integrals

in d dimensions. Here, we neither attempt to solve these equations explicitly nor to rewrite them in such a form that the reduction of a given index is manifest. Instead, we adopt a brute force strategy and use computer algebra to explicitly solve the equations for a given range of indices. An efficient algorithm for solving such recurrence relations has been described in [10]. First, a criterion which selects a simpler integral out of any two integrals is chosen. Typically, integrals with lower values of the indices are considered to be simpler. The above equations are then solved for a very limited range of indices, using Gauss's elimination method. The calculation is repeated after supplementing the chosen set of equations with a few relations involving higher index values. By iterating this procedure, the equations (12) can be solved for the entire index range needed in a given calculation. The advantage of this brute force method is that it immediately generalizes to integrals involving more complicated propagators (e.g. those of Wilson fermions) or to higher loops.

It is possible and advantageous to choose very convergent master integrals, e.g. the integrals

$$H(\{\vec{1}\}, -n) = \int_{-\pi}^{\pi} d^4k \left(\frac{\widehat{k}^2}{4} \right)^{n-\delta} \quad (13)$$

with $n = 0, 1, 2, 3$. The expansion of these integrals in δ is then obtained by expanding the integrand and it is trivial to numerically evaluate them to arbitrary precision. The tadpole integral with two powers of the boson propagator, for example, is

$$\begin{aligned} H(\{\vec{1}\}, 2) &= \left(-\frac{3}{\delta^2} + \frac{703}{72\delta} - \frac{2795}{144} \right) H(\{\vec{1}\}, 0) \\ &+ \left(\frac{21}{2\delta^2} - \frac{3763}{144\delta} + \frac{13313}{288} \right) H(\{\vec{1}\}, -1) \\ &+ \left(-\frac{65}{8\delta^2} + \frac{4993}{288\delta} - \frac{16801}{576} \right) H(\{\vec{1}\}, -2) \\ &+ \left(\frac{27}{16\delta^2} - \frac{207}{64\delta} + \frac{681}{128} \right) H(\{\vec{1}\}, -3) + \mathcal{O}(\delta). \end{aligned}$$

Note that the $1/\delta^2$ -pole is spurious: its coefficient vanishes.

The result for the hard part of the massive tadpole in Eq. (6) has the form

$$\begin{aligned} G^{\text{hard}}(m) &= b_1 + m^2 \left(-b_2 + \frac{1}{16\pi^2} \left(\frac{1}{\delta} + \ln 4 \right) \right) \\ &+ m^4 \left(b_3 - \frac{1}{128\pi^2} \left(\frac{1}{\delta} - \ln 4 \right) \right) + \mathcal{O}(m^6). \end{aligned}$$

Note that the hard part is an analytic function of m . All nonanalytic dependence on masses and momenta arises in the soft part. As mentioned earlier, the $1/\delta$ -poles cancel between the hard and the soft part. The constants b_1, b_2, b_3 are obtained by numerically evaluating the master integrals. It turns out that these three constants are sufficient to express the hard part of any bosonic one-loop integral to any order in the expansion around the continuum limit. Their numerical values are given in [6].

5. Staggered fermions

The strategy outlined in the previous two sections did not rely on the specific form of the propagator. In [6], we have applied the technique to HQET and to QCD with Wilson fermions and calculated a number of QCD one-loop self-energies. The recursion relations between the hard integrals are more complicated than in the bosonic case and the number of master integrals is larger. There are seven master integrals for HQET and sixteen for QCD with Wilson fermions.

We now show how to use the method to calculate loop integrals for staggered fermions, which are a common choice to put fermions on the lattice. The staggered fermion action is obtained by reducing the number of components of the fermion field after spin diagonalizing the naive lattice fermion action. In perturbative calculations it is convenient to work with the naive fermion action and perform the staggering only at the end of the calculation. The fermion doubling inherent in the naive discretization of the fermion action manifests itself in the appearance of multiple soft regions. The sixteen zeros of the propagator denominator of naive fermions,

$$D_F = \sum_{\mu} \frac{1}{4} \sin^2 k_{\mu} = \sum_{\mu} \frac{\eta_i^2}{(1 + \eta_i^2)^2}, \quad (14)$$

in the Brillouin zone give rise to sixteen propagating fermions. Correspondingly loop integrals which involve naive fermion propagators have sixteen soft regions. One of those is the region where all components of the loop momentum η_i are small. The fifteen additional doubler contributions arise after transforming one or several components of the integration momentum as $\eta_i \rightarrow 1/\eta_i$ and then expanding the resulting integrand around small η_i . For purely fermionic integrals, the sixteen soft contributions are all equal. In integrals with both boson and fermion propagators, the boson propagator is far off-shell for the doubler contributions and shrinks to a point upon expanding.

The contribution of the hard part is obtained as usual, by expanding the loop integral in external momenta and particle masses. The lattice tadpole integrals that occur in the case at hand are

$$H(\{a_i\}, n, m) = \prod_{i=1}^4 \int \frac{d\eta_i}{(1 + \eta_i^2)^{a_i}} \frac{1}{D_B^n} \frac{1}{D_F^{m+\delta}}.$$

The regulator has to be on the fermion propagator in order to regulate both the singularities at $\eta_\mu = 0$ and $\eta_\mu = \infty$. The integration-by-parts and partial fractioning relations for this class of integrals are

$$\begin{aligned} 0 &= \left\{ 1 + 2 a_i (\mathbf{a}_i - 1) + 2 \mathbf{a}_i (\mathbf{a}_i - 1) n \mathbf{n} \right. \\ &\quad \left. + 2 \mathbf{a}_i (\mathbf{a}_i - 1) (2 \mathbf{a}_i - 1) (m + \delta) \mathbf{m} \right\} H, \\ 0 &= \left\{ 1 - \sum_{i=1}^d \mathbf{a}_i (1 - \mathbf{a}_i) \mathbf{m} \right\} H, \\ 0 &= \left\{ 1 - \sum_{i=1}^d (1 - \mathbf{a}_i) \mathbf{n} \right\} H. \end{aligned} \quad (15)$$

These relations are sufficient to reduce all integrals H to sixteen convergent master integrals.

6. Summary and conclusion

We have applied methods developed for the evaluation of continuum loop integrals to calculations in lattice perturbation theory. The technique of asymptotic expansions is used to expand

lattice loop integrals around the continuum limit. After performing the expansion, all nontrivial dependence on momenta and masses is encoded in continuum loop integrals. The only genuine lattice integrals left are massless tadpole integrals. With the help of integration-by-part relations we then reduce all tadpole integrals to a small number of master integrals. Except for the numerical evaluation of the master integrals, the entire calculation is performed analytically.

The technique we have presented does not depend on a specific form of the lattice action and after illustrating it with a simple bosonic integral, we have discussed how to apply it to the case of staggered fermions. Since the techniques we have been using were developed for multi-loop integrals, we hope that this method will also be useful beyond one loop.

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