SLAC-PUB-9054
November 2001

Vlasov equation with Coherent Synchrotron Radiation *

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*Work supported by Department of Energy contract DE–AC03–76SF00515.

1 Abstract

The coherent synchrotron radiation (CSR) can cause a bunch instability leading to the short wave length density modulation as it was considered in the previous publication [1]. Here, consideration of the problem is refined. The Vlasov equation for the instability is corrected in two respects taking into account the energy spread in a bunch and the delay time. Parameters defining the corrections are established and effects are estimated for two examples of PEP-II and LCLS chicane.

2 Introduction

In the recent paper [1], the possibility of the longitudinal micro-bunching driven by the coherent synchrotron radiation (CSR) was studied. It was shown, that there is a self-consistent regime where the microbunching within a bunch produces CSR which, in turn, produces the microbunching. For the case of the wave length small compared to the bunch length, the threshold of instability was obtained analyzing the Vlasov equation for a coasting beam. In this paper, this result is refined in two respects: first, the longitudinal CSR wake is defined taking into account the energy spread in a bunch and, secondly, the Vlasov equation is re-derived to take into account the retardation time. Parameters defining the magnitude of the corrections due to these effects are determined. In conclusion, the correction are estimated for PEP-II storage ring and for the LCLS chicane.

3 The Vlasov equation for the CSR

Let us consider a bunch of particles with the bunch centroid located at the azimuth $s = ct$ along a flat trajectory of a constant radius R. The bunch centroid trajectory will be the reference trajectory. Coordinates (z, δ) of a particle within a bunch define the shift of the particle along the reference trajectory from the bunch centroid ($z > 0$ in the head of a bunch) and the offset of the relative energy, $\delta = \Delta E/E$. Consider a particle with coordinates z_0, δ_0 at $t = 0$. A particle trajectory $z_{tr}(s, z_0, \delta_0)$, $\delta_{tr}(s, z_0, \delta_0)$ defines the position of a particle $r(s, z_0, \delta_0)$ at the moment $t = s/c$:

$$r(s, z_0, \delta_0) = \hat{r}[s + z_{tr}(s, z_0, \delta_0)](R + D\delta_{tr}[s, z_0, \delta_0]). \tag{1}$$

Here $\hat{r}(s)$ is the unit vector along the radius and $D = D(s)$ is the dispersion function at the position s. For simplicity, we assume that $D(s)$ changes slowly within the retardation length $(R^2\sigma)^{1/3}$ where σ is the rms bunch length. This allows us to treat the s-dependence of D adiabatically and consider D as a constant. The velocity of a relativistic particle $|v| = c$ is $\beta(s, z_0, \delta_0) = v/c = dr(s, z_0, \delta_0)/ds$, or

$$\beta(s, z_0, \delta_0) = \hat{\phi}(s + z_{tr})(1 + \eta\delta_{tr})(1 + \frac{dz_{tr}}{ds}) + \hat{r}(s + z_{tr})D\frac{d\delta_{tr}}{ds}. \tag{2}$$

Here we introduce $\eta = D/R$, and the unit vector $\hat{\phi}$ tangential to the trajectory. The unit vectors are related:

$$\frac{d\hat{r}(s)}{ds} = \frac{\hat{\phi}(s)}{R}, \quad \frac{d\hat{\phi}(s)}{ds} = -\frac{\hat{r}(s)}{R}. \tag{3}$$

The velocity of a relativistic particle is constant, $\beta^2 = 1$, what is consistent with the equation of motion

$$\frac{dz_{tr}}{ds} = \frac{-\eta\delta_{tr}}{1 + \eta\delta_{tr}}, \quad \frac{d\delta_{tr}}{ds} = 0. \tag{4}$$

The normalized distribution function $\rho(s, r, v)$, $\int drdv\rho = 1$, is defined by the trajectory $r(s, z_0, \delta_0)$, $v(s, z_0, \delta_0)$ with the initial conditions (z_0, δ_0) and by the normalized initial distribution of particles, $f_0(z_0, \delta_0)$,

$$\rho(s, r, v) = \int dz_0 d\delta_0 f_0(z_0, \delta_0)\delta[r - r(s, z + 0, \delta_0)]\delta[v - v(s, z_0, \delta_0)]. \tag{5}$$

Changing variables z_0, δ_0 to $z = z_{tr}(s, z_0, \delta_0)$, $\delta = \delta_{tr}(s, z_0, \delta_0)$, $dzd\delta = dz_0 d\delta_0$, and defining the distribution function $f(z, \delta, s) = f_0[z_{tr}(s, z_0, \delta_0), \delta_{tr}(s, z_0, \delta_0)]$, we get

$$\rho(s, r, v) = \int dzd\delta f(z, \delta, s)\delta[r - \hat{r}(s + z)(R + D\delta)]\delta[v - \hat{\phi}(s + z)]. \tag{6}$$

Integrating Eq. (6) over v, we obtain the density $\rho(r, s)$ and the current density $j(r, s)$ in terms of the distribution function of the bunch $f(z, \delta, s)$:

$$\rho(r, s) = \int dv\rho(s, r, v) = \int dzd\delta f(z, \delta, s)\delta[r - \hat{r}(s + z)(R + D\delta)], \tag{7}$$

$$j(r, s) = \int dvv\rho(s, r, v) = \int dzd\delta f(z, \delta, s)\hat{\phi}(s+z)\delta[r-\hat{r}(s+z)(R+D\delta)]. \tag{8}$$

Let us introduce coordinate r_n, r_t as projections of the vector r on the unit vectors at s,

$$r = \hat{r}(s)r_n + \hat{\phi}(s)r_t; \quad \nabla = \hat{r}(s)\frac{\partial}{\partial r_n} + \hat{\phi}(s)\frac{\partial}{\partial r_t}.. \tag{9}$$

It is easy to see from Eq. (6), that the continuity equation

$$\frac{\partial \rho(r,s)}{\partial s} + \nabla j(r,s) = 0, \tag{10}$$

requires that the distribution function $f(z,\delta,s)$ satisfies the Liouville's equation

$$\frac{\partial f(z,\delta,s)}{\partial s} - \frac{\eta\delta}{1+\eta\delta}\frac{\partial f}{\partial z} = 0. \tag{11}$$

Here we used relations

$$\hat{r}(s+z) = \hat{r}(s)\cos[z/R] + \hat{\phi}\sin[z/R]; \quad \hat{\phi}(s+z) = -\hat{r}(s)\sin[z/R] + \hat{\phi}\cos[z/R]. \tag{12}$$

The Vlasov equation takes into account the variation of the energy due to the radiation field,

$$\frac{\partial f(z_l,\delta_l,s)}{\partial s} - \frac{\eta\delta_l}{1+\eta\delta_l}\frac{\partial f}{\partial z_l} + \frac{d\delta_l}{ds}\frac{\partial f}{\partial \delta_l} = 0, \tag{13}$$

where, for a bunch with the bunch population N_b,

$$\frac{d\delta_l}{ds} = \frac{eN_b}{E}\beta(s)\cdot\mathcal{E}(r_l,s), \tag{14}$$

and $r_l = \hat{r}(s+z_l/R)(R+D\delta_l)$ is the vector defining the test (leading) particle with coordinates z_l, δ_l.

The field $\mathcal{E}(r,s)$ is well known [2],

$$\begin{gathered}\mathcal{E}(r,s) = -\tfrac{\partial}{\partial s}A(r,s) - \nabla\Phi(r,s),\\ A(r,s) = e\int \tfrac{dr'}{|r'-r|}j(r',s-|r'-r|),\ \Phi(r,s) = e\int \tfrac{dr'}{|r'-r|}\rho(r',s-|r'-r|),\end{gathered} \tag{15}$$

where ρ and j are defined by Eqs. (7) and (8). Combining Eqs. (15),(7),(8), and changing integration over r' to integration over $\xi = r' - r$, the field can be written in the form

$$\begin{aligned}\mathcal{E}(r,s) = -e\int dzd\delta \int \tfrac{d\xi}{|\xi|}\{&\tfrac{\partial}{\partial s}[\hat{\phi}(s+z-|\xi|)f(z,\delta,s-|\xi|)\delta(r+\xi-\hat{r}(s+z-|\xi|)(R+D\delta))]\\ &+\tfrac{\partial}{\partial r}[f(z,\delta,s-|\xi|)\delta(r+\xi-\hat{r}(s+z-|\xi|)(R+D\delta))]\}_{r=r_l}.\end{aligned} \tag{16}$$

The first term can be simplified using Eq. (11) and integrating over z by parts. Then,

$$\mathcal{E}(r,s) = -e\int dzd\delta\int\frac{d\xi}{|\xi|}f(z,\delta,s-|\xi|)U(\xi), \tag{17}$$

where

$$U(\xi) = \{[\frac{\hat{\phi}(s+z-|\xi|)}{1+\eta\delta}\frac{\partial}{\partial z} - \frac{\hat{r}(s+z-|\xi|)}{(1+\eta\delta)R} + \frac{\partial}{\partial r}]\delta[r+\xi-\hat{r}(s+z-|\xi|)(R+D\delta)]\}_{r->r_l}. \tag{18}$$

Integration over ξ can be now carried out using formula

$$\int d\xi F(\xi)\delta[r+\xi-\hat{r}(s+z-|\xi|)(R+D\delta)] = \frac{F(\xi)|\xi|}{|\xi|+(1+\eta\delta)[\xi_t\cos(\frac{z-|\xi|}{R})-\xi_n\sin(\frac{z-|\xi|}{R})]}. \tag{19}$$

After integration, $\xi(r)=\hat{r}(s)\xi_n(r)+\hat{\phi}(s)\xi_t(r)$ depends on r and is defined by the equation

$$\xi(r) = -r+\hat{r}(s+z-|xi(r)|)(R+D\delta). \tag{20}$$

Derivatives in U can be calculated using the following formula:

$$\frac{\partial f[\xi(r)]}{\partial r_\alpha} = (\frac{\partial f[\xi(r)]}{\partial \xi_\beta})(\frac{\partial \xi_\beta(r)}{\partial r_\alpha}), \tag{21}$$

where $(\partial\xi_\beta(r)/\partial r_\alpha)$ can be defined differentiating Eq. (20). That gives the system of four linear equations

$$M_{\beta,\mu}\frac{\partial \xi_\mu}{\partial r_\alpha} = -\delta_{\alpha,\beta}. \tag{22}$$

The coefficients

$$M_{\beta\mu} = \delta_{\beta\mu}+\frac{1+\eta\delta}{|\xi|}\xi_\mu\hat{\phi}_\beta(s+z-|\xi|) \tag{23}$$

depend on $\xi(r)$, Eq. (20), taken at $r=r_l$. After such a substitution, Eq. (20) defines the retardation length $\xi(z,z_l,\delta,\delta_l)$. For small $|\xi|<<R$, Eq. (20) can be simplified expanding all terms in the series over ξ. Eq. (20) is reduced to the polynomial equation for $x=|\xi|/R$,

$$x^4 = 12[(\delta_l+\delta)\eta x^2-(\delta_l-\delta)\eta^2+2x(z_l-z)/R]. \tag{24}$$

Let us compare terms in Eq. (24) for typical $z\simeq z_l\simeq\sigma$ and $\delta\simeq\delta_l\simeq\delta_0$ where σ and δ_0 are the rms bunch length and the rms relative energy spread, respectively. For small δ_0,

$$(\sigma/R) >> (\eta\delta_0)^{3/2}, \tag{25}$$

terms proportional to δ, δ_l are negligibly small and the retardation time is given by the well known formula

$$\xi = [24R^2(z_l-z)]^{1/3}. \tag{26}$$

Because $\xi>0$ by casuality, the solution exists only for $z_l>z$, i.e. for the test particle ahead of the radiating particle. In the opposite extreme case, $(\sigma/R)<<(\eta\delta)^{3/2}$,

$$\xi = R\sqrt{\eta(\delta+\delta_l)}. \tag{27}$$

Eqs. (13)-(19) define the Vlasov equation:

$$\frac{\partial f(z_l,\delta_l,s)}{\partial s}-\frac{\eta\delta_l}{1+\eta\delta_l}(\frac{\partial f}{\partial z_l})-\frac{r_eN_b}{\gamma}\frac{\partial f(z_l,\delta_l,s)}{\partial\delta_l}\int dzd\delta f(z,\delta,s-|\xi|)W(z_l-z,\delta,\delta_l)=0, \tag{28}$$

where the wake W per unit length calculated using MATHEMATICA is

$$W=-\frac{36}{R^2}\frac{9x^2(z_l-z)/R-6\eta(\delta_l-\delta)(z_l-z)/R+4\eta(\delta_l+2\delta)x^3}{[x^3-6(\delta+\delta_l)\eta x-6(z_l-z)/R]^3}. \tag{29}$$

In the limit of a small energy spread Eq. (25), $x=[24(z_l-z)/R]^{1/3}$, and Eq. (29) gives the well known result [3], [4]

$$W(z_l-z)=-\frac{2}{3^{4/3}}\frac{1}{(z_l-z)^{4/3}R^{2/3}}. \tag{30}$$

This wake was used in the analysis of CSR instability [1]. For a monochromatic bunch $\delta_0\to 0$, the finite relativistic factor γ has to be taken into account [3] to retain the finite wake at small $(z_l-z)/R<1/\gamma^3$. Eq. (29) shows that the finite energy spread plays the same role† and, in a sense, the finite energy spread is equivalent to effective $\gamma_{ef}=1/\sqrt{\eta\delta}$.

4 Effect on beam dynamics

For the bunch length satisfying condition Eq. (25), the effect of the finite energy spread can be neglected. As examples, we consider two cases: of the PEP-II and of the LCLS chicane. The relevant parameters are listed in the table. The energy spread listed for the chicane is the correlated energy spread δ_{corr}, the uncorrelated δ is much smaller, $\delta=1.5\,10^{-5}$.

Parameter	Symbol	PEP-II	Chicane
energy	E, GeV	3.1	4.5
bunch population	N_b	10^{11}	$6.5\,10^9$
bend radius	R,m	13.5	15
momentum compaction	α	$1.23\,10^{-3}$	
dispersion	D, m		0.2
rms length	σ, m	1.110^{-2}	2.010^{-4}
energy spread	δ_0	$0.77\,10^{-3}$	$0.76\,10^{-2}$

For both examples, the condition Eq. (25) is well satisfied. In particular, for the chicane, σ/R is larger than $[(D/R)\delta_{corr}]^{3/2}$ by an order of magnitude. This allows us to use the usual wake Eq. (refeq160).

†This comment belongs to G. Stupakov

However, even in this case the Vlasov equation Eq. (28) is still different from the one used in our preliminary study [1] due to retardation length in the argument of the integral term. Following the same approach as in reference [1], let us use the dimensionless variable $p = \delta/\delta_0$ and change the norm $\int dpdzf(z,p,s) = 1$. The Vlasov equation takes the following form:

$$\frac{\partial f(z_l,\delta_l,s)}{\partial s} - \frac{\eta\delta_l}{1+\eta\delta_l}\frac{\partial f}{\partial z_l}$$

$$-\frac{r_e N_b}{\gamma}\frac{\partial f(z_l,\delta_l,s}{\partial \delta_l}\int dzd\delta f(z,\delta,s-[24R^2(z_l-z)]^{1/3})W(z_l-z,\delta,\delta_l) = 0, \quad (31)$$

where $W(z_l - z)$ is given by Eq. (30). To study the beam dynamics at the threshold of instability, consider a perturbation with the frequency ω and the wave vector k,

$$f(z,p,s) = f_0(p) + f_{\omega,k}(p)e^{-i\omega s/c+ikz}. \quad (32)$$

The amplitude $f_{\omega,k}(p)$ satisfy the linearized Vlasov equation,

$$(\omega + kc\eta\delta_0 p)f_{\omega,k}(p) = -i\frac{N_b r_e c}{\gamma\delta_0}\frac{df_0}{dp}Z_{ef}(k,\omega)\int dp' f_{\omega,k}(p'). \quad (33)$$

Here Z_{eff} is not just the Fourier component of the wake $W(z_l - z)$ but is defined also by the retardation:

$$Z_{eff}(k,\omega) = -2i(\frac{k}{3R^2})^{1/3}\int_0^\infty \frac{dx}{x^{1/3}}[1-\frac{1}{3}\frac{\omega}{c}(\frac{24R^2}{kx^2})^{1/3}]e^{-ix+i(\omega/c)(24R^2x/k)^{1/3}}. \quad (34)$$

The integral has logarithmic divergence at small $x = k(z_l - z)$ which is regularized by the energy spread Eq. (29).

Eq. (33) leads to the dispersion equation

$$1 = -\frac{ir_e c}{\gamma\delta_0}Z_{eff}(k,\omega)\int\frac{dpd\rho_0/dp}{\omega + ck\eta\delta_0 p}. \quad (35)$$

This equation is the same as the dispersion equation of the reference [1] where, however, the impedance $Z(\omega) = iA(k/R^2)^{1/3}$, $A = 1.63i - 0.94$, is replaced by $Z_{eff}(k,\omega) = iAF(g)(k/R^2)^{1/3}$. The correction factor F is defined by the integral

$$F(g) = \frac{3\sqrt{3}\Gamma(1/3)}{\pi g^2(1-i\sqrt{3})}\int ydye^{iy-i(y/g)^3}[1-\frac{1}{3}\frac{g^3}{y^2}], \quad (36)$$

where $g = (\omega/c)(24R^2/k)^{1/3}$.

For small $g << 1$,

$$F(g) = 1 + \frac{g\sqrt{3}\Gamma(1/3)}{\pi(1-i\sqrt{3})}[1.192 + i\frac{\pi}{6} - \ln g_{min}]. \quad (37)$$

For large g, Re$F(g)$ growth approximately linearly with g, while Im$F(g)$ changes sign, see Fig. 1.

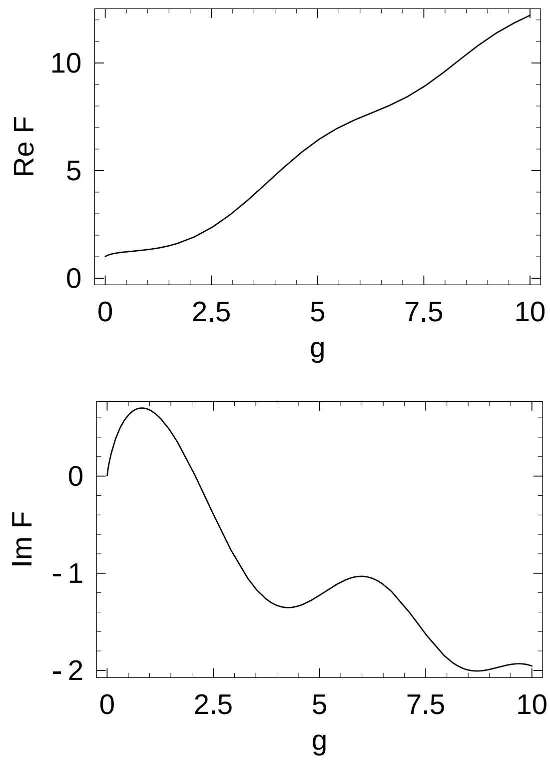

Figure 1: *Correction due to the retardation time,* $F(g)$ *vs* $g = (\omega/c)(24R^2/k)^{1/3}$.

The threshold of instability defined in the reference [1] neglecting retardation. The beam is unstable for the wave vector k

$$(kR)^{2/3} < 1.6\Lambda, \quad \Lambda = \frac{r_e N_b}{\eta\gamma\delta_0^2\sigma\sqrt{2\pi}}. \tag{38}$$

At the threshold, $\mathrm{Im}\,\omega = 0$ and $\mathrm{Re}\,(\omega/c) \simeq k\eta\delta_0$, what corresponds to

$$g_{th} \simeq \eta\delta_0(24)^{1/3}(kR)^{2/3}. \tag{39}$$

The maximum g for an unstable beam is, therefore, limited: $g < 1.6(24)^{1/3}\Lambda\eta\delta_0$, or

$$g_{th} \leq \frac{2r_e N_b}{\gamma\delta_0\sigma\sqrt{2\pi}}. \tag{40}$$

For PEP parameters, this quantity is of the order of $g_{th} \simeq 0.02$ and the correction due to retardation is negligibly small. This is also true for chicane parameters provided the correlated δ_0 is used in the estimate. The uncorrelated energy spread would give large g_{th}, but one may argue that then N_b is defined by the number of particles within the correlation length, which is small.

5 Conclusion

The Vlasov equation for the CSR wake is derived taking into account the energy spread in the bunch and the retardation time. The energy spread modifies the wake. This effect depends on the parameter $(\eta\delta_0)^{3/2}(R/\sigma)$. For the examples considered above, this parameter is much less than one and the correction is small. Another effect, the retardation, does not change the form of the dispersion equation but re-define the effective impedance. Impedance is modified by the additional factor $F(g)$ which depends on the parameter $g \simeq 2r_e N_b/[\gamma\delta_0\sigma\sqrt{2\pi}]$. At large N_b or small energy spread, this parameter may be large. This, however, does not happen into the examples given above. Therefore, our previous results [1] remain valid.

6 Acknowledgments

I thank G. Stupakov for useful discussions.

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