# Testing QCD with Hypothetical Tau Leptons [*]

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## Abstract

We construct new tests of perturbative QCD by considering a hypothetical $\tau$ lepton of arbitrary mass, which decays hadronically through the electromagnetic current. We can explicitly compute its hadronic width ratio directly as an integral over the $e^+e^-$ annihilation cross section ratio, $R_{e^+e^-}$. Furthermore, we can design a set of commensurate scale relations and perturbative QCD tests by varying the weight function away from the form associated with the $V - A$ decay of the physical $\tau$. This method allows the wide range of the $R_{e^+e^-}$ data to be used as a probe of perturbative QCD.

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The hadronic width of the $\tau$ lepton is potentially one of the most important sources for the high precision determination of the coupling $\alpha_{\overline{MS}}$ of QCD [1, 2]. The perturbative QCD (PQCD) analysis of the $\tau$ width has been refined by constructing moments of hadronic decay distributions which minimize sensitivity to the low energy part of the hadronic spectrum [3]. However, it is still uncertain whether the $\tau$ mass is sufficiently high to trust PQCD, particularly due to the strong distortion of hadronic final state interactions [4].

In this paper we construct new renormalization scheme-independent tests of PQCD which we can apply, not only to the physical $\tau$ lepton, but also to a hypothetical $\tau$ lepton of arbitrary mass which decays hadronically through the vector current. Such hypothetical $\tau$ leptons, with masses $M < M_\tau$, have already been considered in Ref. [3]. We can obtain empirical values for the hypothetical lepton's hadronic width and moments directly as integrals over the measured $R_{e^+e^-} = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$. As we shall show, these tests are fundamental properties of QCD which can serve as necessary conditions for the applicability of perturbation theory.

Quantum field theoretic predictions which relate physical observables cannot depend on theoretical conventions such as the choice of renormalization scheme or scale. The most well-known example is the "generalized Crewther relation" [5] in which the leading twist PQCD corrections to the Bjorken sum rule at a given lepton momentum transfer $Q^2$ are inverse to the QCD corrections to $R_{e^+e^-}$ at a corresponding CM energy squared, $s^* = s^*(Q^2)$, independent of renormalization scheme. The ratio of the scales $s^*/Q^2$ has been computed to NLO in PQCD. Such leading-twist predictions between observables are called "commensurate scale relations" and are identical for conformal and nonconformal theories [6].

Another important example is the commensurate scale relation between the PQCD correction to the $\tau$ lepton's width ratio, $R_\tau = \Gamma(\tau^- \to \nu_\tau + \text{hadrons})/\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e)$, and those to $R_{e^+e^-}$. Assuming for now $f$ massless flavors, PQCD yields

$$R_{e^+e^-}(\sqrt{s}) = (3 \sum_f q_f^2) \left[ 1 + \frac{\alpha_R(\sqrt{s})}{\pi} \right], \tag{1}$$

where $\alpha_R$ can be written as a series in $\alpha_s/\pi$ in some renormalization scheme. Note that $\alpha_R$ is an effective charge [7] because it satisfies the Gell-Mann-Low renormalization group equation with the same coefficients $\beta_0$ and $\beta_1$ as the usual coupling $\alpha_s$ (differing only through the third and higher coefficients of the $\beta$-function). Similarly we can

define an effective charge $\alpha_\tau$ as follows

$$R_\tau(M_\tau) = R_\tau^0(M_\tau)\left[1 + \frac{\alpha_\tau(M_\tau)}{\pi}\right]. \tag{2}$$

Leading-twist QCD predicts

$$\alpha_\tau(M_\tau) = \alpha_R(\sqrt{s^*}) \tag{3}$$

to all orders in perturbation theory. The ratio of the commensurate scales is known in NLO PQCD:

$$\frac{\sqrt{s^*}}{M_\tau} = \exp\left[-\frac{19}{24} - \frac{169}{128}\frac{\alpha_R(M_\tau)}{\pi} + \cdots\right]. \tag{4}$$

This result was originally obtained in [6] by using NNLO predictions for $\alpha_R$ and $\alpha_\tau$ obtained in the $\overline{MS}$ scheme and eliminating $\alpha_{\overline{MS}}$. However, as we shall show here, the QCD prediction for $\sqrt{s^*}/M_\tau$ also follows from the fact that both effective charges evolve with universal $\beta_0$ and $\beta_1$ coefficients. The fact that $R_\tau$ can be expressed as [‡]

$$\begin{aligned} R_\tau(M_\tau) &= \frac{2}{(\sum_f q_f^2)} \\ &\times \int_0^{M_\tau^2}\frac{d\,s}{M_\tau^2}\left(1 - \frac{s}{M_\tau^2}\right)^2\left(1 + \frac{2s}{M_\tau^2}\right)R_{e^+e^-}(\sqrt{s}) \end{aligned} \tag{5}$$

implies, by the mean value theorem, that $\alpha_R$ and $\alpha_\tau$ are related by a scale shift. However, the prediction for the ratio $\sqrt{s^*}/M_\tau$ in Eq. (4) is a specific property of PQCD.

A definitive empirical test of the commensurate relation, Eq. (4), is problematic since there is only one $\tau$ lepton in nature, and its mass seems uncomfortably low for tests of leading-twist QCD. However, we can construct new tests of PQCD by considering a hypothetical $\tau$ lepton of arbitrary mass $M$ which decays hadronically through the vector current. Then we can explicitly compute its hadronic width ratio as an integral over the measured $R_{e^+e^-}$. Furthermore, we can design a set of commensurate scale relations and PQCD tests by varying the weight function away from the form associated with the $V - A$ decays of the physical $\tau$. Thus we can use the full range of the $R_{e^+e^-}$ as a novel test of PQCD. As we shall show, such a test must also take into account specific effects attributable to the $s\bar{s}, c\bar{c}, b\bar{b}$ quark thresholds. Also, following

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[‡]We have used $|V_{ud}|^2 + |V_{us}|^2 = 1$, as in [8]. Note that in order to include NNLO corrections in $\alpha_\tau$, we must modify the $O(\alpha_s^3)$ coefficient of $\alpha_R$ by setting $(\sum_f q_f)^2 = 0$.

[9] we shall smear the annihilation data in energy in order to eliminate resonances and other distortions of final state interactions. By smearing $R_{e^+e^-}$ over a range of energy, $\Delta E$, we focus the physics to the time $\Delta t = 1/\Delta E$ where an analysis in terms of PQCD quark and gluon subprocesses is appropriate. Therefore, this method can also be interpreted as an additional test of duality. Scheme-independent relations between $R_{e^+e^-}$ and $\tau$ decay have also been recently discussed in [2].

Given $\alpha_R$, we can construct new effective charges as follows:

$$\alpha_f(M) \equiv \frac{\int_0^{M^2} \frac{ds}{M^2} f\left(\frac{s}{M^2}\right) \alpha_R(\sqrt{s})}{\int_0^{M^2} \frac{ds}{M^2} f\left(\frac{s}{M^2}\right)}. \tag{6}$$

We can choose $f(x)$ to be any smooth, integrable function of $x = s/M^2$. ( For the particular choice, $f_\tau(x) = (1-x)^2(1+2x)$, $\alpha_R^f$ is simply $\alpha_\tau$.) The mean value theorem then implies

$$\alpha_f(M) = \alpha_R(\sqrt{s_f^*}), \qquad 0 \le s_f^* \le M^2. \tag{7}$$

Dimensional analysis ensures that $\sqrt{s_f^*} = \lambda_f M$, where $\lambda_f$ possibly depends on $\alpha_R$. To obtain an estimate for $\lambda_f$ we consider the running of $\alpha_R$ up to third order:

$$\frac{\alpha_R(\sqrt{s})}{\pi} = \frac{\alpha_R(M)}{\pi} - \frac{\beta_0}{4} \ln\left(\frac{s}{M^2}\right) \left(\frac{\alpha_R(M)}{\pi}\right)^2 + \tag{8}$$

$$+ \frac{1}{16}\left[\beta_0^2 \ln^2\left(\frac{s}{M^2}\right) - \beta_1 \ln\left(\frac{s}{M^2}\right)\right] \left(\frac{\alpha_R(M)}{\pi}\right)^3 \ldots$$

We substitute for $\alpha_R$ in Eq. (6) to find

$$\begin{aligned}
\frac{\alpha_f(M)}{\pi} &= \frac{\alpha_R(M)}{\pi} - \frac{\beta_0}{4}\left(\frac{I_1}{I_0}\right)\left(\frac{\alpha_R(M)}{\pi}\right)^2 \\
&+ \frac{1}{16}\left[\beta_0^2\left(\frac{I_2}{I_0}\right) - \beta_1\left(\frac{I_1}{I_0}\right)\right]\left(\frac{\alpha_R(M)}{\pi}\right)^3 \ldots,
\end{aligned} \tag{9}$$

where $I_l = \int_0^1 f(x)(\ln x)^l dx$. By setting $s = s^*$ in Eq. (8) and comparing with Eq. (9), we extract

$$\lambda_f = \exp\left[\frac{I_1}{2I_0} + \frac{\beta_0}{8}\left(\left(\frac{I_1}{I_0}\right)^2 - \frac{I_2}{I_0}\right)\frac{\alpha_R(M)}{\pi}\right]. \tag{10}$$

Note that if $f(x)$ is positive on the interval $[0,1]$, then $I_1/I_0$ is negative as expected. Using $f_\tau(x)$, Eq. (10) is nothing but Eq. (4). Also, since $\lambda_f$ is a constant to leading

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order, $\alpha_f$ should satisfy the same RG equation as $\alpha_R$ with the same coefficients $\beta_0$ and $\beta_1$. In other words, $\alpha_f$ is an effective charge.

We can now study integrals over $R_{e^+e^-}$ data with different weight functions $f(x)$ and vary $M$ to see whether we obtain the PQCD behavior. In general, the weight function $f(x)$ should be chosen to suppress the low energy region, where non-perturbative effects are important. Thus, in the following, we will set $f(x) = x^k$, where $k$ is some positive number. In such a case, we have that

$$\alpha_k(M) = \alpha_R(\lambda_k\, M) \quad \text{with} \quad \lambda_k = e^{I_{1k}/2I_{0k}}, \tag{11}$$

where $I_{1k} = \int_0^1 x^k \ln x\, d\,x$ and $I_{0k} = \int_0^1 x^k\, d\,x$. Note that as k increases, $I_{1k}/2I_{0k} \to 0$, and, therefore, $\sqrt{s^*} \to M$. For very large k, we lose sensitivity to the details of PQCD. It is particularly interesting to use such a test to probe the energy region close to the $\tau$ mass $M_\tau$.

The main difficulty in comparing with $R_{e^+e^-}$ data is that we can no longer consider massless flavors and that we observe hadrons instead of quarks.

Following [9] the effect of quark masses can be approximately taken into account if we use:

$$R_{e^+e^-}(\sqrt{s}) = 3\sum_1^f q_i^2 \frac{v_i(3 - v_i^2)}{2}\left[1 + g(v_i)\frac{\alpha_R(\sqrt{s})}{\pi}\right]$$

$$\equiv R_0(\sqrt{s}) + R_{Sch}(\sqrt{s})\frac{\alpha_R(\sqrt{s})}{\pi} \tag{12}$$

$$g(v) = \frac{4\pi}{3}\left[\frac{\pi}{2v} - \frac{3 + v}{4}\left(\frac{\pi}{2} - \frac{3}{4\pi}\right)\right] \tag{13}$$

where $v_i = \sqrt{1 - 4m_i^2/s}$ is the velocity of the initial quarks in their CM frame. The $v_i(3 - v_i^2)/2$ factor is the parton model mass dependence and $g(v)$ is a QCD modification [10] of the Schwinger positronium corrections [11]. In principle, all these corrections spoil the relation in Eq. (11). However these factors are unity for energies well above their corresponding thresholds.

Nevertheless, we still cannot compare directly with the data since there is no direct correspondence between quark and hadronic thresholds. To obtain a meaningful comparison we have to smear both the PQCD results and the data. Following [9] we define smeared quantities as follows:

$$\bar{R}(\sqrt{s}) = \frac{\Delta}{\pi}\int_0^\infty \frac{R(\sqrt{s'})}{(s - s')^2 - \Delta^2}\,d\,s' \tag{14}$$
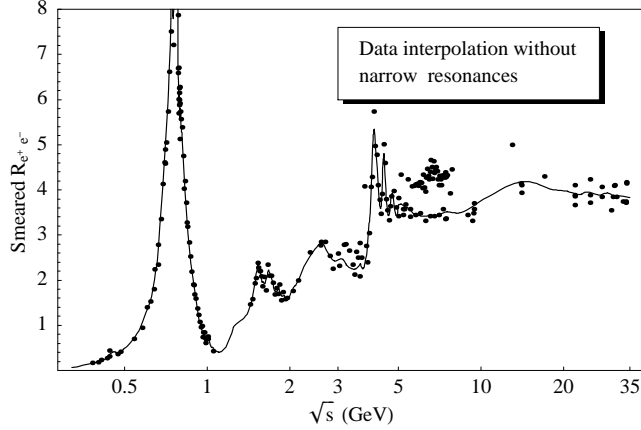
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**Figure 1:** Interpolation of the central values of $R_{e^+e^-}$ data [13]. Narrow resonances are taken into account using their Breit-Wigner form. Note that there seems to be a discrepancy in the central values of experiments between 5 and 10 GeV, that above 20 GeV we have two or three clearly different central values at the same $\sqrt{s}$, and that the point at 13 GeV is much higher than other nearby data.

Note that in the $\Delta \to 0$ limit, we recover the original quantity. In what follows we use the standard value $\Delta = 3\,\mathrm{GeV}^2$ [9, 12]. The smearing effect can be seen by comparing Fig.1, which shows an interpolation of the $R_{e^+e^-}$ data, [13], with Fig.2. For completeness, we also include in Fig.2 the smeared results from NLO PQCD and from the naive parton model ($\alpha_R = 0$).

In order to integrate over $R_{e^+e^-}$, we need to interpolate, *but not fit*, the data. Note that *any fit using the QCD functional dependence will always satisfy the com-*
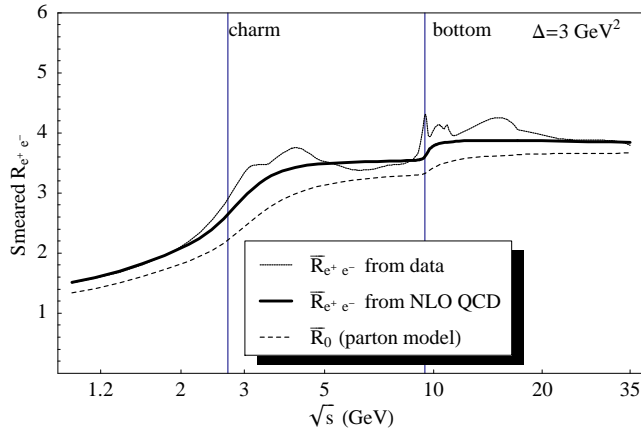


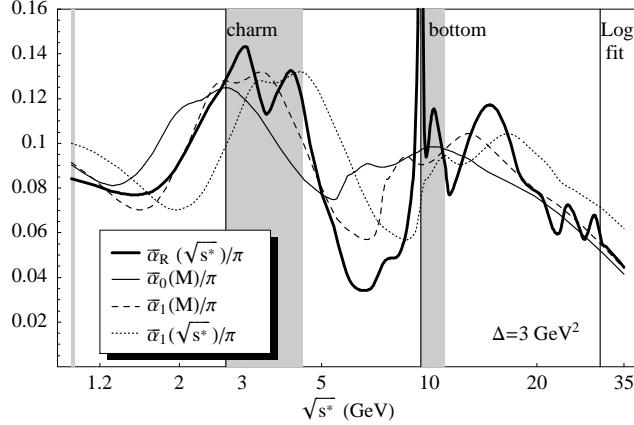**Figure 2:** Effect of smearing on $R_{e^+e^-}$.

**Figure 3:** Comparison between $\bar{\alpha}_R(\sqrt{s^*})$ and different $\bar{\alpha}_k$ moments at $M = \sqrt{s^*}/\lambda_k$. The dotted line shows how the agreement is spoilt if we do not shift $\sqrt{s^*}$ to $M$.

*mensurate scale relations*, even if its quality is poor. To avoid this bias, we have interpolated the central values of the data by means of "$r$-term simple moving averages" up to 30 GeV (to avoid electroweak contributions). That is, if we have a series of raw data $z_1, ...z_n$, we obtain the new set of smoothed data $\sum_{j=0}^{r-1} w_j z_{t-j}$ for $t = r, ...n$, with $\sum_{j=0}^{r-1} w_j = 1$. We have used $r$ ranging from 2 to 6 for different energy regions and our moving averages are "simple" because all the weights $w_j$ are equal. Finally, the resulting smoothed data have been interpolated using cubic-splines. In addition, the narrow resonances that do not appear in Fig.1 are implemented using the Breit-Wigner formula.

We have thus eliminated the QCD biases up to 30 GeV. Above that energy we have matched a logarithmic function whose functional dependence is inspired by QCD, but its contribution in the smearing integrals is negligible for small $\sqrt{s}$.

Unfortunately, we cannot extract directly the effective charges from their corresponding smeared ratios since they are multiplied by other functions inside the smearing integral. However, using Eqs. (12) and (14), we define smeared charges:

$$\bar{\alpha}_R(\sqrt{s}) = \frac{\bar{R}_{e^+e^-}(\sqrt{s}) - \bar{R}_0(\sqrt{s})}{\bar{R}_{Sch}(\sqrt{s})}, \tag{15}$$

and similarly for $\bar{\alpha}_k$. In the massless $\Delta \to 0$ limit we recover the standard effective charges. We expect the smeared charges to satisfy Eq. (10) in energy regions where the threshold corrections can be neglected.

In Fig.3 we compare $\bar{\alpha}_R$ at $\sqrt{s^*}$ with $\bar{\alpha}_k$ moments at $M = \sqrt{s^*}/\lambda_k$. For $\alpha_0$
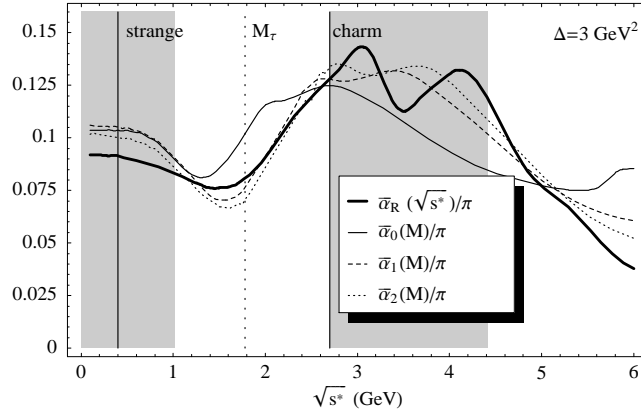
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**Figure 4:** Comparison between $\bar{\alpha}_R(\sqrt{s^*})$ and different $\bar{\alpha}_k$ moments at $M = \sqrt{s^*}/\lambda_k$ in the low energy region.

the agreement is poor, since the low energy region is not suppressed enough. But for $\alpha_1$ we find a reasonable agreement in several regions, and we also show how this agreement disappears if we do not shift the argument of $\alpha_1$ from $\sqrt{s^*}$ to $M = \sqrt{s^*}/\lambda_1$. Starting from higher energies, we find above 30 GeV that commensurate scale relations are satisfied almost identically, which is not surprising since above that energy we have fitted with a QCD inspired behavior. From 15 GeV up to 30 GeV different experiments have measured rather different central values at very similar, or even the same, energies. The smooth interpolation of these points produces artificial oscillations around the mean values of the data. As far as these oscillations are centered on the $\alpha_k$ curves, there is a reasonable agreement, given the quality of the data. In the region between 5 and 10 GeV there seems to be some controversy about the compatibility between different experiments (see Fig.1 and Ref. [14]). It has become standard not to use the older data (which is higher) as we have done in Fig.1. Although the more recent data may be compatible within their experimental errors with the QCD expectations, their central values are systematically lower, which is why Eq. (11) does not seem to hold. Once there are more accurate data, the tests we are proposing, together with a thorough error analysis, will shed light on this situation.

The low energy region is shown in Fig.4 in more detail. Taking into account that we are only using LO QCD and central data values, the agreement between the shaded regions looks quite satisfactory. This is encouraging for the real $\tau$ lepton, which sits in

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a region where PQCD results may be applicable since it is primarily sensitive to the light $u, d, s$ flavors. Nevertheless, by looking at energies $\sqrt{s} \sim 1.5$ GeV, our results seem to support the claims that the $R_{e^+e^-}$ data could be 6-7% lower than the QCD expectations in that region [2].

The commensurate scale relations connecting the moments of the lepton hadronic decay spectrum to $R_{e^+e^-}$ derived here are basic scheme-independent tests of PQCD, depending only on the the universal terms of the $\beta$ function. We have seen, however, that a direct comparison with data is problematic because of several factors such as the distortions of narrow and broad resonances, the physical effects of the quark pair thresholds and the imprecision of much of the $R_{e^+e^-}$ data. Smearing the data over an energy range helps but does not totally remove the effects due to final-state interactions. Quark threshold distortions are partially alleviated by using the Schwinger corrections at small velocity, but the domain of non-relativistic velocity introduces its own complications, including sensitivity of the running coupling to the soft $\alpha_s \, m_q$ scale [15]. Remarkably, the mass range of the physical $\tau$ lepton is potentially clear of the finite quark mass effect since it is well below the $c\bar{c}$ threshold. However, it is clear that higher precision measurements of $R_{e^+e^-}$ throughout the energy domain below the $Z^0$ boson are needed.

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