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$\mathcal{O}(\alpha^2 \ln(m_\mu/m_e))$ Corrections to Electron Energy Spectrum in Muon Decay

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$\mathcal{O}(\alpha^2 \ln(m_\mu/m_e))$ corrections to electron energy spectrum in muon decay are computed using perturbative fragmentation function approach. The magnitude of these corrections is comparable to anticipated precision of the TWIST experiment at TRIUMF where Michel parameters will be extracted from the measurement of the electron energy spectrum in muon decay.

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I. INTRODUCTION

The muon decay into an electron and a pair of neutrinos, $\mu \to e\nu_\mu\bar{\nu}_e$, is a classical process in particle physics. Although the high energy frontier has moved up from the energy scales comparable to the muon mass, precision physics of muons remains an interesting and inspiring source of information about the Standard Model (SM) and its possible extensions [1].

Among very different experiments with muons that include the measurement of muon anomalous magnetic moment, muon lifetime, $\mu \to e\gamma$ branching ratio, and the muon to electron conversion rate in muonic atoms, our attention will be focused on the TWIST experiment [2, 3] at TRIUMF, where the electron energy spectrum in muon decays is planned to be measured to determine Michel parameters with the precision of $\sim 10^{-4}$. In order to confront these measurements with the SM predictions and look for the signs of New Physics, one needs an adequately accurate calculation of the electron energy spectrum within the SM.

Calculations of the electron energy spectrum in muon decay have a long and interesting history that dates back to the very early days of QED and the physics of weak interactions (see e.g. Ref. [4] for historical recollection). In spite of tremendous progress in precision calculations, the $\mathcal{O}(\alpha^2)$ radiative corrections to total muon lifetime have been computed only recently [5], and the calculation of similar corrections to electron energy spectrum has even not been attempted. One reason for this is that, in contrast to total lifetime, the electron energy spectrum in muon decay can not be computed for the vanishing mass of the electron since terms enhanced by the large logarithm $\ln(m_\mu/m_e)$ are present. These terms (excluding the ones that are related to the on–shell definition of the fine–structure constant commonly used in QED) cancel out in the calculation of the total rate which, for this reason, becomes somewhat simpler.

At order $\mathcal{O}(\alpha^2)$, corrections to the spectrum have double–logarithmic $\mathcal{O}(\ln^2(m_\mu^2/m_e^2))$ and single–logarithmic $\mathcal{O}(\ln(m_\mu^2/m_e^2))$ enhanced terms and it is the purpose of this Letter to present the calculation of those. The double–logarithmic terms were computed recently in Ref. [6]. It was pointed out there that the single–logarithmic $\mathcal{O}(\ln(m_\mu^2/m_e^2))$ terms are required to match the precision of the TWIST experiment. Motivated by these considerations, we decided to perform this calculation. In order to accomplish this, we make use of perturbative fragmentation function approach borrowed to a large extent from QCD studies of heavy quark fragmentation in e^+e^- collisions.

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II. PRELIMINARIES

According to QCD factorization theorems [7], which can be transformed to QED in a straightforward way, a differential cross section for producing a particle of a given type with a certain fraction of the initial energy can be written as a convolution of the hard scattering cross section computed with massless partons and the fragmentation function that describes the probability that a massless parton of a given type fragments to the physical particle in the final state that is being observed.

If one considers the process in which a heavy quark (i.e. $m_Q \gg \Lambda_{\rm QCD}$) is produced with large energy and its energy is measured, one can identify the *massive* quark with the physical particle in the final state, in the sense of the preceding discussion. It has been shown in QCD that in this case the perturbative fragmentation function can be defined and that this function absorbs all the terms that are singular in the limit when the mass of the heavy quark goes to zero [8, 9, 10].

Specializing to the case of muon decay, and working along these ideas, it is possible to write down the formula for the electron energy spectrum in muon decay in the following way:

$$\frac{{\rm d}\Gamma}{{\rm d}x}(x,m_\mu,m_e) = \sum_{j=e,\gamma} \int\limits_x^1 \frac{{\rm d}z}{z}\, \frac{{\rm d}\hat{\Gamma}_j}{{\rm d}z}(z,m_\mu,\mu_f)\mathcal{D}_j\left(\frac{x}{z},\mu_f,m_e\right), \tag{1}$$

where $z = 2E/m_\mu$ is the fraction of energy carried away by a parton j in muon decay, x is the same quantity for the observed physical massive electron, ${\rm d}\hat{\Gamma}_j/{\rm d}z$ is the energy distribution of the massless parton of type j in muon decay computed in the $\overline{\rm MS}$ scheme, $\mathcal{D}_j$ is the fragmentation function of the parton j fragmenting into the massive electron, and μ_f stands for the factorization scale. Note that the terms that are suppressed by the ratio of the electron to muon masses, m_e^2/m_μ^2, can not be described by Eq. (1). However, since these terms are known, both for Born and $\mathcal{O}(\alpha)$ corrected electron energy spectrum [11, 12], Eq. (1) is quite adequate for anticipated level of experimental precision.

As we mentioned earlier, partonic decay rate ${\rm d}\hat{\Gamma}_j/{\rm d}z$ has to be computed in the $\overline{\rm MS}$ scheme. This is important in that this requirement goes beyond standard ultraviolet renormalization, since ${\rm d}\hat{\Gamma}_j/{\rm d}z$ is not finite because of collinear singularities. These singularities are removed from ${\rm d}\hat{\Gamma}_j/{\rm d}z$ by conventional renormalization in the $\overline{\rm MS}$ scheme and large collinear logarithms associated with them are absorbed into the fragmentation function $\mathcal{D}_j$.

The perturbative expansion for the energy distribution of the massless partons reads

$$\frac{1}{\Gamma_0}\frac{{\rm d}\hat{\Gamma}_j}{{\rm d}z}(z,m_\mu,\mu_f) = A_j^{(0)}(z) + \frac{\bar{\alpha}(\mu_f)}{2\pi}\hat{A}_j^{(1)}(m_\mu,\mu_f,z) + \left(\frac{\bar{\alpha}(\mu_f)}{2\pi}\right)^2 \hat{A}_j^{(2)}(m_\mu,\mu_f,z), \tag{2}$$

where $\Gamma_0 = G_F^2 m_\mu^5/(96\pi^3)$, $A_j^{(0)}(z) = z^2(3-2z)\delta_{je}$, $\bar{\alpha}(\mu_f)$ is the $\overline{\rm MS}$ renormalized fine structure constant, and we have neglected terms of order $\mathcal{O}(\alpha^3)$ and higher. The $\overline{\rm MS}$ fine structure constant will be later converted into the on–shell coupling fine structure constant $\alpha \approx 1/137.036$.

Before giving explicit expressions for perturbative coefficients, we would like to describe a simple idea (previously exploited in QCD studies) that allows us to compute the $\alpha^2 \ln(m_\mu/m_e)$ enhanced terms without doing explicit two–loop calculation. Since ${\rm d}\hat{\Gamma}_j/{\rm d}z$ is computed for massless partons, it has just two energy scales, the mass of the muon m_μ and the factorization scale μ_f. This means that the only logarithms that arise there are the logarithms of the form $\ln(m_\mu/\mu_f)$. Therefore, by choosing $\mu_f \sim m_\mu$, we effectively eliminate large logarithms from perturbative coefficients of the energy spectrum computed with massless partons and move all the large logarithms to the fragmentation function $\mathcal{D}_j$. On the other hand, the fact that the fragmentation function is process–independent and also satisfies the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) evolution equation as far as its dependence on the factorization scale is concerned, allows us to use some formulas known from previous QCD studies and compute large logarithmic corrections to the electron energy spectrum at order $\mathcal{O}(\alpha^2)$ without performing explicit two–loop calculations.

Let us consider the fragmentation function $\mathcal{D}_j(x,\mu_f,m_e)$ which describes the probability that the massless parton j converts into the physical electron of the mass m_e. This function satisfies the DGLAP evolution equation:

$$\frac{{\rm d}\mathcal{D}_i(x,\mu_f,m_e)}{{\rm d}\ln\mu_f^2} = \sum_j \int\limits_x^1 \frac{{\rm d}z}{z} P_{ji}(z,\bar{\alpha}(\mu_f))\mathcal{D}_j\left(\frac{x}{z},\mu_f,m_e\right), \tag{3}$$

where P_{ji} stands for usual time–like splitting functions which, to the order we work to, can be written as

$$P_{ji}(x,\bar{\alpha}(\mu_f)) = \frac{\bar{\alpha}(\mu_f)}{2\pi}P_{ji}^{(0)}(x) + \left(\frac{\bar{\alpha}(\mu_f)}{2\pi}\right)^2 P_{ji}^{(1)}(x) + \mathcal{O}\left(\bar{\alpha}^3\right). \tag{4}$$

Eq. (3) can be solved as power series in $\bar{\alpha}$ if the initial condition for the function $\mathcal{D}_j$ at some scale μ_0 is provided. This initial condition can be obtained from QCD studies of the heavy quark fragmentation [8] which, when translated to QED, imply that the fragmentation of the massless electron into the physical electron is described by

$$\begin{aligned} \mathcal{D}_e^{\rm ini}(x,\mu_0,m_e) &= \delta(1-x) + \frac{\bar{\alpha}(\mu_0)}{2\pi} d_1(x,\mu_0,m_e) + \mathcal{O}\left(\alpha^2\right), \\ d_1(x,\mu_0,m_e) &\equiv d_1(x) = \left[\frac{1+x^2}{1-x}\left(\ln\frac{\mu_0^2}{m_e^2} - 2\ln(1-x) - 1\right)\right]_+, \end{aligned} \tag{5}$$

and the fragmentation function of the photon fragmenting into the physical electron is

$$\mathcal{D}_\gamma^{\rm ini}(x,\mu_0,m_e) = \frac{\bar{\alpha}(\mu_0)}{2\pi}\left(x^2+(1-x)^2\right)\ln\frac{\mu_0^2}{m_e^2} + \mathcal{O}\left(\alpha^2\right). \tag{6}$$

As will be clear from the following discussion, the $\mathcal{O}(\alpha^2)$ terms in the initial conditions for fragmentation functions are not needed for our purposes since our choice of the initial scale $\mu_0 \sim m_e$ guarantees that no large logarithms appear in the initial condition for $\mathcal{D}_j$. One should also notice that at the lowest order $\mathcal{O}(\alpha^0)$ the fragmentation function does not contain large logarithm and, for this reason, the second order coefficient in $\mathrm{d}\hat{\Gamma}_j/\mathrm{d}z$ is not needed as well. On the other hand, the order $\mathcal{O}(\alpha)$ coefficients in $\mathrm{d}\hat{\Gamma}_j/\mathrm{d}z$ have to be known exactly and they read

$$\begin{aligned} \hat{A}_e^{(1)}(z) &= \left(2z^2(2z-3)\ln\left[\frac{z}{1-z}\right] + 2z + \frac{8}{3}z^3 + \frac{5}{6} - 4z^2\right)\ln\left(\frac{m_\mu^2}{\mu_f^2}\right) \\ &+ 2z^2(2z-3)\left(4\zeta_2 - 4\mathrm{Li}_2(z) + 2\ln^2 z - 3\ln z\ln(1-z) - \ln^2(1-z)\right) \\ &+ \left(\frac{5}{3} - 2z - 13z^2 + \frac{34}{3}z^3\right)\ln(1-z) + \left(\frac{5}{3} + 4z - 2z^2 - 6z^3\right)\ln z \\ &+ \frac{5}{6} - \frac{23}{3}z - \frac{3}{2}z^2 + \frac{7}{3}z^3, \end{aligned} \tag{7}$$

$$\begin{aligned} \hat{A}_\gamma^{(1)}(z) &= \left(\ln\frac{m_\mu^2}{\mu_f^2} + \ln(1-z)\right)\left(\frac{1}{z} - \frac{5}{3} + 2z - 2z^2 + \frac{2}{3}z^3\right) + \ln z\left(\frac{2}{z} - \frac{10}{3} + 4z\right) \\ &- \frac{1}{z} + \frac{1}{3} + \frac{35}{12}z - 2z^2 - \frac{1}{4}z^3. \end{aligned} \tag{8}$$

With these preliminary remarks out of the way, we can proceed with the computation of the fragmentation function and use it to calculate the electron energy spectrum in muon decay.

III. FRAGMENTATION FUNCTION

In this Section we discuss the computation of the fragmentation function. For this purpose, we need to solve the DGLAP equation (3) in a way consistent with the initial conditions. Since we solve this equation perturbatively, we can express the running fine structure constant in the $\overline{\rm MS}$ scheme through the fine structure constant defined in the on–shell scheme, because the use of the on–shell renormalization scheme is the common practice when dealing with QED corrections. To order $\mathcal{O}(\alpha^2)$, the well known relation between the $\overline{\rm MS}$ and the on–shell coupling constants reads

$$\bar{\alpha}(\mu_f) = \alpha + \frac{\alpha^2}{3\pi}\ln\frac{\mu_f^2}{m_e^2}. \tag{9}$$

Solving Eq. (3) by iterations, we obtain

$$\begin{aligned} \mathcal{D}_e(x,\mu_f,m_e) &= \delta(1-x) + \frac{\alpha}{2\pi}\left(LP_{ee}^{(0)}(x) + d_1(x,\mu_0,m_e)\right) + \left(\frac{\alpha}{2\pi}\right)^2\left(L^2\left[\frac{1}{2}P_{ee}^{(0)}\otimes P_{ee}^{(0)}(x)\right.\right. \\ &\left.\left. + \frac{1}{3}P_{ee}^{(0)}(x) + \frac{1}{2}P_{\gamma e}^{(0)}\otimes P_{e\gamma}^{(0)}(x)\right] + L\left[P_{ee}^{(0)}\otimes d_1(x) + P_{ee}^{(1)}(x)\right]\right) + \mathcal{O}\left(\alpha^2 L^0, \alpha^3\right), \end{aligned} \tag{10}$$

$$\mathcal{D}_\gamma(x,\mu_f,m_e) = \frac{\alpha}{2\pi}LP_{e\gamma}^{(0)}(x) + \mathcal{O}\left(\alpha^2\right), \tag{11}$$

where $L = \ln(\mu_f^2/\mu_0^2)$, and the convolution operation is defined in the standard way:

$$A \otimes B(x) = \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}z' \; \delta(x - zz') A(z) B(z') = \int_x^1 \frac{\mathrm{d}z}{z} \; A(z) B\left(\frac{x}{z}\right). \tag{12}$$

We now give explicit expressions for the splitting functions used in Eqs. (10,11). At leading order they read

$$P_{ee}^{(0)}(x) = \left[\frac{1+x^2}{1-x}\right]_+, \qquad P_{\gamma e}^{(0)}(x) = \frac{1+(1-x)^2}{x}, \qquad P_{e\gamma}^{(0)}(x) = x^2 + (1-x)^2. \tag{13}$$

At next–to–leading order the time–like splitting functions have been derived for QCD in Refs. [13, 14, 15, 16] and, by choosing appropriate color structures, can be translated to QED. Since, experimentally, one will probably distinguish events with one and more electrons in the final state, we decided to split the corresponding second order function $P_{ee}^{(1)}(x)$ into four parts in the same way as in Ref. [17]:

$$P_{ee}^{(1)}(x) = P_{ee}^{(1,\gamma)}(x) + P_{ee}^{(1,\mathrm{NS})}(x) + P_{ee}^{(1,\mathrm{S})}(x) + P_{ee}^{(1,\mathrm{int})}(x). \tag{14}$$

Here $P_{ee}^{(1,\gamma)}(x)$ is determined by the set of Feynman diagrams with pure photonic corrections (i.e. no additional electrons in the final state or closed electron loops in virtual corrections); $P_{ee}^{(1,\mathrm{NS})}(x)$ is related to corrections due to non–singlet real and virtual e^+e^- pairs; $P_{ee}^{(1,\mathrm{S})}(x)$ stands for the singlet pair production contribution; and $P_{ee}^{(1,\mathrm{int})}(x)$ describes the interference of the singlet and non–singlet pairs. These functions read

$$\begin{aligned}
P_{ee}^{(1,\gamma)}(x) &= \delta(1-x)\left(\frac{3}{8} - 3\zeta_2 + 6\zeta_3\right) + \frac{1+x^2}{1-x}\Big(2\ln x \ln(1-x) - 2\ln^2 x - 2\mathrm{Li}_2(1-x)\Big) \\
&\quad + \frac{1}{2}(1+x)\ln^2 x + 2x\ln x - 3x + 2, && (15) \\
P_{ee}^{(1,\mathrm{NS})}(x) &= \delta(1-x)\left(-\frac{4}{3}\zeta_2 - \frac{1}{6}\right) - \frac{20}{9}\left[\frac{1}{1-x}\right]_+ - \frac{2}{3}\frac{1+x^2}{1-x}\ln x - \frac{2}{9} + \frac{22}{9}x, && (16) \\
P_{ee}^{(1,\mathrm{S})}(x) &= (1+x)\ln^2 x + \left(-5 - 9x - \frac{8}{3}x^2\right)\ln x - 8 - \frac{20}{9x} + 4x + \frac{56}{9}x^2, && (17) \\
P_{ee}^{(1,\mathrm{int})}(x) &= \frac{1+x^2}{1-x}\left(2\mathrm{Li}_2(1-x) + \frac{3}{2}\ln x\right) - \frac{7}{2}(1+x)\ln x - 7 + 8x, && (18)
\end{aligned}$$

where we have used

$$\zeta_n \equiv \sum_{k=1}^{\infty} \frac{1}{k^n}, \qquad \zeta_2 = \frac{\pi^2}{6}, \qquad \mathrm{Li}_2(x) \equiv -\int_0^x \mathrm{d}z \, \frac{\ln(1-z)}{z}. \tag{19}$$

Finally, we give explicit formulas for various convolutions, which appear in Eq. (10):

$$\begin{aligned}
P_{ee}^{(0)} \otimes P_{ee}^{(0)}(x) &= \delta(1-x)\left(\frac{9}{4} - 4\zeta_2\right) + \left[\frac{1}{1-x}\Big(6 + 8\ln(1-x)\Big)\right]_+ \\
&\quad - \frac{4}{1-x}\ln x + (1+x)[3\ln x - 4\ln(1-x)] - x - 5, && (20) \\
P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)}(x) &= \frac{1-x}{3x}(4 + 7x + 4x^2) + 2(1+x)\ln x, && (21) \\
P_{ee}^{(0)} \otimes d_1(x) &= P_{ee}^{(0)} \otimes P_{ee}^{(0)}(x)\left(\ln\frac{\mu_0^2}{m_e^2} - 1\right) + \delta(1-x)\left(\frac{21}{4} - 8\zeta_3\right) \\
&\quad + \left[\frac{1}{1-x}\Big(7 + 8\zeta_2 - 6\ln(1-x) - 12\ln^2(1-x)\Big)\right]_+ + \frac{8}{1-x}\ln x \ln(1-x) \\
&\quad + (1+x)[6\ln^2(1-x) - 6\ln x\ln(1-x) - 2\mathrm{Li}_2(1-x) - 4\zeta_2] + 2x\ln x \\
&\quad + (7-x)\ln(1-x) - \frac{11}{2} - \frac{3}{2}x. && (22)
\end{aligned}$$

Using these results in Eqs.(10,11), one obtains explicit expressions for the fragmentation functions $\mathcal{D}_{e,\gamma}(x, \mu_f, m_e)$.

IV. ELECTRON ENERGY SPECTRUM

To obtain the electron energy spectrum we have to convolute the fragmentation functions Eqs. (10,11) with $\mathrm{d}\hat{\Gamma}_j/\mathrm{d}z$, and we have given all the necessary ingredients to do that in the previous Sections. Before presenting corrections to the electron energy spectrum, let us note that the dependence on the factorization scale cancels out explicitly in the final result up to the terms that are not enhanced by any large logarithm and for this reason are beyond the scope of this paper.

As we explained earlier, we split the final result into four different pieces: pure photonic radiative corrections, non–singlet pair radiative corrections, singlet pair radiative corrections, and corrections due to the interference of singlet and non-singlet pairs. Following this separation, we write the result for the electron energy spectrum in muon decay as

$$\frac{1}{\Gamma_0}\frac{\mathrm{d}\Gamma}{\mathrm{d}x} = \Delta^{(\gamma)} + \Delta^{(\mathrm{NS})} + \Delta^{(\mathrm{S})} + \Delta^{(\mathrm{int})}. \tag{23}$$

Let us start with pure photonic radiative corrections. Computing the convolutions, we arrive at the following result:

$$\Delta^{(\gamma)} = f_0(x) + \frac{\alpha}{2\pi}f_1(x) + \left(\frac{\alpha}{2\pi}\right)^2\left[\frac{1}{2}f_2^{(0,\gamma)}(x)\ln^2\left(\frac{m_\mu^2}{m_e^2}\right) + f_2^{(1,\gamma)}(x)\ln\left(\frac{m_\mu^2}{m_e^2}\right) + \dots\right], \tag{24}$$

where the dots stand for $\mathcal{O}(\alpha^2)$ terms without the logarithmic enhancement as well as higher order terms in the expansion of the electron energy spectrum in the fine structure constant. The $\mathcal{O}(\alpha^0)$ energy spectrum is given by $f_0(x) = x^2(3-2x)$. The $\mathcal{O}(\alpha)$ correction to it, $f_1(x)$, was calculated in Ref. [11]. The coefficient of the double–log term is given by

$$\begin{aligned}
f_2^{(0,\gamma)}(x) &= 4x^2(3-2x)\left[\frac{1}{2}\ln^2 x + \ln^2(1-x) - 2\ln x\ln(1-x) - \mathrm{Li}_2(1-x) - \zeta_2\right] \\
&+ \left(\frac{10}{3} + 8x - 16x^2 + \frac{32}{3}x^3\right)\ln(1-x) + \left(-\frac{5}{6} - 2x + 8x^2 - \frac{32}{3}x^3\right)\ln x \\
&+ \frac{11}{36} + \frac{17}{6}x + \frac{8}{3}x^2 - \frac{32}{9}x^3,
\end{aligned} \tag{25}$$

and is therefore in agreement with recent results in Ref.[6]. The coefficient of the single–log enhanced term for pure photonic corrections, which is one of the new results in this Letter, reads

$$\begin{aligned}
f_2^{(1,\gamma)}(x) &= 2x^2(3-2x)\Big(-2\mathrm{Li}_3(x) - 2\mathrm{S}_{1,2}(x) + 2\mathrm{Li}_2(x)\ln(1-x) + 2\mathrm{Li}_2(x)\ln x \\
&+ 5\ln x\ln^2(1-x) - 5\ln^2 x\ln(1-x) + 2\ln^3 x - 2\zeta_2\ln(1-x) - 2\zeta_2\ln x + 7\zeta_3\Big) \\
&+ \mathrm{Li}_2(x)\left(\frac{10}{3} + 14x - 40x^2 + \frac{92}{3}x^3\right) + \ln x\ln(1-x)\left(\frac{25}{3} + 32x - 54x^2 + \frac{92}{3}x^3\right) \\
&+ \ln^2(1-x)\left(-12x - 4x^2 + 8x^3\right) + \ln^2 x\left(-\frac{25}{12} - 5x + 22x^2 - \frac{70}{3}x^3\right) \\
&+ \ln(1-x)\left(-\frac{17}{3} - \frac{53}{3}x + \frac{64}{3}x^2 - 12x^3\right) + \ln x\left(-\frac{3}{4} + \frac{37}{6}x + \frac{4}{3}x^2 + \frac{44}{9}x^3\right) \\
&+ \zeta_2\left(-\frac{10}{3} - 2x + 35x^2 - \frac{98}{3}x^3\right) + \frac{211}{216} - \frac{287}{12}x + \frac{83}{3}x^2 - \frac{559}{54}x^3,
\end{aligned} \tag{26}$$

where we used

$$\mathrm{Li}_3(x) \equiv \int_0^x \mathrm{d}z\,\frac{\mathrm{Li}_2(z)}{z}, \qquad \mathrm{S}_{1,2}(x) \equiv \frac{1}{2}\int_0^x \mathrm{d}z\,\frac{\ln^2(1-z)}{z}. \tag{27}$$

The correction due to non–singlet electron–positron pairs, including effects of the running coupling constant, reads

$$\Delta^{(\mathrm{NS})} = \left(\frac{\alpha}{2\pi}\right)^2\left[\frac{1}{3}f_2^{(0,\mathrm{NS})}(x)\ln^2\left(\frac{m_\mu^2}{m_e^2}\right) + f_2^{(1,\mathrm{NS})}(x)\ln\left(\frac{m_\mu^2}{m_e^2}\right) + \dots\right], \tag{28}$$

with

$$f_2^{(0,\mathrm{NS})}(x) \;=\; 2x^2(3-2x)\ln\frac{1-x}{x}+\frac{5}{6}+2x-4x^2+\frac{8}{3}x^3, \tag{29}$$

$$\begin{aligned} f_2^{(1,\mathrm{NS})}(x) \;=&\; 2x^2(3-2x)\left(-2\mathrm{Li}_2(1-x)-\frac{2}{3}\ln x\ln(1-x)+\frac{2}{3}\ln^2(1-x)-\ln^2 x-\frac{2}{3}\zeta_2\right)\\ &+\ \ln(1-x)\left(\frac{10}{9}-\frac{4}{3}x-\frac{46}{3}x^2+12x^3\right)+\ln x\left(\frac{5}{9}+\frac{4}{3}x+8x^2-\frac{76}{9}x^3\right)\\ &-\ \frac{11}{6}-\frac{19}{3}x+\frac{100}{9}x^2-\frac{64}{9}x^3. \end{aligned} \tag{30}$$

Next, we derive the result for the singlet pair correction. Writing

$$\Delta^{(\mathrm{S})}=\left(\frac{\alpha}{2\pi}\right)^2\left[\frac{1}{2}f_2^{(0,\mathrm{S})}(x)\ln^2\left(\frac{m_\mu^2}{m_e^2}\right)+f_2^{(1,\mathrm{S})}(x)\ln\left(\frac{m_\mu^2}{m_e^2}\right)+\ldots\right], \tag{31}$$

we obtain

$$f_2^{(0,\mathrm{S})}(x) \;=\; \frac{2}{3x}+\frac{17}{9}+3x-\frac{14}{3}x^2-\frac{8}{9}x^3+\left(\frac{5}{3}+4x+4x^2\right)\ln x, \tag{32}$$

$$\begin{aligned} f_2^{(1,\mathrm{S})}(x) \;=&\; \Big[\mathrm{Li}_2(1-x)+\ln x\ln(1-x)\Big]\left(\frac{5}{3}+4x+4x^2\right)+\ln^2 x\left(\frac{5}{2}+6x+4x^2\right)\\ &+\ \ln(1-x)\left(\frac{17}{9}+\frac{2}{3x}+3x-\frac{14}{3}x^2-\frac{8}{9}x^3\right)+\ln x\left(\frac{8}{9}+\frac{4}{3x}-\frac{5}{6}x-\frac{19}{3}x^2\right)\\ &-\ \frac{1}{3x}-\frac{67}{9}+\frac{43}{18}x+\frac{77}{18}x^2+\frac{10}{9}x^3. \end{aligned} \tag{33}$$

Finally, for the interference term one finds

$$\Delta^{(\mathrm{int})}=\left(\frac{\alpha}{2\pi}\right)^2\left[f_2^{(1,\mathrm{int})}(x)\ln\left(\frac{m_\mu^2}{m_e^2}\right)+\ldots\right], \tag{34}$$

and

$$\begin{aligned} f_2^{(1,\mathrm{int})}(x) \;=&\; 2x^2(3-2x)\Big(2\mathrm{Li}_3(1-x)-4\mathrm{S}_{1,2}\,(1-x)-2\mathrm{Li}_2(1-x)\ln x\Big)\\ &+\ \mathrm{Li}_2(1-x)\left(\frac{5}{3}+4x-26x^2+\frac{52}{3}x^3\right)+\ln^2 x\left(-9x^2+\frac{26}{3}x^3\right)\\ &+\ \ln x\left(-\frac{5}{3}-\frac{5}{3}x-\frac{28}{3}x^2\right)-\frac{62}{9}+\frac{41}{3}x-\frac{55}{3}x^2+\frac{104}{9}x^3. \end{aligned} \tag{35}$$

V. CONCLUSIONS

By applying techniques developed for perturbative QCD to Quantum Electrodynamics, we computed $\mathcal{O}(\alpha^2)$ correction to the electron energy spectrum in unpolarized muon decay, keeping all the terms enhanced by the logarithms of the muon to electron mass ratio. The double logarithmic corrections $\mathcal{O}(\alpha^2(\ln^2 m_\mu^2/m_e^2))$ are in agreement with recent result in Ref. [6]. The single logarithmic corrections $\mathcal{O}(\alpha^2\ln(m_\mu^2/m_e^2))$, the new result of this Letter, are important to match the precision requirements of the TWIST experiment. To illustrate significance of single logarithmic terms, in Fig. 1 we plot both double and single $\mathcal{O}(\alpha^2)$ logarithmic corrections defined as

$$\begin{aligned} \delta_2^{(0,\mathrm{tot})}(x)&=\frac{1}{f_0(x)}\left(\frac{\alpha}{2\pi}\right)^2\left[\frac{1}{2}f_2^{(0,\gamma)}(x)+\frac{1}{3}f_2^{(0,\mathrm{NS})}(x)+\frac{1}{2}f_2^{(0,\mathrm{S})}(x)\right]\ln^2\left(\frac{m_\mu^2}{m_e^2}\right),\\ \delta_2^{(1,\mathrm{tot})}(x)&=\frac{1}{f_0(x)}\left(\frac{\alpha}{2\pi}\right)^2\left[f_2^{(1,\gamma)}(x)+f_2^{(1,\mathrm{NS})}(x)+f_2^{(1,\mathrm{S})}(x)+f_2^{(1,\mathrm{int})}(x)\right]\ln\left(\frac{m_\mu^2}{m_e^2}\right), \end{aligned} \tag{36}$$

as the function of the energy fraction carried away by electron in muon decay.

As follows from Fig. 1, the $\mathcal{O}\left(\alpha^2 \ln(m_\mu^2/m_e^2)\right)$ corrections computed in this Letter are required for the theoretical prediction with the precision 10^{-4}, which is the benchmark precision for the TWIST experiment. Moreover, within the acceptance region of the TWIST experiment, $0.3 \lesssim x \lesssim 0.98$, the magnitude of $\mathcal{O}\left(\alpha^2 \ln(m_\mu/m_e)\right)$ corrections is in fact comparable to the magnitude of (naively the largest) $\mathcal{O}\left(\alpha^2 \ln^2(m_\mu/m_e)\right)$ terms.

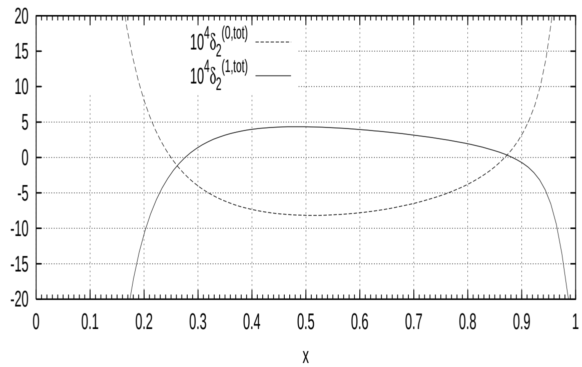

FIG. 1: Double and single logarithmic corrections as a function of x.

The fact that sub–leading logarithmically enhanced corrections are larger than the precision of the TWIST experiment, seems to indicate that full calculation of the $\mathcal{O}(\alpha^2)$ corrections to the electron energy spectrum is very desirable and that without such a calculation an intrinsic theory uncertainty can not be pushed below few $\times\, 10^{-4}$.

Complete calculation of $\mathcal{O}(\alpha^2)$ correction to the electron energy spectrum in muon decay is a very difficult task and it remains to be seen if it can be done. In spite of that, it is possible to extend the analysis of this Letter in a number of ways to further improve on the theory prediction. First of all, it is straightforward to compute the electron energy spectrum in polarized muon decay, by using techniques described in this Letter. Furthermore, $\mathcal{O}\left(\alpha^3 \ln^3(m_\mu^2/m_e^2)\right)$ corrections can be obtained from the DGLAP equation. Also, resummation of the corrections that are singular in the limit $x \to 1$ can be performed and the influence of this resummation on theoretical predictions for the spectrum can be explored. These studies as well as detailed discussion of the present theoretical uncertainty in the electron energy spectrum in polarized muon decay will be presented elsewhere.

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