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Non-Commutativity and Unitarity Violation in Gauge Boson Scattering *

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Abstract

We examine the unitarity properties of spontaneously broken non-commutative gauge theories. We find that the symmetry breaking mechanism in the non-commutative Standard Model of Chaichian *et al.* leads to an unavoidable violation of tree-level unitarity in gauge boson scattering at high energies. We then study a variety of simplified spontaneously broken non-commutative theories and isolate the source of this unitarity violation. Given the group theoretic restrictions endemic to non-commutative model building, we conclude that it is difficult to build a non-commutative Standard Model under the Weyl-Moyal approach that preserves unitarity.

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1 Introduction

The possibility of non-commuting space-time coordinates is an intriguing one which arises naturally in string theory and gives rise to a rich phenomenology[1, 2]. As such, noncommutative quantum field theory (NCQFT) has the potential to provide an attractive and motivated theory of physics beyond the Standard Model (SM). However, while a consistent noncommutative (NC) version of QED has been developed, a NC analog of the full SM has proven to be problematic for reasons detailed below. Possible NC gauge groups and matter representations are theoretically restricted, and as we will show, the basic difficulty lies in constructing a viable symmetry breaking mechanism which reduces the NC gauge group to that of the SM. We investigate a variety of models and demonstrate that the NC symmetry breaking mechanism in a leading candidate for a NCSM[3] leads to an unavoidable violation of tree-level unitarity in gauge boson scattering.

NCQFT can be realized[4] in the string theoretic limit where string endpoints propagate in the presence of background fields. In this case, the endpoints of strings no longer commute and their coordinate operators obey the relation

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} = \frac{i}{\Lambda_{NC}^2} c_{\mu\nu} \,, \qquad (1)$$

where $\theta_{\mu\nu}(c_{\mu\nu})$ is a constant, real, anti-symmetric matrix and Λ_{NC} represents the scale where NC effects set in. The most likely value for the NC scale is that of the string or fundamental Planck scale which, in principle, can be of order a TeV. Space-space noncommutativity occurs in the presence of background magnetic fields with $\theta_{ij} \equiv (\hat{c}_B)_{ij}/\Lambda_B^2 \neq 0$, whereas space-time NC theories are related to background electric fields with $\theta_{0i} \equiv (\hat{c}_E)_{0i}/\Lambda_E^2 \neq 0$. Note that the unit vectors $\hat{c}_{B,E}$ are frame independent and hence Lorentz invariance will be violated at the $\Lambda_{B,E}$ NC scales[5]. These theories conserve CPT and have been claimed to be unitary if the condition $\theta_{\mu\nu}\theta^{\mu\nu} \geq 0$ is satisfied[6]. While this condition might hold for unbroken NC theories, we will show below that it is not sufficient to guarantee unitarity for all choices of Higgs representations.

There are two approaches for constructing quantum field theories on NC spaces. The first relates fields in the NC and the usual R^4 spaces via the Weyl-Moyal correspondence. In this case, ordinary fields are replaced by their NC counterparts and the ordering of fields in the NC space is given by the Moyal star product, defined by

$$\begin{aligned}\hat{A}(\hat{x})\hat{B}(\hat{x}) &= A(x) * B(x) \\ &= A(x) \exp\{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\} B(x)\,,\end{aligned} \tag{2}$$

where the hatted quantities correspond to those in the NC space. NCQFT is formulated as that of ordinary QFT with star products substituting for products and commutators replaced by Moyal brackets,

$$[A, B]_{MB} = A * B - B * A\,, \tag{3}$$

with $\int d^4x[A(x), B(x)]_{MB} = 0$. Only $U(n)$ Lie algebras are closed under Moyal brackets[7] and hence NC model building is restricted in this case. The simplest version of a NCSM thus requires a gauge group at least as large as $U(3) \times U(2) \times U(1)$.

The second approach relies on the Seiberg-Witten map which represents the NC fields in terms of the corresponding fields in ordinary R^4 as a power series expansion in $\theta_{\mu\nu}$. While this procedure allows for generalizations of the star product to be constructed for a broader set of gauge theories[8], it generates an infinite set of higher dimensional operators. It is hence difficult to employ and is not necessarily attractive as a model building option. We neglect this possibility here.

NC extensions have been shown to preserve renormalizability at the one- to two-loop level in the cases of unbroken[9] ϕ^4 and Yang-Mills theories, as well as for spontaneously broken[10] $U(1)$ and $U(2)$ at one-loop.

Ordinary gauge transformations for $U(n)$ gauge theories must be modified[11] to include NC generalizations. Gauge invariance requires non-abelian like gauge couplings, even for $U(1)$ theories, and dictates that matter only be placed in the singlet, (anti-)fundamental, or adjoint representations. In addition, some interaction vertices pick up momentum dependent phase factors. In the NC version of QED, this induces 3- and 4-point self-couplings for photons and restricts QED interactions to particles of charge $q = 0, \pm e$. Despite these limitations, $2 \to 2$ scattering processes at high energies in QED provide clean observables[12] for NCQED effects. In addition, several low-energy QED processes[13], *e.g.,* Lamb shift, $g-2$, can probe NC interactions.

There are clear difficulties in formulating a NCSM based on the Weyl-Moyal approach due to the restrictions imposed from group theory and gauge invariance: (i) charge quantization, *e.g.*, NCQED cannot contain fractionally charged particles, and (ii) performing the NC $U(n)$ symmetry breaking without generating a mass for the gauge fields of the $SU(n)$ component of the $U(n)$. A first attempt to build a NCSM by Chaichian *et al.*[3] is based on the gauge group $U(3)_c \times U(2)_L \times U(1)$ and at first appears to resolve these issues. However, we find that the model proposed by these authors leads to unitarity violation at high energies in $2 \to 2$ gauge boson scattering. By studying similar scattering processes in more simplified NC gauge theories, we have isolated the cause of the unitarity breakdown in this model to be the choice of NC symmetry breaking mechanism.

A review of the Chaichian *et al.* model and our calculations of gauge boson scattering within it is given in the next section. Our study of potential unitarity violation in $2 \to 2$ gauge scattering in a variety of simpler NC models is presented in Section III, and a discussion of our results is given in Section IV.

2 The Non-commutative Standard Model

As was discussed in the introduction, group theory and gauge invariance enforce strict constraints on the construction of NC models following the conventional Moyal approach. In addition to the requirement that theories be built out of products of $U(n)$ factors and that only the singlet (S), fundamental (F), anti-fundamental ($\bar{F}$) or adjoint (A) representations are allowed for matter fields (*i.e.*, $Q = 0, \pm 1$ for the case of $U(1)$), Chaichian *et al.* have shown [14] that a "No Go" theorem is operative. This theorem states that for any gauge group consisting of a product of simple group factors, matter fields can only transform nontrivially under at most two of the simple groups. In particular, if a field transforms under one factor as a F then it must transform under the second as an $\bar{F}$. Note that while SM matter fields are already in fundamental representations, the left-handed quark doublet(Q_L) transforms nontrivially under all *three* SM group factors. This clearly complicates the construction of the NCSM.

The minimal group structure for the NCSM which can contain $SU(3)_c \times SU(2)_L \times U(1)_Y$ in the commutative limit is $U(3)_c \times U(2)_L \times U(1)$; this has both positive and negative features for NCSM construction. On the positive side, the SM fractional hypercharges may receive contributions arising from the $U(1)$ factor as well as from the two $U(1)$ subgroups, $U(1)_{c,L}$, contained in $U(3)_c$ and $U(2)_L$ so that the constraint of $U(1)$ charge $Q = 0, \pm 1$ assignments can be satisfied. However, the increase in group rank by two implies that there are now two new additional neutral weak bosons in the theory that will have couplings to SM matter fields. These two new states must be made sufficiently massive as to avoid present Tevatron direct search constraints [15] as well as those arising from precision electroweak data [16]; we thus expect their masses to be greater than a TeV or so. In addition, spontaneous symmetry breaking must take place in at least two steps to recover the correct phenomenological structure and have the correct SM commutative limit. Effectively, this means that the $U(1) \times U(1)_c \times U(1)_L$ symmetry must first break to $U(1)_Y$ *without* breaking the $SU(3)_c$ or $SU(2)_L$ groups themselves.

Field	$U(3)$	$U(2)$	$U(1)$
Q_L	$\bar{F}$	F	S
L_L	S	F	$\bar{F}$
e_R	S	S	$\bar{F}$
u_R	$\bar{F}$	S	F
d_R	$\bar{F}$	S	S
h	S	F	S

Table 1: Quantum number assignments for the SM fermion and Higgs fields in the NCSM using the notation discussed in the text.

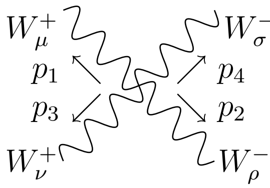
In a recent paper, Chaichian [3] have apparently either overcome or satisfied all of the above obstacles and constructed a NC version of the SM. The SM matter content of this proposed construction is given in Table 1. As can be seen the representation content explicitly satisfies the No-Go theorem. Next, the authors perform the breaking of the $U(3)_c \times U(2)_L \times U(1)$ symmetry in several steps: first, two linear combinations of the $U(1)$'s within $U(1)_c \times U(1)_L \times U(1)$ must be broken. As this is a product of three group factors the No-Go theorem requires that two Higgs fields are necessary to break this symmetry down to $U(1)_Y$. It is clear that these Higgs fields cannot correspond to any of the usual $SU(n)$ representations since then their vev's would not only break the $U(1)$ subgroups of the $U(n)$'s but the $SU(n)$ factors as well. To get around this, Chaichian *et al.* [3] have considered Higgs fields in a new representation, which they call Higgsac (H) fields. They transform only under the $U(1)$ subgroup of $U(n)$, so that when they acquire a vev the $SU(n)$ subgroup remains unbroken. Essentially, a Higgsac breaks $U(1)_c \times U(1)_L$ to $U(1)'$ and a second Higgsac then produces the breaking $U(1)' \times U(1)$ to $U(1)_Y$. Given the two distinct Higgsac vev's, the masses of the two heavy gauge bosons are *uncorrelated* and arbitrary. To be more specific, denoting the original weak eigenstate $U(1)_c \times U(1)_L \times U(1)$ gauge fields by $G^{0'}, W^{0'}$ and B, respectively, the weak and the unprimed mass eigenstate fields are related by: $G^{0'} = c_{23}G^0 + s_{23}(c_{11}W^0 + s_{11}Y)$, $W^{0'} = -s_{23}G^0 + c_{23}(c_{11}W^0 + s_{11}Y)$, and $B = -s_{11}W^0 + c_{11}Y$. Here W^0, G^0 are the new massive gauge bosons while Y is the massless boson coupling to hypercharge and $c_{ij}(s_{ij}) = \cos\delta_{ij}(\sin\delta_{ij})$, with δ_{ij} being appropriate mixing angles. As mentioned above we anticipate that these new gauge bosons are more massive than about ~ 1 TeV. Given the gauge couplings $g_{1,2,3}$ for the appropriate $U(n)$ groups, the mixing angles can be expressed as $\tan\delta_{23} = 2g_2/3g_3$ and $2c_{23}\tan\delta_{11} = g_1/g_2$. In addition, consistency with the fermion couplings of the SM in the commutative limit implies that $g_3 = g_s$, the usual QCD coupling, $g_2 = g$, the usual weak coupling, $g_1 = 2g'/c_{11}$, with g' being the usual hypercharge coupling of the SM, and also the relation $s_{11}c_{23} = \tan\theta_W$.

The last step of the symmetry breaking is accomplished through the vev of the isodoublet field h listed in the Table. It generates the ordinary fermion masses in the usual manner as well as the conventional $W^\pm$ and Z masses with the identification $e = g\sin\theta_W$. There is however one additional effect: mixing is induced between the SM Z field and the more massive G^0, W^0 states which is of order $m_Z^2/m_{G,W}^2$; for TeV or heavier states we expect this mixing to be quite small, ≤ 0.01, and can be safely neglected on most occasions. Given the values of the couplings, all of the mixing angles are fixed so that the only free parameters in the gauge sector of the model are the masses of the G^0 and W^0 states. Of course the scales $\Lambda_{E,B}$ associated with the NC physics, as defined in the introduction, still remain arbitrary.

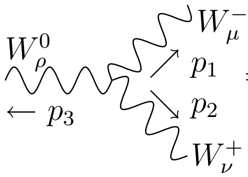
Before examining the phenomenology of this model, we may first want to check whether or not the theory is unitary at tree-level. The classic test for this in the SM is the scattering of pairs of longitudinal $W^{\pm}$, *i.e.*, $W_L^+ W_L^- \to W_L^+ W_L^-$. This process provides a particularly useful test in the NCSM case as it is independent of how the fermions are embedded into the theory. In the SM at amplitude level, the leading terms from a typical diagram behave as (s^2/m_W^4); such terms cancel among the contributions from the s- and t-channel γ and Z exchanges and the 4-point graph due to gauge invariance and the natural relationship $m_W = m_Z \cos\theta_W$. The sub-leading s/m_W^2 contributions also cancel when s- and t-channel Higgs boson exchanges are included yielding a result which does not grow with s and is unitary. In the NCSM, as we will see below, the leading terms still cancel because of the gauge symmetry, but the subleading terms remain so that unitarity is not satisfied. Let us now discuss these cancellations in some detail.

To be as explicit and straightforward as possible, we will separately calculate both the leading and next-to-leading s/m_W^2 terms for each of the contributing diagrams to demonstrate how the required cancellations fail to occur in the NCSM. For simplicity, these calculations will be performed in the limit that the weak scale mixing induced between the Z and the heavy states W^0 and G^0 can be neglected. The additional terms that are generated by such mixing are subleading to the ones included below by factors of order $m_Z^2/m_{G,W}^2$ and will have no influence upon our results.

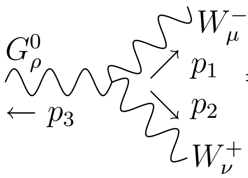
Using the couplings derived in [9] for NC $U(n)$ gauge theories, we can now generate all of the Feynman rules for this model. The relevant ones for $W_L^+ W_L^- \to W_L^+ W_L^-$ are presented in Fig. 1. Here, we have introduced the wedge product which is defined as $p_i \wedge p_j = \frac{1}{2} p_i^\mu p_j^\nu \theta_{\mu\nu}$, and the direction of the momenta are as labeled. The kinematic phases at each vertex, which arise from the Fourier transformation of the interaction term into momentum space, mentioned above are explicitly apparent. To be specific, we will perform our calculation in



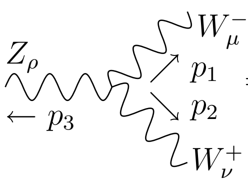
$$= ig^2 \Big\{ \left[\cos\left(p_1 \wedge p_2 + p_3 \wedge p_4\right) + \cos\left(p_1 \wedge p_4 + p_2 \wedge p_3\right)\right] g_{\mu\nu} g_{\rho\sigma} - \cos\left(p_1 \wedge p_4 + p_2 \wedge p_3\right) g_{\mu\rho} g_{\nu\sigma} - \cos\left(p_1 \wedge p_2 + p_3 \wedge p_4\right) g_{\mu\sigma} g_{\nu\rho} \Big\}$$



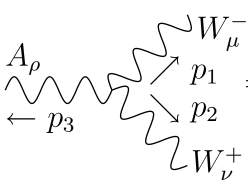
$$= -g\, c_{23} c_{11} \sin\left(p_1 \wedge p_2\right) \left[(p_1 - p_2)_\rho g_{\mu\nu} + (p_2 - p_3)_\mu g_{\nu\rho} + (p_3 - p_1)_\nu g_{\mu\rho}\right]$$



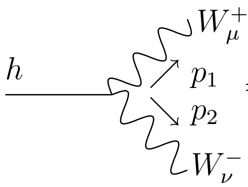
$$= g\, s_{23} \sin\left(p_1 \wedge p_2\right) \left[(p_1 - p_2)_\rho g_{\mu\nu} + (p_2 - p_3)_\mu g_{\nu\rho} + (p_3 - p_1)_\nu g_{\mu\rho}\right]$$



$$= -g \left\{ i c_W \cos\left(p_1 \wedge p_2\right) - c_{23} s_{11} s_W \sin\left(p_1 \wedge p_2\right) \right\} \Big[(p_1 - p_2)_\rho g_{\mu\nu} + (p_2 - p_3)_\mu g_{\nu\rho} + (p_3 - p_1)_\nu g_{\mu\rho} \Big]$$



$$= -g \left\{ i s_W \cos\left(p_1 \wedge p_2\right) + c_{23} s_{11} c_W \sin\left(p_1 \wedge p_2\right) \right\} \Big[(p_1 - p_2)_\rho g_{\mu\nu} + (p_2 - p_3)_\mu g_{\nu\rho} + (p_3 - p_1)_\nu g_{\mu\rho} \Big]$$

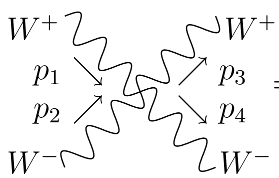


$$= ig\, m_W e^{i\, p_1 \wedge p_2} g_{\mu\nu}$$

Figure 1: NCSM Feynman rules relevant for $W_L^+ W_L^- \to W_L^+ W_L^-$.

the center-of-mass frame. This is sufficiently general as all Lorentz violation is isolated in the wedge products which we never need to explicitly evaluate. We now turn to the individual contributions.

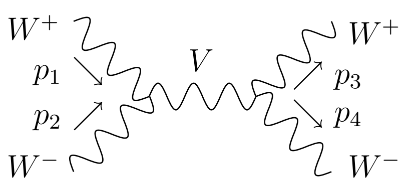
- The four-point graph gives the following $\mathcal{O}\left(s^2/m_W^4, s/m_W^2\right)$ contributions to the amplitude:



$$
\begin{aligned}
&= \left(ig^2 s^2/16m_W^4\right) \Big\{ \left(1+c_\theta\right)^2 \left[\cos\left(p_1 \wedge p_2 - p_3 \wedge p_4\right) + \cos\left(p_1 \wedge p_3 - p_2 \wedge p_4\right)\right] \\
&\quad - \left(1-c_\theta\right)^2 \cos\left(p_1 \wedge p_2 - p_3 \wedge p_4\right) - 4\cos\left(p_1 \wedge p_3 - p_2 \wedge p_4\right) \Big\} \\
&\quad + \left(ig^2 s/2m_W^2\right) \Big\{ \left(1-c_\theta\right)\cos\left(p_1 \wedge p_3 - p_2 \wedge p_4\right) \\
&\quad -2\, c_\theta \cos\left(p_1 \wedge p_2 - p_3 \wedge p_4\right) \Big\} ,
\end{aligned}
$$

where we have introduced the abbreviation $\cos(\theta) = c_\theta$, with θ being the scattering angle between the p_1 and p_3 three-momenta.

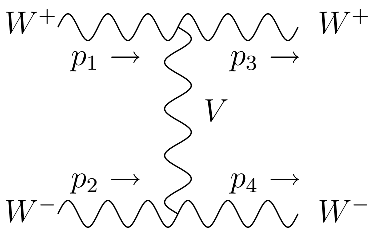
- The amplitude for the s-channel exchange for a generic gauge boson V is given by



$$
= \left(is^2/4m_W^4\right) A_V^{21} A_V^{34}\, c_\theta + \left(ism_V^2/4m_W^4\right) A_V^{21} A_V^{34}\, c_\theta \ ,
$$

where the quantities A_V^{ij} are obtained from the coefficients of the gauge 3-point couplings in Fig. 1, with the ij denoting the momentum ordering in the wedge product. For example, $A_{G^0}^{21} = g\, s_{23} \sin\left(p_2 \wedge p_1\right)$ and $A_\gamma^{21} = -igs_W \cos\left(p_2 \wedge p_1\right) - gc_{23}s_{11}c_W \sin\left(p_2 \wedge p_1\right)$. For the full amplitude we must sum over all possible intermediate states $V = \gamma, Z, G^0$ and W^0.

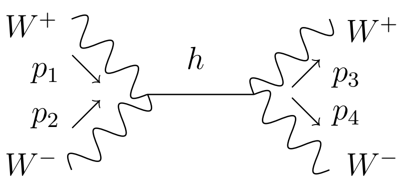
- The t-channel exchange of a generic gauge boson V yields the following expression:



$$= -\,(is^2/16m_W^4)\,A_V^{13}A_V^{42}\;(1-c_\theta)\,(3+c_\theta) + (is/8m_W^2)\,A_V^{13}A_V^{42}\Big\{-16c_\theta + (m_V^2/m_W^2)\,(3+c_\theta)\Big\}\,,$$

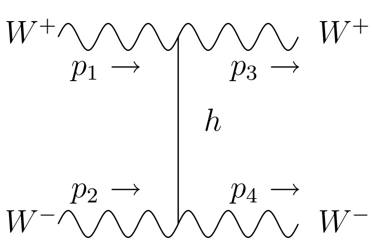
where the A_V^{ij} are as defined above. Again, to obtain the full amplitude we must sum over $V=\gamma, Z, G^0$ and W^0.

- The s-channel SM Higgs exchange yields



$$= -\,(ig^2 s/4m_W^2)\,\exp\left(ip_1\wedge p_2 - ip_3\wedge p_4\right)\;,$$

- similarly the t-channel Higgs exchange gives the corresponding result



$$= (ig^2 s/8m_W^2)\,(1-c_\theta)\,\exp\left(-ip_1\wedge p_3 + ip_2\wedge p_4\right)\;.$$

Note that there are no Higgsac contributions as they do not couple to the SM gauge fields.

Summing these expressions, using the relations

$$p_1\wedge p_3 - p_2\wedge p_4 = p_1\wedge p_2 - p_3\wedge p_4\;, \tag{4}$$

which follows from momentum conservation and the antisymmetry of the wedge product, and $c_{23}s_{11} = \tan\theta_W \equiv t_W$, and setting $m_W^2 = m_Z^2 c_W^2$, we find that the $\mathcal{O}\left(s^2/m_W^4\right)$ do indeed cancel as expected. The remaining $\mathcal{O}\left(s/m_W^2\right)$ contributions only partially cancel, however, and the remainder can be combined to yield

$$
\begin{aligned}
i\mathcal{M} \;=\; & \frac{ig^2}{8}\frac{s}{m_W^2}\Bigg\{6\cos\left(p_1\wedge p_3 - p_2\wedge p_4\right) - 3\exp\left(-ip_1\wedge p_3 + ip_2\wedge p_4\right) \\
& -3\Bigg[\cos\left(p_1\wedge p_3\right)\,\cos\left(p_2\wedge p_4\right) + t_W^4\sin\left(p_1\wedge p_3\right)\,\sin\left(p_2\wedge p_4\right) \\
& +i\,t_W^2\sin\left(p_1\wedge p_3 - p_2\wedge p_4\right)\Bigg] - 3\left[\frac{m_{G^0}^2}{m_W^2}s_{23}^2 + \frac{m_{W^0}^2}{m_W^2}c_{23}^2c_{11}^2\right]\sin\left(p_1\wedge p_3\right)\,\sin\left(p_2\wedge p_4\right) \\
& +c_\theta\Bigg[4\cos\left(p_1\wedge p_2 - p_3\wedge p_4\right) - \exp\left(-ip_1\wedge p_2 + ip_3\wedge p_4\right) \\
& -\Bigg(\cos\left(p_1\wedge p_3\right)\,\cos\left(p_2\wedge p_4\right) + 2\cos\left(p_1\wedge p_2\right)\,\cos\left(p_3\wedge p_4\right) + t_W^4\sin\left(p_1\wedge p_3\right)\,\sin\left(p_2\wedge p_4\right) \\
& +2\,t_W^4\sin\left(p_1\wedge p_2\right)\,\sin\left(p_3\wedge p_4\right) - i\,t_W^2\sin\left(p_1\wedge p_3 - p_2\wedge p_4\right)\Bigg) \\
& -\left[\frac{m_{G^0}^2}{m_W^2}s_{23}^2 + \frac{m_{W^0}^2}{m_W^2}c_{23}^2c_{11}^2\right]\left\{\sin\left(p_1\wedge p_3\right)\,\sin\left(p_2\wedge p_4\right) + 2\sin\left(p_1\wedge p_2\right)\,\sin\left(p_3\wedge p_4\right)\right\}\Bigg]\Bigg\}\,.
\end{aligned}
\tag{5}
$$

One can trivially check that in the commutative SM limit this expression indeed vanishes. Remember that in the SM, the mass relationship between the $W^\pm$ and Z fields as well as the Higgs exchanges ensures the cancellations. Although the above expression is rather complicated, it is clear that it cannot be made to vanish in the general NC case. This is most easily seen by the presence of the large and most dangerous terms proportional to $m_{G^0}^2$ and $m_{W^0}^2$; recall that these masses are arbitrary functions of the two Higgsac vevs. Here there are no corresponding relationships between the new gauge boson masses themselves or with those of the $W^\pm$ and Z and there are no Higgsac contributions with couplings proportional to m_{G^0} and/or m_{W^0} to help the cancellations. As we see from the above calculation, the $m_{G^0}^2$

and $m_{W^0}^2$ terms arise from the s- and t-channel vector boson exchanges when the propagators are expanded to leading- and next-to-leading order in s. This means that such terms are unavoidable in this scenario and unitarity must fail.

It is instructive to ask at what scale tree-level unitarity is violated in this model. Clearly, this is dependent on the value of the center-of-mass energy, since the theory is only unitary for all values of $\sqrt{s}$ when $\Lambda_{NC} \to \infty$. To be specific, let us examine the $W_L^+ W_L^-$ scattering process in the center-of-mass frame and assume for simplicity that the non-commutativity is of the space-space type. It is clear as noted above that the most dangerous terms are proportional to the squares of the new gauge boson masses. Keeping only these leading terms we find the constraint

$$\Lambda_B > 0.83[(\sqrt{s})^6 \ \sin^2\gamma \ (m_{W^0}^2 + 0.19 m_{G^0}^2)]^{1/4}\,, \tag{6}$$

where all quantities are in TeV. Here γ is the angle between the momentum p_1 and the unit vector $\hat{c}_B$. These bounds can be quite severe; taking $\sin\gamma = 1$ and $m_{W^0} = m_{G^0}$, one finds that $\Lambda_B > 0.86(9.6, 27.3)\sqrt{m_{G^0}}$ TeV for $\sqrt{s} = 1(5, 10)$ TeV, which is a strong constraint for new gauge boson masses in the few TeV range. Similar strong bounds are to be expected in the cases of space-time or mixed non-commutativity.

In order to elucidate the origin of the apparent problems that we have just encountered, it would be beneficial to examine a parallel set of processes in somewhat more simplified NC gauge theories that are more tractable. By using a set of test models we hope to cleanly isolate the basic causes of the unitarity failure in the NCSM and probe the issue in a more general way. We hope to answer the question of just when are spontaneously broken NC gauge theories non-unitary.

3 Test Models

In this section we examine gauge boson scattering in a variety of test cases of spontaneously broken NC theories. We start with the simplest possibility, that of spontaneously broken NC $U(1)$, and then work our way towards more realistic models adding one layer of complications at a time. In this manner, we hope to isolate the source of unitarity violation discovered above and determine its consequences for future NC model building. Since the unitarity failure encountered above occurred in the pure gauge/Higgs sector, we need not worry about the fermion content of any of the test models that we examine below and thus need only consider the process of gauge boson scattering.

3.1 Spontaneously Broken NC $U(1)$

We first examine the simplest example of a spontaneously broken NC theory, that of NC $U(1)$ which is broken by a complex scalar field transforming as a fundamental under the gauge group. A one-loop analysis of this model was first performed in [10], to which the reader is referred for a more detailed discussion. The physical spectrum after symmetry breaking consists of a massive real scalar Higgs field h and a massive vector field Z. This theory was shown by explicit calculation to be one loop renormalizable[10], and we therefore expect it to be unitary at tree-level unlike the NCSM whose one-loop properties are unknown. To test whether or not tree-level unitarity is satisfied we calculate the high energy limit of the scattering process $Z_LZ_L \to Z_LZ_L$. As in the NCSM case above, we need to check if the potential unitarity violating terms of $\mathcal{O}\left(s^2/m_Z^4, s/m_Z^2\right)$ vanish after summing over all the diagrams.

The Feynman rules relevant to this calculation are given in Fig. 2, and the diagrams contributing to $Z_LZ_L \to Z_LZ_L$ are presented in Fig. 3. Summing these seven diagrams following the same procedure as in the case of the NCSM and keeping only the $\mathcal{O}\left(s^2/m_Z^4, s/m_Z^2\right)$ terms, yields the amplitude

$$
\begin{aligned}
i\mathcal{M} &= \frac{ig^2}{2}\frac{s}{m_Z^2}\Big\{\cos(p_1\wedge p_3 - p_2\wedge p_4) + \cos(p_1\wedge p_4 - p_2\wedge p_3) - 2\,c_{(12)}c_{(34)} \\
&\quad + c_\theta\Big[\cos(p_1\wedge p_4 + p_2\wedge p_3) - \cos(p_1\wedge p_3 + p_2\wedge p_4) - 10\,\Big(s_{(12)}s_{(34)} \\
&\quad + s_{(14)}s_{(23)} - s_{(13)}s_{(24)}\Big)\Big]\Big\}\ , \qquad (7)
\end{aligned}
$$

where c_θ is defined as in the previous section. We have introduced the shorthand notation $\sin\left(p_i\wedge p_j\right) = s_{(ij)}$, $\cos\left(p_i\wedge p_j\right) = c_{(ij)}$; it is important that these should not be confused with the abbreviations used for the mixing angles in the NCSM. We see that the $\mathcal{O}(s^2/m_Z^4)$ terms cancel trivially as expected and as in the NCSM case above. Use of the relations

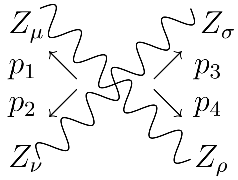
$$
\begin{aligned}
p_1\wedge p_2 - p_3\wedge p_4 &= p_1\wedge p_3 - p_2\wedge p_4 \\
p_1\wedge p_2 + p_3\wedge p_4 &= p_1\wedge p_4 - p_2\wedge p_3 \\
p_1\wedge p_3 + p_2\wedge p_4 &= -p_1\wedge p_4 - p_2\wedge p_3\ , \qquad (8)
\end{aligned}
$$

which follow directly from momentum conservation and the antisymmetry of the wedge product, and the consequent identity

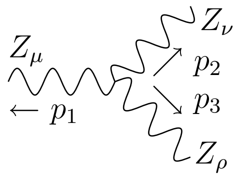
$$
s_{(12)}s_{(34)} = s_{(13)}s_{(24)} - s_{(14)}s_{(23)}\ , \qquad (9)
$$

then leads to a cancellation of the $\mathcal{O}(s/m_Z^2)$ terms. Unitarity is thus preserved at high energies as expected for this scenario. For the sake of completeness, we also examined the process $hZ_L \to hZ_L$, and found that the leading $\mathcal{O}(s/m_Z^2)$ terms similarly cancel.

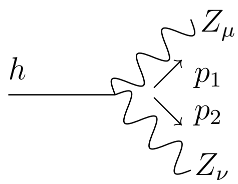
This calculation demonstrates that it is possible to construct a spontaneously broken NC theory that preserves tree-level unitarity in processes involving the scattering of longitudinal gauge bosons.



$$= -4ig^2 \Big\{ \sin(p_1 \wedge p_2) \sin(p_3 \wedge p_4) \left(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma} \right) + \sin(p_3 \wedge p_1) \sin(p_2 \wedge p_4) \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\nu} g_{\rho\sigma} \right) + \sin(p_1 \wedge p_4) \sin(p_2 \wedge p_3) \left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) \Big\}$$



$$= -2g \sin(p_1 \wedge p_2) \left[(p_1 - p_2)_\rho g_{\mu\nu} + (p_2 - p_3)_\mu g_{\nu\rho} + (p_3 - p_1)_\nu g_{\mu\rho} \right]$$



$$= 2ig\, m_Z \cos(p_1 \wedge p_2) g_{\mu\nu}$$

Figure 2: NC $U(1)$ Feynman rules relevant for $Z_L Z_L \to Z_L Z_L$.

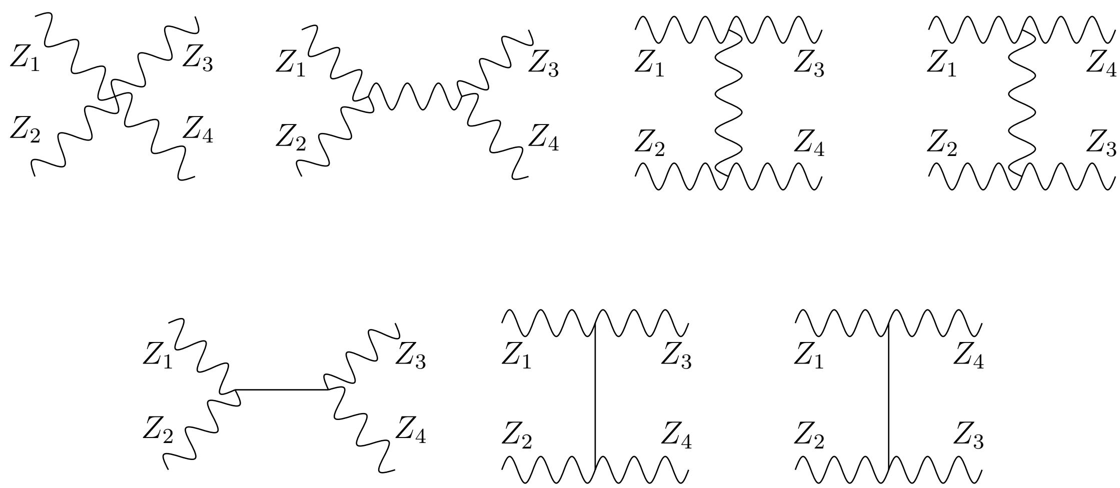


Figure 3: Diagrams contributing to the scattering process $Z_L Z_L \to Z_L Z_L$ in NC $U(1)$; the subscripts denote the momentum carried by the field.

3.2 $U(1) \times U(1)$

Perhaps $U(1)$ is too simple of an example to reveal the potential unitarity failure of spontaneously broken NC theories. We next extend our investigation to a NC $U(1) \times U(1)$ theory; this example contains a direct product gauge group and mixing between the gauge boson of each group, which introduces additional complexities beyond the model of Sec. 3.1. The Lagrangian density defining this theory is

$$\mathcal{L} = -\frac{1}{4} B_{\mu\nu} * B^{\mu\nu} - \frac{1}{4} C_{\mu\nu} * C^{\mu\nu} + (D_\mu \phi)^\dagger * D^\mu \phi - V(\phi) \ , \tag{10}$$

where $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + ig\,[B_\mu, B_\nu]_{MB}$ and similarly for $C_{\mu\nu}$ with $g \to g'$, $D_\mu \phi = \partial_\mu \phi + ig\, B_\mu * \phi - ig^{'} \phi * C_\mu$, and the potential $V(\phi)$ is chosen so that its minimum is at $\phi_0 = \nu$. Expanding ϕ around ν, and introducing the suggestive notation

$$s_W = \frac{g}{\sqrt{g^2 + g'^2}} \ , \ c_W = \frac{g^{'}}{\sqrt{g^2 + g'^2}} \ , \ e = \sqrt{g^2 + g'^2} \ , \tag{11}$$

we find a physical spectrum of fields consisting of a real massive scalar Higgs h and the following gauge bosons:

$$\begin{aligned} A &= c_W B + s_W C \ , \ \ m_A^2 = 0 \ ; \\ Z &= s_W B - c_W C \ , \ \ m_Z^2 = 2e^2 \nu^2 \ . \end{aligned} \tag{12}$$

We again consider the scattering process $Z_L Z_L \to Z_L Z_L$, and determine whether the unitarity violating $\mathcal{O}\,(s^2/m_Z^4, s/m_Z^2)$ terms vanish. We present the necessary Feynman rules for this scenario in Fig. 4, and the contributing diagrams to this process in Fig. 5. Note that exchanges of multiple gauge bosons in the s- and t-channels are now present since an AZZ 3-point coupling is induced through mixing. In deriving these Feynman rules we have used the trigonometric relations

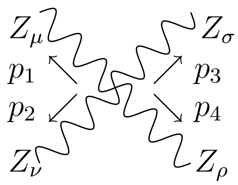
$$s_W^4 - c_W^4 = s_W^2 - c_W^2 \ , \ \ s_W^6 + c_W^6 = 1 - 3s_W^2 c_W^2 \ . \tag{13}$$

Summing the diagrams in Fig. 5, and keeping only $\mathcal{O}\left(s^2/m_Z^4, s/m_Z^2\right)$ contributions, we arrive at the amplitude

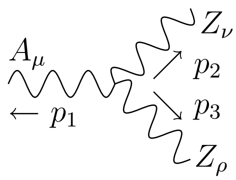
$$\begin{aligned}
i\mathcal{M} \ = \ & -ie^2 \frac{s^2}{m_Z^4}\Big\{ \Big[\Big(-1 + 3s_W^2 c_W^2\Big) + s_W^4 - s_W^2 c_W^2 + c_W^4\Big] \Big(c_\theta s_{(12)} s_{(34)} \\
& + \frac{1}{4}(-3 + 2c_\theta + c_\theta^2\,)\, s_{(13)} s_{(24)} + \frac{1}{4}(-3 - 2c_\theta + c_\theta^2\,)\, s_{(14)} s_{(23)}\Big)\Big\} \\
& + \frac{ie^2}{2}\frac{s}{m_Z^2}\Big\{ 4\left(1 - 3s_W^2 c_W^2\right)\left(s_{(13)} s_{(24)} + s_{(14)} s_{(23)}\right) \\
& - 3\left(1 - 4s_W^2 c_W^2\right)\left(s_{(13)} s_{(24)} + s_{(14)} s_{(23)}\right) - 2c_{(12)} c_{(34)} + c_{(13)} c_{(24)} + c_{(14)} c_{(23)} \\
& + c_\theta \Big[\left(11 - 32 s_W^2 c_W^2\right)\left(s_{(13)} s_{(24)} - s_{(14)} s_{(23)}\right) - 2\left(1 - 4s_W^2 c_W^2\right) s_{(12)} s_{(34)} \\
& - 8\left(1 - 3s_W^2 c_W^2\right) s_{(12)} s_{(34)} + c_{(14)} c_{(23)} - c_{(13)} c_{(24)}\Big]\Big\} \ .
\end{aligned} \tag{14}$$

The $\mathcal{O}\left(s^2/m_Z^4\right)$ terms have vanished using the relation $s_W^4 + c_W^4 = 1 - 2s_W^2 c_W^2$ and the $\mathcal{O}\left(s/m_Z^2\right)$ contributions also cancel by using the relations in Eqs. 8 and 9. Thus the presence of product groups and multiple gauge bosons is *not* the crucial feature causing the lack of tree-level unitarity in the NCSM. Again, for completeness, we have also studied the process $AZ_L \to AZ_L$, and found that the leading $\mathcal{O}\left(s/m_Z^2\right)$ terms cancel.

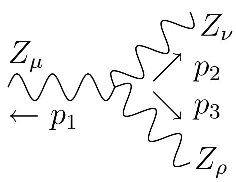
We therefore conclude that the additional complexities of direct product gauge groups and gauge boson mixing do not destroy our ability to maintain unitarity in NC processes containing longitudinal vector particles.



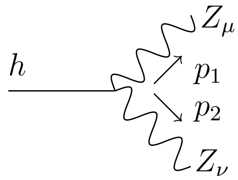
$$= -4ie^2\,(1-3s_W^2c_W^2)\Big\{\sin(p_1\wedge p_2)\sin(p_3\wedge p_4)\,(g_{\mu\sigma}g_{\nu\rho}-g_{\mu\rho}g_{\nu\sigma})+ \sin(p_3\wedge p_1)\sin(p_2\wedge p_4)\,(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\nu}g_{\rho\sigma})+ \sin(p_1\wedge p_4)\sin(p_2\wedge p_3)\,(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\sigma}g_{\nu\rho})\Big\}$$



$$= -2e\,s_Wc_W\sin(p_1\wedge p_2)\,[(p_1-p_2)_\rho g_{\mu\nu}+(p_2-p_3)_\mu g_{\nu\rho}+(p_3-p_1)_\nu g_{\mu\rho}]$$



$$= -2e\,(s_W^2-c_W^2)\sin(p_1\wedge p_2)\,[(p_1-p_2)_\rho g_{\mu\nu}+(p_2-p_3)_\mu g_{\nu\rho}+(p_3-p_1)_\nu g_{\mu\rho}]$$



$$= 2ie\,m_Z\cos(p_1\wedge p_2)g_{\mu\nu}$$

Figure 4: NC $U(1)\times U(1)$ Feynman rules relevant for $Z_LZ_L\to Z_LZ_L$.

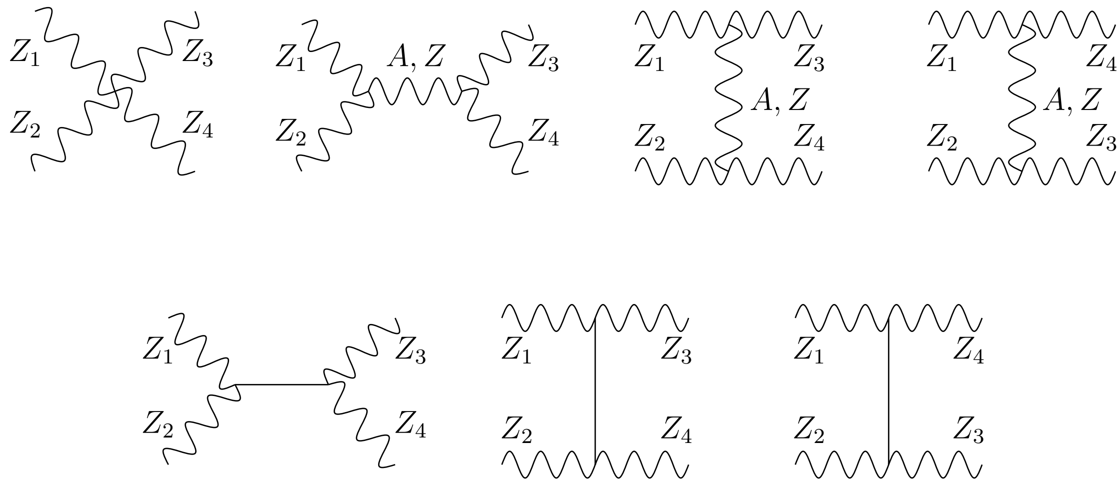


Figure 5: Diagrams contributing to the scattering process $Z_L Z_L \to Z_L Z_L$ in NC $U(1)\times U(1)$; the subscripts denote the momentum carried by the field.

3.3 $U(2)$: Fundamental + Adjoint Breaking

Perhaps unitarity was satisfied for the previous two test cases because both $U(1)$ and $U(1)\times U(1)$ are abelian gauge groups. To address this issue, we study NC $U(2)$ broken by both a fundamental and an adjoint Higgs; this example furnishes the breaking of a non-abelian gauge group together with a more complicated example of mixing than that found in the $U(1)\times U(1)$ case. It contains a two step symmetry breaking process akin to that used in the NCSM of [3] as well as a spectrum which closely resembles that in [3]. The theory is defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left(B_{\mu\nu} * B^{\mu\nu}\right) + \left(D_\mu \phi_F\right)^\dagger * D^\mu \phi_F - V\left(\phi_F\right) + \mathrm{Tr}\left(D_\mu \phi_A * D^\mu \phi_A\right) - V'\left(\phi_A\right) \ , \quad (15)$$

where ϕ_F is the fundamental Higgs, ϕ_A the matrix valued adjoint Higgs, and B_μ the matrix valued $U(2)$ gauge field. The field strength tensor and covariant derivatives are

$$B_{\mu\nu} \ = \ \partial_\mu B_\nu - \partial_\nu B_\mu + g\left[B_\mu, B_\nu\right]_{MB} \ ,$$

$$
\begin{aligned}
D_\mu \phi_F &= \partial_\mu \phi_F + ig\, B_\mu * \phi_F \ , \\
D_\mu \phi_A &= \partial_\mu \phi_A + g\, [B_\mu, \phi_A]_{MB} \ .
\end{aligned} \tag{16}
$$

We choose the potentials $V(\phi_F), V'(\phi_A)$ so that their symmetry breaking minima occur at

$$
\phi_{F,0} = \nu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \ \phi_{A,0} = \frac{\nu'}{\sqrt{2}} \frac{\sigma_3}{2} \ . \tag{17}
$$

The gauge sector of this theory consists of the following particles:

$$
\begin{aligned}
A &= \frac{1}{\sqrt{2}} (B_0 + B_3) \ , \ m^2 = 0 \\
Z &= \frac{1}{\sqrt{2}} (B_0 - B_3) \ , \ m_Z^2 = g^2 \nu^2 \\
W^\pm &= \frac{1}{\sqrt{2}} (B_1 \mp i B_2) \ , \ m_W^2 = m_F^2 + m_A^2 \ , \\
& \qquad m_F^2 = \frac{g^2 \nu^2}{2} \ , \ m_A^2 = \frac{g^2 \nu'^2}{2} \ .
\end{aligned} \tag{18}
$$

Our calculation will involve the contributions from the two neutral scalars ϕ_0, ϕ_3 from $\phi_A = \phi_\mu\, \sigma^\mu/2$, and the scalar h from the expansion $\phi_F = \nu + h/\sqrt{2} + i\sigma/\sqrt{2}$. We note in particular the relation $m_Z^2 = 2m_F^2$, which will be used later in this section.

As above, we examine the process $W_L^+ W_L^- \to W_L^+ W_L^-$. The pertinent Feynman rules for this calculation are presented in Fig. 6, and the relevant diagrams in Fig. 7. Summing these contributions, we find that the leading terms again cancel leaving the following $\mathcal{O}(s/m_W^2)$ contribution to the amplitude: (Note that the $\mathcal{O}(s^2/m_W^4)$ terms cancel after straightforward manipulations and so have not been displayed for this case.)

$$
i\mathcal{M} = \frac{ig^2}{16} \frac{s}{m_W^2} \Big\{ 8 \left(c_{(13)} c_{(24)} + s_{(13)} s_{(24)} \right) - 3 \frac{m_Z^2}{m_W^2} \exp\left(-ip_1 \wedge p_3 + ip_2 \wedge p_4 \right)
$$

$$-4\frac{m_F^2}{m_W^2}\exp\left(ip_1\wedge p_2 - ip_3\wedge p_4\right) + 2\frac{m_F^2}{m_W^2}\exp\left(-ip_1\wedge p_3 + ip_2\wedge p_4\right)$$

$$-16\frac{m_A^2}{m_W^2}\left(c_{(12)}c_{(34)} + s_{(12)}s_{(34)}\right) + 8\frac{m_A^2}{m_W^2}\left(c_{(13)}c_{(24)} + s_{(13)}s_{(24)}\right)$$

$$+c_\theta\Big[-16\left(c_{(12)}c_{(34)} + s_{(12)}s_{(34)}\right) + 24\left(c_{(13)}c_{(24)} + s_{(13)}s_{(24)}\right)$$

$$-2\frac{m_Z^2}{m_W^2}\exp\left(ip_1\wedge p_2 - ip_3\wedge p_4\right) - \frac{m_Z^2}{m_W^2}\exp\left(-ip_1\wedge p_3 + ip_2\wedge p_4\right)$$

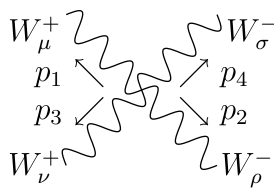
$$-2\frac{m_F^2}{m_W^2}\exp\left(-ip_1\wedge p_3 + ip_2\wedge p_4\right) - 8\frac{m_A^2}{m_W^2}\left(c_{(13)}c_{(24)} + s_{(13)}s_{(24)}\right)\Big]\Big\} \qquad (19)$$

Although messy in appearance this potentially unitarity violating contribution can be shown to vanish through the use of the relations in Eq. 8 and the identities $m_Z^2 = 2m_F^2$, $m_W^2 = m_F^2 + m_A^2$. Here we see that the existence of certain mass relationships can be crucial in obtaining unitarity. We have also verified the cancellation of these dangerous terms in the scattering processes $hZ_L \to hZ_L$, $hW_L^+ \to hW_L^+$, and $W_T^+W_L^- \to W_T^+W_L^-$.

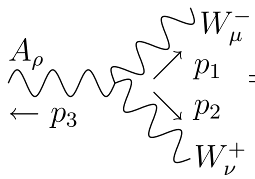
The additional features present in this simplified theory do not ruin the delicate cancellations required to maintain unitarity; the problem with the NCSM of [3] must lie elsewhere, even though this toy model has features similar to those of the NCSM.

3.4 $U(2)$: Fundamental Higgs + Higgsac

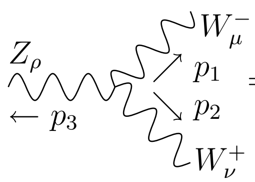
We now examine NC $U(2)$ broken by both a fundamental Higgs and a Higgsac. Although this model is identical to that studied in Sec. 3.3 upon replacement of the Higgsac with an adjoint Higgs, we will find that the dangerous $\mathcal{O}\left(s/m_W^2\right)$ terms now remain in the amplitude for $W_L^+W_L^- \to W_L^+W_L^-$. This theory is also quite similar to the NCSM of [3], the only difference here being that the Higgsac breaks a $U(1)$ subgroup of $U(2)$ rather than a $U(1)$ subgroup of $U(2)\times U(1)$. The success of the model of Sec. 3.3 therefore allows us to conclude



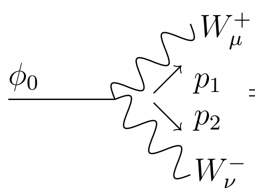
$$= ig^2 \Big\{ \left[\cos\left(p_1 \wedge p_2 + p_3 \wedge p_4\right) + \cos\left(p_1 \wedge p_4 + p_2 \wedge p_3\right)\right] g_{\mu\nu} g_{\rho\sigma} \\ -\cos\left(p_1 \wedge p_4 + p_2 \wedge p_3\right) g_{\mu\rho} g_{\nu\sigma} \\ -\cos\left(p_1 \wedge p_2 + p_3 \wedge p_4\right) g_{\mu\sigma} g_{\nu\rho} \Big\}$$



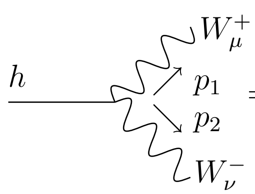
$$= \frac{ig}{\sqrt{2}} e^{i\, p_1 \wedge p_2} \left[(p_1 - p_2)_\rho g_{\mu\nu} + (p_2 - p_3)_\mu g_{\nu\rho} + (p_3 - p_1)_\nu g_{\mu\rho}\right]$$



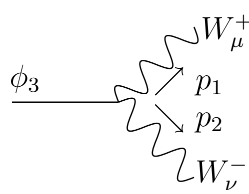
$$= -\frac{ig}{\sqrt{2}} e^{-i\, p_1 \wedge p_2} \left[(p_1 - p_2)_\rho g_{\mu\nu} + (p_2 - p_3)_\mu g_{\nu\rho} + (p_3 - p_1)_\nu g_{\mu\rho}\right]$$



$$= 2g\, m_A \sin\left(p_1 \wedge p_2\right) g_{\mu\nu}$$



$$= ig\, m_F e^{i\, p_1 \wedge p_2} g_{\mu\nu}$$



$$= 2ig\, m_A \cos\left(p_1 \wedge p_2\right) g_{\mu\nu}$$

Figure 6: NC $U(2)$ Feynman rules relevant for $W_L^+ W_L^- \to W_L^+ W_L^-$.

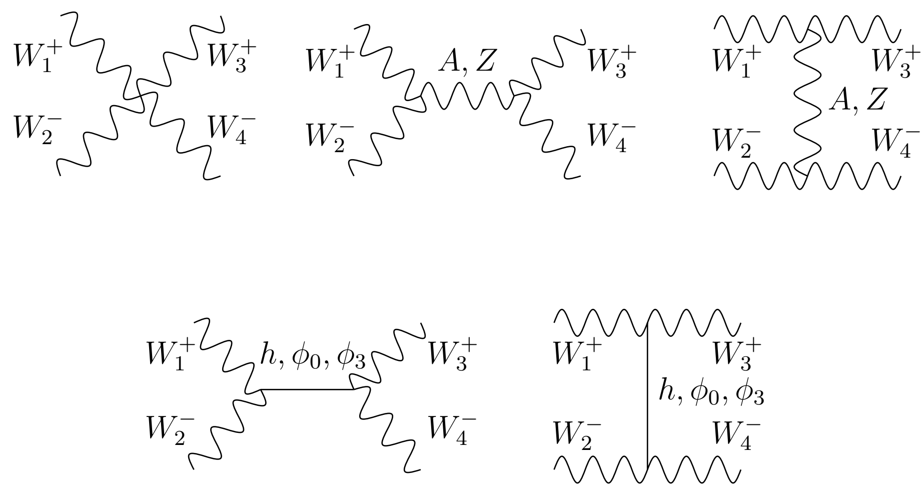


Figure 7: Diagrams contributing to the scattering process $W_L^+ W_L^- \to W_L^+ W_L^-$ in NC U(2); the subscripts denote the momentum carried by the field.

that the problem with the theory proposed in [3] is the use of the Higgsacs to break the NC $U(1)$ subgroups of $U(3) \times U(2) \times U(1)$.

We begin with the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left(B_{\mu\nu} * B^{\mu\nu}\right) + \left(D_\mu \phi_F\right)^\dagger * D^\mu \phi_F - V\left(\phi_F\right) + \left(D_\mu \phi_H\right)^\dagger * D^\mu \phi_H - V'\left(\phi_H\right) \ . \tag{20}$$

The field strength tensor $B_{\mu\nu}$ and the covariant derivative of the fundamental Higgs ϕ_F are the same as given in Eq. 16 of Sec. 3.3., and the covariant derivative of the Higgsac is given by

$$D_\mu \phi_H = \partial_\mu \phi_H + \frac{ig}{2} B_\mu^0 * \phi_H \ . \tag{21}$$

Our conventions for the Higgsac covariant derivative differ slightly from those in [3] in that we have not included a factor of two in the interaction term arising from the trace of the SU(2) identity matrix. This is consistent with our normalization of the triple gauge couplings. We choose the same minimum for ϕ_F as in Eq. 17, and the Higgsac minimum at $\phi_{H,0} = \nu'$. The spectrum of Sec. 3.3 consisted of a massless boson A and three massive bosons $Z, W^\pm$; the effect of the Higgsac here is to induce mixing between A and Z, leading to the following gauge sector spectrum:

$$\begin{aligned}
W^{\pm} &= \frac{1}{\sqrt{2}}\left(B_1 \mp iB_2\right)\ ,\ \ m_W^2 = \frac{g^2\nu^2}{2}\ , \\
Z^{+} &= c_\alpha Z + s_\alpha A\ ,\ \ m_+^2 = m_W^2\left(1 + \sec\left(\beta\right) + \tan\left(\beta\right)\right)\ , \\
Z^{-} &= c_\alpha A - s_\alpha Z\ ,\ \ m_-^2 = m_W^2\left(1 - \sec\left(\beta\right) + \tan\left(\beta\right)\right)\ .
\end{aligned} \tag{22}$$

Here s_α, c_α are shorthand for the mixing angles $\sin\left(\alpha\right), \cos\left(\alpha\right)$, and $\tan\left(\beta\right)$ is the ratio of the Higgsac and fundamental Higgs vevs; the explicit expressions are given by

$$\begin{aligned}
\tan\left(\beta\right) &= \frac{\nu'^2}{2\nu^2}\ , \\
\alpha &= \frac{\beta}{2}\ .
\end{aligned} \tag{23}$$

Upon taking $\beta \to 0$ we recover the case where NC U(2) is broken by just a fundamental Higgs; the Higgsac decouples. We know from the calculations in Sec. 3.3 that in this limit the theory is unitary. To test the validity of using the Higgsac to break the NC $U(1)$ subgroup we must see whether we maintain a unitary theory for all values of β.

We again study the process $W_L^+W_L^- \to W_L^+W_L^-$. This calculation requires the $W^\pm$ four-point coupling and the $h - W^+ - W^-$ vertex, which are given Fig. 6, as well as the $Z^+ - W^+ - W^-$ and $Z^- - W^+ - W^-$ vertices, which are presented in Fig. 8. The contributing diagrams are displayed in Fig. 9. Notice that since the Higgsac couples only to $Z^\pm$, it does not enter into this calculation; this is to be contrasted with the model of Sec. 3.3, in which the addition of an adjoint Higgs generated additional diagrams which were required to maintain unitarity in $W_L^+W_L^- \to W_L^+W_L^-$. Summing the contributions in Fig. 9, we find that the $\mathcal{O}\left(s^2/m_W^4\right)$ terms cancel as expected as these terms arise from the pure gauge sector and are not affected by the details of the symmetry breaking. The $\mathcal{O}\left(s/m_W^2\right)$ terms again split into those which are proportional to c_θ and those which are not, as in the previous examples. These must cancel independently; for simplicity we list only the c_θ independent terms below:

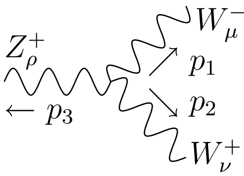
$$\begin{aligned} i\mathcal{M} \;=\; & \frac{ig^2}{16}\frac{s}{m_W^2}\Bigg\{6\exp\left(-ip_1\wedge p_3+ip_2\wedge p_4\right)-3\frac{m_+^2}{m_W^2}\Big[c_\alpha^2\exp\left(-ip_1\wedge p_3+ip_2\wedge p_4\right) \\ & +s_\alpha^2\exp\left(ip_1\wedge p_3-ip_2\wedge p_4\right)-2c_\alpha s_\alpha\left(c_{(13)}c_{(24)}-s_{(13)}s_{(24)}\right)\Big] \\ & -3\frac{m_-^2}{m_W^2}\Big[c_\alpha^2\exp\left(ip_1\wedge p_3-ip_2\wedge p_4\right)+s_\alpha^2\exp\left(-ip_1\wedge p_3+ip_2\wedge p_4\right) \\ & +2c_\alpha s_\alpha\left(c_{(13)}c_{(24)}-s_{(13)}s_{(24)}\right)\Big]\Bigg\}+c_\theta\left[\ldots\right]\ . \end{aligned} \tag{24}$$

In obtaining this expression we have used the relations in Eq. 8. It is clear that this expression does not vanish for general β. As an example, let us examine an arbitrary case with $\tan(\beta)=\sqrt{3}$, $s_\alpha=1/2$, $c_\alpha=\sqrt{3}/2$; with this choice of parameters the amplitude takes the very simple form

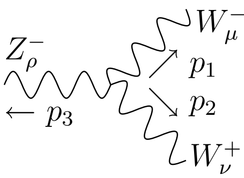
$$i\mathcal{M}=12\sqrt{3}\,s_{(13)}s_{(24)}+c_\theta\left[\ldots\right]\ . \tag{25}$$

This clearly vanishes in the commutative limit when $U(2)$ splits into the product of independent group factors $SU(2)\times U(1)$, but not in a general NC setting. In addition, we have found that a similar lack of cancellation of $\mathcal{O}\left(s/m_-^2\right)$ terms occurs in the process $hZ_L^-\to hZ_L^-$. Since the difference between the previous test case, where unitarity was maintained, and the one presented here was to trade an adjoint Higgs for a Higgsac in the symmetry breaking, we thus conclude that the use of the Higgsac representation in NC symmetry breaking induces unitarity violations in processes involving longitudinal gauge bosons.

As can be seen from the above discussion, our survey of toy models has allowed us to isolate the source of the tree-level unitarity breakdown in the proposed NCSM: the breaking



$$= -\frac{ig}{\sqrt{2}}\left(c_\alpha\, e^{-ip_1\wedge p_2} - s_\alpha\, e^{ip_1\wedge p_2}\right)\left[(p_1-p_2)_\rho g_{\mu\nu} + (p_2-p_3)_\mu g_{\nu\rho} + (p_3-p_1)_\nu g_{\mu\rho}\right]$$



$$= \frac{ig}{\sqrt{2}}\left(c_\alpha\, e^{ip_1\wedge p_2} + s_\alpha\, e^{-ip_1\wedge p_2}\right)\left[(p_1-p_2)_\rho g_{\mu\nu} + (p_2-p_3)_\mu g_{\nu\rho} + (p_3-p_1)_\nu g_{\mu\rho}\right]$$

Figure 8: Additional Feynman rules necessary for $W^+_L W^-_L \to W^+_L W^-_L$ in NC $U(2)$ broken by both a fundamental Higgs and Higgsac.

of the symmetry by the Higgsac fields.

4 Discussion and Conclusion

The straightforward construction of a NC version of the SM is made difficult by the requirements of gauge invariance and the constraints that arise from group theory. Once these conditions are met we expect any realistic perturbative NC model that reduces to the SM in the commutative limit to possess a number of essential properties in order to be a viable theory and make phenomenological predictions: renormalizability, anomaly freedom, tree-level unitarity, *etc.* In this paper we have explicitly shown that the version of the NCSM constructed by Chaichian *et al.*[3] does not satisfy the requirement of tree-level unitarity for gauge boson scattering at high energies. In order to track down the origin of this unitarity violation we have examined a number of simpler NC gauge models with various gauge structures wherein the symmetry was spontaneously broken by various Higgs representations. It was clear from this study that the use of the Higgsac representations to break the gauge symmetries is the source of this unitarity violation. In the SM the mass relationship between

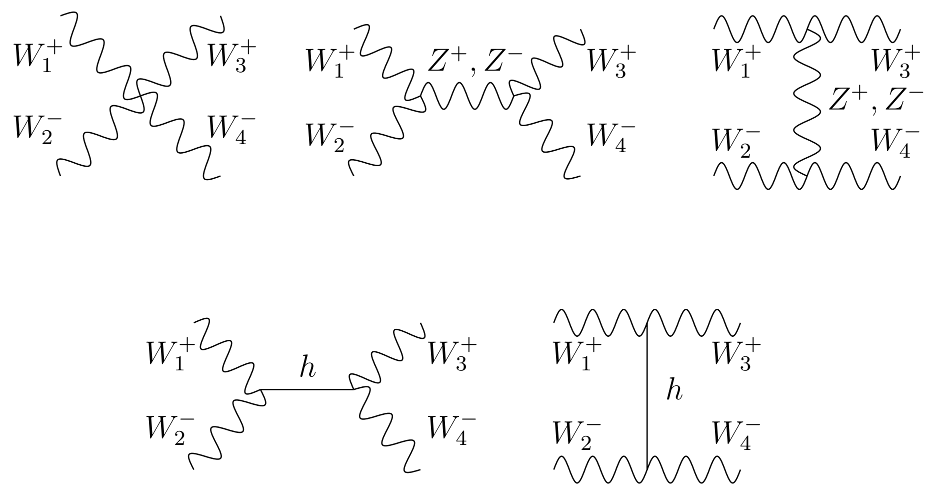


Figure 9: Diagrams contributing to the scattering process $W_L^+ W_L^- \to W_L^+ W_L^-$ in NC $U(2)$ broken by both a fundamental Higgs and Higgsac; the subscripts denote the momentum carried by the field.

the $W^\pm$ and Z fields as well as the Higgs exchanges ensures cancellations of the leading and sub-leading unitarity violating terms for this process. In the version of the NCSM discussed here there are no relationships between the new gauge boson masses themselves or with those of the $W^\pm$ and Z and there are no Higgsac contributions to induce cancellations. Since these $m^2_{G^0}$ and $m^2_{W^0}$ terms are unavoidable in this approach unitarity must fail and this version of the NCSM becomes phenomenologically unacceptable.

If Higgsac representations cannot be used in the symmetry breaking of the NCSM then we are faced with a severe model construction problem. The use of products of $U(n)$ groups, required by gauge invariance and algebraic closure in the Moyal approach, necessitates the breaking of their $U(1)$ subgroups as the first stage of symmetry breaking since they cannot be identified with the $U(1)_Y$ gauge symmetry of the SM. As noted earlier, ordinary Higgs representations such as fundamentals or adjoints will not break these $U(1)$'s but will instead break the $SU(n)$ subgroups; this must be avoided. Using larger NC gauge groups does not help the situation as fundamental and adjoint vevs only induce the breaking pattern

$U(n) \to U(n-1)$ instead of breaking to $SU(n-1)$.

We thus conclude that it is difficult, if not impossible, to build a phenomenologically viable non-commutative Standard Model under the Weyl-Moyal approach.

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