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The Loop Group of E_8 and K-Theory from $11d$

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We examine the conjecture that an $11d$ E_8 bundle, appearing in the calculation of phases in the M-Theory partition function, plays a physical role in M-Theory, focusing on consequences for the classification of string theory solitons. This leads for example to a classification of IIA solitons in terms of that of LE_8 bundles in $10d$. Since $K(\mathbb{Z}, 2)$ approximates LE_8 up to π_{14}, this reproduces the K-Theoretic classification of IIA D-branes while treating NSNS and RR solitons more symmetrically and providing a natural interpretation of G_0 as the central extension of $\tilde{L}E_8$.

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1. Introduction and Motivation

In this note we study the classification of solitons in string theory and M-Theory. Our starting point is the intersection of two suggestive results. First, as argued by Witten [1][2] and more extensively by Diaconescu, Moore and Witten [3][4], certain subtle phases in the M-Theory partition function suggest a connection to an E_8 gauge theory over a $12d$ manifold Z bounded by Y. This follows from the fact that E_8 bundles in $12d$ are specified topologically by their Chern-Simons 3-form [5], so that the calculation of these M-Theory phases as sums over topologically distinct M-Theory 3-form configurations takes a natural form in terms of the index theory of $12d$ E_8 bundles. That this E_8 index theory result agreed precisely with a very different calculation based on IIA K-Theory led Diaconescu, Moore and Witten to suggest a deeper connection between the M-Theory 3-form and the Chern-Simons 3-form of a $12d$ E_8 bundle. Since the index calculation depends only on $\partial Z = Y$, the physical data lies in the restriction of the $12d$ bundle to an E_8 bundle in $11d$.

Secondly, it is commonly believed that the K-Theory of $\mathbb{C}P^\infty \sim K(\mathbb{Z},2)$ bundles classifies D-Brane configurations in Type IIA string theory, as argued in [6][7] and phrased in terms of $K(\mathbb{Z},2)$ in [8]. However, the physical connection of the group $K(\mathbb{Z},2)$ to M-Theory is unclear. Moreover, as fleshed out in a beautiful paper by Maldacena, Moore and Seiberg [9], the Atiyah-Hirzebruch Spectral Sequence (AHSS) construction of the K-Theoretic classification of Type II RR solitons involves anomaly cancellation conditions in an intimate and beautiful way. How this relates to the proposal of [8] is again unclear.

These lines of reasoning beg to be connected. As a first hint, note that $K(\mathbb{Z},2)$ and LE_8 are homotopically identical up to π_{14}. [3],[4] Thus the classification of LE_8 bundles over 10-manifolds agrees with that of $\mathbb{C}P^\infty$ bundles. Further, up to important questions of central extension and torsion which we address below, the classification of LE_8 bundles over 10-manifolds is precisely the classification of E_8 bundles over 11-manifolds with a compatible circle action. Thus the classification of solitons and the cancellation of anomalies in

[3] LE_8 denotes the loop group of E_8, and $\tilde{L}E_8$ its centrally extended generalization. We describe their low-dimensional topology below; for a complete discussion, see eg [10].

[4] We are deeply indebted to Petr Hořava for insightful discussions during early stages of this work suggesting looking at the loop group of E_8 as an M-Theoretic alternative to the stringy picture of $K(\mathbb{Z},2)$ arising from an infinite number of unstable D9-branes [11]. For a discussion of possible relations between these two pictures and their implications for supersymmetry and $11d$ dynamics, see [12].

M-Theory and IIA (and Heterotic, as we shall see), as well as the relationship between these as revealed by the AHSS, can all be phrased in terms of a single E_8 structure in $11d$. That an $11d$ E_8 bundle ties together so many pieces of the M-Theory puzzle strongly supports the conjecture that an $11d$ E_8 bundle plays a physical role in M-Theory, and should be reflected in its fundamental degrees of freedom.

Taking this seriously thus leads us to conjecture that the classification of RR and NSNS solitons in IIA derives from the classification of LE_8 bundles over 10-manifolds. This generalizes the accepted K-Theoretic classification of RR solitons (and adds to growing evidence that K-Theory at least approximately respects IIB S-duality, suggesting that K-Theory plays some role even beyond weak coupling) while leading to novel predictions about the complete classification of IIA solitons, including the interpretation of the cosmological "constant"[5] G_0 of (massive) IIA as the central charge of $\tilde{L}E_8$, and several constraints relating torsion in M-Theory, $\tilde{L}E_8$ and IIA.

In the remainder of this note we present further motivation for these conjectures and show how such a framework reproduces and extends the familiar classification of solitons in M-Theory and its $10d$ descendants[6]. Of course, $11d$ SUSY does not to play well with gauge bundles, and it is difficult to see how a dynamical bundle can coexist with 32 supercharges. (For further thoughts along these lines see eg [14][12].) However, objects to which the E_8 gauge connection couples in M-Theory and the string theory generically violate at least half of the supercharges, so we might expect to see gauge bundle information only in situations with reduced supersymmetry. In any case, the resolution is unclear, so we restrict ourselves in the following to studying the soliton classification, leaving questions of dynamics and SUSY to future work. We begin by reviewing the topological classification of E_8 bundles over 11-manifolds.

2. The Topological Classification of E_8 Bundles in $11d$

E_8 has exceptionally simple low-dimensional topology. In particular, its only non-trivial homotopy group below dimension 15 is $\pi_3(E_8) = \mathbb{Z}$. The basic non-trivial E_8 bundle is thus that over an S^4 whose transition functions on the S^3 equator lie in $\pi_3(E_8)$.

[5] Since the dilaton is not constant in the presence of $D8$-branes, this should properly be called a cosmological term rather than a cosmological constant.

[6] For earlier thoughts on the role of E_8 in M-Theory, see eg [13][14][15]. See also [12][16]for related current work

Due to the absence of other relevant homotopy classes, E_8 bundles over manifolds of dimension $3 < d < 16$ are topologically classified entirely by the transition functions on the S^3 equators of S^4's in the 4-skeleton of the base manifold [5]. These are measured by the restriction of the first Pontrjagin class p_1, which is the exterior derivative of the Chern-Simons 3-form C_3 on each coordinate patch [5], to the given S^4. E_8 bundles over 11-manifolds are thus topologically classified by the specification of a 3-form C_3, a remarkable fact that depends crucially on the simple low-dimensional topology of E_8.

The basic monopole in such bundles is thus a codimension 5 object supporting 4-form flux such that the integral of p_1 over an S^4 linking the defect is the monopole number,

$$\int_{S^4} \frac{G_4}{2\pi} = n \in \mathbb{Z}, \tag{2.1}$$

where $G_4 = dC_3 = dTr\ (A \wedge F + \frac{2}{3} A \wedge A \wedge A)$. There is also a codimension 4 instanton such that the integral of p_1 over a transverse 4-plane is non-zero. Such a bundle can be trivialized inside and outside any 3-sphere in this plane, with the transition functions on this linking S^3 classified by $\pi_3(E_8)$. If we restrict to configurations which are compactly supported in the transverse plane, the integral of p_1 over the transverse 4-plane is thus an integer counting instanton number. Such an instanton can be produced by considering a monopole-antimonopole pair whose fluxlines run from one to the other; the integral of p_1 over a transverse 4-plane between them is thus quantized, with the choice of orientation specifying whether this plane links the monopole or antimonopole and thus fixing the sign. If the flux takes delta-function support in the transverse plane, this is a zero-radius instanton Poincare dual to the first Pontrjagin class of the bundle.

Due to the magic of E_8,

$$p_2 = p_1 \wedge p_1 = \frac{G_4 \wedge G_4}{16\pi^2},$$

a relation that would not hold had we considered for example $U(N)$ bundles. Thus p_2 does not reveal any new topology not already contained in G_4. However, since we can always pull the codimension 5 defects to infinity, p_2 can represent a charge in compactly supported cohomology. For example, consider a bundle such that the integral of p_2 over some 8-plane is non-zero; this reveals the presence of a codimension 8 object Poincare dual to p_2. Since we can express p_2 as the exterior derivative of a 7-form G_7, we can relate this integral over an 8-plane to an integral over its "S^7 at infinity" (again, we are looking at compactly supported cohomology) to get

$$\int_{\mathbb{R}^8} p_2 = \int_{S^7} \frac{G_7}{2\pi} = k \in \mathbb{Z},$$

so the codimension 8 objects are quantized and localized. There is again an associated codimension 7 "instanton" (properly, this is an intersection of codimension 4 instantons) such that the integral of G_7 over a transverse 7-plane is non-zero. Instanton number is quantized in a more subtle way here, since there is no homotopy class directly counting these instantons. However, since these codimension 7 instantons can be constructed as the flux stretching between a codimension 8 monopole-antimonopole pair, a quantization condition applies.

The role of these codimension 7 and 8 objects is more transparent when we consider the first non-trivial AHSS differential for such bundles,

$$d_4 = G_4 \cup + [\text{Torsion}]. \tag{2.2}$$

Ignoring torsion for the moment, this differential enforces for example the condition

$$d * G_4 = G_4 \wedge G_4.$$

This reflects the fact that the G_7 whose exterior derivative is p_2 really is the dual of G_4. Physically, this equation requires a codimension 5 object wrapping a 4-cycle supporting k units of G_4 flux to be the endpoint of k codimension 8 objects.

This classification has an immediate reading in terms of the conjecture discussed above. The codimension 5 monopole is the $M5$-brane, the codimension 8 the $M2$-brane, while the codimension 4 and 7 instantons are the M-Theory $MF6$ and $MF3$ Fluxbranes discussed by Gutperle and Strominger[17]. Moreover, the AHSS differential precisely effects the $11d$ supergravity equation of motion $d * G_4 = G_4 \wedge G_4$, which implies that an $M5$ wrapping a 4-cycle supporting k units of G_4 flux must be the endpoint of k $M2$-branes, a familiar result, and ensures the Dirac quantization of the $M2$ and $MF3$ branes.

Returning briefly to (2.2), the torsion terms can be studied by checking when the sign of the Pfaffian of the Dirac operator can be made well defined for the fermion contribution to a path integral describing an open M2-brane via the inclusion of some chiral 2-form. In particular if the M2-brane wraps a circle we recover the familiar obstruction $W_3 + H$

from [18]. We reserve further discussion of $11d$ torsion until Section 6; about $10d$ torsion we will say more shortly.

At this point it is clear that the soliton spectrum of the various perturbative string theories should be reproduced by compactifying the base manifolds of our $11d$ E_8 bundles, since it has precisely reproduced the M-Theory solitons from which they descend. Explicitly studying the dimensional reduction of the E_8 bundle will reveal several interesting details, including an intrinsically $10d$ classification of IIA solitons treating NSNS and RR solitons largely symmetrically, to which we now turn.

3. Type IIA and K-Theory from LE_8

Consider an E_8 bundle F over an 11-manifold Y with a circle action that commutes with the transition functions. Let X be the $10d$ space of orbits of the circle action. Sections of F thus define sections of an LE_8 bundle $E \to X$.

Let's pause to review the topology[7] of LE_8. By the canonical homotopy-lowering map, $\pi_p(LE_8) = \mathbb{Z}$ for $p = 2, 14, 22, ...$, and trivial otherwise. The low-dimensional cohomology is similarly simple,

$$H^{even}(LE_8) = \mathbb{Z} \qquad H^{odd} = 0.$$

Since $H_2(LE_8) = \mathbb{Z}$, LE_8 admits a central extension given by a single positive integer. This centrally extended Kac-Moody algebra has a canonically associated group manifold, both of which we shall denote by $\tilde{L}E_8$ in a heinous abuse of notation. The topology of $\tilde{L}E_8$ differs from that of LE_8 in several important ways. In particular, $\pi_2(\tilde{L}E_8)$ is trivial[8], and its low-dimensional cohomology is consequently different from that of LE_8.

We now return to our $10d$ and $11d$ bundles. For every soliton or defect in F there is a soliton or defect in E. However, the $10d$ bundle has a generalization which does not lift, measured by the integer central extension of $\tilde{L}E_8$. Since $\pi_3(E_8) = \mathbb{Z} \neq \pi^*(\pi_2(\tilde{L}E_8))$, where π^* is the pullback along the circle fibration projection map, the central extension of $\tilde{L}E_8$ obstructs a lift to $11d$. Correspondingly, Type IIA string theory has a single

[7] For a more extensive discussion of such (possibly centrally extended) loop algebras and the topology of their canonically associated group manifolds, see [10].

[8] The triviality of $\pi_2(\tilde{L}G)$ depends only on G being simple and simply connected. This is essentially the statement that LG admits a single universal central extension of which all others are cosets; see [10] for an extensive discussion of the topology of centrally extended algebras.

integer, the 0-form field strength G_0, which is the obstruction to lifting to M-Theory. Domain walls over which this integer jumps, $D8-$branes, similarly cannot be lifted. We thus conjecture that the central extension k of this $\tilde{L}E_8$ bundle over $10d$ measures the cosmological "constant" of (massive) IIA, G_0, as

$$G_0 = k. \tag{3.1}$$

That a lift is indeed possible when $G_0 = 0$ fixes a possible additive constant to zero[9].

The distinct topology of the centrally extended $\tilde{L}E_8$ implies that the spectrum of stable, consistent D-branes is altered in the presence of $D8$-branes. In particular, characteristic classes which are torsion when the central extension is non-vanishing will reveal instabilities of various brane configurations in the presence of G_0 which may be stable in the absence thereof, or vice-versa. We are thus led to study the complete topology of $\tilde{L}E_8$, including torsion, which will provide explicit, testable predictions about the (in)stability of brane configurations in massive IIA[19].

Since the homotopy and cohomology groups of LE_8 agree with those of $PU(\infty) = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ up to[10] dimension 14, the classification of RR solitons via LE_8 bundles differs from that of Bouwknegt and Mathai [8] only in phenomena related to high (greater than 14) dimensional topology[11]. Remarkably, the same $\tilde{L}E_8$ structure also serves to classify the NS-NS solitons, as we now discuss.

3.1. NS-NS Solitons from LE_8

Since $\pi_2(LE_8) = \mathbb{Z}$, the primary $10d$ LE_8 defect is codimension 4, i.e. $(5 + 1)$ dimensional as in $11d$. An S^3 linking k such defects, or more generally any S^3 supporting k units of H-flux as in the SU(2) WZW model, has LE_8 instanton number equal to k. By

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[9] Notice that this proposal is reminiscent to the situation in AdS/CFT, and particularly $AdS_3 \times S^3 \times T^4$ in which the cosomological constant on the AdS_3 is determined by the central charge of the $\hat{sl}_2$ affine Lie algebra of the boundary WZW model. We thank Liat Maoz for reminding us of this relationship.

[10] $K(\mathbb{Z}, 2)$ is by definition the space whose homotopy classes are all trivial except for $\pi_2(K(\mathbb{Z}, 2)) = \mathbb{Z}$. It is realized for example by $\mathbb{C}P^\infty$ which appears in the consideration *a la* Sen of D-brane classification via non-trivial tachyon bundles associated with the gauge bundles over D-$\bar{D}$ pairs.

[11] Bouwknegt and Mathai [8] argue that IIA D-branes are classified by the K-Theory of the algebra of sections of a vector bundle associated to a $PU(\infty) = K(\mathbb{Z}, 2)$ principal bundle, roughly.

this we mean that the bundle can be trivialized on the northern and southern hemispheres and the transition function is the element k of $\pi_2(LE_8)$. The defect is characterized by the fact that, at the defect itself, the LE_8 picture breaks down because the circle orbits are not closed. This $10d$ defect is the reduction of an $11d$ defect transverse to the S^1. This is precisely the IIA $NS5$-brane arising from a transverse $M5$-brane.

Similarly, a fundamental IIA string is an $11d$ codimension 8 soliton whose embedding is invariant with respect to the circle action. In particular, the $11d$ bundle is then invariant with respect to the circle action, so transition functions of the $10d$ bundle consist of zero-modes in LE_8, that is, they inhabit an E_8 subgroup. In fact the transition functions in $10d$ are just the embedding of those in $11d$ into LE_8, and so the fundamental string is, like the $M2$-brane, Poincare dual to the square of the first Pontrjagin class (the second Pontrjagin class) of this E_8 sub-bundle of the LE_8 bundle. This is however not to say that the rest of the LE_8 is unimportant - in particular, the dynamics of the $M2$-brane need not respect the circle action, so those of the fundamental string need not restrict themselves to the zero mode subgroup at finite coupling.

3.2. RR Solitons from LE_8

Let's quickly return to the classification of RR solitons via LE_8 bundles. The $D4$-brane arises as an $11d$ 5-defect whose embedding and field configuration are invariant under the circle action. Similarly to the F-string it can be realized with an $E_8 \subset LE_8$. It is characterized by the fact that each linking S^4 has E_8 instanton number one. The $D2$-brane is a $2+1$-soliton transverse to the circle, and is Poincare dual to $d * G_4$, a 7-form related to p_2 of the E_8 bundle by the canonical dimension lowering map between characteristic classes of a space and its loop space. The $D6$-brane arises from a non-trivial circle fibration, such that the π_2 of LE_8 lifts to the π_3 of E_8 via a Hopf fibration, while the $D0$-brane arises as usual as a momentum mode along the S^1 fibers. In both cases the associated flux arises from the KK gauge field, the branes representing trivial E_8 fibrations over the 11-fold.

Finally, as discussed above, $D8$-brane number is connected to the central extension of $\tilde{L}E_8$. Thus, while the $D8$-brane does not appear to have a simple geometric interpretation in terms of an $11d$ E_8 soliton, it has a deep connection to the $\tilde{L}E_8$ structure in $10d$. This connection may provide insight into the connection between $11d$ gravity and the E_8 structure[12].

3.3. *Fluxbranes from LE_8*

The $11d$ E_8 origin of IIA Fluxbranes is similarly automatic; its reading in terms of LE_8 follows naturally. The simplest example is the direct dimensional reduction of the codimension-4 E_8 fluxtube, which gives the NS-NS $F6$ in IIA. Similarly, a codimension-4 fluxtube which wraps the M-Theory circle remains a codimension-4 fluxtube - this is the IIA RR $F5$-brane. Analogously, the codimension-8 fluxtube reduces transversely to the RR $F3$-brane and, wrapping the M-Theory circle, to the NSNS $F2$-brane. The $F1$ and $F7$ arise as fluxtubes associated to the nontrivial bundles of the $D0$ and $D6$ branes, respectively. Thus we realize the full spectrum of RR and NSNS Fp-Branes discussed by Gutperle and Strominger [17] in terms of LE_8, as expected.

3.4. *K-Theory from LE_8 and Indiscretions regarding Torsion*

We have seen how the classification of both NSNS and RR solitons in Type IIA arises from the classification of LE_8 bundles in $10d$, these derived from a fundamental E_8 structure in M-Theory. Due to the remarkable topology of LE_8, this reproduces the conjectured K-Theoretic classification for RR charges and fields. We would now like to connect this construction with the AHSS approximation to the K-Theoretic classification. In the remainder of this section we will use the language of M-branes and D-branes for simplicity and clarity; in light of the above discussion, it should be clear that the entire discussion can be phrased explicitly in terms of $11d$ E_8 bundle information.

The classifying group of solitons in M-Theory is a refinement of cohomology obtained by taking the quotient with respect to a series of differentials that reflect the fact that some configurations are anomalous and so should not be included, while others are related by dynamical processes and so must carry the same conserved charges(see eg [9]). For example, an $M5$-brane wrapping a 4-cycle that supports k units of G_4 flux leads to an anomaly that, neglecting torsion, can be canceled if k $M2$-branes end on the $M5$. Thus some $M5$-brane wrappings are anomalous and some $M2$-brane configurations (such as k $M2$'s and the vacuum) are equivalent, this following from the $11d$ supergravity equation of motion

$$d * G_4 = G_4 \wedge G_4.$$

The left hand side of this equation is the intersection number of $M2$-branes with a sphere linking the $M5$, and the right is roughly the integral of the G_4 flux over the 4-cycle wrapped by the 5-brane. Both of these numbers are measured in units of the 8-form Poincare dual to the $M2$-branes. In the absence of $M2$-branes ending on the $M5$'s, this supergravity constraint is summarized[12] by requiring that the following "differential" annihilate the G_4 flux

$$d_4 G_4 = G_4 \wedge G_4 + [\text{Torsion}].$$

We expect that the torsion terms are nontrivial because, for example, G_4 is half-integral when the $M5$ brane wraps a 4-cycle with non-vanishing w_4 [20]. Also, as we will soon see, its dimensional reduction is nontrivial.

The classification for IIA follows from dimensional reduction of this M-Theory story. There are three distinct classes of reductions of this constraint to IIA, reflecting three possible locations of the M-Theory circle x^{11} in the above scenario. First consider an $M5$-brane wrapping x^{11} which is not in the 4-cycle, so that the anomaly-canceling $M2$-branes do wrap x^{11}. This leads to an anomaly condition requiring F-string insertions on a $D4$ as follows. The $M5$-brane wraps x^{11} and so the G_4 flux that it generates has no 11 component; it is thus not Kaluza-Klein reduced. Similarly, the 4-cycle does not wrap and so the G_4 supported on the 4-cycle is not reduced. Thus the $10d$ anomaly condition arising from this situation is identical to the $11d$ condition:

$$d_4 G_4 = G_4 \wedge G_4 + [\text{Torsion}],$$

now a $10d$ constraint with G_4 identified with the 4-form RR fieldstrength.

Next consider the case in which both the $M5$-brane and the 4-cycle wrap x^{11}, yielding a $D4$ with $D2$ insertions as follows. The G_4 flux sourced by the $M5$-brane is still not reduced, but now the 4-cycle is reduced to a 3-cycle, the G_4 flux it supports dimensionally reduced to the 3-form H. The resulting anomaly constraint is thus

$$d_3 G_4 = H \wedge G_4 + [\text{Torsion}].$$

This is a well-known differential from the AHSS for twisted K-Theory [21], which was seen to be the relevant constraint in [9]. In particular the torsion correction was seen to be $Sq^3 G_4$.

The final case involves an $M5$-brane not wrapping x^{11}, reducing to an $NS5$-brane with $D2$-brane insertions. In this case the 4-form flux is dimensionally reduced to H while the flux in the 4-cycle is not reduced, yielding the constraint

$$d_4 H = G_4 \wedge H + [\text{Torsion}].$$

[12] This was seen in type II in [9].

The torsion in this case is as yet poorly understood.

Combining these three constraints, as well as the AHSS conditions on other RR fluxes, we hope to arrive at a K-Theoretic classification of both NSNS and RR charged objects in IIA. We expect this classification to be T-dual to the S-duality covariant classification in [22]. Independently of our proposal, it would be interesting to better understand the $11d$ lifts of the other constraints on RR fluxes.

For example, anomaly cancellation on a $D2$-brane in IIA wrapping a 3-cycle C with k units of H flux requires k $D0$-brane insertions. Lifting this to M-Theory we learn that, while we know of no restrictions on what cycle an $M2$-brane may wrap, if it wraps a 3-cycle C such that

$$\int_{C\times S^1} \frac{G_4}{2\pi} = k \neq 0$$

then k units of momentum around x^{11} must be absorbed by the brane. To get an intuitive understanding of the physics at work[13], let us pretend that C is a 2-cycle times the time direction, with a constant H flux density, and then KK reduce on the 2-cycle. Before reducing, this corresponds to a constant flux of $D0$-branes incident on the $D2$-brane in IIA, while in M-Theory this corresponds to a steady injection of p^{11} into the $M2$. KK reducing, the G_4-flux reduces to an electric field along the circle, while the $M2$-brane reduces to a particle charged under this field. This flux drives the charged particle to accelerate around the circle with a constant acceleration, that is, to absorb p^{11} at a constant rate. The anomaly condition lifted to M-Theory is thus simply $F = ma$! Although we do not understand the deep connection of the M-Theory E_8 bundle to gravity, this relation between G_4 and p^{11} is perhaps a significant clue.

4. The Heterotic String and the Small Instanton Transition

Consider now an E_8 bundle over an 11-fold $X = M \times S^1/\mathbb{Z}_2$. The bulk bundle naturally restricts to two $10d$ E_8 bundles, one over each of the two boundary components. At this point the realization of the various objects in Heterotic string theory in terms of instantons of the E_8 bundle follows naturally from the beautiful arguments of [24]. For example, an $M2$-brane stretching between the two boundary components is precisely the strongly-coupled fundamental Heterotic string. Moreover, anomaly considerations descend

[13] See also the beautiful discussion in [23], which addresses an analogous effect for dielectric branes in a non-compact geometry.

naturally. In 11-d, there is a mod 2 relation between the Pontrjagin classes of the E_8 bundle, $w(F \to Y)$, and that of the base manifold's tangent bundle, $w(TY)$ - thus for example $G_4 = w_4(TY)/2$. This condition reduces on the induced bundle over the orbifold fixed point to the $10d$ condition, which arises from a gravitational anomaly [24][25].

It is easy to see the Heterotic 5-brane arising from the bulk E_8 bundle. Recall that the $11d$ E_8 5-defect is defined such that a 4-sphere linking the 5-defect has instanton number one. Consider a parallel $11d$ 5-$\bar{5}$ pair separated a finite distance in a transverse direction, y, of $\mathbb{R}^{(10,1)}$. For every point y_p there is a $10d$ bundle given by the restriction of the $11d$ bundle to the $10d$ slice $y = y_p$. Since any 4-plane in the slice $y = y_p$, with y_p between the two defects, links one or the other of the defects, the $10d$ bundles over points between the two 5-defects have instanton number ± 1, the sign fixed by choice of orientation, while the $10d$ bundles over points not between the two defects have instanton number zero. Since the $10d$ bundles over points between the $11d$ defects are non-singular, their instantons are "large". The singular $10d$ bundles which contain the $11d$ 5-defects, by contrast, contain "small" instantons. These are the Heterotic 5-branes.

Next consider a similar configuration with the two defects pulled away to infinity, leaving a single codimension-4 instanton stretched along the coordinate y and taking compact support in the transverse 4-plane. If we pinch the instanton over a point $y = y_*$, we can nucleate a $5 - \bar{5}$ pair at y_* and move them away to infinity, leaving behind no flux in the interval between them. From the point of view of the $11d$ bundle, this is a completely continuous process respecting all conserved charges and symmetries. From the point of view of the induced $10d$ bundle over any point $y = y_o \neq y_*$, however, things look rather odd; the originally large and fluffy instanton shrinks to a singular "small" instanton and then disappears altogether!

Now consider an E_8 bundle over the 11-manifold $Y = \mathbb{R}^{(9,1)} \times (\mathbb{R}/\mathbb{Z}_2)$, where the y coordinate along which the $11d$ instanton is extended has been orbifolded by a Z_2 reflection. If we repeat the pinching-transition over the point $y = 0$, which from the point of view of the covering space is completely continuous and respects all conservation laws, as well as the orbifold symmetry, we find a transition in the orbifold theory in which a "large" instanton in the boundary bundle shrinks to a singular "small" instanton before disappearing from the boundary and moving into the bulk as an $11d$ 5-defect, i.e. an M5-brane. This is precisely the Heterotic small instanton transition studied near one boundary component, as read by the $11d$ E_8 bundle. Note that, while the number of boundary instantons $n_{\partial Y}$ is not conserved, $n_{\partial Y} + n_Y$ is.

5. Speculations about E_8 Bundles and $11d$ SUSY

Since objects to which the E_8 gauge connection couples in M-Theory and string theory violate at least half of the 32 $11d$ supercharges, we should perhaps expect to see gauge bundle information only in situations with reduced supersymmetry. It is thus reasonable to wonder if the gauge connections inhabit representations of only a sub-algebra of the $11d$ superalgebra, representations that in particular contain neither gravitons nor gravitinos. The Chern-Simons 3-form of this connection can then be set equal to the 3-form in the $11d$ supermultiplet, for example via a Lagrange multiplier[14],

$$\delta S \sim \int_{M^{11}} \alpha(C_3^M - C_3^{E_8}).$$

It is worth keeping in mind that both the M-Theory 3-form and the E_8 Chern-Simons form respect an abelian gauge symmetry, since for example under a local E_8 gauge transformation with gauge parameter Λ the CS-3-form transforms as $C \to C + d\mathrm{Tr}(\Lambda F)$, so this action is in fact gauge invariant and respects all the requisite symmetries.

Of course, not all bundles in the same topological equivalence class correspond to BPS solitons. Rather, the bundles in each equivalence class are related by a change in boundary conditions which does not change the topology; in the associated SUGRA class, this corresponds (roughly, as the equations of motion are non-linear) to a shift by a solution to the vacuum equations of motion. However, since the topological classification of bundles is precisely the classification by charge (at least up to torsion terms), there is some choice of background fields which does not affect the topological class and yields precisely the BPS soliton. In particular we attribute an array of classical moduli, such as the size of Heterotic instantons, to precisely such a freedom of choice of boundary conditions.

6. Conclusions and Open Questions

We have argued that the topological classification of E_8 bundles in $11d$, which naturally reproduces the soliton spectrum of M-Theory, reproduces when reduced on $S^1/\mathbb{Z}_2$ the spectrum of Heterotic $E_8 \times E_8$, while reduced on S^1 reproduces the spectrum and

[14] We particularly thank Eva Silverstein for discussions on this topic.

K-classification of RR and NSNS solitons in Type IIA[15]. Remarkably, while there appears to be no simple dynamical role for E_8 in Type IIA, there does appear to be a deep role for its loop group LE_8 in the K-Theoretic classification of IIA solitons, including in an important way its central extension. The relevance of E_8 bundles even for perturbative string theories with no dynamical gauge bosons suggests an important role for E_8 in the construction of the fundamental degrees of freedom of M-Theory.

The most obvious open question is how, precisely, an $11d$ gauge theory fits with $11d$ supersymmetry. This is extremely confusing. Perhaps a natural place to look for hints to this puzzle is in Heterotic $E_8 \times E_8$, where the gauge boson couples in an intricate but natural and beautiful way. Extending this story to $11d$ would be an exciting advance.

Another obvious omission in our presentation is the absence of torsion terms in (2.2). That this is an important omission is clear from any geometry where, for example, an $M5$-brane lies inside not an S^4 but some orbifold thereof. Following [3], one thus expects the torsion terms to include some $\mathbb{Z}$ lift of sq^4; however, as there is no canonical lift of the $\mathbb{Z}_2$ Steenrod squares of even rank, identifying the correct "derivation" is somewhat delicate. In the language of Witten, and in the orientable case, one might expect the fourth AHSS differential to take the form $d_4 = \lambda + G_4 \cup$. However, the sign in front of λ is not obvious. It could of course be fixed by comparison with the 5-brane anomaly, but would still leave ambiguous the correct torsion terms in non-orientable cases, where some lift of the $\mathbb{Z}_2$ Steenrod square sq^4 must obtain.

One avenue of approach might be to identify a canonical lift for the special case of 11-folds with compatible circle actions. As a first guess, define

$$\tilde{Sq}^4 = \pi^*(Sq_3),$$

where π^* is the pullback of the projection of the S^1 fibration. From various Adem relations one can argue that this restricts correctly to sq^4 if $\pi^*(\beta) = sq^2$. A case where one might test this possibility would be an $M5$-brane wrapping $SU(3)/SO(3) \equiv M_5$, whose anomaly requires an $M2$-brane to end upon the $M5$-brane. Reducing on an S^1 to a $D4$-$D2$, the anomaly arises from Sq^3 in the $D4$-brane worldvolume, which is canceled by the incident

[15] While we of course do not have a candidate for what the complete K-Theory of $\tilde{L}E_8$ bundles is, it should be identical to that of the universal classifying group $K(\mathbb{Z}, 2)$ up to corrections involving topology well above $11d$, as discussed above. One might for example attempt to generalize Rosenberg's K-Theory, [21].

$D2$. Pulling back along the S^1 fibration, Sq^3 should lift to a $\mathbb{Z}$-graded rank-four differential which measures the correct $10d$ anomaly under bundle projection. It would be interesting to explicitly check when, if ever, such a non-trivial pullback exists, and when it does whether it restricts to the $\mathbb{Z}_2$-graded sq^4. We leave such questions to future work.

Finally, it would be particularly interesting to revisit the beautiful and delicate calculations of Diaconescu, Moore and Witten in [3], who showed that the cancellation of anomalies in IIA and M-Theory agree, though the structures underlying the calculations in the two cases were apparently unrelated. DMW read this unlikely agreement as strong evidence for the conjecture that RR fields and charges in IIA are indeed classified by K-Theory. We expect that the IIA computation will take a natural form in terms of $\hat{E}_8$ bundles, and that in this language the relation to anomaly cancellation in M-Theory will be immediate. This would be interesting to check directly.

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