Pion pole contribution to hadronic light-by-light scattering and muon anomalous magnetic moment

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We derive an analytic result for the pion pole contribution to the light-by-light scattering correction to the anomalous magnetic moment of the muon, $a_\mu = (g_\mu - 2)/2$. Using the vector meson dominance model (VMD) for the pion transition form factor, we obtain $a_\mu^{\text{LBL},\pi^0} = +56 \times 10^{-11}$.

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The recent measurement of the muon anomalous magnetic moment of the muon by the E821 experiment in Brookhaven [1],

$$a_\mu = 116\ 592\ 020(160) \times 10^{-11} \tag{1}$$

significantly deviates from the Standard Model. Theoretical predictions, which include five-loop QED and two-loop electroweak effects, rely on experimental data and theoretical models to account for the non-perturbative hadronic contributions. Depending on the treatment of the latter, the discrepancy between the experiment and the theory can be as large as $2.6\,\sigma$. After the release of the E821 result (1), the hadronic effects came under renewed scrutiny and the reliability of various estimates has been disputed [2, 3]. The main focus of those discussions was the hadronic vacuum polarization which modifies the photon propagator and has been evaluated using data on e^+e^- annihilation into hadrons and the τ lepton hadronic decays [4, 5, 6].
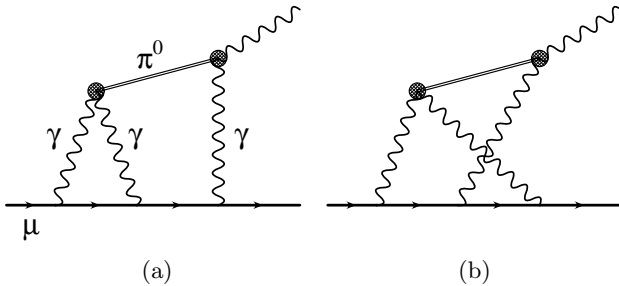

FIG. 1: Pion pole contributions to the muon $g - 2$.

Very recently it has been pointed out [7, 8] that a significant part of the discrepancy between the theory and the experiment may be due to the theoretical evaluation of the pion pole contribution to hadronic light-by-light scattering, which influences $g_\mu - 2$ via diagrams shown in Fig. 1. Namely, it has been claimed that although the magnitude of those contributions computed in [9, 10, 11, 12, 13] is correct, they had been taken with an incorrect (negative) sign.

Since the pion pole contribution is the largest among the hadronic light-by-light scattering contributions to a_μ, the change in sign has important implications for the comparison of the experimental result with the theoretical prediction. If the correct sign is positive, the theory and experiment are in much better agreement (about $1\,\sigma$ is removed from the reported discrepancy). Motivated by this, we have recalculated the pion pole contribution to the light-by-light scattering correction to a_μ. Using the vector dominance model for the pion transition form factor, we obtain

$$a_\mu^{\text{LBL},\pi^0} = +56 \times 10^{-11}, \tag{2}$$

thereby confirming the result of Ref. [7]. The purpose of this Letter is to describe the calculation leading to Eq. (2). We do not address the validity of the model. Our objective is to obtain an analytical (rather than numerical) result and settle the issue of its sign.

At low energies, the interaction of a neutral pion with photons is described by the Wess-Zumino-Witten Lagrangian,

$$\mathcal{L}_{\text{WZW}} = -\frac{\alpha N_c}{12\pi F_\pi} F_{\mu\nu} \widetilde{F}^{\mu\nu} \pi^0, \tag{3}$$

where $N_c = 3$ is the number of colors and $F_\pi \simeq 92.4$ MeV is the pion decay constant. Since this is a non-

renormalizable interaction, employing it in loop calculations results in ultraviolet divergences. While this is not a problem in principle, since divergent contributions can be absorbed into higher dimensional operators of the chiral Lagrangian, in practice this precludes any numerical estimate of the corresponding contributions because, as in the case of muon anomalous magnetic moment, the relevant counterterms are not known. In this situation one resorts to models to obtain a finite result. A simple and commonly adopted option is to introduce a form factor into the $\pi^0\gamma\gamma$ interaction vertex, which damps the contributions of highly virtual photons. This results in the following $\pi^0\gamma\gamma$ interaction vertex:

$$\frac{\alpha N_c}{3\pi F_\pi} F_{\pi^0\gamma\gamma}(q_1^2, q_2^2) i\epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta, \qquad (4)$$

where $q_{1,2}$ denote the momenta of the two outgoing photons.

The transition form factor $F_{\pi^0\gamma\gamma}(q_1^2, q_2^2)$ depends on the adopted model. For our purpose it is sufficient to use the simplest of these models (VMD) where it is assumed that the transition form factor reads

$$F_{\pi^0\gamma\gamma}(q_1^2, q_2^2) = \frac{M^2}{M^2 - q_1^2} \frac{M^2}{M^2 - q_2^2}, \qquad (5)$$

where the parameter M is phenomenologically determined to be close to the mass of the ρ meson $M \approx m_\rho \simeq 769$ MeV.

The simple structure of the form factor opens a way for analytic calculations. Initially, the diagram has three widely separated scales:

$$m_\pi^2 - m_\mu^2 \ll m_\mu^2 \ll M^2, \qquad (6)$$

and one can take advantage of this hierarchy to construct an expansion in $\delta \equiv (m_\pi^2 - m_\mu^2)/m_\mu^2$ and m_μ^2/M^2 using asymptotic expansions. The expansion in δ is especially simple and reduces to the Taylor expansion of the pion propagator. The expansion in m_μ^2/M^2 is the so-called Large Mass Expansion [14, 15, 16].

Consider first the diagram in Fig. 1(a). The lines denoted by γ-propagators are in fact products of the massless photon propagators $1/k_i^2$ and of the form factor terms $M^2/(k_i^2 - M^2)$. The vector meson mass M in the latter sets the hard scale for the momentum integrals, whereas the muon mass sets the soft scale. (After the Taylor expansion in δ these are the only mass scales present.) We can now obtain four possible combinations of soft/hard integration momenta in the two loops of Fig. 1(a). The leading quadratic logarithm of the muon and ρ mass ratio is obtained by adding contributions with both momenta soft, both momenta hard, and the one with the momentum in the triangle (left-hand side) loop hard, and the right-hand side loop soft. The fourth, soft-hard combination contributes only in the next order, suppressed by m_μ^2/M^2.

The non-planar diagram in Fig. 1(b) can be evaluated in a similar manner, with the only difference that a fifth momentum region appears. Namely, the two integration momenta may be hard, but their difference in the pion propagator may be soft. In this case the diagram factorizes into a product of two one-loop vacuum diagrams.

Integrations in all four regions can be carried out analytically, and coefficients in both expansion parameters, m_μ^2/M^2 and δ, can be obtained to an arbitrary order. Technically, the calculations involve integrations of two-loop single-scale diagrams of the type of either a muon propagator or a massive vacuum diagram. The most complicated topology arises when both loop momenta are soft, of order m_μ. This leads to two basic on-shell two loop diagrams of the self-energy type which can be evaluated using the recent results of Ref. [17].

The result can be written as

$$a_\mu^{\text{LBL},\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \frac{m_\mu^2}{F_\pi^2} \left(\frac{N_c}{\pi}\right)^2 X_{\pi^0}, \qquad (7)$$

and we find

$$\begin{aligned}
X_{\pi^0} &= \frac{1}{48}L^2 + \left(\frac{1}{96} - \frac{\pi}{48\sqrt{3}}\right)L - \frac{277}{10368} \\
&+ \frac{\pi}{24\sqrt{3}}S_2 - \frac{17\pi}{3456\sqrt{3}} + \frac{19}{128}S_2 - \frac{\zeta_3}{288} - \frac{11\pi^2}{15552} \\
&+ \frac{m_\mu^2}{M^2}\left[\frac{155}{1296}L^2 - \left(\frac{65}{1296} + \frac{\pi}{16\sqrt{3}}\right)L - \frac{11915}{62208}\right. \\
&\left. + \frac{\pi}{24\sqrt{3}}S_2 + \frac{\pi}{36\sqrt{3}} + \frac{39}{64}S_2 - \frac{1}{288}\zeta_3 + \frac{347\pi^2}{93312}\right] \\
&+ \delta\left[\left(\frac{1}{72} - \frac{\pi}{72\sqrt{3}}\right)L - \frac{1}{1296} + \frac{5}{96}\frac{\pi}{\sqrt{3}}S_2 - \frac{11\pi}{864\sqrt{3}}\right. \\
&\left. - \frac{1}{384}S_2 - \frac{1}{216}\zeta_3 + \frac{53\pi^2}{31104}\right] + \mathcal{O}\left(\frac{m_\mu^4}{M^4}, \delta^2\right), \quad (8)
\end{aligned}$$

where $L = \log(M/m_\mu)$, $\zeta_3 \simeq 1.202057$ is the Riemann zeta function and $S_2 = \frac{4}{9\sqrt{3}}\text{Cl}_2\left(\frac{\pi}{3}\right) \simeq 0.260434$.

Substituting $\alpha = 1/137.036$, $N_c = 3$, $M = 769$ MeV, $m_\mu = 105.66$ MeV, $m_\pi = 134.98$ MeV, and $F_\pi = 92.4$ MeV into Eq. (7) we obtain the result [18] in Eq. (2).

The result (2) is free from numerical errors, which does not mean that it represents the exact contribution of the pion pole to $g_\mu - 2$. The dependence of that result on the model and form factor parameters has been discussed in detail in [7].

Our result can be checked in several ways, in particular to ensure the correct treatment of the Feynman rules in a computer code [19]. For example, using the WZW Lagrangian one can evaluate the vacuum polarization contribution to $g_\mu - 2$ where the virtual photon splits into π^0 and γ, shown in Fig. 2. The contribution of this diagram to a_μ should be positive, since it can be related via dispersion relations to the cross section $\sigma(e^+e^- \to \pi^0\gamma)$ [21]. Indeed, we find

$$a_\mu^{\text{vp},\pi^0\gamma} = \left(\frac{\alpha}{\pi}\right)^3 \frac{m_\mu^2}{F_\pi^2} \left(\frac{Nc}{\pi}\right)^2 X_{\pi^0\gamma}, \qquad (9)$$
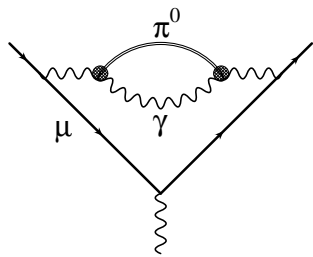
FIG. 2: Vacuum polarization contribution of the neutral pion.

where

$$X_{\pi^0\gamma} = \frac{L}{1296} + \frac{181}{15552} - \frac{\pi}{96\sqrt{3}} + \frac{7\pi^2}{7776} + \mathcal{O}\left(\frac{m_\mu^2}{M^2}, \delta\right).$$

(10)

Including several more terms in the m_μ/M and δ expansions, we obtain $a_\mu^{\mathrm{vp},\pi^0\gamma} \simeq 3.7 \times 10^{-11}$, a positive contribution.

It is also useful to compute the pole contribution of a (hypothetical) scalar meson "σ" with a mass close to m_μ. By analogy with other diagrams which can be evaluated for scalar and pseudoscalar contributions [22, 23, 24], one might expect that the leading ultraviolet logarithm L^2 in the scalar contribution has equal magnitude but opposite sign, compared to the pseudoscalar π^0 [25]. We have confirmed this by an explicit calculation, substituting $F^{\mu\nu}$ and σ for $\widetilde{F}^{\mu\nu}$ and π^0 in (3), and we find

$$a_\mu^{\mathrm{LBL},\sigma} = \left(\frac{\alpha}{\pi}\right)^3 \frac{m_\mu^2}{F_\pi^2} \left(\frac{Nc}{\pi}\right)^2 X_\sigma,$$

$$X_\sigma = -\frac{L^2}{48} + \frac{L}{48}\frac{\pi}{\sqrt{3}} - \frac{L}{288} + \frac{485}{10368} - \frac{5}{96}\frac{\pi}{\sqrt{3}}S_2$$
$$- \frac{29}{3456}\frac{\pi}{\sqrt{3}} - \frac{35}{15552}\pi^2 + \frac{\zeta_3}{288} - \frac{5}{128}S_2 + \mathcal{O}\left(\frac{m_\mu^2}{M^2}, \delta\right).$$

(11)

We see that the leading logarithm is indeed the same as in (8) except for the sign.

Let us also note that the calculation presented in this Letter can easily be extended to provide analytic results for the pion pole contribution when other models (see e.g. [7]) for the pion transition form factors are used. Those other form factors can be represented as linear combinations of heavy propagators $1/(q_1^2 - M^2)$ and $1/(q_2^2 - M^2)$, and the technique described in this Letter is applicable.

Our result confirms the positive sign of the pion pole contribution to the muon $g_\mu - 2$. We would like to note that this effect was found to be positive already several years ago in a correct numerical calculation [26]. It was an apparently simple algebraic mistake which led to the sign error in later works [27]. We hope that our analytical result will help better understand the dependence of this effect on the ultraviolet regulator (M) and thus place the theory of the muon $g - 2$ on firmer footing, as we await the release of the new Brookhaven experimental result.

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