Wake field of the e-cloud *

S. Heifets Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA

1 Abstract

The wake field of the cloud is derived analytically taking into account the finite size of the cloud and nonlinearity of the electron motion.

Contributed to the 23d Advanced ICFA Beam Dynamics Workshop on High Luminosity e⁺e⁻ Colliders, Ithaca, NY, 15-19 October 2001

^{*}Work supported by Department of Energy contract DE-AC03-76SF00515.

2 Introduction

The analytic expression for the effective transverse wake field caused by the electron cloud in a positron storage ring is derived. The derivation includes the frequency spread in the cloud, which is the main effect of the nonlinearity of electron motion in the cloud. This approach allows calculation of the Q-factor and study the tune spread in a bunch.

3 Wake field of the cloud

Derivation of the effective wake field induced by the electron cloud is complicated by the substantial nonlinearity of the motion of electrons in the cloud. A simple estimate of the wake field is known [2], [3], and was recently used to study the emittance blow up [6]. The wake was obtained in these papers in a linear approximation and for equal transverse sizes of the beam and of the cloud what is, certainly, wrong in reality. The estimate is based on an assumption that the wake is defined, mostly, by the electrons in the close proximity to the beam. This argument is correct but not obvious because the roll-off of the force of interaction at large distances from the beam may be compensated by the large number of distant electrons. In this paper the wake is derived in somewhat more rigorous way which, hopefully, may clarify validity of the assumptions assumed in the linear approximation. Our derivation is valid for the arbitrary ratio of the size of the cloud to the rms size of the bunch, allowing us to define the Q-factor of the wake, and to study the tune spread induced by the e-cloud.

Let us consider a flat Gaussian bunch with transverse rms $\sigma_x \gg \sigma_y$ and the bunch rms length σ_z . A slice at the distance z > 0 from the head of the bunch is at s = ct - z in the ring at the moment t. Let us use notation y(t, z) for the vertical displacement of a positron in a slice z and Y(t, s) for an electron at location s. The bunch can be flat or round. Equation of motion of an electron of the cloud in the first case is

$$\frac{d^2 Y(t,s)}{dt^2} = 2r_0 c^2 \lambda_b(s-ct) BE(X, Y-y_c(t,ct-s)),$$
(1)

where r_0 is the classical radius of a particle of the cloud, $\lambda_b(z)$ is the linear bunch density, $y_c(t, z)$ is the displacement of the centroid of a slice z = ct - s, and X can be considered as a constant defined by initial location of the electron. For a round beam, Y can be understood as the displacement of a particle along the radius. The X dependence in this case should be omitted.

The explicit form of the vertical force BE produced by a Gaussian bunch is given by Bassetti-Erskin formula [7],

$$BE(x,y) = h\sqrt{\pi} \operatorname{Re} \left[W(u+ivp) - W(pu+iv)e^{-(1-p^2)(u^2+v^2)} \right],$$

$$u = hx, \quad v = \frac{hy}{p}, \quad p = \frac{\sigma_y}{\sigma_x}, \quad h = \frac{1}{\sqrt{2[\sigma_x^2 - \sigma_y^2]}},$$
 (2)

where $W(z) = Erf[-iz]e^{-z^2}$. It is convenient sometimes to use the integral representation,

$$BE(X,Y) = -\left(\frac{Y}{\sigma_y(\sigma_x + \sigma_y)}\right)S_0(X,Y),\tag{3}$$

where

$$S_0(X,Y) = \left(\frac{1+p}{2p}\right) \int_0^\infty \frac{d\mu}{(1+\mu)^{3/2}\sqrt{1+\mu/p^2}} e^{-\frac{\mu}{1+\mu}\frac{Y^2}{2\sigma_y^2} - \frac{\mu X^2}{2\sigma_x^2(\mu+p^2)}}.$$
 (4)

Note that $S_0(0,0) = 1$ and $S(r) = 2\sigma_{\perp}^2/r^2$ for a round beam.

Motion of a particle in a slice z of a bunch is described by a similar equation. This equation, averaged over the transverse Gaussian distribution of the slice, describes the betatron motion of the slice centroid:

$$\frac{d^2y_c(t,z)}{dt^2} + \omega_\beta^2 y_c(t,z) = \frac{2r_e c^2}{\gamma} \frac{dN_e}{ds} \int dX dY BE(X, y_c(t,z) - Y) \rho_\Sigma(X, Y, t, ct - z).$$
(5)

Here the density ρ_{Σ} of electrons in the cloud can be obtained from the normalized to one distribution function

$$\rho_{\Sigma}(X,Y,t,s) = \int d\dot{X} d\dot{Y} \rho_{\Sigma}(X,\dot{X},Y,\dot{Y},t,s), \qquad (6)$$

where $\rho_{\Sigma}(X, \dot{X}, Y, \dot{Y}, 0, s)$ is Gaussian initial distribution with the rms $\Sigma_{x,y}$.

Dependence on the offset y_c in Eq. (5) comes from the dependence of the factor BE and from the dependence of ρ_{Σ} . The amplitude of the bunch centroid is always small compared to the dimensions of the cloud. In the linearized Eq. (5), the term given by the expansion of the factor BE over y_c gives the tune shift

$$\Delta\omega_{\beta} = -\frac{r_e c^2}{\gamma\omega_{\beta}} \frac{dN_e}{ds} \int dX_0 dY_0 \left(\frac{\partial BE(X,Y)}{\partial Y}\right)_{Y=Y_{tr}} \rho_{\Sigma}^{(0)}(X_0,Y_0,0,s), \quad (7)$$

where s = ct - z, $\rho_{\Sigma}^{(0)}(X_0, Y_0, 0, s)$ is the cloud density at t = 0, and $Y_{tr} = Y_{tr}(X_0, Y_0, t, s)$ is a trajectory of an electron of the cloud with the initial conditions X_0, Y_0 at t = 0. Eq. (7) defines G,

$$\Delta\omega_{\beta} = \frac{2\pi r_e c^2 n_e}{\gamma \omega_{\beta}} G(\frac{\Omega_0 z}{c}). \tag{8}$$

If $Y_{tr} = Y_0$, G is constant,

$$G = \frac{\Sigma_x \Sigma_y}{\sqrt{\Sigma_y^2 + \sigma_y^2} [\sqrt{\Sigma_x^2 + \sigma_x^2} + \sqrt{\Sigma_y^2 + \sigma_y^2}]}.$$
(9)

In the following, we will neglect this effect and put $y_c \to 0$ in the argument of *BE* in the right-hand-side (RHS) of Eq. (5). The RHS is defined then by the distribution function of the cloud $\rho_{\Sigma}(X, \dot{X}, Y, \dot{Y}, t, s)$, which satisfies the continuity equation

$$\frac{\partial \rho_{\Sigma}}{\partial t} + \dot{Y}\frac{\partial \rho_{\Sigma}}{\partial Y} + F_{Y}\frac{\partial \rho_{\Sigma}}{\partial \dot{Y}} + (X - > Y) = 0, \qquad (10)$$

where F_Y is the RHS of Eq. (1). Let us linearize Eq. (10) expanding the force F_Y over y_c and taking $\rho_{\Sigma}(t,s) = \rho^{(0)}(t,s) + \rho^{(1)}$. The first part, $\rho^{(0)}$ describes a perturbation of the cloud density by the bunch with the zero offset. Note that $\rho^{(0)}(t,s)$ is an even function of Y provided it is even at t = 0. Because BE(X,Y) is odd function of Y, $\rho^{(0)}$ does not contribute to Eq. (5). The equation for the second part, $\rho^{(1)}(Y, \dot{Y}, t, s)$, is

$$\frac{\partial \rho^{(1)}}{\partial t} + \dot{Y}\frac{\partial \rho^{(1)}}{\partial Y} + F_Y^{(0)}\frac{\partial \rho^{(1)}}{\partial \dot{Y}} = y_c(t, ct-s)\frac{\partial \rho^{(0)}}{\partial \dot{Y}}\frac{\partial F_Y^{(0)}}{\partial Y}, \tag{11}$$

where

$$F_Y^{(0)} = -2r_0 c^2 \lambda_b (ct - s) BE(X, Y).$$
(12)

Introduce new variables Y_0, Y_0 ,

$$Y = Y_{tr}(Y_0, \dot{Y}_0, t, s), \quad \dot{Y} = \dot{Y}_{tr}(Y_0, \dot{Y}_0, t, s), \tag{13}$$

and the function $f(Y_0, \dot{Y}_0, t, s)$, which is related to $\rho^{(1)}$ by

$$\rho^{(1)}(Y, \dot{Y}, t, s) = f(Y_0, \dot{Y}_0, t, s)|_{Y_0 = Y_{tr}(Y, -t, s)}.$$
(14)

The function f satisfies equation

$$\frac{\partial f}{\partial t} = y_c(t, ct - s) \frac{\partial \rho^{(0)}}{\partial \dot{Y}} \frac{\partial F_Y^{(0)}}{\partial Y}.$$
(15)

The arguments Y and \dot{Y} in the RHS have to be expressed in terms of Y_0 , \dot{Y}_0 using Eq. (13). Then, the RHS is a function of Y_0 , \dot{Y}_0 and can be written as Poisson brackets $y_c \{F_Y^{(0)}, \rho^{(0)}\}_{Y,\dot{Y}} = y_c \{F_Y^{(0)}, \rho^{(0)}\}_{Y_0,\dot{Y}_0}$. Thus, for $t > t_0 = s/c$,

$$f(Y_0, \dot{Y}_0, t, s) = \int_{t_0}^t dt' y_c(t', ct'-s) \{ F_Y^{(0)}[Y_{tr}(Y_0, \dot{Y}_0, t'-t_0, s), ct'-s], \rho_{\Sigma}^{(0)}[Y_0, \dot{Y}_0, 0, s] \}_{Y_0, \dot{Y}_0}$$
(16)

Here we used the identity: $\rho^{(0)}[Y_{tr}(Y_0, \dot{Y}_0, t', s), t', s] = \rho^{(0)}_{\Sigma}[Y_0, \dot{Y}_0, 0, s].$

Eqs. (16, 14) define the RHS of Eq. (5):

$$RHS = -\frac{2r_ec^2}{\gamma} \frac{dN_e}{ds} \int dX_0 dY_0 d\dot{X}_0 d\dot{Y}_0 \int_0^t dt' y_c(t', c(t'-t) + z)$$
$$BE[X, Y_{tr}(Y_0, \dot{Y}_0, t, s)] \{ F_Y^{(0)}[Y_{tr}(Y_0, \dot{Y}_0, t'-t_0, s), ct'-s], \rho^{(0)}[Y_0, \dot{Y}_0, 0, s] \}_{Y_0, \overset{\ell}{X}_0}$$

Integrating by parts and changing variable t' to z', t' = t + (z' - z)/c, it can be transformed to the form

$$RHS = -\frac{2r_ec}{\gamma} \frac{dN_e}{ds} 2r_0 c^2 \int_0^z dz' y_c (t + \frac{z'-z}{c}, z') \lambda_b(z') \int dX_0 dY_0 d\dot{X}_0 d\dot{Y}_0 \rho^{(0)} [Y_0, \dot{Y}_0, 0, s] \{BE[X, Y_{tr}(Y_0, \dot{Y}_0, \frac{z'}{c}, s)], BE[Y_{tr}(Y_0, \dot{Y}_0, t + \frac{z}{c}, s)]\}_{Y_0, \dot{Y}_0},$$
(18)

where s = ct - z. Let us compare Eq. (18) with the standard form of the force due to the transverse wake field per unit length W:

$$\frac{d^2 y_c(t,z)}{dt^2} + \omega_\beta^2 y_c(t,z) = \frac{r_e c^2}{\gamma} \int^z dz' W(z'-z) y_c(t+\frac{z'-z}{c},z') \lambda_b(z'), \quad (19)$$

where $\lambda_b(z)$ is the linear density of a bunch normalized to the bunch population, $\int dz \lambda_b(z) = N_b$. Comparison defines the effective wake of the cloud per unit length:

$$W(z, z') = 4r_0 c \frac{dN_e}{ds} \int dX_0 dY_0 d\dot{X}_0 d\dot{Y}_0 \rho_{\Sigma}^{(0)}[Y_0, \dot{Y}_0, 0, s] \{BE[X, Y_{tr}(Y_0, \dot{Y}_0, z/c, s)], BE[X, Y_{tr}(Y_0, \dot{Y}_0, z'/c, s)]\}_{Y_0, \dot{Y}_0}.$$
 (20)

This formula gives the wake in terms of the trajectories of electrons in the field of a bunch with the zero offset.

Let us calculate the effective wake neglecting anharmonicity of the oscillations in the cloud but taking into account dependence of the frequency on amplitudes. This will allow us to calculate the *Q*-factor of the wake. In this approximation,

$$Y_{tr}(Y_0, \dot{Y}_0, t, s) = Y_0 \cos[\psi] + \frac{\dot{Y}_0}{\Omega} \sin[\psi], \qquad (21)$$

where $d\psi/dt = \Omega(X_0, Y_0, ct - s)$. The Poisson in this approximation give:

$$\{BE[X, Y_{tr}(Y_0, \dot{Y}_0, t, s), t, s], BE[X, Y_{tr}(Y_0, \dot{Y}_0, t + \frac{z'-z}{c}, s), t', s]\}_{Y_0, \dot{Y}_0}$$
$$= \left[\frac{\sin[\psi(z)]Cos[\psi(z')]}{\Omega(z)} - \frac{\sin[\psi(z')]Cos[\psi(z)]}{\Omega(z')}\right] \left(\frac{\partial BE}{\partial Y_{tr}}\right)_z \left(\frac{\partial BE}{\partial Y_{tr}}\right)_{z'}.$$
(22)

Here $d\psi(z)/dz = \Omega(z)/c$, $(\frac{\partial BE}{\partial Y_{tr}})_z$ and $(\frac{\partial BE}{\partial Y_{tr}})_{z'}$ have arguments $Y_{tr}(Y_0, \dot{Y}_0, z/c, s)$ and $Y_{tr}(Y_0, \dot{Y}_0, z'/c, s)$, respectively.

After substitution of Eq. (22) into Eq. (20), the initial velocity \dot{Y}_0 can be put to zero. This is justified because the potential well of an electron in

the field of a bunch is deeper than the average potential well of a beam by a large factor equal to the ratio of the bunch spacing to the rms σ_z .

The frequency Ω in 1D case is derived from the Hamiltonian $H(Y, \dot{Y}, t, s) = \frac{\dot{Y}^2}{2} + U$, where U(X, Y, t, s) is related to the force $F = -(\partial U/\partial Y) = 2r_0\lambda_b(s-ct)c^2BE(X,Y)$. We can distinguish two extreme cases: a sharp edge bunch (low bunch current), $\Omega_0\sigma_z/c << 1$, and a adiabatic bunch $\Omega_0\sigma_z/c >> 1$ where Ω_0 is frequency of linear oscillations,

$$\left(\frac{\Omega_0}{c}\right)^2 = \frac{\lambda_b(0)r_0}{\sigma_y(\sigma_x + \sigma_y)}.$$
(23)

In the first case, the energy H and Ω are defined by initial Y_0 and Y_0 ,

$$\frac{\Omega_0}{\Omega} = \frac{2\Omega_0}{\pi} \int_0^{Y_0} \frac{dY}{\sqrt{2[U(Y_0) - U(Y)]}}.$$
(24)

In the second case, the adiabatic-invariant $J(H) = \int (dY/2\pi) \sqrt{2[H - U(Y, t)]}$ is constant, and $\frac{1}{\Omega} = \frac{1}{dJ/dH}$ is again given by Eq. (24).

Electrons of the cloud after interaction with a sharp edge bunch change their velocity and are accelerated to the speed above the average velocity before interaction. For adiabatic bunch it does not take place. For this reason, the maximum energy of the electrons does not increase proportional to the beam current but is limited by the condition $\Omega \sigma_z/c \simeq 1$.

Eq. (24) can be simplified noticing that the main contribution to the integral is given by coordinates Y in the vicinity of Y_0 . Expanding $U(Y_0) - U(Y)$ around Y_0 , we get

$$\frac{\Omega(z)}{\Omega_0} = \sqrt{S_0(X, Y_0) \frac{\lambda_b(z)}{\lambda_b(0)}}.$$
(25)

Here $\lambda_b(0)$ is the maximum linear density of a bunch, $\lambda_b(0) = N_b/(\sigma_z \sqrt{2\pi})$. The first factor, $\sqrt{S_0(X, Y_0)}$, is shown in Fig. 1. The error introduced by this approximation was checked numerically and is small, $\Delta\Omega/\Omega \simeq 0.2$.

Eqs. (20), (22), and Eq. (25) define the wake per unit length as a function of $\zeta = \Omega_0(z'-z)/c$, proportional to the distance z'-z between leading and trailing slices, and $\zeta_0 = \Omega_0 z/c$, the position of the leading slice from the head of the bunch:

$$W(z, z') = \frac{8n_e}{\lambda_b(1+p)} (\frac{\Omega_0}{c}) W_{eff}(\zeta, \zeta_0).$$
(26)

Here,

$$W_{eff}(z,z') = \int_0^\infty dx \int_0^\infty dy e^{-\frac{x^2}{2}(\frac{\sigma_x}{\Sigma_x})^2 - \frac{Y^2}{2}(\frac{\sigma_y}{\Sigma_y})^2} \left[\frac{\sin[\psi(z)]Cos[\psi(z')]}{\Omega(z)/\Omega_0} - \frac{\sin[\psi(z')]Cos[\psi(z)]}{\Omega(z')/\Omega_0}\right] \\ [S_0(x,y_z) - y_z^2 S_1(x,y_z)] [S_0(x,y_z') - y_{z'}^2 S_1(x,y_z')],$$
(27)



Figure 1: Frequency Ω/Ω_0 vs initial amplitudes $0 < X/\sigma_x < 3, 0 < Y/\sigma_y < 5$.

where $y_z = y \cos[\psi(z)]$, and $y'_z = y \cos[\psi(z')]$.

In the integrals we used dimensionless $x = X/\sigma_x$, $y = Y/\sigma_y$. The functions $S_0(x, y)$ and $S_1(x, y)$ in this variables are

$$S_{0;1}(X,Y) = \left(\frac{1+p}{2p}\right) \int_0^\infty \frac{d\mu}{(1+\mu)^{3/2}\sqrt{1+\mu/p^2}} e^{-\frac{\mu}{1+\mu}\frac{y^2}{2} - \frac{\mu x^2}{2(\mu+p^2)}} [1;\frac{\mu}{1+\mu}].$$
(28)

The wake Eq. (27) is a weak function of parameters p, z, and the ratio $\sum_{x,y}/\sigma_{x,y}$. The wake W_{eff} Eq. (25)-(28) calculated for parameters z' = 0, p = 0.2, $\sum_x/\sigma_x = \sum_y/\sigma_y = 5$ is shown in Fig.2. The calculations were carried out with MATHEMATICA interpolating functions $S_{0,1}$ and $\Omega(X,Y)$ and, then, carrying out double integrals in Eq. (27).

For the nominal LER PEP-II parameters, Table I, the average cloud density $n_e = 4.75 \, 10^6$, $\Omega_y/(2\pi) = 14.0$ GHz, the number of oscillations within the bunch rms $\Omega_y \sigma_z/(2\pi c) = 0.6$, and the amplitude of the wake field is 695 V/pC/cm what corresponds to the shunt impedance 4.7 MOhm/m. This should be compared with the resistive wall transverse wake

$$W_x(s) = \frac{4\delta_0}{b^3} \sqrt{\frac{2\pi R}{s}},\tag{29}$$

where δ_0 is the skin depth at the revolution harmonics. For PEP-II parameters, $\delta_0 \simeq 0.17$ mm, and $W_x = 2.0 \ V/pC/cm$ at s = 1 cm.

The wake, see Fig. 2, can be approximated by the wake of a single mode with frequency $\mu\Omega_0$,

$$W_{eff}(\zeta) = W_{max} \sin(\mu \zeta) e^{-\frac{\mu \zeta}{2Q}}.$$
(30)



Figure 2: Effective wake $W_{eff}(\zeta, 0)$ of the cloud as function of $\zeta = \Omega_0 z/c$.

Dependence of the factor $W_{eff}(\zeta, 0)$ on parameters is illustrated in Figs.4-6. The best fit in all cases was for $\mu = 0.9$.

The wake shown in Fig. 3-4 was calculated for a long bunch $\Omega_0 \sigma_z/c >> 1$ and for the leading slice at the head of the bunch, z' = 0. Fig. 5 shows wakes generated by a leading slice z' = 0 for several values of $\Omega_0 \sigma_z/c$. For a short bunch, the wake is mostly linear. Initial slope is the same in all cases. Fig. 6 shows wake for different positions of the leading slice within a bunch.

4 Summary

The wake field induced by the beam interaction with the electron cloud can cause the head-tail instability. The effective wake of the e-cloud is given in terms of electron trajectories in the filed of the beam with the zero offset. Neglecting anharmonicity of the motion but taking into account the amplitude dependence of the frequencies of electron oscillations, we obtain expression for the effective wake driven by the e-cloud. Dependence of the wake on the beam parameters is in a good agreement with the tracking simulations [3].

References

- [1] Proposed upgrade of the B-factory, Snowmass, 2001
- K. Ohmi, "Beam-Photoelectron Interactions in Positron Storage Rings," Phys. Rev. Lett. 75 1526 (1995).
- [3] K. Ohmi and F. Zimmermann, EPAC 2000, Vienna, Austria, June 2000.

Parameter	Symbol	Value
Energy, LER	E, GeV	3.1
average radius	$R,\!\mathrm{m}$	350
bend radius, LER	$ ho_c,\mathrm{m}$	13.752
relat.factor	γ	$6.103\ 10^3$
momentum compaction	α	$1.23 \ 10^{-3}$
bunch length	σ_l , cm	1.1
relative energy spread	δ	7.710^{-4}
$\operatorname{emittance,nm}$	$\epsilon_{x,y}$	49.5/1.2
tune	$Q_{y,x}$,	38.57/36.6
average x beta, m	β_x	9.370
average y beta,m	β_y	12.47
synchrotron tune	ν_s	0.0251
vertical half gap	$b~{ m cm}$	2.5
Bunch population	N_b	1.010^{11}
number of bunches	n_b	692

Table 1: Main Parameters of the LER PEP-II

- [4] J.B. Murphy, Workshop on Broadband Impedance Measurements and Modeling, SLAC, Stanford, February 2000
- [5] S. Heifets, G. Stupakov, Microbunching by Coherent Synchrotron Radiation, SLAC-PUB-8988, August 2001.
- [6] K. Ohmi, F. Zimmermann, E. Perevedentsev, Study of the fast head-tail instability caused by electron cloud, CERN-SL-2001-011 AP, presented at HEACC2001, Tsukuba, March 2001
- [7] M. Bassetti and G. Erskine, CERN ISR TH/80-06 (1980).
- [8] K. Ohmi and F. Zimmermann, "Head-Tail Instability Caused by Electron Clouds in Positron Rings," Phys. Rev. Lett. 85 3821 (2000).
- [9] K. Satoh and Y.H. Chin, "Transverse Mode Coupling in a Bunched Beam," Nucl. Instr. Meth. 207, 309 (1983).
- [10] A. W. Chao, Physics of Collective Beam Instabilities in High Energy Accelerators (Wiley-Interscience Publication, New York, (1993).
- [11] J.B. Murphy, S. Krinsky and R. Gluckstern, Proceedings of 1995 IEEE PAC (1995),
- [12] Ya.S. Derbenev, J. Rossbach, E.L. Saldin, V.D. Shiltsev, Microbunch Radiative Tail-Head Interaction, Preprint TESLA-FFI 95-05, September 1995.



Figure 3: $W_{eff}(\zeta, 0)$ vs $p = \sigma_y / \sigma_x$ for fixed $\Sigma_{x,y} / \sigma_{x,y} = 5$.

[13] S. Heifets, G. Stupakov, Nonlinear stage of the microwave instability, SLAC-PUB-8758, August 2001.



Figure 4: Dependence of $W_{eff}(\zeta, 0)$ on $\Sigma_{x,y}/\sigma_{x,y}$ for fixed p = 0.2.



Figure 5: $W_{eff}(\zeta, 0)$ for several values of $\Omega_0 \sigma_z/c$. p = 0.2, $\Sigma_{x,y}/\sigma_{x,y} = 5$.



Figure 6: $W_{eff}(\zeta, z_{head})$ for several values of $\Omega_0 z'/c$, where z' is location of the leading slice. Other parameters are: $\Omega_0 \sigma_z/c = 5$, p = 0.2, $\Sigma_{x,y}/\sigma_{x,y} = 5$.