On the use of the reciprocal basis in neutral meson mixing

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In the presence of CP violation, the effective Hamiltonian matrix describing a neutral meson anti-meson system does not commute with its hermitian conjugate. As a result, this matrix cannot be diagonalized by a unitary transformation and one needs to introduce a reciprocal basis. Although known, this fact is seldom discussed and almost never used. Here, we use this concept to highlight a parametrization of the Hamiltonian matrix in terms of physical observables, and we show that using it reduces a number of long and tedious derivations into simple matrix multiplications. These results have a straightforward application for propagation in matter. We also comment on the (mathematical) relation with neutrino oscillations.

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I. INTRODUCTION

We are interested in the effective 2×2 Hamiltonian matrix describing the mixing in the $P^0 - \overline{P^0}$ systems, where P stands for K, D, B_d, or B_s. We denote this 2×2 matrix by $\boldsymbol{H} = \boldsymbol{M} - i/2\boldsymbol{\Gamma}$ where

$$\boldsymbol{M} = \left(\boldsymbol{H} + \boldsymbol{H}^\dagger\right)/2 \quad \text{and} \quad -i\boldsymbol{\Gamma}/2 = \left(\boldsymbol{H} - \boldsymbol{H}^\dagger\right)/2, \tag{1}$$

describe the hermitian and anti-hermitian parts of $\boldsymbol{H}$, respectively. Both $\boldsymbol{M}$ and $\boldsymbol{\Gamma}$ are hermitian. Matrices satisfying $[\boldsymbol{H}, \boldsymbol{H}^\dagger] = 0$ are called "normal" matrices. It is easy to show that

$$\left[\boldsymbol{H}, \boldsymbol{H}^\dagger\right] = 0 \Leftrightarrow [\boldsymbol{M}, \boldsymbol{\Gamma}] = 0. \tag{2}$$

Moreover, a matrix is normal if and only if it can be diagonalized by a unitary transformation. It is often stated that non-unitary transformations arise whenever $\boldsymbol{H}$ is not hermitian. This is not the case. What is relevant is whether $\boldsymbol{H}$ is normal or not. Indeed, if $\boldsymbol{\Gamma} \neq 0$ then $\boldsymbol{H}$ is not hermitian; however, $\boldsymbol{H}$ can still be diagonalized by a unitary matrix as long as $[\boldsymbol{M}, \boldsymbol{\Gamma}] = 0$.

In section II we introduce the concept of 'reciprocal basis' and we show that the presence of T violation in the $P^0 - \overline{P^0}$ system forces us to use such a basis. The physical observables are defined in section III and they are used in section IV to parametrize $\boldsymbol{H}$ exclusively in terms of measurable quantities. The time evolution of the $P^0 - \overline{P^0}$ system is discussed in section V. Section VI explains why the $P^0 - \overline{P^0}$ should be considered as intermediate states, and section VII shows an error which arises when one does not use the reciprocal basis. Matter effects are then considered in section VIII. This differs from all previous analyses of matter effects in that no use is made of the Good equations; here the time evolution is obtained in a trivial way. In section IX we compare the mixing in the $P^0 - \overline{P^0}$ system with the mixing in the neutrino sector. To this end, we start by showing how the equation describing the time evolution and its solution would look if we had chosen a different reference frame. We present our conclusions in section X. For completeness, appendix A contains some elementary notions of collision theory, which are needed to describe the evolution in the presence of interactions with matter. Appendix B contains two other parametrizations of the physical observables commonly found in the literature, the first of which is most convenient for the comparison with the neutrino sector.

II. THE RECIPROCAL BASIS

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A. Definition

Why do we change basis at all? One reason is that the time evolution of the state $|\psi(t)\rangle$ describing the $P^0 - \overline{P^0}$ mixed state, which is given by

$$i\frac{d}{dt}|\psi(t)\rangle = \boldsymbol{H}|\psi(t)\rangle, \tag{3}$$

becomes trivial in the basis in which $\boldsymbol{H}$ is diagonal. Eq. (3) and $\boldsymbol{H}$ have been written in the $P^0 - \overline{P^0}$ rest frame and t is the proper time.

We denote the (complex) eigenvalues of $\boldsymbol{H}$ by $\mu_a = m_a - i/2\Gamma_a$ and $\mu_b = m_b - i/2\Gamma_b$, corresponding to the eigenvectors

$$\begin{pmatrix} |P_a\rangle \\ |P_b\rangle \end{pmatrix} = \begin{pmatrix} p_a & q_a \\ p_b & -q_b \end{pmatrix} \begin{pmatrix} |P^0\rangle \\ |\overline{P^0}\rangle \end{pmatrix} = \boldsymbol{X}^T \begin{pmatrix} |P^0\rangle \\ |\overline{P^0}\rangle \end{pmatrix}. \tag{4}$$

As a result, the matrix $\boldsymbol{H}$ is diagonalized through

$$\boldsymbol{X}^{-1}\boldsymbol{H}\boldsymbol{X} = \begin{pmatrix} \mu_a & 0 \\ 0 & \mu_b \end{pmatrix}, \tag{5}$$

where

$$\boldsymbol{X}^{-1} = \frac{1}{p_a q_b + p_b q_a} \begin{pmatrix} q_b & p_b \\ q_a & -p_a \end{pmatrix}. \tag{6}$$

As stated above, $\boldsymbol{H}$ is normal if and only if $\boldsymbol{X}$ is unitary. This is what one learns in algebra.

So, why do (most) people worry about performing non-unitary transformations? The reason is that one would like the mass basis $\{|P_a\rangle, |P_b\rangle\}$ to retain a number of the nice (orthogonality) features of the $\{|P^0\rangle, |\overline{P^0}\rangle\}$ flavor basis. Among these: the orthogonality conditions

$$\begin{aligned} \langle P^0|\overline{P^0}\rangle &= \langle\overline{P^0}|P^0\rangle = 0, \\ \langle P^0|P^0\rangle &= \langle\overline{P^0}|\overline{P^0}\rangle = 1; \end{aligned} \tag{7}$$

the fact that $|P^0\rangle\langle P^0|$ and $|\overline{P^0}\rangle\langle\overline{P^0}|$ are projection operators; the completeness relation

$$|P^0\rangle\langle P^0| + |\overline{P^0}\rangle\langle\overline{P^0}| = 1; \tag{8}$$

and the decomposition of the effective Hamiltonian as

$$\begin{aligned} \mathcal{H} &= |P^0\rangle H_{11}\langle P^0| + |P^0\rangle H_{12}\langle\overline{P^0}| + |\overline{P^0}\rangle H_{21}\langle P^0| + |\overline{P^0}\rangle H_{22}\langle\overline{P^0}| \\ &= \left(|P^0\rangle,\ |\overline{P^0}\rangle \right) \boldsymbol{H} \begin{pmatrix} \langle P^0| \\ \langle\overline{P^0}| \end{pmatrix}. \end{aligned} \tag{9}$$

All these relations involve the basis of flavor eigenkets $\{|P^0\rangle, |\overline{P^0}\rangle\}$ and the basis of the corresponding bras $\{\langle P^0|, \langle\overline{P^0}|\}$. The problem is that, when $\boldsymbol{H}$ is not normal, we *cannot* find similar relations involving the basis of mass eigenkets $\{|P_a\rangle, |P_b\rangle\}$ and the basis of the corresponding bras, $\{\langle P_a|, \langle P_b|\}$. In particular, it is easy to see from the diagonalization in Eq. (5) that the analogue of Eq. (9) is

$$\begin{aligned} \mathcal{H} &= |P_a\rangle\mu_a\langle\tilde{P}_a| + |P_b\rangle\mu_b\langle\tilde{P}_b| \\ &= \left(|P_a\rangle,\ |P_b\rangle \right) \begin{pmatrix} \mu_a & 0 \\ 0 & \mu_b \end{pmatrix} \begin{pmatrix} \langle\tilde{P}_a| \\ \langle\tilde{P}_b| \end{pmatrix}. \end{aligned} \tag{10}$$

This does not involve the bras $\langle P_a|$ and $\langle P_b|$,

$$\begin{pmatrix} \langle P_a| \\ \langle P_b| \end{pmatrix} = \boldsymbol{X}^\dagger \begin{pmatrix} \langle P^0| \\ \langle\overline{P^0}| \end{pmatrix}, \tag{11}$$

but rather the so called 'reciprocal basis'

$$\begin{pmatrix} \langle \tilde{P}_a| \\ \langle \tilde{P}_b| \end{pmatrix} = \boldsymbol{X}^{-1} \begin{pmatrix} \langle P^0| \\ \langle \overline{P^0}| \end{pmatrix}. \tag{12}$$

The reciprocal basis may also be defined by the orthogonality conditions

$$\begin{aligned} \langle \tilde{P}_a|P_b\rangle &= \langle \tilde{P}_b|P_a\rangle = 0, \\ \langle \tilde{P}_a|P_a\rangle &= \langle \tilde{P}_b|P_b\rangle = 1. \end{aligned} \tag{13}$$

Moreover, $|P_a\rangle\langle\tilde{P}_a|$ and $|P_b\rangle\langle\tilde{P}_b|$ are projection operators, and the partition of unity becomes

$$|P_a\rangle\langle\tilde{P}_a| + |P_b\rangle\langle\tilde{P}_b| = 1. \tag{14}$$

If $\boldsymbol{H}$ is not normal, then $\boldsymbol{X}$ is not unitary, and $\{\langle P_a|, \langle P_b|\}$ in Eq. (11) do not coincide with $\left\{\langle \tilde{P}_a|, \langle \tilde{P}_b|\right\}$ in Eq. (12). Another way to state this fact is to note that $\boldsymbol{H}$ is normal ($\boldsymbol{X}$ is unitary) if and only if its right-eigenvectors coincide with its left-eigenvectors.

That these features have an impact on the $K^0 - \overline{K^0}$ system, was pointed out long ago by Sachs [1,2] and by Enz and Lewis [3]. More recently, they have been stressed by Alvarez-Gaumé *et al.* [4], and by Branco, Lavoura and Silva in their book "CP violation" [5]. Still, we have found that they are not common knowledge. This is unfortunate since there are a number of results that usually require considerable algebra which become trivial once the matrix formulation discussed here is implemented. Moreover, one can express the matrix elements of $\boldsymbol{H}$, written in the $P^0 - \overline{P^0}$ basis, in terms of observable quantities. This is what we show here.

B. The relation to CP violation

We will now show that the reciprocal basis is required by the observation of T and CP violation in the mixing in the neutral meson systems. The discrete symmetries have the following effects on the matrix elements of $\boldsymbol{H}$:

$$\begin{aligned} \mathcal{CPT} \text{ conservation} &\Rightarrow H_{11} = H_{22}, \\ \mathcal{T} \text{ conservation} &\Rightarrow |H_{12}| = |H_{21}|, \\ \mathcal{CP} \text{ conservation} &\Rightarrow H_{11} = H_{22} \text{ and } |H_{12}| = |H_{21}|. \end{aligned} \tag{15}$$

The 1964 discovery that $|H_{12}| \neq |H_{21}|$ in the kaon system [6] means that there is T and CP violation in $K^0 - \overline{K^0}$ mixing. Moreover, since the $(1,1)$ entry in the matrix $[\boldsymbol{H}, \boldsymbol{H}^\dagger]$ is given by $|H_{12}|^2 - |H_{21}|^2$, this experimental result also implies that the matrix $\boldsymbol{H}$ is not normal and, thus, that we are forced to deal with non-unitary matrices in the neutral kaon system.

For the other neutral meson systems, $|H_{12}| \neq |H_{21}|$ has not been established experimentally. Nevertheless, the Standard Model predicts that, albeit the difference is small, $|H_{12}| \neq |H_{21}|$ does indeed hold. As before, this implies CP violation in the mixing and forces the use of the reciprocal basis in all the neutral meson systems.

III. OBSERVABLES IN THE $P^0 - \overline{P^0}$ MIXING

Let us start by introducing some notation. We define

$$\begin{aligned} \mu &= m - i\Gamma/2 \equiv (\mu_a + \mu_b)/2, \\ \Delta\mu &= \Delta m - i\Delta\Gamma/2 \equiv \mu_a - \mu_b. \end{aligned} \tag{16}$$

Sometimes it is convenient to trade the eigenvalue difference for $x - iy \equiv \Delta\mu/\Gamma$. We may write the mixing matrix $\boldsymbol{X}$ in terms of new parameters

$$\theta = \frac{\frac{q_a}{p_a} - \frac{q_b}{p_b}}{\frac{q_a}{p_a} + \frac{q_b}{p_b}}, \tag{17}$$

and

$$\frac{q}{p} = \sqrt{\frac{q_a q_b}{p_a p_b}} \tag{18}$$

Notice that we have not defined the quantities q and p separately; we only define the ratio q/p. With this notation the mixing matrix may be re-written as

$$\boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ \frac{q}{p}\sqrt{\frac{1+\theta}{1-\theta}} & -\frac{q}{p}\sqrt{\frac{1-\theta}{1+\theta}} \end{pmatrix} \begin{pmatrix} p_a & 0 \\ 0 & p_b \end{pmatrix}, \tag{19}$$

$$\boldsymbol{X}^{-1} = \begin{pmatrix} p_a^{-1} & 0 \\ 0 & p_b^{-1} \end{pmatrix} \begin{pmatrix} \frac{1-\theta}{2} & \frac{p}{q}\frac{\sqrt{1-\theta^2}}{2} \\ \frac{1+\theta}{2} & -\frac{p}{q}\frac{\sqrt{1-\theta^2}}{2} \end{pmatrix}. \tag{20}$$

We point out that these transformation matrices involve the normalization constants p_a and p_b. Finally, it will also prove convenient to define

$$\delta = \frac{1 - \left|\frac{q}{p}\right|^2}{1 + \left|\frac{q}{p}\right|^2}, \tag{21}$$

meaning that $|q/p| = \sqrt{\frac{1-\delta}{1+\delta}}$.

The fact that the trace and determinant are invariant under the general similarity transformation in Eq. (5) implies that

$$\begin{aligned} \mu &= (H_{11} + H_{22})/2, \\ \Delta\mu &= \sqrt{4H_{12}H_{21} + (H_{22} - H_{11})^2}. \end{aligned} \tag{22}$$

Moreover, from

$$\begin{aligned} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} p_a \\ q_a \end{pmatrix} &= \mu_a \begin{pmatrix} p_a \\ q_a \end{pmatrix}, \\ \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} p_b \\ -q_b \end{pmatrix} &= \mu_b \begin{pmatrix} p_b \\ -q_b \end{pmatrix}. \end{aligned} \tag{23}$$

we find that

$$\begin{aligned} \frac{q_a}{p_a} &= \frac{\mu_a - H_{11}}{H_{12}} = \frac{H_{21}}{\mu_a - H_{22}}, \\ \frac{q_b}{p_b} &= \frac{H_{11} - \mu_b}{H_{12}} = \frac{H_{21}}{H_{22} - \mu_b}, \end{aligned} \tag{24}$$

leading to

$$\begin{aligned} \theta &= \frac{H_{22} - H_{11}}{\mu_a - \mu_b}, \\ \delta &= \frac{|H_{12}| - |H_{21}|}{|H_{12}| + |H_{21}|}, \end{aligned} \tag{25}$$

and $q/p = \sqrt{H_{21}/H_{12}}$. We see that $\mathrm{Re}\,\theta$ and $\mathrm{Im}\,\theta$ are CP- and CPT-violating, while δ is CP- and T-violating.

Although $\boldsymbol{H}$ contains eight real numbers, only seven are physically meaningful. Indeed, one is free to change the phase of the kets $|P^0\rangle$, $|\overline{P^0}\rangle$, $|P_a\rangle$, and $|P_b\rangle$, as

$$\begin{aligned} |P^0\rangle &\to e^{i\gamma}|P^0\rangle, \\ |\overline{P^0}\rangle &\to e^{i\overline{\gamma}}|\overline{P^0}\rangle, \\ |P_a\rangle &\to e^{i\gamma_a}|P_a\rangle, \\ |P_b\rangle &\to e^{i\gamma_b}|P_b\rangle. \end{aligned} \tag{26}$$

Under these transformations

$$
\begin{aligned}
H_{12} &\to e^{i(\overline{\gamma}-\gamma)} H_{12}, \\
H_{21} &\to e^{i(\gamma-\overline{\gamma})} H_{21}, \\
q/p &\to e^{i(\gamma-\overline{\gamma})} q/p,
\end{aligned} \tag{27}
$$

while H_{11}, H_{22}, μ, $\Delta\mu$, θ, and δ do not change. Therefore, the relative phase between H_{12} and H_{21} is physically meaningless and $\boldsymbol{H}$ contains only seven observables. Similarly, *the phase of q/p is also unphysical.* As a result, we have four observables in the eigenvalues, μ and $\Delta\mu$, and three in the mixing matrix, θ and δ (or, alternatively, $|q/p|$).

IV. PARAMETRIZING $\boldsymbol{H}$ WITH MEASURABLE QUANTITIES

Eqs. (22) and (25) give the measurable mixing and eigenvalue parameters in terms of the H_{ij} matrix elements which one can calculate in a given model. Given the current and upcoming experimental probes of the various neutral meson systems, it seems much more appropriate to do precisely the opposite; that is, to give the H_{ij} matrix elements in terms of the experimentally accessible quantities. Such expressions would give M_{ij} and Γ_{ij} in a completely model independent way, with absolutely no assumptions. One could then calculate these quantities in any given model; if they fit in the allowed ranges the model would be viable.

Surprisingly, this is not is done in most expositions of the $P^0 - \overline{P^0}$ mixing. The reason is simple. Eqs. (22) and (25) are non-linear in the H_{ij} matrix elements. Thus, inverting them by brute force would entail a tedious calculation. With the matrix manipulation discussed here this inversion is straightforward. Indeed, Eq. (5) can be trivially transformed into [7]

$$
\begin{aligned}
\boldsymbol{H} &= \boldsymbol{X} \begin{pmatrix} \mu_a & 0 \\ 0 & \mu_b \end{pmatrix} \boldsymbol{X}^{-1} \\
&= \begin{pmatrix} \mu - \frac{\Delta\mu}{2}\theta & \frac{p}{q}\frac{\sqrt{1-\theta^2}}{2}\Delta\mu \\ \frac{q}{p}\frac{\sqrt{1-\theta^2}}{2}\Delta\mu & \mu + \frac{\Delta\mu}{2}\theta \end{pmatrix},
\end{aligned} \tag{28}
$$

where we have used Eqs. (19) and (20). This equation expresses in a very compact form the relation between the quantities which are experimentally accessible and those which are easily calculated in a given theory. Expanding it, we find

$$
\begin{aligned}
M_{11} &= m - \mathrm{Re}\,\theta\,\frac{\Delta m}{2} - \mathrm{Im}\,\theta\,\frac{\Delta\Gamma}{4}, \\
M_{22} &= m + \mathrm{Re}\,\theta\,\frac{\Delta m}{2} + \mathrm{Im}\,\theta\,\frac{\Delta\Gamma}{4}, \\
\frac{q}{p}M_{12} &= \frac{1}{2(1+\delta)}\left[\mathrm{Re}(\sqrt{1-\theta^2})\left(\Delta m - i\delta\,\frac{\Delta\Gamma}{2}\right) + \mathrm{Im}(\sqrt{1-\theta^2})\left(\frac{\Delta\Gamma}{2} + i\delta\,\Delta m\right)\right],
\end{aligned} \tag{29}
$$

and

$$
\begin{aligned}
\Gamma_{11} &= \Gamma - \mathrm{Re}\,\theta\,\frac{\Delta\Gamma}{2} + \mathrm{Im}\,\theta\,\Delta m, \\
\Gamma_{22} &= \Gamma + \mathrm{Re}\,\theta\,\frac{\Delta\Gamma}{2} - \mathrm{Im}\,\theta\,\Delta m, \\
\frac{q}{p}\Gamma_{12} &= \frac{1}{1+\delta}\left[\mathrm{Re}(\sqrt{1-\theta^2})\left(\frac{\Delta\Gamma}{2} + i\delta\,\Delta m\right) - \mathrm{Im}(\sqrt{1-\theta^2})\left(\Delta m - i\delta\,\frac{\Delta\Gamma}{2}\right)\right].
\end{aligned} \tag{30}
$$

We would argue that this is the best way to quote the experimental results. The impact of any assumption made about the physical observables, such as CPT or T conservation, is transparent in Eqs. (29) and (30).

A few remarks are in order. Firstly we note that Eqs. (19) and (20) involved the overall normalization factors p_a and p_b, but that these cancel in the multiplication on the right hand side of Eq. (28). Secondly, although M_{12}, Γ_{12} and q/p are not rephasing invariant, we can see from Eqs. (27) that $q/p\,M_{12}$, $q/p\,\Gamma_{12}$ and $M_{12}\,\Gamma_{12}^*$ are indeed physically meaningful. Thirdly, the equations involving Γ are needed also for the unitarity conditions [8]

$$\sum_g |A_g|^2 = \Gamma_{11} = \Gamma\,(1 - y\,\mathrm{Re}\,\theta + x\,\mathrm{Im}\,\theta)\ ,$$

$$\sum_g \left|\bar{A}_g\right|^2 = \Gamma_{22} = \Gamma\,(1 + y\,\mathrm{Re}\,\theta - x\,\mathrm{Im}\,\theta)\ ,$$

$$\sum_g \frac{q}{p} A_g^* \bar{A}_g = \frac{q}{p}\Gamma_{12} = \Gamma\,\frac{(y + i\delta x)\,\mathrm{Re}(\sqrt{1-\theta^2}) - (x - i\delta y)\,\mathrm{Im}(\sqrt{1-\theta^2})}{1+\delta}, \tag{31}$$

where $A_g = \langle g|T|P^0\rangle$, $\bar{A}_g = \langle g|T|\overline{P^0}\rangle$, and the sums run over all the available decay modes g.

V. TIME EVOLUTION

The time evolution of the neutral meson system is easily obtained using Eqs. (10) and (14), and the fact that $|P_a\rangle\langle\tilde{P}_a|$ and $|P_b\rangle\langle\tilde{P}_b|$ are projection operators. We find,

$$\begin{aligned}\exp\left(-i\mathcal{H}t\right) &= e^{-i\mu_a t}|P_a\rangle\langle\tilde{P}_a| + e^{-i\mu_b t}|P_b\rangle\langle\tilde{P}_b|. \\ &= \left(\ |P_a\rangle,\ |P_b\rangle\ \right)\begin{pmatrix} e^{-i\mu_a t} & 0 \\ 0 & e^{-i\mu_b t}\end{pmatrix}\begin{pmatrix}\langle\tilde{P}_a| \\ \langle\tilde{P}_b|\end{pmatrix}.\end{aligned} \tag{32}$$

It is now trivial to write the evolution operator back in the flavor basis. Indeed, using Eqs. (4) and (12), we find

$$\begin{aligned}\exp\left(-i\mathcal{H}t\right) &= \left(\ |P^0\rangle,\ |\overline{P^0}\rangle\ \right)\boldsymbol{X}\begin{pmatrix} e^{-i\mu_a t} & 0 \\ 0 & e^{-i\mu_b t}\end{pmatrix}\boldsymbol{X}^{-1}\begin{pmatrix}\langle P^0| \\ \langle\overline{P^0}|\end{pmatrix} \\ &= \left(\ |P^0\rangle,\ |\overline{P^0}\rangle\ \right)\begin{pmatrix} g_+(t) - \theta\, g_-(t) & \frac{p}{q}\sqrt{1-\theta^2}\,g_-(t) \\ \frac{q}{p}\sqrt{1-\theta^2}\,g_-(t) & g_+(t) + \theta\, g_-(t)\end{pmatrix}\begin{pmatrix}\langle P^0| \\ \langle\overline{P^0}|\end{pmatrix},\end{aligned} \tag{33}$$

where

$$g_\pm(t) \equiv \frac{1}{2}\left(e^{-i\mu_a t} \pm e^{-i\mu_b t}\right) = e^{-imt}\,e^{-\Gamma t/2}\begin{cases}\cos\left(\frac{\Delta\mu\, t}{2}\right) \\ -i\sin\left(\frac{\Delta\mu\, t}{2}\right)\end{cases}. \tag{34}$$

This corresponds to the usual expressions for the time evolution of a state which starts out as P^0 or $\overline{P^0}$

$$\begin{aligned}|P^0(t)\rangle &= \exp\left(-i\mathcal{H}t\right)|P^0\rangle = \left[g_+(t) - \theta\, g_-(t)\right]|P^0\rangle + \frac{q}{p}\sqrt{1-\theta^2}\,g_-(t)\,|\overline{P^0}\rangle, \\ |\overline{P^0}(t)\rangle &= \exp\left(-i\mathcal{H}t\right)|\overline{P^0}\rangle = \frac{p}{q}\sqrt{1-\theta^2}\,g_-(t)|P^0\rangle + \left[g_+(t) + \theta\, g_-(t)\right]|\overline{P^0}\rangle,\end{aligned} \tag{35}$$

respectively. At this point it is important to emphasize the fact that, in deriving this result, no assumptions were made about the form of the original matrix $\boldsymbol{H}$. This observation will become important once we consider the evolution in matter.

VI. NEUTRAL MESONS AS INTERMEDIATE STATES

Because there is CP violation in $P^0 - \overline{P^0}$ mixing, there is no selection rule allowing us to choose a final state f to which P_a (or P_b) can decay while P_b (P_a) cannot. That is, all calculations must involve the full transition chain [9]

$$i \to X\{P_a, P_b\} \to Xf, \tag{36}$$

with both neutral meson eigenstates as intermediate states, in order to be *formally correct*. Obviously, one could ignore this problem. Still, as we show in section VII, one will be lead into incorrect results if the reciprocal basis is not used as the 'out' bra.

Recently, Amorim, Santos, and Silva [10] have highlighted a very important point about the transition chain in Eq. (36). They showed that this evolution can be fully parametrized by the usual quantities λ_f and $\lambda_{\bar{f}}$, describing the decays $\{P^0, \overline{P^0}\} \to f, \bar{f}$, supplemented by two new quantities ξ_i and $\xi_{\bar{i}}$, describing the production mechanism $i, \bar{i} \to \{P^0, \overline{P^0}\}$. (Although they applied these results only to the case in which $i, \bar{i} \to \{P^0, \overline{P^0}\}$ represents a decay, their formalism is valid in all generality.) The new quantities ξ_i and $\xi_{\bar{i}}$ may entail new sources of CP violation, just like λ_f and $\lambda_{\bar{f}}$ do. They are absent from the decays $B \to J/\psi K \to J/\psi [f]_K$ studied previously [11] because, in those cases, the initial B^0 meson can only decay to one of the kaon's flavor eigenstates. However, they are crucial for the decays $B^\pm \to D + X^\pm \to [f]_D + X^\pm$ [12], and, in general, whenever the initial state i can produce (or, in particular, decay into) both flavor eigenstates of the intermediate neutral meson system, P^0 and $\overline{P^0}$.

Let us consider the decay chain $i \to X\{P_a, P_b\} \to Xf$. The complete amplitude for this process involves the amplitude for the initial decay into XP_a or XP_b, the time-evolution amplitude for this state, given by Eq. (32), and finally the amplitude for the decay into Xf. Suppressing the reference to X, we find

$$A\,(i \to P_{a,b} \to f) = \langle f|T|P_a\rangle\, e^{-i\,\mu_a t}\, \langle \tilde{P}_a|T|i\rangle + \langle f|T|P_b\rangle\, e^{-i\,\mu_b t}\, \langle \tilde{P}_b|T|i\rangle. \tag{37}$$

This is an exact expression. However, sometimes it is possible to choose a final state f and to set the experimental conditions in such a way as to maximize the importance of $i \to XP_a \to Xf$ relative to $i \to XP_b \to Xf$. In that case we may make the approximation

$$\begin{aligned} A\,(i \to P_{a,b} \to f) &\approx A\,(i \to P_a \to f) \\ &= \langle f|T|P_a\rangle\, e^{-i\,\mu_a t}\, \langle \tilde{P}_a|T|i\rangle \\ &= \langle f|T|P_a\rangle\, e^{-i\,\mu_a t}\, \left[\langle \tilde{P}_a|P^0\rangle\langle P^0|T|i\rangle + \langle \tilde{P}_a|\overline{P^0}\rangle\langle \overline{P^0}|T|i\rangle\right], \end{aligned} \tag{38}$$

where we have used the partition of unity $|P^0\rangle\langle P^0| + |\overline{P^0}\rangle\langle \overline{P^0}| = 1$ to derive the last line. When one uses the approximation in Eq. (38), one talks about 'the decay $i \to XP_a$',[1] and writes

$$\begin{aligned} A\,(i \to XP_a) &= \langle \tilde{P}_a|P^0\rangle\, A(i \to XP^0) + \langle \tilde{P}_a|\overline{P^0}\rangle\, A(i \to X\overline{P^0}) \\ &= \frac{1}{2}\left[p^{-1} A(i \to XP^0) + q^{-1} A(i \to X\overline{P^0})\right], \end{aligned} \tag{39}$$

where, in the last line, we have assumed the CPT-invariant case:

$$\begin{aligned} \langle \tilde{P}_a| &= \frac{1}{2}\left(p^{-1}\langle P^0| + q^{-1}\langle \overline{P^0}|\right), \\ \langle \tilde{P}_b| &= \frac{1}{2}\left(p^{-1}\langle P^0| - q^{-1}\langle \overline{P^0}|\right). \end{aligned} \tag{40}$$

Therefore, the ratio of the two component amplitudes in Eq. (39) is given by $q^{-1}/p^{-1} = p/q$, and not by q^*/p^*—as would have been the case if we had used $\langle P_H|$ instead of $\langle \tilde{P}_H|$. The difference between q^{-1}/p^{-1} and q^*/p^* only disappears in the limit $|q/p| = 1$. We will now show that this has a formal impact in the study of the decay $B_d \to J/\psi K_S$.

VII. ON THE NEED FOR THE RECIPROCAL BASIS IN $B_D \to J/\psi K_S$

This decay is so important that it is surprising how many times it is calculated without even mentioning that the use of the reciprocal basis is *required* in order to obtain the *exact* result. We repeat, in this decay the use of the reciprocal basis is not a convenient calculational tool. It is unavoidable when one wishes to obtain the result without approximations.

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[1] Nevertheless, strictly speaking, it is Eq. (37) which expresses the correct way to think about decays into neutral-meson eigenstates [3,9]. As we stressed above, the point is that, since CP is violated, there is no final state f that can be obtained only from P_a and not from P_b. There will always be a non-zero amplitude for the decay path $i \to XP_b \to Xf$.

The first observation we should make is that what one looks for experimentally is the decay chain $B_d \to J/\psi K \to J/\psi(\pi\pi)_K$, and that both intermediate K_S and K_L contribute to this decay. The following argument should make it clear why the intermediate K_L must contribute. Consider the decay chain $B_d \to J/\psi K \to J/\psi(\pi\pi)_K$, but where we have chosen to look only for kaons which live a proper time $\tau \gg \tau_S$ before they decay. Clearly, for these kaons, the K_S component will have disappeared before the decay, and all $\pi\pi$ final states must have come from an intermediate K_L. This explains why, in general, one must use Eq. (37). However, in the experiments searching for $B_d \to J/\psi K_S$ one is looking at kaon proper times $\tau \leq 10\tau_S$. Therefore, in these experiments the decay path $B_d \to J/\psi K_L \to J/\psi(\pi\pi)_K$ is very suppressed with respect to the decay path $B_d \to J/\psi K_S \to J/\psi(\pi\pi)_K$ both due to the huge ratio $A(K_S \to \pi\pi)/A(K_L \to \pi\pi)$ and to the time interval probed. This leads us to Eq. (38) and, ignoring the normalizations p_a and p_b, allows us to talk about the decay $B_d \to J/\psi K_S$ as in Eq. (39).

Having established under which circumstances we may (to good approximation) talk about the decay $B_d \to J/\psi K_S$, we are now in position to describe the upcoming measurement of CP violation in this decay. These experiments will determine the imaginary part of

$$\lambda_{B_d \to J/\psi K_S} \equiv \frac{q_{Bd}}{p_{Bd}} \frac{A(\overline{B_d} \to J/\psi K_S)}{A(B_d \to J/\psi K_S)}. \tag{41}$$

We wish to calculate $A(B_d \to J/\psi K_S)$ and $A(\overline{B_d} \to J/\psi K_S)$. We recall that the decays $B_d \to J/\psi \overline{K^0}$ and $\overline{B_d} \to J/\psi K^0$ are forbidden to leading order in the SM, and, to simplify the problem, we consider the CPT-conserving case, in which

$$\begin{aligned} |K_S\rangle &= p_K |K^0\rangle - q_K |\overline{K^0}\rangle, \\ \langle K_S| &= p_K^* \langle K^0| - q_K^* \langle \overline{K^0}|, \\ \langle \tilde{K}_S| &= \frac{1}{2}\left[p_K^{-1}\langle K^0| - q_K^{-1}\langle \overline{K^0}|\right]. \end{aligned} \tag{42}$$

The question is whether one should use $\langle \tilde{K}_S|$ or $\langle K_S|$ in the final state. That is, we wish to know whether to use

$$\begin{aligned} A\left(B_d \to J/\psi K_S\right) &= \langle \tilde{K}_S|K^0\rangle\, A(B_d \to J/\psi K^0) + \langle \tilde{K}_S|\overline{K^0}\rangle\, A(B_d \to J/\psi \overline{K^0}) \\ &= \frac{1}{2}\left[p_K^{-1} A(B_d \to J/\psi K^0) - q_K^{-1} A(B_d \to J/\psi \overline{K^0})\right] \\ &= \frac{1}{2} p_K^{-1} A(B_d \to J/\psi K^0), \end{aligned} \tag{43}$$

and

$$A\left(\overline{B_d} \to J/\psi K_S\right) = -\frac{1}{2} q_K^{-1} A(\overline{B_d} \to J/\psi \overline{K^0}) \tag{44}$$

or, alternatively, use

$$\begin{aligned} A\left(B_d \to J/\psi K_S\right) &= \langle K_S|K^0\rangle\, A(B_d \to J/\psi K^0) + \langle K_S|\overline{K^0}\rangle\, A(B_d \to J/\psi \overline{K^0}) \\ &= p_K^* A(B_d \to J/\psi K^0) - q_K^* A(B_d \to J/\psi \overline{K^0}) \\ &= p_K^* A(B_d \to J/\psi K^0), \end{aligned} \tag{45}$$

and

$$A\left(\overline{B_d} \to J/\psi K_S\right) = -q_K^* A(\overline{B_d} \to J/\psi \overline{K^0}). \tag{46}$$

In the first case we obtain

$$\lambda_{B_d \to J/\psi K_S} \equiv -\frac{q_{Bd}}{p_{Bd}} \frac{A(\overline{B_d} \to J/\psi \overline{K^0})}{A(B_d \to J/\psi K^0)} \frac{p_K}{q_K}, \tag{47}$$

in the second we obtain

$$\lambda_{B_d \to J/\psi K_S} \equiv -\frac{q_{Bd}}{p_{Bd}} \frac{A(\overline{B_d} \to J/\psi \overline{K^0})}{A(B_d \to J/\psi K^0)} \frac{q_K^*}{p_K^*}. \tag{48}$$

From the previous section, we know that the first expression is the correct one. And, in deriving it, we had to know what the reciprocal basis was and that it had to be used. Nevertheless, since $|q_K/p_K|$ only differs from one at order 10^{-3} and we are looking for a large effect in $\lambda_{B_d \to J/\psi K_S}$, this detail, although needed for an *exact formal* derivation, is numerically insignificant. This explains why it has gone largely unnoticed [13].

VIII. MATTER EFFECTS IN THE $P^0 - \overline{P^0}$ EVOLUTION

We now wish to study how the time evolution of the $P^0 - \overline{P^0}$ changes in the presence of matter. It should be clear that the matter effects will change the specific form of $\boldsymbol{H}$ but, since we have considered the most general such matrix, all the derivations presented above should still apply. It remains to relate the parameters in matter and in vacuum.

We will denote the matrices, matrix elements and eigenvalues in vacuum by unprimed quantities and their analogue in matter by primed quantities. For example, when kaons transverse matter, they are subject to strong interactions which conserve strangeness but which treat the K^0 and $\overline{K^0}$ differently.[2] This effect may be parametrized by a new effective Hamiltonian

$$\boldsymbol{H}_{\rm nuc} = \begin{pmatrix} \chi & 0 \\ 0 & \bar{\chi} \end{pmatrix}, \tag{49}$$

which must be added to the Hamiltonian in vacuum. Notice that this parametrization is completely general. It describes any strangeness-conserving interaction whatsoever. It is also important to notice that our original evolution equation, Eq. (3), and vacuum Hamiltonian $\boldsymbol{H}$ have been written in the $P^0 - \overline{P^0}$ rest frame. Before we add $\boldsymbol{H}_{\rm nuc}$ to $\boldsymbol{H}$ we must ensure that $\boldsymbol{H}_{\rm nuc}$ is also expressed in the rest frame. This point is discussed in appendix A.

The full Hamiltonian in matter becomes

$$\boldsymbol{H}' = \boldsymbol{H} + \boldsymbol{H}_{\rm nuc}. \tag{50}$$

Now, we have already studied the most general effective Hamiltonian, and Eq. (28) relates such an Hamiltonian written in the flavor basis with the corresponding eigenvalues and mixing parameters. Therefore, relating the observables in vacuum and in matter becomes another simple exercise. Eqs. (28), (49) and (50) yield

$$\begin{pmatrix} \mu' - \frac{\Delta\mu'}{2}\theta' & \frac{p'}{q'}\frac{\sqrt{1-\theta'^2}}{2}\Delta\mu' \\ \frac{q'}{p'}\frac{\sqrt{1-\theta'^2}}{2}\Delta\mu' & \mu' + \frac{\Delta\mu'}{2}\theta' \end{pmatrix} = \begin{pmatrix} \mu - \frac{\Delta\mu}{2}\theta & \frac{p}{q}\frac{\sqrt{1-\theta^2}}{2}\Delta\mu \\ \frac{q}{p}\frac{\sqrt{1-\theta^2}}{2}\Delta\mu & \mu + \frac{\Delta\mu}{2}\theta \end{pmatrix} + \begin{pmatrix} \chi & 0 \\ 0 & \bar{\chi} \end{pmatrix}. \tag{51}$$

A few features are worth mentioning. Firstly, $H'_{12} = H_{12}$ and $H'_{21} = H_{21}$. As a result, $q'/p' = q/p$. In particular, the CP- and T-violating parameter δ, which depends on $|q'/p'| = |q/p|$, is the same in vacuum and in the presence of matter. Therefore, the parameters in vacuum and in matter are related through,

$$\begin{aligned} \mu' &= \mu + \frac{\chi + \bar{\chi}}{2}, \\ \Delta\mu' &= \sqrt{(\Delta\mu)^2 + 2\theta\,\Delta\mu\,\Delta\chi + (\Delta\chi)^2} = \Delta\mu\,\sqrt{1 + 4r\,\theta + 4r^2}, \\ \theta' &= \frac{\Delta\mu\,\theta + \Delta\chi}{\sqrt{(\Delta\mu)^2 + 2\theta\,\Delta\mu\,\Delta\chi + (\Delta\chi)^2}} = \frac{\theta + 2r}{\sqrt{1 + 4r\,\theta + 4r^2}}, \end{aligned} \tag{52}$$

where $\Delta\chi = \bar{\chi} - \chi$, and we have introduced the 'regeneration parameter' $r = \Delta\chi/(2\Delta\mu)$. It will also prove convenient to find

$$\sqrt{1-\theta'^2} = \frac{\Delta\mu\,\sqrt{1-\theta^2}}{\sqrt{(\Delta\mu)^2 + 2\theta\,\Delta\mu\,\Delta\chi + (\Delta\chi)^2}} = \sqrt{\frac{1-\theta^2}{1 + 4r\,\theta + 4r^2}}. \tag{53}$$

Secondly, it is clear from Eq. (51), and also from Eqs. (52) and (53), that the flavor-diagonal matter effects considered here act just like violations of CPT. Thirdly, we expect the matter effects to be much larger than any (necessarily small) CPT-violation that there might be already present in vacuum. Therefore, we may take $\theta = 0$ to get

$$\begin{aligned} \mu' &= \mu + \frac{\chi + \bar{\chi}}{2}, \\ \Delta\mu' &= \sqrt{(\Delta\mu)^2 + (\bar{\chi} - \chi)^2} = \Delta\mu\,\sqrt{1 + 4r^2}, \\ \theta' &= \frac{\bar{\chi} - \chi}{\sqrt{(\Delta\mu)^2 + (\bar{\chi} - \chi)^2}} = \frac{2r}{\sqrt{1 + 4r^2}}, \end{aligned} \tag{54}$$

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[2] The total cross-section for $\overline{K^0}$ interacting with a nucleus is larger than that for K^0 on the same nucleus. For example, $\overline{K^0}p \to \Lambda\pi^+$ takes place but there is no corresponding reaction for K^0.

and $\sqrt{1-\theta'^2} = 1/\sqrt{1+4r^2}$. We stress that Eq. (51) is completely general, as will be the time evolution based on it.

The time evolution in matter is now trivial to find. It is given in Eqs. (33) [or, alternatively, in Eqs. (35)] and (34), with the unprimed quantities substituted by the primed quantities. This solution had been found for the kaon system by Good [14], building on earlier work by Case [15]. Recent re-derivations may be found in references [16] and [17]. In all these articles, the authors write a new evolution equation obtained by combining the diagonalized form of $\boldsymbol{H}$ with the new term $\boldsymbol{H}_{\rm nuc}$ written in the $\{K_L, K_S\}$ basis. Thus, they would seem to be solving a new complicated set of equations: the so-called 'Good equations'. In the method presented here, we have made no reference to 'new' differential equations. We had already solved the most general evolution equation once and for all, Eqs. (35); and we had seen how $\boldsymbol{H}$ could be written in terms of observables, Eq. (28). All we had to do was to refer back to those results.

It should also be pointed out that this matrix formulation is very useful whenever we have non-uniform materials. For example, one might wish to study an experiment in which a kaon beam traverses vacuum, matter, and then vacuum again before it decays. Or a beam that traverses copper, carbon and then tungsten. In the matrix formulation, all we have to do is multiply three evolution matrices

$$\exp\left[-i\mathcal{H}t_1\right]\exp\left[-i\mathcal{H}(t_2-t_1)\right]\exp\left[-i\mathcal{H}(t_3-t_2)\right], \tag{55}$$

each given by Eq. (33).

IX. ON THE (MATHEMATICAL) RELATION WITH NEUTRINO OSCILLATIONS

A. Boosted frames

As we have mentioned before, the evolution equation (3) in which our study is based has been written in the rest frame of the $P^0 - \overline{P^0}$ system. We denote this explicitly by

$$i\frac{d}{dt_{\rm rest}}|\psi(t_{\rm rest})\rangle = \boldsymbol{H}|\psi(t_{rest})\rangle. \tag{56}$$

The advantage of doing this is that in the rest frame the energy is given simply by $E = m$. As a result, the time parameter which appears in the solutions presented in Eqs. (35) or (B7), through the time dependent functions $g_\pm(t)$ defined in Eq. (34), is really $t_{\rm rest}$.

Now imagine that we wished to have Eq. (56) given in a boosted frame (named the lab frame from now on). In that case we would start by noticing that both the energy and the time are altered in the boosted frame. They become

$$\begin{aligned} E_{\rm lab} &= m\,\gamma, \\ t_{\rm lab} &= \gamma\, t_{\rm rest} = \frac{E_{\rm lab}}{m}\, t_{\rm rest}. \end{aligned} \tag{57}$$

Ignoring the matrix structure for the time being, Eq. (56) would change schematically into

$$i\frac{d}{dt_{\rm lab}}|\psi(t_{\rm lab})\rangle = m\,\gamma\,|\psi(t_{\rm lab})\rangle = E_{\rm lab}|\psi(t_{\rm lab})\rangle, \tag{58}$$

as it had to. Now, if the boost is much larger that the mass, $p \gg m$, we may use

$$E_{\rm lab} = \sqrt{p^2+m^2} \sim p + \frac{m^2}{2p} + \cdots \sim p + \frac{m^2}{2E} + \cdots. \tag{59}$$

However, we do not need to do this. We have already found the solution to Eq. (56) in the rest frame. In order to change it into the lab frame all we have to do is to substitute $t_{\rm rest}$ in the time evolution functions of Eq. (34) by $t_{\rm rest} = t_{\rm lab}/\gamma$. We notice that $1/\gamma = m/E$. Therefore, when written in terms of $t_{\rm lab}$ the time evolution functions of Eq. (34) become

$$g_\pm(t_{\rm rest}) = e^{-im^2 t_{\rm lab}/E}\, e^{-\Gamma\, m\, t_{\rm lab}/(2E)} \begin{cases} \cos\left(\frac{m}{E}\frac{\Delta\mu}{2}\, t_{\rm lab}\right) \\ -i\sin\left(\frac{m}{E}\frac{\Delta\mu}{2}\, t_{\rm lab}\right) \end{cases} \tag{60}$$

And, using $\Delta m = m_a - m_b$ and $m = (m_a + m_b)/2$, we realize that the argument of the trigonometric functions is given by

$$\frac{m}{E}\frac{\Delta\mu}{2}\, t_{\rm lab} = \frac{m_a^2 - m_b^2}{4E}\, t_{\rm lab} - \frac{i}{4}\frac{\Delta\Gamma\, m}{E}\, t_{\rm lab}. \tag{61}$$

B. A neutrino-like oscillation

For the comparison with neutrinos, it is most convenient to use the parametrization of the CP-violating quantities discussed in the first subsection of appendix B. To obtain relations that mimic those in the neutrino system, we compute the probability that P^0 becomes $\overline{P^0}$ using Eqs. (B7), (60), (61), setting $\mathrm{Im}\,\phi_R = \mathrm{Im}\,\theta_R = 0$, and letting $\Gamma = \Delta\Gamma \sim 0$ (another way of thinking about this limit is to suppose that we are performing an experiment in a time scale much smaller than the mesons' decay time). We find

$$\left|\langle\overline{P^0}|P^0(t_{\rm lab})\rangle\right|^2 = \sin^2\theta_R\,\sin^2\left(\frac{m_a^2-m_b^2}{4E}\,t_{\rm lab}\right). \tag{62}$$

If, instead, the experiment is performed in matter, we obtain

$$\begin{aligned}\left|\langle\overline{P^0}|P^0(t_{\rm lab})\rangle\right|^2 &= \left|\sin^2\theta_R'\right|\,\sin^2\left(\frac{m_a^2-m_b^2}{4E}\,t_{\rm lab}\right),\\ &= \frac{\sin^2\theta_R}{|1-4r\,\cos\theta_R+4r^2|}\,\sin^2\left(\frac{m_a^2-m_b^2}{4E}\,t_{\rm lab}\right)\\ &= \frac{\sin^2\theta_R}{\left|(\cos\theta_R-2r)^2+\sin^2\theta_R\right|}\,\sin^2\left(\frac{m_a^2-m_b^2}{4E}\,t_{\rm lab}\right),\end{aligned} \tag{63}$$

where we have used Eq. (B8) in getting to the second line, and

$$r = \frac{\Delta\chi}{2\Delta\lambda} = \frac{\Delta\chi}{2\Delta m} = \frac{\Delta\chi/\gamma}{2\Delta m/\gamma} = \frac{E}{m_a^2-m_b^2}\frac{\Delta\chi}{\gamma}. \tag{64}$$

Eq. (63) exhibits a resonance structure because the time independent coefficient reaches its maximum if $2r = \cos\theta_R$. For the final step in the connection to neutrinos, we look at this case further by assuming that the imaginary part of r ($\Delta\chi$), which is proportional to $\sigma_{\rm tot}$ in appendix A, is negligible. Then $\Delta\chi$ is real and we may parametrize

$$V \equiv \frac{\mathrm{Re}\Delta\chi}{\gamma}. \tag{65}$$

As a result,

$$r = \frac{E\,V}{m_a^2-m_b^2} \tag{66}$$

is real and the resonance condition, which becomes

$$\frac{2E\,V}{m_a^2-m_b^2} = \cos\theta_R, \tag{67}$$

can be satisfied for θ_R real. Eq. (63) and the resonance condition in Eq. (67) are in exactly the same form as the usual discussions of neutrino oscillations in matter [18].

Although there is this mathematical connection between neutrino oscillations and $P^0-\overline{P^0}$ oscillations, the situations are physically very different. Indeed, it is important to notice that there are no CPT relations between the two neutrino species involved in neutrino oscillation, and the vacuum mixing angle θ_R in Eq. (62) may be small. Eq. (63) shows that, even if θ_R is small, the effective mixing angle in matter will be large when one hits the resonance condition in Eq. (67). In contrast, as we show in appendix B, in the $P^0-\overline{P^0}$ system, the deviation of $\cos\theta_R$ from zero measures violations of CPT. Assuming CPT conservation in the $P^0-\overline{P^0}$ system, $\sin\theta_R = 1$ and the vacuum transition in Eq. (62) already reaches unity (at select times)[3]. Said otherwise, the small mixing angle mixing discussed in neutrino oscillations in vacuum, is (in the connection presented here) the mathematical analogue of large violations of CPT in the $P^0-\overline{P^0}$ system.

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[3]Recall that we have assumed Γ=0 and $\mathrm{Im}\,\phi_R = 0$ (T conservation). When $\Gamma \neq 0$, the right hand side of Eq. (62) appears multiplied by $\exp(-\Gamma t_{\rm rest})$.

X. CONCLUSIONS

We have shown that the presence of T violation in the neutral meson systems implies that the corresponding effective Hamiltonian $\boldsymbol{H}$ does not commute with itself. Therefore, $\boldsymbol{H}$ cannot be diagonalized by an unitary transformation and we must introduce the reciprocal basis. This basis must be used in order to obtain the correct form for some physical observables, such as the parameter λ in the decays $B_d \to J/\psi K_S$. But, working with the reciprocal basis is a blessing rather than a nuisance. We show that using the reciprocal basis has the following advantages:

- the relation between the effective Hamiltonian matrix when written in the mass and flavor basis is simply obtained and easily inverted, thus providing a parametrization of $\boldsymbol{H}$ in terms of measurable quantities;
- one obtains a one line derivation of the evolution of the states;
- propagation in matter is reduced to the case of propagation in vacuum, with the vacuum and matter parameters related in a trivial fashion, without any recourse to the Good equations;
- the propagation in non-uniform media is reduced to a multiplication of evolution matrices.

It is true that some of these results can be obtained without using the reciprocal basis. But, as we have tried to illustrate in the article, this concept is not only needed but, when used, greatly simplifies the various derivations. In addition, we can use this formalism to highlight the similarity between the matter effects in the $P^0 - \overline{P^0}$ systems and the matter effects in neutrino oscillations.

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APPENDIX A: MATTER EFFECTS IN THE REST AND LABORATORY FRAMES

In this appendix we show how $\boldsymbol{H}_{\rm nuc}$ is related to physical cross-sections and what is the form of this relation in the rest and laboratory frames. This is relevant for Eq. (49) and for those wishing to expand on the analogy with the neutrino oscillations discussed at the end of section IX. We follow here the notation of references [22] and [17].

Let us consider the evolution of a coherent wave-packet ϕ with wavenumber k in the laboratory frame,

$$\frac{d}{dz}\phi = ik\,\phi. \tag{A1}$$

In the presence of a block of material at rest in the laboratory frame, the wavenumber suffers a shift given approximately by

$$k' - k \approx \frac{2\pi N}{k} f(0) = N\left(\frac{2\pi}{k}{\rm Re} f(0) + i\frac{\sigma_{\rm tot}}{2}\right), \tag{A2}$$

where N is the density of scattering centers in the medium and $f(0)$ is the elastic forward scattering amplitude. On the last equality, we have used the fact that the imaginary part of $f(0)$ is related to the total cross section $\sigma_{\rm tot}$ by the optical theorem,

$${\rm Im} f(0) = \frac{k}{4\pi}\sigma_{\rm tot}. \tag{A3}$$

We also recall that

$$|f(\theta)|^2 = \frac{1}{2\pi}\frac{d\sigma}{d\cos\theta}. \tag{A4}$$

In this equation (and only here), θ refers to the scattering angle in the laboratory frame.

We conclude that the presence of matter changes the evolution in vacuum by an amount

$$i\frac{d}{dt_{\rm lab}}\phi = -v(k'-k)\phi = -\frac{2\pi N}{k/v}f(0)\,\phi, \tag{A5}$$

where v is the beam velocity in the lab frame and $z = v\,t_{\rm lab}$. To change into the rest frame of the beam we notice that $t_{\rm lab} = \gamma\,t_{\rm rest}$ and $k = m\gamma v$, where $\gamma = 1/\sqrt{k^2+m^2}$, leading to

$$i\frac{d}{dt_{\rm rest}}\phi = -\frac{2\pi N}{m}f(0)\,\phi. \tag{A6}$$

When studying the $P^0 - \overline{P^0}$ systems, we denote by f ($\bar{f}$) the elastic forward scattering amplitude of P^0 ($\overline{P^0}$). Therefore, the new contribution in the $P^0 - \overline{P^0}$ rest frame is given by

$$i\frac{d}{dt_{\rm rest}}|\psi(t_{\rm rest})\rangle = \begin{pmatrix} \chi & 0 \\ 0 & \bar{\chi} \end{pmatrix} |\psi(t_{\rm rest})\rangle, \tag{A7}$$

where [17]

$$\chi = -\frac{2\pi N}{m}f \quad \text{and} \quad \bar{\chi} = -\frac{2\pi N}{m}\bar{f}, \tag{A8}$$

leading to Eq. (49).

APPENDIX B: OTHER PARAMETRIZATIONS FOR T AND CPT VIOLATION

The way we parametrize T and CPT violation in the mixing of neutral mesons is different from the parametrizations used by some other authors. For ease of reference, we collect here formulae summarizing the relationships among different parametrizations.

1. The parameters ϕ_R and θ_R

Some authors (for instance [19]) introduce two complex angles θ_R and ϕ_R by writing

$$\begin{aligned} p_a &= N_a \cos\frac{\theta_R}{2}, \qquad & q_a &= N_a e^{i\phi_R}\sin\frac{\theta_R}{2}, \\ p_b &= N_b \sin\frac{\theta_R}{2}, \qquad & q_b &= N_b e^{i\phi_R}\cos\frac{\theta_R}{2}. \end{aligned} \tag{B1}$$

Then,

$$\begin{aligned} \frac{q}{p} &= e^{i\phi_R}, \\ \delta &= \tanh\left(\mathrm{Im}\,\phi_R\right), \\ \theta &= -\cos\theta_R, \end{aligned} \tag{B2}$$

and $\sqrt{1-\theta^2} = \sin\theta_R$. CPT is violated if and only if $\cos\theta_R \neq 0$. T is violated if and only if $\mathrm{Im}\,\phi_R \neq 0$. Some authors use a particular phase convention and claim that $\mathrm{Re}\,\phi_R \neq 0$ also corresponds to T violation. Clearly this statement is false since the phase of q/p has no physical meaning; we know that there is one and only one T- and CP-violating quantity in $\boldsymbol{H}$.

With this notation, Eqs. (19), (20), (28), (33), and (35) become

$$\boldsymbol{X} = \begin{pmatrix} \cos\frac{\theta_R}{2} & -\sin\frac{\theta_R}{2} \\ e^{i\phi_R}\sin\frac{\theta_R}{2} & e^{i\phi_R}\cos\frac{\theta_R}{2} \end{pmatrix} \begin{pmatrix} N_a & 0 \\ 0 & -N_b \end{pmatrix}, \tag{B3}$$

$$\boldsymbol{X}^{-1} = \begin{pmatrix} N_a^{-1} & 0 \\ 0 & -N_b^{-1} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta_R}{2} & e^{-i\phi_R}\sin\frac{\theta_R}{2} \\ -\sin\frac{\theta_R}{2} & e^{-i\phi_R}\cos\frac{\theta_R}{2} \end{pmatrix}, \tag{B4}$$

$$\boldsymbol{H} = \begin{pmatrix} \mu + \cos\theta_R\,\frac{\Delta\mu}{2} & e^{-i\phi_R}\sin\theta_R\,\frac{\Delta\mu}{2} \\ e^{i\phi_R}\sin\theta_R\,\frac{\Delta\mu}{2} & \mu - \cos\theta_R\,\frac{\Delta\mu}{2} \end{pmatrix}, \tag{B5}$$

$$\exp\left(-i\mathcal{H}t\right) = \left(\,|P^0\rangle,\ |\overline{P^0}\rangle\,\right) \begin{pmatrix} g_+(t) + \cos\theta_R\, g_-(t) & e^{-i\phi_R}\sin\theta_R\, g_-(t) \\ e^{i\phi_R}\sin\theta_R\, g_-(t) & g_+(t) - \cos\theta_R\, g_-(t) \end{pmatrix} \begin{pmatrix} \langle P^0| \\ \langle\overline{P^0}| \end{pmatrix}, \tag{B6}$$

and

$$\begin{aligned} |P^0(t)\rangle &= \left[g_+(t) + \cos\theta_R\, g_-(t)\right]|P^0\rangle + e^{i\phi_R}\sin\theta_R\, g_-(t)\,|\overline{P^0}\rangle, \\ |\overline{P^0}(t)\rangle &= e^{-i\phi_R}\sin\theta_R\, g_-(t)|P^0\rangle + \left[g_+(t) - \cos\theta_R\, g_-(t)\right]|\overline{P^0}\rangle, \end{aligned} \tag{B7}$$

respectively.

Finally, the relation between the matter and vacuum parameters described in Eqs. (52) and (53) become

$$\begin{aligned} \mu' &= \mu + \frac{\chi + \bar{\chi}}{2}, \\ \Delta\mu' &= \sqrt{(\Delta\mu)^2 - 2\cos\theta_R\,\Delta\mu\,\Delta\chi + (\Delta\chi)^2} = \Delta\mu\,\sqrt{1 - 4r\,\cos\theta_R + 4r^2}, \\ \cos\theta_R' &= \frac{\Delta\mu\,\cos\theta_R - \Delta\chi}{\sqrt{(\Delta\mu)^2 - 2\cos\theta_R\,\Delta\mu\,\Delta\chi + (\Delta\chi)^2}} = \frac{\cos\theta_R - 2r}{\sqrt{1 - 4r\,\cos\theta_R + 4r^2}}, \\ \sin\theta_R' &= \frac{\Delta\mu\,\sin\theta_R}{\sqrt{(\Delta\mu)^2 - 2\cos\theta_R\,\Delta\mu\,\Delta\chi + (\Delta\chi)^2}} = \frac{\sin\theta_R}{\sqrt{1 - 4r\,\cos\theta_R + 4r^2}}, \end{aligned} \tag{B8}$$

and $\phi_R' = \phi_R$.

2. The parameters ϵ_S and δ_S

Other authors (for instance [20,21]) use two complex parameters, ϵ_S and δ_S, and write

$$\begin{aligned} \frac{q_a}{p_a} &= \frac{1 - \epsilon_S + \delta_S}{1 + \epsilon_S - \delta_S}, \\ \frac{q_b}{p_b} &= \frac{1 - \epsilon_S - \delta_S}{1 + \epsilon_S + \delta_S}. \end{aligned} \tag{B9}$$

Obviously then,

$$\begin{aligned} \frac{q}{p} &= \sqrt{\frac{\left(1 - \epsilon_S\right)^2 - \delta_S^2}{\left(1 + \epsilon_S\right)^2 - \delta_S^2}}, \\ \delta &= \frac{8\mathrm{Re}\left[\epsilon_S^*\left(1 + \epsilon_S^2 - \delta_S^2\right)\right]}{\left(\left|\left(1 + \epsilon_S\right)^2 - \delta_S^2\right| + \left|\left(1 - \epsilon_S\right)^2 - \delta_S^2\right|\right)^2}, \\ \theta &= \frac{2\delta_S}{1 + \delta_S^2 - \epsilon_S^2}. \end{aligned} \tag{B10}$$

CPT invariance corresponds to $\delta_S = 0$. T invariance corresponds to $\mathrm{Re}\left[\epsilon_S^*\left(1 + \epsilon_S^2 - \delta_S^2\right)\right] = 0$. The authors who use this parametrization, however, always do so in conjunction with the assumption that δ_S and ϵ_S are small. Then,

$$\begin{aligned} \delta &\approx 2\,\mathrm{Re}\,\epsilon_S\,, \\ \theta &\approx 2\delta_S\,. \end{aligned} \tag{B11}$$

Moreover, $\sqrt{1-\theta^2} \approx 1 - 2\delta_S^2$.

It should be kept in mind that the R-parametrization is exact and general, while the S-parametrization is interesting only when using a phase convention $\mathcal{CP}|P^0\rangle = \pm|\overline{P^0}\rangle$, which implies that CP conservation corresponds to vanishing δ_S and ϵ_S.

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