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A MATRIX FORMALISM FOR LANDAU DAMPING*

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Abstract

Existing methods of analysing the effect of bunch to bunch tune shifts on coupled bunch instabilities are applicable to beams with a single unstable mode, or a few non-interacting unstable modes. We present a more general approach that involves computing the eigenvalues of a reduced state matrix. The method is applied to the analysis of PEP-II longitudinal coupled bunch modes, a large number of which are unstable in the absence of feedback.

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ABSTRACT

Existing methods of analysing the effect of bunch to bunch tune shifts on coupled bunch instabilities are applicable to beams with a single unstable mode, or a few non-interacting unstable modes. We present a more general approach that involves computing the eigenvalues of a reduced state matrix. The method is applied to the analysis of PEP-II longitudinal coupled bunch modes, a large number of which are unstable in the absence of feedback.

1 Introduction

The operating current in high current storage rings and particle accelerators is often limited by coupled bunch instabilities, which arise out of the electromagnetic interaction between stored bunches and their surroundings. Coupled bunch instabilities can be cured by introducing a tune spread between the bunches (Landau damping), so that they cannot organize a growing coherent oscillation. A longitudinal tune spread may be introduced by means of a subharmonic cavity, or by adding an off-harmonic term to the klystron drive signal. A radio frequency quadrupole may be used to generate transverse tune spreads.

Landau damping of coupled bunch modes due to bunch to bunch tune spreads has already been analysed by deriving dispersion relations for the coherent tune shift [1)]. Unfortunately, the dispersion relations are not easily soluble if there are unstable modes that couple to each other through the tune spread. Together with decreasing revolution frequencies in new high energy accelerators, the introduction of damped rf cavities with broad HOM resonances necessitates analysis of Landau damping in the presence of a spectrum of unstable coupled bunch modes. The most direct way of doing this for a system of N bunches is to solve for the eigenvalues of the N x N state matrix A. In the case of rings with large N, this becomes computationally infeasible. If we assume slow tune variation around the ring, we can make the eigenvalue problem more manageable by creating an equivalent M x M matrix that models the dynamics of the most unstable modes of A.

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2 Equivalent State Matrix

In general we can write the equations of motion of a linear system as:

$$\dot{X} = AX; \qquad X(t) = X_o\, e^{\lambda t}, \tag{1}$$

where X_o is any eigenvector of A, and λ is the corresponding eigenvalue. Consider N identical evenly spaced rigidly oscillating bunches with oscillation coordinates x_k. For mathematical convenience, we shall consider the x_ks to be complex, so that the state matrix A has size N and the eigenvectors are merely the N Fourier vectors:

$$v_l = [1 \;\; e^{jl\theta} \;\; e^{2jl\theta} \;\; ... \;\; e^{(N-1)jl\theta}]^T/\sqrt{N}; \qquad \theta = 2\pi/N; \qquad l = 1, 2, ...N-1 \tag{2}$$

The corresponding eigenvalues λ_l are given by: $\lambda_l = (-\lambda_r + j\omega_z) + \alpha_l$, if we assume that the coherent eigenvalue shifts α_l ($j\Omega_l$, usually) and the radiation damping rate λ_r are small compared to the longitudinal or transverse oscillation frequency ω_z. We can calculate α_l for each mode l by scaling the effective impedance at the corresponding revolution harmonic [2].

From here on we shall drop the common additive term $(-\lambda_r + j\omega_z)$ from the eigenvalues λ_l, so that we are left with only the part that contains the coherent tune shift of mode l. This merely shifts the eigenvalue spectrum of A, without changing the eigenvectors. We now have:

$$A = \sum_{l=0}^{N-1} \alpha_l\, v_l\, v_l^H, \tag{3}$$

where v_l^H denotes the complex conjugate of v_l^T. If we now add a small tune shift δ_k ($\delta_k \ll \omega_z$) to the tune of each bunch k, we get the following modified matrix:

$$A = \mathrm{diag}(j\delta_0 \;\; j\delta_1 \;\; ... \;\; j\delta_{N-1}) + \sum_{l=0}^{N-1} \alpha_l\, v_l\, v_l^H, \tag{4}$$

The eigenvalues of this A matrix reveal the damping effect of a tune spread on the unstable coupled bunch modes. Unfortunately, if N is very large ($N = 1746$ at PEP-II), the eigenvalue problem becomes computationally difficult, or even infeasible. The next two subsections describe the construction of an equivalent A-matrix of reduced size, whose eigenvalues approximate the most unstable eigenvalues of A.

2.1 Single Unstable Mode

The physics behind the approximation is illustrated by the simple case of a beam with only one unstable coupled bunch mode v_o. Equation 4 reduces to:

$$A = \mathrm{diag}(j\delta_0 \;\; j\delta_1 \;\; ... \;\; j\delta_{N-1}) + \alpha_0\, v_0\, v_0^H$$

Summing the rows of the eigenvalue equation, we get:

$$1 = \alpha_0 \, \langle \frac{1}{\lambda - j\delta_k} \rangle_k, \tag{5}$$

where $\langle u_k \rangle_k$ denotes the mean of u over all k. The common approach at this stage is to make the approximation that the δ_ks are closer to their neighbours than they are to λ, in which case we can replace the discrete averaging in the above equation by an average over a fictitious continuous distribution [3] $\rho(\delta)$:

$$1 \approx \alpha_0 \int_{\delta_{min}}^{\delta_{max}} \frac{\rho(\delta)}{\lambda - j\delta} \; d\delta \tag{6}$$

Physically, this approximation is equivalent to the statement that neighbouring tunes are blurred together by the speed of evolution of the unstable mode. The matrix reduction method inverts this approximation by going from N discrete tunes to M ($M < N$). We could, for example, choose M to be N/2 by averaging pairs of adjacent tunes. We would then have the following average over N/2 fictitious tunes δ^1_m:

$$1 \approx \alpha_0 \, \langle \frac{1}{\lambda - j\delta^1_m} \rangle_m \tag{7}$$

The physical interpretation of this approximation is the same as before, with the criterion that the δ^1_ms are closer to their neighbours than they are to λ. We now have a smaller matrix A^1 of size N/2 x N/2 whose largest eigenvalue is about the same as that of A. We can progressively reduce the size of A as long as the closeness criterion holds, until the eigenvalues become easy to compute.

2.2 Multiple Unstable Modes

In the case where the beam impedance hits more than one coupled bunch mode, we need to transform the state vector X and the state matrix A to the Fourier basis:

$$Y = V^H X; \quad B = V^H A V; \quad BY = \lambda Y, \tag{8}$$

where the columns of V are the normalized Fourier eigenvectors of a beam with no tune spread. With a little manipulation, we can arrive at the following dispersion relation from Equations 4 and 8:

$$Y = CY; \qquad C_{m,n} = \alpha_n \, \langle \frac{e^{j(n-m)2\pi k/N}}{\lambda - j\delta_k} \rangle_k \tag{9}$$

This dispersion relation is hard to solve in its present form. If we assume that δ is a smooth function of k, and therefore so is $\lambda - j\delta$, then the terms far from the main diagonal drop out of C. If there are only a few unstable modes excited by narrow impedance resonances, their eigenvalues can be calculated independently as in the previous section, provided that the mode numbers are not too close, and the unstable unperturbed (no tune spread) eigenvalues are far from degeneracy.

Unfortunately, in the case of rings with low revolution frequency and/or damped rf cavities such as PEP-II, the unstable modes are clustered together, and their interaction through the tune spread must be considered. Let mode p be the most unstable unperturbed eigenmode. Consider the set of unperturbed modes from $(p-q)$ to $(p+r-1)$, where q and $(r-1)$ are larger than the number of non-negligible diagonals above the main diagonal in C. If the modes outside this set are stable or have eigenvalues far from α_p, they do not couple to mode p. We could truncate C so that only the portion that couples modes within the set to each other remains. We could now make use of the smoothness of δ_k to downsample it by a factor $N/(q+r) = N/M$, making sure that the closeness criterion is still satisfied. The obvious next step is to transfer back to the regular basis to get the following equivalent state matrix:

$$A^1 = \mathrm{diag}(j\delta_0^1 \;\, j\delta_1^1 \;\, ... \;\, j\delta_{M-1}^1) + \sum_{m=0}^{M-1} \alpha_m^1 \, v_m \, v_m^H, \tag{10}$$

where $\{\delta_m^1\}$ is the downsampled version of $\{\delta_k\}$ and $\alpha_m^1 = \alpha_{p-q+m}$. The matrix A^1 models the truncated C-matrix. It is most accurate close to row p, if $q \approx r$, while it introduces an artificial “wrap around” coupling between modes at either end of the truncated C-matrix due to the downsampling of δ_k.

We now have a reduced matrix whose eigenvalues approximate those of a Landau damped beam in the general case, if bunch tune variation is smooth.

3 Application to PEP-II

In this section we apply the equivalent matrix method to the study of longitudinal tune spreads in the PEP-II rings. The impedance of the two strongest HOMs in the damped PEP-II rf cavities produces a broad spectrum of unstable longitudinal coupled bunch modes, which are expected to stabilise with feedback. Here we examine the effect of bunch tune spreads as the only longitudinal damping mechanism in the HER and LER. Of course, the effect of bunch by bunch feedback can be added on as an increase in radiation damping. The rings have a harmonic number of 3492, with every other bucket filled. We will assume that all 1746 buckets are equally filled.

The maximum current in the HER is 1A. Based on the measured cavity HOMs, we have a band of roughly 60 unstable modes about mode 770, and another band of roughly 40 unstable modes around mode 682 in the absence of tune spreads. The most unstable mode is at $p = 770$, with $Re(\alpha_p) = 115/s$. With a uniform tune distribution between $\delta = -300$rad/s and $\delta = 300$rad/s, we cannot compute the eigenvalues of the 1746 x 1746 A-matrix directly, so we reduce it by a factor of 6 ($M = 291$). We can choose $q = 145$, $r = 146$. Figure 1(a) shows the eigenvalues (*including radiation damping*) of the unperturbed HER beam and the approximate

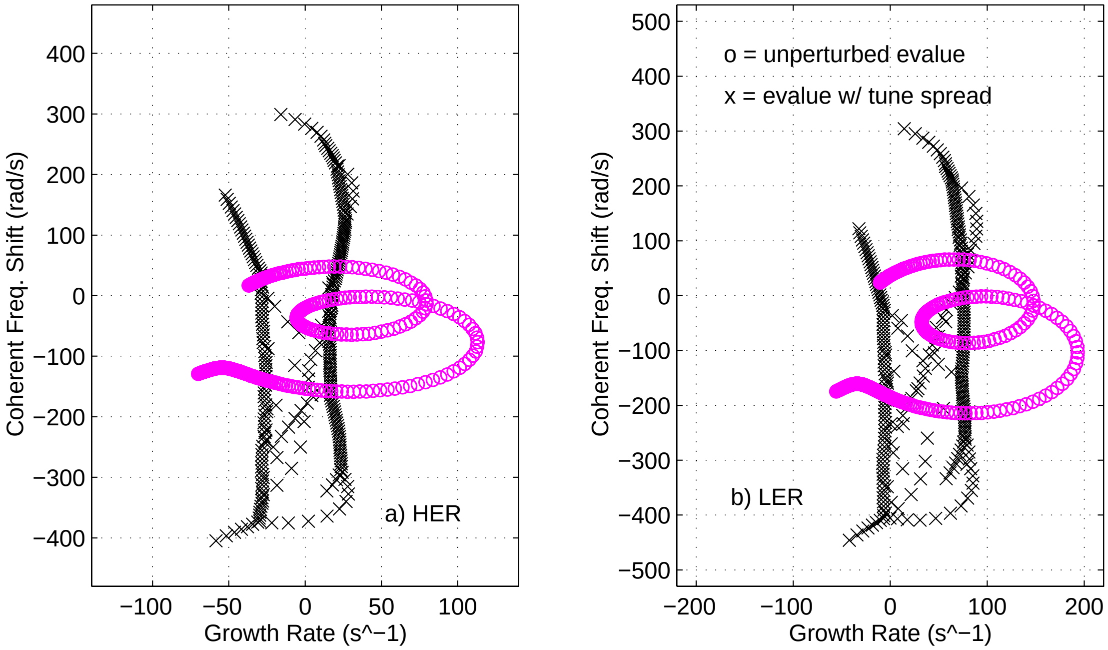

Figure 1: *Eigenvalues of longitudinal coupled bunch modes in PEP-II with and without bunch to bunch tune spread, 1746 bunches: (a) HER, 1A, 600rad/s tune spread (b) LER, 2.25A, 600rad/s tune spread.*

eigenvalues of the beam with a tune spread of 600rad/s across the bunches. The perturbed eigenvalue spectrum, shown with 'x's, is most accurate at its center, since $q \approx r$. We can see from the figure that the most unstable modes are Landau damped down to a growth rate of roughly 20/s.

The LER has a maximum current of 2.25A. Since the cavities in the two rings are identical, the LER is most unstable at the same value of p, with $Re(\alpha_p) = 200/s$. If we assume the same tune distribution as in the case of the HER, we could use the same values of q and r. Figure 1(b) shows the perturbed and unperturbed eigenvalues of the LER longitudinal coupled bunch modes. In this case, the most unstable mode is damped down to a growth rate of 75/s.

4 Summary

Existing methods of analysing the effect of bunch to bunch tune shifts on coupled bunch instabilities are applicable to beams with a single unstable mode, or a few non-interacting unstable modes. Unfortunately, in the case of rings with low revolution frequency and/or damped rf cavities such as PEP-II, we are faced with multiple unstable modes.

We have presented a more general approach to the Landau damping problem that involves computing the eigenvalues of a reduced state matrix. The application of the method to the case of longitudinal coupled bunch modes in PEP-II has shown that a tune spread of 95Hz across the bunches damps the most unstable HER mode from a growth rate of 115/s to a growth rate of 20/s. The corresponding numbers for the LER at full current are 200/s and 75/s respectively, given the same tune spread.

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References

1. Y.H. Chin *et al*, DESY 86-097, (1986).
2. J. Laclare, Proc. 11th International Conference on High Energy Accelerators, Geneva, (Birkhauser, 1980).
3. H.G. Hereward, CERN/PS 65-20, (1965).