hep-th/0206196
SLAC-PUB-9256
SU-ITP-02-24

On Smooth Time-Dependent Orbifolds and Null Singularities

Michal Fabinger and John McGreevy

Department of Physics and SLAC
Stanford University
Stanford, CA 94305/94309

We study string theory on a non-singular time-dependent orbifold of flat space. The orbifold group, which involves only space-like identifications, is obtained by a combined action of a null Lorentz transformation and a constant shift in an extra direction. In the limit where the shift goes to zero, the geometry of this orbifold reproduces an orbifold with a light-like singularity, which was recently studied by Liu, Moore and Seiberg (hep-th/0204168). We find that the backreaction on the geometry due to a test particle can be made arbitrarily small, and that there are scattering processes which can be studied in the approximation of a constant background. We quantize strings on this orbifold and calculate the torus partition function. We construct a basis of states on the smooth orbifold whose tree level string interactions are nonsingular. We discuss the existence of physical modes in the singular orbifold which resolve the singularity. We also describe another way of making the singular orbifold smooth which involves a sandwich pp-wave.

June 2002

Work supported by the Department of Energy contract DE-AC03-76SF00515.

1. Introduction

Most of our present knowledge of string theory pertains to time-independent backgrounds. However, some of the most interesting questions we would like to ask a theory of quantum gravity, namely those related to cosmological singularities and horizons, belong fully to the realm of time-dependent spacetimes. If we want to understand how string theory answers these questions, we need to know how to formulate string theory in such spacetimes.

This is in general a difficult problem. As a simple class of examples, it seems natural to study time-dependent orbifolds of flat Minkowski space by a discrete subgroup of the Poincaré group [1-10] (see also [11]). Many such orbifolds contain closed timelike curves, which raise unpleasant issues. Better in this regard is the model studied by Liu, Moore and Seiberg [9] which is an orbifold by $\mathbf{Z}$ generated by a parabolic element of $SO(1,2)$ and belongs to the class of models described by Horowitz and Steif [2]. The orbifold has a light-like singularity and contains closed light-like curves. It has a null Killing vector, which allows one to use light-cone quantization. (See [10] for a discussion of the stability of the singularity in this orbifold. Various other time-dependent backgrounds of string theory were studied, for example, in [12-18], and more recently in [19-35].)

In the first part of this paper we consider string theory on a very closely-related orbifold, which was recently discussed in [7,8]. The generator of the orbifold group is a parabolic element of $SO(1,2)$ combined with a constant shift in a fourth direction. Its main virtue is that the orbifold group has no fixed points, and therefore the quotient space contains no singularities at all. In the limit where the shift goes to zero we recover precisely the orbifold of [9].

Studying this singular limit provides a new perspective on the interesting null singularity of [9]. Perhaps more significantly, the orbifold with the shift provides a time-dependent string background which has a free world-sheet description, and in which the backreaction is under control. The ability to study the issue of time-dependence separately from complications raised by the presence of singularities is likely to be quite useful.

For example, despite the solvable and smooth nature of the world-sheet theory for this model, it is still not clear how to do string calculations to an arbitrary loop order in the covariant gauge. The orbifold does not have any useful Euclidean continuation, so one is forced to do computations in Lorentzian signature of space-time and therefore also in Lorentzian signature on the world-sheet. Riemann surfaces of genus different from one, however, do not admit any smooth Lorentzian metric, posing an obvious difficulty for covariant calculations.

In the second part of the paper, we ask whether in the $\mathbb{R}^{1,3} \times \mathbb{R}^6$ model of Liu, Moore and Seiberg [9], specifying the initial data for the geometry only on a slice of a constant light-cone time determines that there will be a singularity in the future (or in the past). We find that the answer to this question is negative and we present everywhere non-singular gravitational solutions whose light-cone-time past exactly coincides with that of the singular orbifold of Liu, Moore and Seiberg [9]. In addition to the light-cone-time past, one can, if desired, make also late light-cone-time future coincide with that [9].

While this work was in progress we learned that a similar study was being made [36], and that the smooth time-dependent orbifold of Minkowski space was also suggested by Joe Polchinski and Eva Silverstein.

The organization of this paper is as follows. In §2 we describe the classical geometry of the nonsingular orbifold and discuss some sets of coordinates which will be useful. In §3 we probe the geometry with test masses and show that it does not collapse gravitationally. In §4 we consider the wavefunctions of a scalar field in this background. In §5 we quantize strings on the orbifold in light-cone gauge, and calculate the partition function. In §6 we study tree level amplitudes on the orbifold and show that there is a basis of wavefunctions in which they are nonsingular. In §7 we study the mode of the singular orbifold which turns on the shift. In §8 we resolve the singularity of [9] with a sandwich wave. In §9 we conclude.

2. Classical Geometry

The geometry we will study is a $\mathbf{Z}$ orbifold of flat Minkowski space $\mathbb{R}^{1,3}$ (times $\mathbb{R}^6$ or $\mathbb{R}^{22}$, depending on whether we want to consider superstrings or bosonic strings). In terms of coordinates

$$x^+ = \frac{x^0 + x^1}{\sqrt{2}}, \quad x^- = \frac{x^0 - x^1}{\sqrt{2}}, \quad x = x^2, \quad \chi = x^3, \tag{2.1}$$

the metric is

$$ds^2 = -2x^+x^- + dx^2 + d\chi^2 \tag{2.2}$$

We will write the generator of the orbifold group Γ_L as

$$g_L = \exp\left(ivJ\right)\ \exp\left(iLp^\chi\right), \quad J \equiv \frac{1}{\sqrt{2}} J^{x^0 x} + \frac{1}{\sqrt{2}} J^{x^1 x} \tag{2.3}$$

This corresponds to a composition of a null Lorentz transformation of the (x^+, x, x^-) subspace and a translation by L in the χ-direction. In terms of the spacetime coordinates, g_L acts as

$$\begin{pmatrix} x^+ \\ x \\ x^- \\ \chi \end{pmatrix} \quad \to \quad \begin{pmatrix} x^+ \\ x + vx^+ \\ x^- + vx + \frac{1}{2}v^2 x^+ \\ \chi + L \end{pmatrix} \tag{2.4}$$

For $L = 0$ the orbifold becomes the orbifold studied by Liu, Moore and Seiberg [9], which is singular at $x^+ = 0$. For non-zero L this orbifold is completely smooth and does not have any closed time-like or space-like curves. The spacetime interval $(nvx^+)^2 + L^2$ between points identified by the n-th power of the orbifold group generator (2.3) is strictly positive. The $\mathbb{R}^{1,3}/\Gamma_{L=0}$ orbifold has the property that every spacetime point with $x^+ > 0$ is in the causal future of every point with $x^+ < 0$ [9]. This is no longer true for $L \neq 0$.

The $\mathbb{R}^{1,3}/\Gamma_L$ orbifold in general preserves the subgroup of the four-dimensional Poincaré symmetry group generated by p^χ, J, and $p^+ = -p_-$. For non-zero L, the topology of the spacetime is simply $\mathbb{R}^3 \times S^1$, which can be made manifest by defining a new set of coordinates $\widetilde{x}^+, \widetilde{x}, \widetilde{x}^-, \widetilde{\chi}$ by

$$\begin{aligned}
\widetilde{x}^+ &= x^+ \\
\widetilde{x} &= x - \frac{\chi}{L} v x^+ \\
\widetilde{x}^- &= x^- - \frac{\chi}{L} v x + \frac{1}{2}\frac{\chi^2}{L^2} v^2 x^+ \\
\widetilde{\chi} &= \frac{\chi}{L},
\end{aligned} \tag{2.5}$$

in terms of which the orbifold action becomes

$$(\widetilde{x}^+, \widetilde{x}, \widetilde{x}^-, \widetilde{\chi}) \quad \to \quad (\widetilde{x}^+, \widetilde{x}, \widetilde{x}^-, \widetilde{\chi} + 1). \tag{2.6}$$

The map between (x^+, x, x^-, χ) and $(\widetilde{x}^+, \widetilde{x}, \widetilde{x}^-, \widetilde{\chi})$ is everywhere smooth and one-to-one, and can be easily inverted:

$$\begin{aligned}
x^+ &= \widetilde{x}^+ \\
x &= \widetilde{x} + \widetilde{\chi} v \widetilde{x}^+ \\
x^- &= \widetilde{x}^- + \widetilde{\chi} v \widetilde{x} + \frac{1}{2}\widetilde{\chi}^2 v^2 \widetilde{x}^+ \\
\chi &= L\widetilde{\chi},
\end{aligned} \tag{2.7}$$

The metric now becomes

$$ds^2 = -2d\widetilde{x}^+ d\widetilde{x}^- + d\widetilde{x}^2 + \left[L^2 + (v\widetilde{x}^+)^2\right] d\widetilde{\chi}^2 + 2v\,(\widetilde{x}^+ d\widetilde{x} - \widetilde{x} d\widetilde{x}^+) d\widetilde{\chi}. \tag{2.8}$$

which is non-degenerate everywhere.

Away from $x^+ = 0$ one can define another useful set of coordinates,

$$\begin{aligned}
y^+ &= x^+ \\
y &= \frac{x}{x^+} \\
y^- &= x^- - \frac{1}{2}\frac{x^2}{x^+} \\
\psi &= \chi - \frac{L}{2\pi}\frac{x}{x^+}.
\end{aligned} \tag{2.9}$$

The corresponding identifications and metric are

$$(y^+, y, y^-, \psi) \sim (y^+, y + 2\pi, y^-, \psi), \tag{2.10}$$

$$ds^2 = -2dy^+dy^- + \left((y^+)^2 + \frac{L^2}{(2\pi)^2}\right) dy^2 + d\psi^2 + \frac{L}{\pi} dy d\psi. \tag{2.11}$$

The definition of the y-coordinates in (2.9) is the same as in [9].

3. Backreaction of particles on the geometry

One of the most obvious questions that arise when one considers time-dependent orbifolds is whether the presence of a single particle does not cause the spacetime to gravitationally collapse. Placing one such particle of rest mass m (say $m \neq 0$) in the orbifold corresponds to adding to the universal covering space an infinite number of particles (the original one plus its images) which are boosted with respect to each other. Since the boost of distant particles goes to infinity, one might worry that the mass of a finite number of them might be larger than the corresponding Schwarzschild radius (which would be a clear sign of a large backreaction).

We will see that if $L \neq 0$ this does not happen here, provided m is not too large. Suppose we work in an inertial frame in which the 'original particle' is at rest. At any time x^0 the distance to its n-th image will be no smaller than nL, i.e. it grows at least linearly with n. The velocity of the n-th image is

$$\mathrm{v}_n = \frac{nv}{4 + n^2 v^2} \sqrt{8 + n^2 v^2}, \tag{3.1}$$

and corresponds to energy

$$E_n = m\gamma_n = \frac{m}{\sqrt{1 - \mathrm{v}_n^2}} = m \left(1 + \frac{1}{4} n^2 v^2\right) \sim \frac{1}{4} m\, n^2 v^2. \tag{3.2}$$

Therefore the total energy of the first $2n$ images grows like n^3. Taking into account more and more images, the corresponding Schwarzschild radius will grow like $(n^3)^{1/D-3} = n^{3/D-3}$ in spacetime dimension D. In order for our lower bound nL on the size of the region containing the first $2n$ images to grow faster than the Schwarzschild radius we need $D > 6$. This is of course satisfied in our case where the universal covering space is $\mathbb{R}^{1,3}$ times $\mathbb{R}^6$. As a result, the gravitational backreaction of a particle of small enough mass on the geometry of the orbifold (with large enough L) will be small, and in particular, it will not cause a gravitational collapse. (Of course, we cannot repeat the same argument for the orbifold with $L = 0$.)

For massless particles, we need to modify the argument just very slightly. Pick an inertial frame in which the particle has an energy E_0. The images will be boosted in some direction by the same gamma-factor γ_n as in (3.2). This means that $E_n \leq (1+v)\gamma_n E_0$, leading to the same conclusion as before.

A very similar kind of reasoning shows that for small enough energies (and large enough L), scattering amplitudes of approximately localized particles (or strings) are not affected much by the image particles, and therefore they are inherited from the covering space with just small corrections. This means that there exist scattering amplitudes for which it does make sense to work in the approximation of a fixed background, as we do in §6. Of course, for large enough energies this approximation will break down, just like in any other spacetime.

4. Scalar particle wavefunctions on the smooth $\mathbb{R}^{1,3}/\Gamma_L$ orbifold

First quantized wavefunctions of scalar particles in the $\mathbb{R}^{1,3}/\Gamma_{L\neq 0}$ orbifold are a close analog of the wavefunctions for the $L=0$ case [9]. The wave equation for scalars is

$$\Big[-2\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-}+(\frac{\partial}{\partial x})^2+(\frac{\partial}{\partial \chi})^2\Big]\Psi = m^2\Psi. \tag{4.1}$$

The orbifold group generator (2.3) can be expressed as[1]

$$\begin{gathered}\mathcal{U}(g_L)=\exp(2\pi i\widehat{J})\,\exp(iL\widehat{p}^{\chi}),\\ \widehat{J}=\widehat{x}^+\widehat{p}-\widehat{x}\widehat{p}^+=-i\big(x^+\frac{\partial}{\partial x}+x\frac{\partial}{\partial x^-}\big),\quad \widehat{p}^+=-i\frac{\partial}{\partial x^-},\quad \widehat{p}^{\chi}=-i\frac{\partial}{\partial \chi}.\end{gathered} \tag{4.2}$$

Because $\mathcal{U}(g_L)$ commutes with $\widehat{p}^+$, $\widehat{J}$, and $\widehat{p}^{\chi}$ we may use the corresponding eigenvalues to label the wavefunctions. Up to the case of $p^+=0$, they have the form

$$\begin{aligned}\Psi_{p^+,J,p^{\chi}} &=\sqrt{\frac{p^+}{ix^+}}\,\exp\left[-ip^+x^- - i\frac{m^2+p_\chi^2}{2p^+}x^+ + i\frac{p^+}{2x^+}(x-\xi)^2+ip^{\chi}\chi\right]\\ &=\int_{-\infty}^{\infty}\frac{dp}{\sqrt{2\pi}}\;e^{-ip\xi}\phi_{p^+,p,p^{\chi}}(x^+,x^-,x,\chi)\end{aligned} \tag{4.3}$$

where $\xi=-J/p^+$ and

$$\phi_{p^+,p,p^{\chi}}(x^+,x,x^-,\chi)=\exp\left(-ip^+x^- - ip^-x^+ + ipx + ip^{\chi}\chi\right),\; p^- = \frac{m^2+p^2+p_\chi^2}{2p^+}. \tag{4.4}$$

[1] From now on we will set $v=2\pi$ for simplicity.

In the limit $x^+ \to 0$ the wavefunctions (4.3) become

$$\lim_{x^+\to 0} \Psi_{p^+,J,p^\chi}(x^+, x, x^-, \chi) = \sqrt{2\pi} e^{-ip^+x^- + ip^\chi \chi} \delta(x - \xi), \quad \xi = -J/p^+. \tag{4.5}$$

In the $L = 0$ case a similar statement implied [9] that the wavefunctions of definite p^+ were supported at $x^+ = 0$ on the lattice $x \in \frac{1}{p^+}\mathbf{Z}$. Here the situation is just slightly more complicated. In terms of (4.3) taking the orbifold corresponds to making $2\pi J + Lp^\chi$ quantized:

$$\widetilde{J} \equiv J + \frac{Lp^\chi}{2\pi} \in \mathbf{Z}. \tag{4.6}$$

As a result, the support of the wavefunctions (4.3) at $x^+ = 0$ is $x \in \mathbf{Z}/p^+ - Lp^\chi/2\pi p^+$, i.e. the lattice has a p^χ-dependent shift. This allows one to construct a Fock space basis consisting only of wavefunctions which are smooth everywhere. We will be introduce it in section 6.

5. Strings on the smooth orbifold

The analysis of string propagation on the orbifold $\mathbb{R}^{1,3}/\Gamma_L$ proceeds very similarly to the analysis for $\mathbb{R}^{1,3}/\Gamma_0$ performed in [9].

5.1. Light-cone gauge quantization

In the light-cone gauge $x^+ = y^+ = \tau$, we can express the worldsheet Lagrangian for as

$$\mathcal{L} = -p^+\partial_\tau x_0^- + \frac{1}{4\pi\alpha'}\int_0^{2\pi} d\sigma \left(\alpha' p^+ \left(\partial_\tau x \partial_\tau x + \partial_\tau \chi \partial_\tau \chi\right) - \frac{1}{\alpha' p^+}\left(\partial_\sigma x \partial_\sigma x + \partial_\sigma \chi \partial_\sigma \chi\right)\right). \tag{5.1}$$

Invariance under the choice of origin of the σ-coordinate leads to the constraint [37]

$$\int d\sigma \left(\partial_\sigma \chi \partial_\tau \chi + \partial_\sigma x \partial_\tau x - \frac{1}{2\tau}\partial_\sigma x^2\right) = 0. \tag{5.2}$$

The orbifold action

$$\begin{aligned} &x(\sigma,\tau) \to x(\sigma,\tau) + 2\pi n\tau \\ &x_0^-(\tau) \to x_0^-(\tau) + 2\pi n \int_0^{2\pi} \frac{d\sigma}{2\pi} x(\sigma,\tau) + \frac{(2\pi n)^2}{2}\tau \\ &\chi(\sigma,\tau) \to \chi(\sigma,\tau) + L, \end{aligned} \tag{5.3}$$

leaves both (5.1) and (5.2) invariant.

Because of the invariance of the action (5.1) under constant shifts of x^-, p^+ is a conserved quantity. The equation of motion for p^+ implies

$$P_{x^+} = p^+ \partial_\tau x_0^- = \frac{1}{4\pi\alpha'} \int_0^\ell d\sigma \left((\partial_\tau x)^2 + (\partial_\tau \chi)^2 + (\partial_\sigma x)^2 + (\partial_\sigma \chi)^2 \right) \tag{5.4}$$

where we have rescaled σ to range in $[0, \ell = 2\pi\alpha' p^+)$. The direction of the Killing vector x^+ is changed by the orbifold action, and as a result, the Hamiltonian $P_{x^+}$is not invariant under (5.3).

The mode expansions for the physical worldsheet fields in the w-twisted sector can be written as follows:

$$\begin{aligned}
x(\sigma,\tau) &= \xi + \frac{p}{p^+}\tau + \frac{2\pi w \sigma \tau}{\ell} + \\
&\quad i\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{n\neq 0} \left\{ \frac{\alpha_n}{n} \exp\left[-\frac{2\pi i n(\sigma+\tau)}{\ell} \right] + \frac{\widetilde{\alpha}_n}{n} \exp\left[\frac{2\pi i n(\sigma-\tau)}{\ell} \right] \right\} \\
\chi(\sigma,\tau) &= \chi_0 + \frac{p^\chi}{p^+}\tau + \frac{w\sigma L}{\ell} + \\
&\quad i\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{n\neq 0} \left\{ \frac{a_n}{n} \exp\left[-\frac{2\pi i n(\sigma+\tau)}{\ell} \right] + \frac{\widetilde{a}_n}{n} \exp\left[\frac{2\pi i n(\sigma-\tau)}{\ell} \right] \right\},
\end{aligned} \tag{5.5}$$

where, upon quantization, the oscillators satisfy the usual commutation relations. The solution of the equation of motion for x_0^- is $x_0^- = P_{x^+}\tau/p^+$, up to an additive constant. As in the previous section, $\widetilde{J}$ defined as

$$\widetilde{J} \equiv J + \frac{Lp^\chi}{2\pi} = -\xi p^+ + \frac{Lp^\chi}{2\pi} \tag{5.6}$$

must be quantized. Here, the constraint (5.2) implies

$$\widetilde{J} w = N - \widetilde{N}, \tag{5.7}$$

where N and $\widetilde{N}$ are the usual number operators.

Strings on the $\mathbb{R}^{1,3}/\Gamma_L$ orbifold can also be quantized covariantly in a manner very similar to the covariant quantization of $\mathbb{R}^{1,3}/\Gamma_0$ in [9].

5.2. The torus partition function

The calculation of the torus partition function for bosonic strings on the resolved orbifold $\mathbb{R}^{1,3}/\Gamma_L$ was actually implicitly contained in [9]. It is, however, instructive to see it more explicitly. We wish to calculate

$$Z = \int [\mathbf{dX}][d\chi] e^{iS} \tag{5.8}$$

with

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{ab} \left(\partial_a \mathbf{X}^T(\sigma_1, \sigma_2) \cdot \mathbf{G} \cdot \partial_b \mathbf{X}(\sigma_1, \sigma_2) + \partial_a \chi(\sigma_1, \sigma_2) \partial_b \chi(\sigma_1, \sigma_2) \right). \quad (5.9)$$

Here the coordinates x^+, x and x^- are combined into a single vector $\mathbf{X}$, and $\mathbf{G}$ represents the corresponding part of the space-time metric:

$$\mathbf{X} = \begin{pmatrix} x^+ \\ x \\ x^- \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (5.10)$$

The orbifold action becomes

$$\mathbf{X} \to e^{2\pi \mathcal{J}} \mathbf{X} = (1 + 2\pi \mathcal{J} + 2\pi^2 \mathcal{J}^2) \mathbf{X}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (5.11)$$

The worldsheet metric can be chosen as

$$g_{ab}\, d\sigma^a d\sigma^b = (d\sigma^1 + \tau_+ d\sigma^2)(d\sigma^1 + \tau_- d\sigma^2), \quad \tau_\pm = \tau_1 \pm \tau_2, \quad (5.12)$$

with $\sigma^1, \sigma^2 \in [0, 1)$. Plugging the world-sheet metric into (5.9), we get

$$\begin{aligned} S = \frac{1}{4\pi\alpha' \tau_2} \int d^2\sigma (\tau_+ \tau_- \partial_1 \mathbf{X}^T \cdot \mathbf{G} \cdot \partial_1 \mathbf{X} - 2\tau_1 \partial_1 \mathbf{X}^T \cdot \mathbf{G} \cdot \partial_2 \mathbf{X} + \partial_2 \mathbf{X}^T \cdot \mathbf{G} \cdot \partial_2 \mathbf{X} \\ + (\partial_1 \chi)^2 + (\partial_2 \chi)^2). \end{aligned} \quad (5.13)$$

The worldsheet fields in the (w_1, w_2)-twisted sector can be expanded as

$$\begin{aligned} \mathbf{X}(\sigma^1, \sigma^2) &= \exp\left[2\pi(\sigma^1 w_1 + \sigma^2 w_2)\mathcal{J}\right] \sum_{n_1, n_2 \in \mathbf{Z}} \mathbf{X}_{n_1, n_2} e^{2\pi i (n_1 \sigma^1 + n_2 \sigma^2)} \\ \chi(\sigma^1, \sigma^2) &= \sigma^1 w_1 L + \sigma^2 L \tau_2^{-1}(w_2 - w_1 \tau_1) + \sum_{n_1, n_2 \in \mathbf{Z}} \chi_{n_1, n_2} e^{2\pi i (n_1 \sigma^1 + n_2 \sigma^2)}, \end{aligned} \quad (5.14)$$

implying, in particular,

$$\begin{aligned} \partial_1 \mathbf{X}(\sigma^1, \sigma^2) &= 2\pi w_1 \mathcal{J} \mathbf{X} + \exp\left[2\pi(\sigma^1 w_1 + \sigma^2 w_2)\mathcal{J}\right] \sum_{n_1, n_2 \in \mathbf{Z}} 2\pi i n_1 \mathbf{X}_{n_1, n_2} e^{2\pi i (n_1 \sigma^1 + n_2 \sigma^2)} \\ \partial_2 \mathbf{X}(\sigma^1, \sigma^2) &= 2\pi w_2 \mathcal{J} \mathbf{X} + \exp\left[2\pi(\sigma^1 w_1 + \sigma^2 w_2)\mathcal{J}\right] \sum_{n_1, n_2 \in \mathbf{Z}} 2\pi i n_2 \mathbf{X}_{n_1, n_2} e^{2\pi i (n_1 \sigma^1 + n_2 \sigma^2)}. \end{aligned} \quad (5.15)$$

Let us first see how the modes of $\mathbf{X}$ with non-zero (n_1, n_2) contribute to the partition function. Their contributions to the action will be of the form

$$\sum_{\alpha, \beta = 0,1,2} c_{\alpha\beta}\, \mathbf{X}^T_{-n_1, -n_2} \cdot (\mathcal{J}^T)^\alpha \cdot \mathbf{G} \cdot \mathcal{J}^\beta \cdot \mathbf{X}_{n_1, n_2}. \quad (5.16)$$

All the matrices $(\mathcal{J}^T)^\alpha \cdot \mathbf{G} \cdot \mathcal{J}^\beta$ are of the form

$$\begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} \\ \Upsilon_{21} & \Upsilon_{22} & 0 \\ \Upsilon_{31} & 0 & 0 \end{pmatrix},$$

and if α or β is non-zero, then $\Upsilon_{13} = \Upsilon_{22} = \Upsilon_{31} = 0$. Therefore only terms with $\alpha = \beta = 0$ will contribute to the determinant of the matrix and have any effect on the result of the gaussian integration[2]. Because they are independent of w_1 and w_2, one concludes that the contribution of oscillators of $\mathbf{X}$ to the partition function will be the same as in Minkowski space [9].

As a result, only the zero modes of $\mathbf{X}$ can have any (w_1, w_2)-dependent contribution. Because in this case $(n_1, n_2) = (0, 0)$, the second terms on the RHS of (5.15) vanish, and we get

$$S_{X_{0,0}} = \frac{(2\pi)^2}{4\pi\alpha'\tau_2} \int d^2\sigma (\tau_+\tau_- w_1^2 - 2\tau_1 w_1 w_2 + w_2^2) \mathbf{X}_{0,0}^T \mathcal{J}^T \mathbf{G} \mathcal{J} \mathbf{X}_{0,0}, \tag{5.17}$$

or[3]

$$S_{X_{0,0}} = \frac{\pi}{\alpha'\tau_2}(w_1\tau_+ - w_2)(w_1\tau_- - w_2)(x^+)^2. \tag{5.18}$$

Since in each sector χ is just a periodic boson, whose zero mode contributes an action

$$S_{\chi_{0,0}} = \frac{\pi}{\alpha'\tau_2}(w_1\tau_+ - w_2)(w_1\tau_- - w_2)\frac{L^2}{4\pi^2}, \tag{5.19}$$

we find that the full partition function for the bosons is

$$\begin{aligned} Z =& \frac{iZ^{\text{ghost}} Z^{\perp} L}{\tau_2^2 (\eta(\tau_+)\eta(-\tau_-))^4} \times \\ & \int \frac{d^3x}{(2\pi\sqrt{\alpha'})^4} \sum_{w_1, w_2 \in \mathbf{Z}} \exp\left[\frac{i\pi(w_1\tau_+ - w_2)(w_1\tau_- + w_2)}{\alpha'\tau_2} [(x^+)^2 + \frac{L^2}{4\pi^2}] \right]. \end{aligned} \tag{5.20}$$

Using Poisson resummation, one can show [9] that for $L = 0$ the bosonic contribution to the one-loop cosmological "constant" diverges as $x^+ \to 0$,

$$\Lambda(x^+) \sim \frac{1}{(x^+)^2} \times \int_{\mathcal{F}} \frac{d^2\tau}{(\tau_2)^2} \frac{Z^{\text{tr}}}{|\eta(\tau)|^2} \tag{5.21}$$

For $L \neq 0$, this contribution is resolved to

$$\Lambda(x^+) \sim \frac{1}{(x^+)^2 + \frac{L^2}{4\pi^2}} \times \int_{\mathcal{F}} \frac{d^2\tau}{(\tau_2)^2} \frac{Z^{\text{tr}}}{|\eta(\tau)|^2} + \dots, \tag{5.22}$$

where the dots denote other non-singular terms (which are subleading if L is small). This is consistent with the interpretation [9] of the divergence at $L = 0, x^+ = 0$ as arising from light winding modes.

[2] We wick rotate the timelike modes to define this integral.

[3] To avoid any confusion, we should stress that here x^+ denotes the zero-mode of $x^+(\sigma, \tau)$.

5.3. Extension to superstring backgrounds

The $\mathrm{I\!R}^{1,3}/\Gamma_L$ orbifold can be thought of as a superstring background. In the Green-Schwarz formalism one needs to add six more worldsheet bosons (corresponding to $\mathrm{I\!R}^6$) and the appropriate worldsheet fermion content. Because $\pi_1(\mathrm{I\!R}^{1,3}/\Gamma_L) = \mathbf{Z}$, there are two different spin structures we can choose. One of them makes the orbifold supersymmetric, and corresponds to all the extra worldsheet fields being single-valued [37]. For the non-supersymmetric choice, strings with odd winding numbers will have antiperiodic worldsheet fermions.

The formulas for the bosonic partition function (5.20) can be applied to the superstring. The full partition function will contain a factor of the fermionic partition function, which will vanish if we choose the supersymmetric spin structure.

With the opposite choice of the spin structure, supersymmetry is broken and a one-loop dilaton tadpole is generated, in analogy to [38]. For large enough L, this potential is an everywhere-finite function on spacetime which goes to zero at large $|x^+|$. However, because of the low codimension of its support, this one-loop energy-momentum tensor causes a significant backreaction on the geometry. To avoid this, one could consider including a rotation of the extra planes (x^4-x^5, x^6-x^7, x^8-x^9) into the action of the orbifold group generator g_L. Compensating for the Casimir stress-energy by a small local change of the metric and dilaton near the axes of rotation would be similar to [28].

In this way, one would obtain a non-supersymmetric perturbatively stable time-dependent background. At least in some range of parameters, the leading instability would be nucleation of Witten bubbles [39-47,28]. The fact that such bubbles are possible (in some range of parameters) can be seen quite easily. Given a small enough v, the region around $\widetilde{x} = \widetilde{x}^+ = 0$ looks like a time-independent generalized 'twisted circle' orbifold [e.g. 48-55] (cf. (2.8)), the twist being the rotation in the extra planes (x^4-x^5, x^6-x^7, x^8-x^9). As a result, it must be possible to nucleate a generalized Witten bubble locally. For this reason, there must exist a bubble solution which is extendable to a global solution to Einstein's equations asymptoting to the non-supersymmetric time-dependent orbifold. It is possible that such a solution could be obtained by an alternative Wick rotation of the Euclidean Kerr solution.

6. Tree-level string interactions on the smooth orbifold

The basis of states used in [9] to calculate string amplitudes on the $\mathrm{I\!R}^{1,3}/\Gamma_0$ orbifold is a singular one. Using this basis of states in Minkowski space one would find similar tree-level divergences in special kinematic regimes. However, on the singular orbifold one has no option (as one has in Minkowski space) to use a less singular basis.

On the smooth orbifold, one has a better option. Define a basis of wavefunctions by the following convolution of plane waves:

$$
\begin{aligned}
&\psi_{\chi_0,\widetilde{J},p^+,\vec{p}}(x^+,x,x^-,\chi,\vec{x}) \equiv \\
&\qquad \sqrt{\frac{a}{2\pi}}\int dp^\chi e^{-ip^\chi\chi_0-\frac{a}{4}(p^\chi)^2}\int\frac{dp}{\sqrt{2\pi}}\ e^{-ip\xi}\phi_{p^+,p,p^\chi}\ (x^+,x,x^-,\chi)e^{i\vec{p}\cdot\vec{x}},
\end{aligned}
\tag{6.1}
$$

where in this integral, the variable $\xi = -J/p^+$ is considered as a function of p^χ and $\widetilde{J}$, via the equation

$$
\xi \equiv \frac{1}{p^+}(-\widetilde{J}+\frac{L}{2\pi}p^\chi);
$$

and $\phi_{p^+,p,p^\chi}\ (x^+,x,x^-,\chi)$ is an on-shell plane wave as in (4.4), with p^- now including also a contribution from $\vec{p}$.

Performing the two integrals in (6.1) one finds

$$
\begin{aligned}
\psi_{\chi_0,\widetilde{J},p^+,\vec{p}}(x^+,x,x^-,\chi,\vec{x}) &= e^{ip^+x^-+i\frac{x^+}{2p^+}(\vec{p}^2+m^2)+i\vec{p}\cdot\vec{x}+\frac{ip^+}{2x^+}\left(x+\widetilde{J}/p^+\right)^2} \times \\
&\times \frac{\sqrt{a}|p^+|}{\sqrt{2\left(\frac{L}{2\pi}\right)^2-iax^+p^+-2(x^+)^2}} \times \\
&\times \exp\left[\frac{-x^+p^+}{-2i\left(\frac{L}{2\pi}\right)^2+x^+p^+a+2i(x^+)^2}\left(\chi-\chi_0-\frac{L}{2\pi x^+}\left(x+\widetilde{J}/p^+\right)\right)^2\right]
\end{aligned}
\tag{6.2}
$$

We note that this convolution (which diagonalizes $\widetilde{J}$) would not be sensible on the orbifold without the shift by L. As one can see from the fact that the magnitude of this wave remains finite near $x^+ \to 0$, these wavefunctions lack the focusing properties of those in the singular orbifold. (Intuitively, they are gaussian linear superpositions of J-eigenfunctions (4.3), (4.5) which are focused on different points.)

As one might expect from this observation, the tree-level divergences in string theory amplitudes involving untwisted states on the singular orbifold are absent in the resolved case using this basis of states. For example for tachyons the four-point amplitude for scattering these states is obtained, as in [9], by convolving the four-point amplitude for plane-wave tachyons on the covering space with the kernel which produces the wavefunctions from plane waves, and replacing delta functions with Kronecker deltas. The amplitudes in the $\widetilde{J}$ basis (6.2) may be obtained from those in the J-basis (equation (6.16) of [9]) by a convolution integral over p^χ considering the quantum number $J = \widetilde{J} - \frac{L}{2\pi}p^\chi$ to be a function of p^χ.

Amplitudes involving generic momenta are smooth [9]. A divergence was found in a kinematic regime which explored the Regge region of the amplitude on the covering space. The dangerous part of the amplitude occurs when $p_1^+ = p_3^+$, near very large q in the following integral:

$$\mathcal{A}_4 \sim \int^\infty \frac{dq}{|q|} \int dk \;\; e^{\frac{1}{2}ik\chi^t - \frac{a}{8}k^2} \;\; q^{4-\alpha'(k^2+\vec{p}_t^2)} \;\; \exp\left(-iq\sqrt{\mu_{12}}\xi_t\right) \tag{6.3}$$

where $k \equiv p_3^\chi - p_1^\chi$, $\widetilde{J}_t \equiv \widetilde{J}_3 - \widetilde{J}_1$, $\chi^t = \chi_0^1 - \chi_0^3$ and

$$\xi_t \equiv \xi_3 - \xi_1 = \frac{1}{p_1^+}(-\widetilde{J}_t + \frac{L}{2\pi}k) \tag{6.4}$$

Performing the integral over k one finds

$$\mathcal{A}_4 \sim \int^\infty \frac{dq}{|q|} \frac{1}{\sqrt{a+8\alpha' \ln q}} \exp\left[-\frac{2(\frac{\sqrt{\mu_{12}}}{p_1^+}\frac{L}{2\pi}q - \frac{1}{2}\chi^t)^2}{a+8\alpha' \ln q} + \ln q(4-\alpha'\vec{p}_t^2) + i\frac{\sqrt{\mu_{12}}}{p_1^+}\widetilde{J}_t q \right] \tag{6.5}$$

which is well-behaved near $q \to \infty$ because of the $-L^2q^2/\ln q$ term in the exponential.

The divergences in four-point functions found in [9] are therefore absent in the resolved orbifold. One might have worried that these divergences were associated purely with the time-dependent nature of the background and the identifications which include very large boosts. This calculation demonstrates that this is not the case.

7. On the nature of the resolution by the shift

The light-like orbifold singularity of [9] is very different from any time-like singularity in string theory, such as the A_k singular limit of the supersymmetric $\mathbb{C}^2/\mathbf{Z}_{k+1}$ orbifold. In particular, if one wants to obtain the A_k singularity as a limit of some smooth geometry, one has to send the curvature to infinity. In this sense the A_k singularity is really a curvature singularity. This is not true in the case of the $\mathbb{R}^{1,3}/\Gamma_0$ orbifold, because it can be thought of as the $L \to 0$ limit of the $\mathbb{R}^{1,3}/\Gamma_L$ orbifold, which is smooth and flat everywhere.

It is important that the group $\Gamma_0 = \mathbf{Z}$ (2.4) is not finite. If one added a shift by L in an extra direction, for example, to the orbifold group generator of $\mathbb{C}^2/\mathbf{Z}_{k+1}$, the group would become $\mathbf{Z}$ and one would obtain a (generalized) 'twisted circle' orbifold [48-55]. In that case, sending L to zero would not lead to the original $\mathbb{C}^2/\mathbf{Z}_{k+1}$. Points identified by the $(k+1)$-th power of the orbifold group generator would be very close to each other, and for any non-zero L, they would be distinct. If we wanted the get some $\mathbb{C}^2/\mathbf{Z}_{k+1}$ orbifold in the limit $L \to 0$, we would be forced to perform a T-duality.

7.1. On the possibility of a local resolution by the shift mode

Because the $\mathbb{R}^{1,3}/\Gamma_0$ orbifold is a limit of the smooth $\mathbb{R}^{1,3}/\Gamma_L$ orbifold, it is natural to ask whether there are any physical modes in the $\mathbb{R}^{1,3}/\Gamma_0$ orbifold which would make L locally non-zero and smooth out the null singularity.

To address this question we calculate the distance in the configuration space between $L = 0$ and some fixed non-zero $L = L_f$. More precisely, we will calculate the Zamolodchikov metric for the CFT operator which increases L by δL in the light-cone gauge at some fixed non-zero light-cone time, and then integrate it over L. In the time-independent context, a similar calculation for a marginal operator would give the distance in the moduli space.

Let us first use the y and ψ coordinates (2.9), (2.11). The worldsheet action (with the period of σ being $2\pi\alpha' p^+$)

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \left((\tau^2 + \frac{L^2}{4\pi^2})[(\partial_\tau y)^2 - (\partial_\sigma y)^2)] + [(\partial_\tau\psi)^2 - (\partial_\sigma\psi)^2] + \frac{L}{\pi}[\partial_\tau y\partial_\tau\psi - \partial_\sigma y\partial_\sigma\psi] \right), \tag{7.1}$$

leads to the correlation functions

$$\begin{aligned}
\langle y(\sigma,\tau)y(\sigma',\tau')\rangle &= -\frac{\alpha'}{2\tau\tau'}\ln|(\tau-\tau')^2 - (\sigma-\sigma')^2| \\
\langle y(\sigma,\tau)\psi(\sigma',\tau')\rangle &= \frac{\alpha' L}{4\pi\tau\tau'}\ln|(\tau-\tau')^2 - (\sigma-\sigma')^2| \\
\langle \psi(\sigma,\tau)\psi(\sigma',\tau')\rangle &= -\frac{\alpha'}{2}\left(1 + \frac{L^2}{4\pi^2\tau\tau'}\right)\ln|(\tau-\tau')^2 - (\sigma-\sigma')^2|.
\end{aligned} \tag{7.2}$$

The operator which, when added to the worldsheet action, changes L by δL is

$$\mathcal{O}_L\delta L = \frac{1}{4\pi\alpha'}\left(\frac{L}{2\pi^2}((\partial_\tau y)^2 - (\partial_\sigma y)^2) + \frac{1}{\pi}(\ \partial_\tau y\ \partial_\tau\psi - \partial_\sigma y\ \partial_\sigma\psi)\right)\delta L. \tag{7.3}$$

Using (7.2), one finds that the Zamolodchikov metric, defined as

$$\mathcal{G}_{L,L}\ \delta L\ \delta L \equiv \lim_{\epsilon\to 0}\epsilon^4\langle\mathcal{O}_L(\sigma,\tau)\mathcal{O}_L(\sigma+\epsilon,\tau)\rangle\ (\delta L)^2, \tag{7.4}$$

is

$$\mathcal{G}_{L,L}\ \delta L\ \delta L = \frac{1}{2\pi^2(y^+)^2}\ (\delta L)^2, \tag{7.5}$$

The resulting distance between $L = 0$ and $L = L_f$ is therefore

$$\Delta\mathcal{D} = \frac{L_f}{\sqrt{2}\pi|y^+|}. \tag{7.6}$$

An easy way to arrive at (7.5) is to translate the Zamolodchikov metric (7.4) into the usual metric on the space of metrics [56]. The marginal operator (7.3) can be written in a general form

$$\mathcal{O}_L \delta L = \frac{1}{4\pi\alpha'} \delta G_{\mu\nu} \; (\partial_\tau Y^\mu \partial_\tau Y^\nu - \partial_\sigma Y^\mu \partial_\sigma Y^\nu), \tag{7.7}$$

where $\mu, \nu = 2, 3$ and where we have defined $Y^2 = y$ and $Y^3 = \psi$. The coordinates y and ψ, defined in (2.9), are linear combinations of the cartesian coordinates x and χ whose coefficients depend on $\tau = x^+$. We can express this fact as

$$Y^\mu(\sigma, \tau) = e^\mu_\alpha(\tau) X^\alpha(\sigma, \tau), \tag{7.8}$$

where $\alpha = 2, 3$ and $X^2 = x, X^3 = \chi$. This allows us to rewrite the expectation value of interest as

$$\begin{aligned}\langle \mathcal{O}_L(\sigma, \tau) \mathcal{O}_L(\sigma + \epsilon, \tau) \rangle \; (\delta L)^2 = \frac{1}{(4\pi\alpha')^2} \delta G_{\mu\nu} \; e^\mu_\alpha(\tau) e^\nu_\beta(\tau) \; \delta G_{\mu'\nu'} \; e^{\mu'}_{\alpha'}(\tau) e^{\nu'}_{\beta'}(\tau) \; \times \\ \times \; \langle [\partial_\tau X^\alpha \partial_\tau X^\beta - \partial_\sigma X^\alpha \partial_\sigma X^\beta]|_{\sigma,\tau} [\partial_\tau X^{\alpha'} \partial_\tau X^{\beta'} - \partial_\sigma X^{\alpha'} \partial_\sigma X^{\beta'}]|_{\sigma+\epsilon,\tau} \rangle + \ldots\end{aligned} \tag{7.9}$$

The dots here denote terms containing τ-derivatives of various e^μ_α. It is important to note that such terms do not contribute to the limit (7.4) and can be ignored. Evaluation of the expectation values on the right-hand side of (7.9) is identical to the standard calculation in flat Minkowski space, giving

$$\begin{aligned}\langle [\partial_\tau X^\alpha \partial_\tau X^\beta - \partial_\sigma X^\alpha \partial_\sigma X^\beta]|_{\sigma,\tau} [\partial_\tau X^{\alpha'} \partial_\tau X^{\beta'} - \partial_\sigma X^{\alpha'} \partial_\sigma X^{\beta'}]|_{\sigma+\epsilon,\tau} \rangle \\ = \frac{2\alpha'^2}{\epsilon^4} (\eta^{\alpha\alpha'} \eta^{\beta\beta'} + \eta^{\alpha\beta'} \eta^{\alpha'\beta}).\end{aligned} \tag{7.10}$$

The Zamolodchikov metric then becomes

$$\mathcal{G}_{L,L} \; \delta L \; \delta L = \frac{1}{4\pi^2} \; \delta G_{\mu\nu} \; e^\mu_\alpha \eta^{\alpha\alpha'} e^{\mu'}_{\alpha'} \; \delta G_{\mu'\nu'} \; e^{\mu'}_\beta \eta^{\beta\beta'} e^{\nu'}_{\beta'}, \tag{7.11}$$

or

$$\mathcal{G}_{L,L} \; \delta L \; \delta L = \frac{1}{4\pi^2} \; \delta G_{\nu\mu} \; G^{\mu\mu'} \; \delta G_{\mu'\nu'} \; G^{\nu'\nu}. \tag{7.12}$$

Plugging

$$G_{\mu\nu} = \begin{pmatrix} \tau^2 + \frac{L^2}{4\pi^2} & \frac{L}{2\pi} \\ \frac{L}{2\pi} & 1 \end{pmatrix}, \; G^{\mu\nu} = \frac{1}{\tau^2} \begin{pmatrix} 1 & -\frac{L}{2\pi} \\ -\frac{L}{2\pi} & \frac{L^2}{4\pi^2} \end{pmatrix}, \; \delta G_{\mu\nu} = \begin{pmatrix} \frac{L}{2\pi^2} & \frac{1}{2\pi} \\ \frac{1}{2\pi} & 0 \end{pmatrix} \delta L \tag{7.13}$$

into (7.12), we get,

$$\mathcal{G}_{L,L} \; \delta L \; \delta L = \frac{(\delta L)^2}{2\pi^2 \tau^2}. \tag{7.14}$$

Upon translating from light-cone gauge, $y^+ = \tau$, this gives (7.5).

The reader may be puzzled that even in the completely smooth orbifold $\mathbb{R}^{1,3}/\Gamma_L$ we have found a quantity which diverges near $y^+ = 0$. This is an artifact of our choice of coordinates on field space. If we chose instead to modify L using the coordinates x^+, x, x^- (2.1), (2.2), and $\widetilde{\chi} = \chi/L$, we would write

$$\widetilde{\mathcal{O}}_L \delta L = \frac{1}{2\pi\alpha'}[(\partial_\tau \widetilde{\chi})^2 - (\partial_\sigma \widetilde{\chi})^2] L \; \delta L. \tag{7.15}$$

For the metric on field space we find (just like in the case of changing the size of an ordinary Kaluza-Klein circle in string theory),

$$\widetilde{\mathcal{G}}_{L,L} \delta L \delta L \sim \frac{(\delta L)^2}{L^2}. \tag{7.16}$$

Therefore, for any nonzero L *or* nonzero x^+, we have found a fluctuating mode which turns on the shift. However, when both $L = 0$ and $x^+ = 0$, the kinetic term for any such mode diverges, and the fluctuations freeze out. Further, as with any physical description which uses constant-value surfaces of a null coordinate x^+ as initial-data slices, objects with $p^+ = 0$ are subtle. The singularity at $x^+ = 0$ is such an object. Begin with any spacelike hypersurface on which to specify initial data with $L = 0$. If we try to extend it as much as possible, it will inevitably touch the locus $x^+ = 0$. At $x^+ = 0$ the L-mode which resolves the orbifold does not fluctuate. Therefore with this initial data, L will continue to vanish along $x^+ = 0$, and the singularity will persist.

We should point out that our discussion above is not completely conclusive. The most important issue is whether for $L = 0$ we are allowed to do any perturbative calculations near the singularity at all. It is natural to expect that in the near singularity region any string probe would cause a large backreaction and make the singularity locally space-like, in the spirit of [10]. For this reason, the applicability of CFT techniques is questionable if L and $|x^+|$ are both small.

8. Replacing the $\mathbb{R}^{1,3}/\Gamma_0$ singularity with a sandwich wave

In this section, we would like to ask whether it is possible to have a smooth future evolution of the geometry if we specify the initial data on the slice of a constant light-cone time y^+ to be exactly those of the (singular) $\mathbb{R}^{1,3}/\Gamma_0$ orbifold. In other words, is the presence of the singularity inevitable once such initial data on $y^+ = y^+_{in} < 0$ are given?

As we will see, the answer is that the singularity at $y^+ = 0$ can be evaded easily (if $|y^+_{in}| \gg \sqrt{\alpha'}$) and we will construct the corresponding gravitational solutions. The $y^+ = y^+_{in}$ hypersurface is not a Cauchy surface, and of course, there can be gravitational waves coming from the region where the data have not been specified.

The appropriate classical gravitational solutions are smooth orbifolds of a certain type of four-dimensional Ricci-flat "sandwich waves" [57] described in [58] (times $\mathbb{R}^6$). These waves belong to the family of plane-fronted waves with parallel rays (pp-waves), as can be seen from [59]. They consist of three regions: (I) $y^+ < A$, where the spacetime is exactly flat, (II) $y^+ \in (A, B)$ with the Riemann tensor non-vanishing, and (III) $y^+ > B$, where the spacetime is flat again. The support of the Riemann tensor is therefore perfectly localized between A and B.

We will parametrize the geometry in all the three regions by one set of coordinates. In addition to y^+, we will also use coordinates y^-, y and $\widehat{\chi}$. The metric in region II can be written as

$$ds_{II}^2 = -2dy^+dy^- + N_y^2 \cosh^2[\Omega(y^+ - M)]dy^2 + N_{\widehat{\chi}}^2 \cos^2[\Omega(y^+ - M)]d\widehat{\chi}^2, \tag{8.1}$$

where $M, N_y, N_{\widehat{\chi}}$ and Ω are some constants. This metric can be matched onto flat Minkowski space along $y^+ = A$ and $y^+ = B$. In order to do so, one has to perform a coordinate change to obtain the following form of the Minkowski metric in regions I and III

$$\begin{aligned} ds_I^2 &= -2dy^+dy^- + (\alpha_I + y^+\beta_I)^2 dy^2 + (\gamma_I + y^+\delta_I)^2 d\widehat{\chi}^2 \\ ds_{III}^2 &= -2dy^+dy^- + (\alpha_{III} + y^+\beta_{III})^2 dy^2 + (\gamma_{III} + y^+\delta_{III})^2 d\widehat{\chi}^2. \end{aligned} \tag{8.2}$$

The appropriate coordinate transformation relating (8.2) and

$$ds^2 = -2dUdV + (d\widetilde{X}^2)^2 + (d\widetilde{X}^3)^2 \tag{8.3}$$

is

$$\begin{aligned} U &= y^+ \\ V &= y^- + \frac{1}{2}\beta(\alpha + y^+\beta)y^2 + \frac{1}{2}\delta(\gamma + y^+\delta)\widehat{\chi}^2 \\ \widetilde{X}^2 &= (\alpha + y^+\beta)y \\ \widetilde{X}^3 &= (\gamma + y^+\delta)\widehat{\chi} \end{aligned} \tag{8.4}$$

The matching conditions requiring that the metric and its first derivatives are smooth lead to certain simple algebraic relations between the coefficients in (8.1) and (8.2), which we will not write down explicitly. In addition, there is a constraint resulting from the requirement that the metric have no coordinate singularity in region II.

Intuitively, the distance between two points separated by a constant coordinate distance in y changes linearly with y^+ in region I, then it follows a hyperbolic cosine in region II, and finally in region III, it changes linearly again. (See fig. 1.) A similar statement is true for the $\widehat{\chi}$ coordinate, with the cosine not being hyperbolic.

Note that the metric (8.1), (8.2) in all the three regions is invariant under constant shifts of y. This allows us to consider a **Z** orbifold with the orbifold group action generated by $y \to y + 2\pi$,

$$(y^+, y, y^-, \widehat{\chi}) \sim (y^+, y + 2\pi, y^-, \widehat{\chi}). \tag{8.5}$$

In particular for region I, we may choose $\alpha_I = 0$ and $\beta_I = 1$:

$$ds_I^2 = -2dy^+dy^- + (y^+)^2 dy^2 + (\gamma_I + y^+\delta_I)^2 d\widehat{\chi}^2. \tag{8.6}$$

Using a coordinate transformation in the spirit of (8.4), we can write this also as

$$ds_I^2 = -2dy^+dy^- + (y^+)^2 dy^2 + d\chi^2. \tag{8.7}$$

In this way, we obtain exactly the same metric and identifications as in the case of the $\mathbb{R}^{1,3}/\Gamma_0$ orbifold, expressed in terms of coordinates y^+, y, y^-, χ defined by (2.1) and (2.9) with $L = 0$.

If we start with (8.7) (or equivalently with (8.6)), what will be the geometry after the wave passes, i.e. in region III? This depends on the strength of the wave we choose and on the light-cone time B where the wave is glued to flat space.

If we choose $B - A$ small, the y-circle will continue to shrink after the wave passes (slower than before), and eventually it will form a singularity. On the other hand, an appropriate choice of parameters ($B > M$) can make the y-circle expand again without going through a singularity. This can be seen as follows.

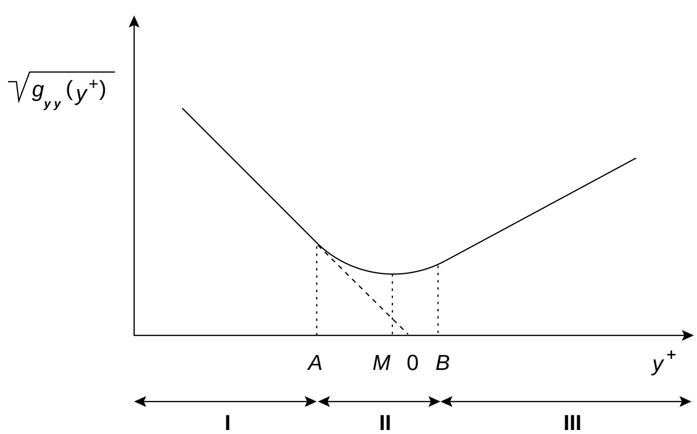

Fig. 1: The circumference of the y-circle is proportional to $\sqrt{g_{yy}}$. In the non-singular case it first linearly contracts just like in the $\mathbb{R}^{1,3}/\Gamma_0$ orbifold, then it follows a hyperbolic cosine, and eventually, it linearly expands (or stays constant, in the marginal case).

The matching conditions at $y^+ = A$ ($A < 0$) imply

$$\Omega A \tanh[\Omega(A - M)] = 1, \quad N_y^2 = \frac{A^2}{\cosh^2[\Omega(A - M)].} \tag{8.8}$$

This means that if we choose some definite value of $\Omega > 1/|A|$, the value of $A - M$ and N_y^2 will be uniquely determined. In addition, in order to avoid $\widehat{\chi}$-coordinate singularities between $y^+ = A$ and $y^+ = M$, we need to satisfy

$$\Omega|A - M| < \frac{\pi}{2}, \tag{8.9}$$

which using (8.8) is equivalent to

$$\Omega > \frac{1}{\tanh(\frac{\pi}{2})|A|}. \tag{8.10}$$

The condition that the null singularity is removed by the sandwich wave is that function $\sqrt{g_{yy}^{III}} = |\alpha_{III} + y^+ \beta_{III}|$ not vanish inside region III. If we make the natural choice $\alpha_{III} + \beta_{III} B > 0$, this implies that $\beta_{III} \geq 0$. By the matching condition at $y^+ = B$ (analogous to a combination of equations (8.8))

$$N_y^2 \sinh[\Omega(B - M)] \cosh[\Omega(B - M)] = (\alpha_{III} + \beta_{III} B) \beta_{III},$$

the requirement $\beta_{III} \geq 0$ is equivalent to $B \geq M$. Of course, this can also be seen very intuitively from fig. 1.

We see that if we choose Ω consistent with (8.9), determine M and N_y^2 from (8.8), and take $B > M$ (and $\Omega(B-M) < \pi/2$ to avoid $\widehat{\chi}$-coordinate singularities inside $y^+ \in [M, B]$), we obtain a wave where the circumference of the y-circle first shrinks and then it expands without ever vanishing. The necessary curvature scale is at least of order $1/|A|$. For the marginal choice $B = M$, the light-cone time future of the geometry will be a static Kaluza-Klein compactification on a circle.

9. Conclusion

In this paper, we have studied a completely smooth time-dependent orbifold of Minkowski space. Strings in this background are very well-behaved, and the string amplitudes appear to define a set of S-matrix observables. In particular, perturbative corrections to the background are well-controlled by the value of the dilaton at large $|x^+|$. This is true even in certain nonsupersymmetric generalizations, with large enough $L \gg \sqrt{\alpha'}$. Such cases seem to provide interesting and simple examples of time-dependent backgrounds where nonperturbative instabilities such as bubbles of nothing [39] can be the leading effects.[4]

Acknowledgements

We would like to thank Allan Adams, Jiří Bičák, Steve Giddings, Matthew Headrick, Simeon Hellerman, Matthew Kleban, Albion Lawrence, Xiao Liu, Liam McAllister, Luboš Motl, Mohammad Sheikh-Jabbari, Steve Shenker, and Eva Silverstein for very useful and helpful discussions. J.M. is grateful to the Harvard Theory Group for hospitality. The work of M.F. was supported by the DOE under contract DE-AC03-76SF00515, by the A.P. Sloan Foundation, and by a Stanford Graduate Fellowship. The work of J.M. was supported in part by National Science Foundation grant PHY00-97915.

[4] A similar situation for time-independent orbifolds was found in [28].

References

[1] E. Witten, "(2+1)-Dimensional Gravity As An Exactly Soluble System," Nucl. Phys. B **311**, 46 (1988).

[2] G. T. Horowitz and A. R. Steif, "Singular String Solutions With Nonsingular Initial Data," Phys. Lett. B **258**, 91 (1991).

[3] A. A. Tseytlin, "Exact string solutions and duality," arXiv:hep-th/9407099; C. Klimčík and A. A. Tseytlin, unpublished (1994); A. A. Tseytlin, unpublished (2001).

[4] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt and N. Turok, "From big crunch to big bang," arXiv:hep-th/0108187; N. Seiberg, "From big crunch to big bang - is it possible?," arXiv:hep-th/0201039.

[5] L. Cornalba and M. S. Costa, "A New Cosmological Scenario in String Theory," arXiv:hep-th/0203031.

[6] N. A. Nekrasov, "Milne universe, tachyons, and quantum group," hep-th/0203112.

[7] J. Figueroa-O'Farrill and J. Simón, "Generalized supersymmetric fluxbranes," JHEP **0112**, 011 (2001) [arXiv:hep-th/0110170]

[8] J. Simón, "The geometry of null rotation identifications," arXiv:hep-th/0203201.

[9] H. Liu, G. Moore and N. Seiberg, "Strings in a time-dependent orbifold," arXiv:hep-th/0204168.

[10] A. Lawrence, "On the instability of 3D null singularities," arXiv:hep-th/0205288.

[11] V. Balasubramanian, S. F. Hassan, E. Keski-Vakkuri and A. Naqvi, "A space-time orbifold: A toy model for a cosmological singularity," arXiv:hep-th/0202187.

[12] D. Amati and C. Klimčík, "Nonperturbative Computation Of The Weyl Anomaly For A Class Of Nontrivial Backgrounds," Phys. Lett. B **219**, 443 (1989).

[13] G. T. Horowitz and A. R. Steif, "Space-Time Singularities In String Theory," Phys. Rev. Lett. **64**, 260 (1990).

[14] H. J. de Vega and N. Sanchez, "Space-Time Singularities In String Theory And String Propagation Through Gravitational Shock Waves," Phys. Rev. Lett. **65**, 1517 (1990).

[15] C. R. Nappi and E. Witten, "A Closed, expanding universe in string theory," Phys. Lett. B **293**, 309 (1992) [arXiv:hep-th/9206078].

[16] C. R. Nappi and E. Witten, "A WZW model based on a nonsemisimple group," Phys. Rev. Lett. **71**, 3751 (1993) [arXiv:hep-th/9310112].

[17] E. Kiritsis and C. Kounnas, "String Propagation In Gravitational Wave Backgrounds," Phys. Lett. B **320**, 264 (1994) [Addendum-ibid. B **325**, 536 (1994)] [arXiv:hep-th/9310202].

[18] E. Kiritsis, C. Kounnas and D. Lüst, "Superstring gravitational wave backgrounds with space-time supersymmetry," Phys. Lett. B **331**, 321 (1994) [arXiv:hep-th/9404114].

[19] E. Silverstein, "(A)dS backgrounds from asymmetric orientifolds," [arXiv:hep-th/0106209].

[20] I. I. Kogan and N. B. Reis, "H-branes and chiral strings," Int. J. Mod. Phys. A **16**, 4567 (2001) [arXiv:hep-th/0107163].

[21] A. Adams, J. Polchinski and E. Silverstein, "Don't panic! Closed string tachyons in ALE space-times," JHEP **0110**, 029 (2001) [arXiv:hep-th/0108075].

[22] P. Berglund, T. Hübsch and D. Minic, "de Sitter spacetimes from warped compactifications of IIB string theory," arXiv:hep-th/0112079.

[23] M. Gutperle and A. Strominger, "Spacelike branes," JHEP **0204**, 018 (2002) [arXiv:hep-th/0202210].

[24] A. Buchel, "Gauge / gravity correspondence in accelerating universe," arXiv:hep-th/0203041.

[25] A. Sen, "Rolling tachyon," JHEP **0204**, 048 (2002) [arXiv:hep-th/0203211].

[26] C. M. Chen, D. V. Gal'tsov and M. Gutperle, "S-brane solutions in supergravity theories," arXiv:hep-th/0204071.

[27] M. Kruczenski, R. C. Myers and A. W. Peet, "Supergravity S-branes," JHEP **0205**, 039 (2002) [arXiv:hep-th/0204144].

[28] O. Aharony, M. Fabinger, G. Horowitz, and E. Silverstein, "Clean Time-Dependent String Backgrounds from Bubble Baths," arXiv:hep-th/0204158.

[29] S. Elitzur, A. Giveon, D. Kutasov and E. Rabinovici, "From big bang to big crunch and beyond," arXiv:hep-th/0204189.

[30] L. Cornalba, M. S. Costa and C. Kounnas, "A resolution of the cosmological singularity with orientifolds," arXiv:hep-th/0204261.

[31] B. Craps, D. Kutasov and G. Rajesh, "String propagation in the presence of cosmological singularities," arXiv:hep-th/0205101.

[32] S. Roy, "On supergravity solutions of space-like Dp-branes," arXiv:hep-th/0205198.

[33] S. Kachru and L. McAllister, "Bouncing brane cosmologies from warped string compactifications," arXiv:hep-th/0205209.

[34] A. Maloney, E. Silverstein, and A. Strominger, "de Sitter Space in Non-Critical String Theory," arXiv:hep-th/0205316.

[35] N. S. Deger and A. Kaya, "Intersecting S-Brane Solutions of D=11 Supergravity," arXiv:hep-th/0206057.

[36] H. Liu, G. Moore and N. Seiberg, to appear at the same time as this paper.

[37] H. Liu, G. Moore, and N. Seiberg, to appear.

[38] R. Rohm, "Spontaneous supersymmetry breaking in supersymmetric string theories," Nucl. Phys. B **237**, 553 (1984).

[39] E. Witten, "Instability Of The Kaluza-Klein Vacuum," Nucl. Phys. B **195**, 481 (1982).

[40] F. Dowker, J. P. Gauntlett, G. W. Gibbons and G. T. Horowitz, "Nucleation of P-Branes and Fundamental Strings," Phys. Rev. D **53**, 7115 (1996) [arXiv:hep-th/9512154]

[41] F. Dowker, J. P. Gauntlett, G. W. Gibbons and G. T. Horowitz, "The Decay of magnetic fields in Kaluza-Klein theory," Phys. Rev. D **52**, 6929 (1995) [arXiv:hep-th/9507143]

[42] H. A. Shinkai and T. Shiromizu, "Fate of Kaluza-Klein bubble," Phys. Rev. D **62**, 024010 (2000) [arXiv:hep-th/0003066]

[43] M. Fabinger and P. Hořava, "Casimir effect between world-branes in heterotic M-theory," Nucl. Phys. B **580**, 243 (2000) [arXiv:hep-th/0002073]

[44] G. T. Horowitz and L. Susskind, "Bosonic M theory," J. Math. Phys. **42**, 3152 (2001) [arXiv:hep-th/0012037]

[45] S. P. De Alwis and A. T. Flournoy, "Closed string tachyons and semi-classical instabilities," arXiv:hep-th/0201185

[46] D. Birmingham and M. Rinaldi, "Bubbles in anti-de Sitter space," arXiv:hep-th/0205246

[47] V. Balasubramanian and S. F. Ross, "The dual of nothing," arXiv:hep-th/0205290

[48] A. A. Tseytlin, "Melvin solution in string theory," Phys. Lett. B **346**, 55 (1995) [arXiv:hep-th/9411198].

[49] J. G. Russo and A. A. Tseytlin, "Magnetic flux tube models in superstring theory," Nucl. Phys. B **461**, 131 (1996) [arXiv:hep-th/9508068].

[50] M. S. Costa and M. Gutperle, "The Kaluza-Klein Melvin solution in M-theory," JHEP **0103**, 027 (2001) [arXiv:hep-th/0012072].

[51] M. Gutperle and A. Strominger, "Fluxbranes in string theory," JHEP **0106**, 035 (2001) [arXiv:hep-th/0104136].

[52] J. G. Russo and A. A. Tseytlin, "Magnetic backgrounds and tachyonic instabilities in closed superstring theory and M-theory," Nucl. Phys. B **611**, 93 (2001) [arXiv:hep-th/0104238].

[53] L. Motl, "Melvin matrix models," arXiv:hep-th/0107002.

[54] A. A. Tseytlin, "Magnetic backgrounds and tachyonic instabilities in closed string theory," arXiv:hep-th/0108140.

[55] J. R. David, M. Gutperle, M. Headrick and S. Minwalla, "Closed string tachyon condensation on twisted circles," JHEP **0202**, 041 (2002) [arXiv:hep-th/0111212].

[56] P. Candelas, T. Hübsch and R. Schimmrigk, "Relation Between The Weil-Petersson And Zamolodchikov Metrics," Nucl. Phys. B **329**, 583 (1990).

[57] H. Bondi, F. A. E. Pirani and I. Robinson, "Gravitational waves in general relativity III. Exact plane waves," Proc. Roy. Soc. **A251** (1959) 519;

[58] W. Rindler, "Essential Relativity", Revised *Second* Edition, Springer-Verlag, New York, 1977.

[59] J. Ehlers and W. Kundt, "Exact Solutions of the Gravitational Field Equations," in "Gravitation: an introduction to current research," ed. L. Witten, New York, Wiley (1962). See also C. Misner, K. S. Thorne and J. A. Wheeler, "Gravitation," San Francisco, W. H. Freeman (1973), Exercise 35.8.