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One-Loop n-Point Helicity Amplitudes in (Self-Dual) Gravity

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Abstract

We present an ansatz for all one-loop amplitudes in pure Einstein gravity for which the n external gravitons have the same outgoing helicity. These loop amplitudes, which are rational functions of the momenta, also arise in the quantization of self-dual gravity in four-dimensional Minkowski space. Our ansatz agrees with explicit computations via D-dimensional unitarity cuts for $n \leq 6$. It also has the expected analytic behavior, for all n, as a graviton becomes soft, and as two momenta become collinear. The gravity results are closely related to analogous amplitudes in (self-dual) Yang-Mills theory.

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1 Introduction

Gravity and Yang-Mills theory have many similarities at the classical level. Both theories are nonlinear, and possess the respective local symmetries of general coordinate invariance and non-abelian gauge invariance. In the weak field limit, they both admit plane-wave solutions corresponding in the quantum theory to massless particles, gravitons and gluons. Also, a rich set of exact solutions is known for the field equations of each theory restricted to self-dual configurations; for example, the multi-instanton solutions of Yang-Mills theory on S^4 [1], and various gravitational instantons [2].

At the quantum level, however, the two theories behave quite differently. The dimensionless non-abelian coupling of pure gauge theory becomes logarithmically strong in the infrared, leading to confinement of colored quanta, whereas in the ultraviolet the theory is renormalizable and in fact, asymptotically free. On the other hand, the dimensionful nature of Newton's constant means that the infrared behavior of gravity is well-described by the classical limit, but also implies that its ultraviolet behavior is nonrenormalizable by power-counting arguments. Thus, at the quantum level gravity should presumably be regarded as only an effective low-energy limit of some more fundamental theory, such as string theory.

It is nevertheless interesting to examine more carefully the quantum (loop) behavior of gravity, and its connection with gauge theory. The scattering amplitudes of gravitons and of gluons that have been investigated to date have proven to be quite closely related. Classical (tree-level) amplitudes for gravity obey a 'squaring relation', derived by Kawai, Lewellen and Tye (KLT) from string theory, in which each graviton amplitude is given, roughly speaking, by the sums of products of pairs of gluon amplitudes [3, 4]. These n-point tree-level KLT relations, in conjunction with the unitarity of the S-matrix, have recently led to similar relations between four-point amplitudes at the multi-loop level, for the maximally supersymmetric versions of gravity and gauge theory, $N = 8$ supergravity and $N = 4$ super-Yang-Mills theory [5]. Such relations have led to an improved understanding of the ultraviolet behavior of $N = 8$ supergravity in various dimensions.

In this letter we present an ansatz for the one-loop amplitudes in pure Einstein gravity ($\mathcal{L} = -\frac{2}{\kappa^2}\sqrt{-g}R$, $\kappa = \sqrt{32\pi G_N}$) with an arbitrary number of external gravitons, all having positive helicity.[1] These 'all-plus' gravity amplitudes are closely related to the previously computed one-loop all-plus gauge amplitudes [6, 7]. Both sets of amplitudes correspond to self-dual configurations of the field strengths, obeying respectively $R_{\mu\nu\rho\sigma} = \frac{i}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta\rho\sigma}$ and $F_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}F_{\alpha\beta}$, with $\epsilon_{0123} = +1$. Self-dual gravity (SDG) [8] and self-dual Yang-Mills theory (SDYM) [9, 10] have attracted attention through their connection with integrable models, twistor theory and $N = 2$ string theory [10, 11, 12, 13, 14, 15].

Strictly speaking, these self-dual theories are only defined as full quantum theories in four-dimensional space-times with an even number of time dimensions, i.e. signatures $(0, 4)$ (Euclidean space) and $(2, 2)$ (complex time), where the above self-dual constraint lacks an 'i' and is compatible with the reality of the fields. However, a self-dual sector of the full theory can be defined in signature $(1, 3)$ (four-dimensional Minkowski space). At the linearized level, the classical solutions are circu-

[1] Our crossing-symmetric convention is to label outgoing states by their helicity, and incoming states by the reversed helicity (i.e. the helicity they would have if crossed into the final state).

larly polarized plane waves, corresponding to superpositions of states of identical helicity (which may have different momenta). For such solutions in gauge theory, the complexified chromo-electric and chromo-magnetic fields satisfy $E^a_j = -iB^a_j$, where j is a spatial index and a an adjoint gauge index. The gravitational analogs of E^a_j and B^a_j are $E_{jk} \equiv R_{j0k0}$ and $B_{jk} \equiv \frac{1}{2}\epsilon_{j0}{}^{mn}R_{mnk0}$; they similarly satisfy $E_{jk} = -iB_{jk}$ for circularly polarized gravitational waves.

It is nontrivial that the connection between positive helicity amplitudes and self-duality survives the full nonlinear interactions, as demonstrated for gauge theory by Duff and Isham [16]. More recently, several authors [17, 18, 19] have shown that various SDYM Lagrangians [9] (whose classical field equations all solve the SDYM constraints) could be used to compute the tree-level matrix elements for an off-shell gauge current to produce $(n-1)$ identical-helicity on-shell gluons in the full gauge theory. The tree-level scattering amplitudes that result from putting the current on-shell actually vanish for all n by a supersymmetry Ward identity (SWI) [20],

$$\mathcal{M}_n^{\rm SUSY}(1^\pm, 2^+, \ldots, n^+) = 0, \tag{1}$$

where the external states, labeled by their helicity, may be either gauge bosons or gravitons. Eqn. (1) applies also to tree amplitudes for nonsupersymmetric Yang-Mills theory, and gravity, because fermionic superpartners never contribute to tree graphs for bosonic amplitudes.

At one loop, identical-helicity amplitudes in non-supersymmetric theories do not vanish. They are known explicitly for pure Yang-Mills theory for all n [6, 7]. Cangemi [19] and Chalmers and Siegel (CS) [21] showed that these loop amplitudes can also be computed from various SDYM Lagrangians (up to an overall factor of two). The result for the CS Lagrangian,

$$\mathcal{L}_{\rm CS}^{\rm SDYM} = {\rm tr}\Big[\bar\phi\Big(\partial^2\phi + ig(\partial^\alpha{}_+\phi)(\partial_{\alpha+}\phi)\Big)\Big]\,, \tag{2}$$

where two-component spinor notation has been used, agrees with the pure-Yang-Mills result including the factor of two. The action (2) is obtained by truncating the light-cone action for $N=4$ supersymmetric Yang-Mills theory [22]; the '+' spinor is defined with respect to the light-cone direction. The corresponding truncation of the $N=8$ supergravity action contained in ref. [23] gives an action for self-dual gravity [24],

$$\mathcal{L}_{\rm CS}^{\rm SDG} = \bar\phi\Big(\partial^2\phi + \frac{\kappa}{2}(\partial^\alpha{}_+\partial^\beta{}_+\phi)(\partial_{\alpha+}\partial_{\beta+}\phi)\Big)\,. \tag{3}$$

Note that the kinematic structure of the 3-point vertex in eq. (3) is just the square of that in eq. (2), a feature which is consistent with the KLT relations.

For both eqs. (2) and (3), $\bar\phi$ is the loop counting parameter: Tree amplitudes (which all vanish on-shell) have one external $\bar\phi$, one-loop amplitudes have no external $\bar\phi$'s, and there are no $(l>1)$-loop amplitudes.[2] Thus the one-loop all-plus amplitudes for Yang-Mills theory (gravity) are the *only* nonvanishing scattering amplitudes in the CS version of SDYM (SDG).

To construct an ansatz for the n-point one-loop all-plus graviton amplitudes, $\mathcal{M}_n^{\text{1-loop}}(1^+, 2^+, \ldots, n^+)$, we have used a combination of explicit computation (for $n = 4, 5, 6$), and general analytic properties

[2]This does not mean that identical-helicity amplitudes vanish in full gauge theory, or gravity, for $l > 1$, just that the connection with self-dual theories breaks down at that point.

(to infer the all-n result). The analytic properties are very similar to those of the all-plus gauge amplitudes [6, 7]. First of all, the unitarity cuts for the amplitudes are identically zero in four dimensions: Each one-loop cut is a product of two tree amplitudes, one on each side of the cut; but every possible assignment of helicity to the two gravitons crossing the cut leads to the vanishing of at least one of the two tree amplitudes, via eq. (1). Similar reasoning shows that the one-loop all-plus amplitudes do not contain multi-particle poles, of the form $1/(k_{i_1}+k_{i_2}+\cdots+k_{i_m})^2$ with $m>2$. The only permitted kinematic singularities are those where one external momentum becomes soft, or two momenta become collinear. These singularities have a known universal form, which will be described in more detail below. Finally, the loop-momentum integration does not generate any infrared nor ultraviolet divergences. In summary, $\mathcal{M}_n^{\text{1-loop}}(1^+,2^+,\ldots,n^+)$ is a finite rational function of the momenta, totally symmetric in the n arguments, with only soft and collinear singularities.

For the explicit computation, we first used the SWI (1) to replace the one-loop amplitude with a graviton in the loop by that with a massless scalar in the loop [25]. We then calculated the cuts for this scalar loop in an arbitary dimension D, where they are nonvanishing. From the D-dimensional cut information, one can extract the $D=4$ amplitude [26]. Further details of this computation, and an intriguing relation between the D-dimensional all-plus amplitudes and certain $N=8$ supergravity amplitudes, may be found in ref. [27].

From the form of the all-plus amplitudes for $n=4,5,6$, and particularly their factorization properties as a graviton momentum becomes soft, we have arrived at an ansatz for the remaining amplitudes, $n\geq 7$. As an additional check, we have verified that the amplitudes factorize properly as two gravitons become collinear.

2 The Ansatz and Its Soft Limits

To motivate the ansatz for the one-loop all-plus graviton amplitudes, we briefly describe the only other known nontrivial infinite sequence of graviton amplitudes. These are the tree amplitudes with maximal helicity violation (MHV), consistent with eq. (1), for which exactly two gravitons have opposite helicity from the remaining $n-2$. Berends, Giele and Kuijf (BGK)[4] found the following compact expression,[3]

$$\mathcal{M}_n^{\text{tree}}(1^-,2^-,3^+,\ldots,n^+) = -i\left(\frac{\kappa}{2}\right)^{n-2}\langle 1\,2\rangle^8 \times\left[\frac{[1\,2]\,[n-2\;\;n-1]}{\langle 1\;\;n-1\rangle\;N(n)}\left(\prod_{i=1}^{n-3}\prod_{j=i+2}^{n-1}\langle i\,j\rangle\right)\prod_{l=3}^{n-3}\Big(-\langle n^-|\not{K}_{l+1,n-1}|l^-\rangle\Big)\;+\;\mathcal{P}(2,3,\ldots,n-2)\right], \tag{4}$$

where

$$N(n)\equiv\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\langle i\,j\rangle\,, \tag{5}$$

$K_{i,j}^\mu\equiv\sum_{s=i}^{j}k_s^\mu$, and $+\mathcal{P}(M)$ instructs one to sum the quantity inside the brackets over all permutations of the set M.

[3] Our overall phase conventions differ from those of ref. [4] by a '$-i$'.

The spinor inner products [28] are denoted by $\langle i\, j\rangle = \langle i^-|j^+\rangle$ and $[i\, j] = \langle i^+|j^-\rangle$, where $|i^\pm\rangle$ are massless Weyl spinors of momentum k_i, labeled with the sign of the helicity. They are antisymmetric, with norm $|\langle i\, j\rangle| = |[i\, j]| = \sqrt{s_{ij}}$, where $s_{ij} = 2k_i \cdot k_j$, and they carry a relative phase,

$$\frac{[i\, j]}{\langle i\, j\rangle} = -\frac{(k_i^1 - ik_i^2)k_j^+ - (k_j^1 - ik_j^2)k_i^+}{(k_i^1 + ik_i^2)k_j^+ - (k_j^1 + ik_j^2)k_i^+}\,, \tag{6}$$

where $k_i^+ = k_i^0 + k_i^3$.

Because of the permutation sum, the quantity in square brackets in eq. (4) is manifestly symmetric in the labels $2, 3, \ldots, n-2$. In fact it is symmetric under exchange of all labels $1, 2, 3, \ldots, n$, as required by an $N = 8$ SWI [4, 5]. BGK also checked the behavior of their expression as a graviton momentum becomes soft, and found the expected universal behavior [29],

$$\mathcal{M}_n^{\rm tree}(1, 2, \ldots, n^+) \ \stackrel{k_n\to 0}{\longrightarrow}\ \frac{\kappa}{2}\, \mathcal{S}_n \times \mathcal{M}_{n-1}^{\rm tree}(1, 2, \ldots, n-1), \tag{7}$$

where the gravitational soft factor for $k_n \to 0$ is (for positive helicity)

$$\mathcal{S}_n = \frac{-1}{\langle 1\, n\rangle \langle n,\, n-1\rangle} \sum_{i=2}^{n-2} \frac{\langle 1\, i\rangle \langle i,\, n-1\rangle [i\, n]}{\langle i\, n\rangle}\,. \tag{8}$$

Momentum conservation can be used to show that the soft factor is also symmetric under the interchange of legs 1 and $n-1$ with the others.

In general, one might expect the one-loop generalization of eq. (7) to contain an additional term on the right-hand side, proportional to a one-loop correction to the soft factor (8) multiplied by $\mathcal{M}_{n-1}^{\rm tree}$. However, $\mathcal{M}_{n-1}^{\rm tree}$ vanishes for the all-plus helicity configuration, so such a term cannot appear and the soft behavior of the one-loop all-plus amplitudes is identical to that of the tree amplitudes,

$$\mathcal{M}_n^{\text{1-loop}}(1^+, 2^+, \ldots, n^+) \ \stackrel{k_n\to 0}{\longrightarrow}\ \frac{\kappa}{2}\, \mathcal{S}_n \times \mathcal{M}_{n-1}^{\text{1-loop}}(1^+, 2^+, \ldots, (n-1)^+). \tag{9}$$

More generally, for any helicity configuration one can show that $\mathcal{S}_n$ has no loop corrections to all orders of perturbation theory; this will be discussed elsewhere [27].

For the one-loop ansatz, we introduce an off-shell extension of the BGK tree amplitudes,

$$\begin{aligned} h(a, \{1, 2, \ldots, n\}, b) \equiv \frac{[1\, 2]}{\langle 1\, 2\rangle} & \frac{\langle a^-|\not{K}_{1,2}|3^-\rangle \langle a^-|\not{K}_{1,3}|4^-\rangle \cdots \langle a^-|\not{K}_{1,n-1}|n^-\rangle}{\langle 2\, 3\rangle \langle 3\, 4\rangle \cdots \langle n-1,\, n\rangle\ \langle a\, 1\rangle \langle a\, 2\rangle \langle a\, 3\rangle \cdots \langle a\, n\rangle\ \langle 1\, b\rangle \langle n\, b\rangle} \\ & + \mathcal{P}(2, 3, \ldots, n), \end{aligned} \tag{10}$$

and $h(a, \{1\}, b) \equiv 1/(\langle a\, 1\rangle^2 \langle 1\, b\rangle^2)$. Although it is not obvious in this form, h is symmetric under the interchange $a \leftrightarrow b$, and also under the exchange of 1 with any of the labels $2, 3, \ldots, n$. For example,

$$\begin{aligned} h(a, \{1, 2\}, b) &= \frac{[1\, 2]}{\langle 1\, 2\rangle \langle a\, 1\rangle \langle 1\, b\rangle \langle a\, 2\rangle \langle 2\, b\rangle}\,, \\ h(a, \{1, 2, 3\}, b) &= \frac{[1\, 2]\, [2\, 3]}{\langle 1\, 2\rangle \langle 2\, 3\rangle \langle a\, 1\rangle \langle 1\, b\rangle \langle a\, 3\rangle \langle 3\, b\rangle} + \frac{[2\, 3]\, [3\, 1]}{\langle 2\, 3\rangle \langle 3\, 1\rangle \langle a\, 2\rangle \langle 2\, b\rangle \langle a\, 1\rangle \langle 1\, b\rangle} \\ &\quad + \frac{[3\, 1]\, [1\, 2]}{\langle 3\, 1\rangle \langle 1\, 2\rangle \langle a\, 3\rangle \langle 3\, b\rangle \langle a\, 2\rangle \langle 2\, b\rangle}\,. \end{aligned} \tag{11}$$

(To obtain the form of $h(a,\{1,2,3\},b)$ in eq. (11) from eq. (10) requires the Schouten identity, $\langle a\,b\rangle\langle c\,d\rangle = \langle a\,c\rangle\langle b\,d\rangle + \langle a\,d\rangle\langle c\,b\rangle$.) The symmetry properties of h are manifest for all n in an alternative, recursive definition [27].

The relation between h and $\mathcal{M}_n^{\rm tree}$ is

$$\left.\frac{h(n,\{n-1,n-2,\ldots,2\},1)}{\langle n\,1\rangle^2}\right|_{k_1+k_2+\cdots+k_n=0} = (-1)^n \frac{\mathcal{M}_n^{\rm tree}(1^-,2^-,3^+,\ldots,n^+)}{i\,(\kappa/2)^{n-2}\,\langle 1\,2\rangle^8}\,. \tag{12}$$

In this form, momentum conservation only has to be used in one factor in $\mathcal{M}_n^{\rm tree}$, in order to convert it into h. Unlike $\mathcal{M}_n^{\rm tree}$, the quantity $h(a,M,b)$ is defined and has simple analytic properties even off-shell, i.e. without imposing $k_a+K_M+k_b=0$. Here K_M is the sum of the massless momenta in the set M, $K_M^\mu \equiv \sum_{i\in M} k_i^\mu$.

We refer to h as a 'half-soft' function, because it satisfies

$$h(a,M,b) \stackrel{k_m\to 0}{\longrightarrow} -\mathcal{S}_m(a,M,b)\times h(a,M-m,b), \qquad\qquad \text{for } m\in M, \tag{13}$$

where the half-soft factor

$$\mathcal{S}_m(a,M,b) \equiv \frac{-1}{\langle a\,m\rangle\langle m\,b\rangle}\sum_{j\in M}\langle a\,j\rangle\langle j\,b\rangle\frac{[j\,m]}{\langle j\,m\rangle} \tag{14}$$

is analogous to the full soft factor (8), but its sum is restricted to the set M. Equation (13) is easiest to check from the definition (10) in the limit $k_1\to 0$; the sum over $(n-1)!$ permutations breaks up into $n-1$ sums of $(n-2)!$ terms, each of which gives $h(a,\{2,3,\ldots,n\},b)$ times a term in the half-soft factor.

The half-soft functions also solve a recursion relation [27],

$$h(a,P,b) = -\frac{1}{K_P^2}\sum_{\substack{M\cap N=\emptyset\\ M\cup N=P}} h(a,M,b)h(a,N,b)\langle a^-|\not{K}_M\not{K}_N|a^+\rangle\langle b^-|\not{K}_M\not{K}_N|b^+\rangle\,, \tag{15}$$

where the sum is over 'distinct nontrivial partitions' of P into two subsets: M and N must both be non-empty, and (N,M) is not distinct from (M,N). When $b=a$, this recursion relation arises from considering, at tree level, off-shell currents for producing a set P of identical-helicity gravitons with momenta k_i and polarization tensors ε_+ which are in a light-cone gauge with respect to k_a: $\varepsilon_+^{\mu\nu} = \langle a^-|\gamma^\mu|i^-\rangle\langle a^-|\gamma^\nu|i^-\rangle/(2\langle a\,i\rangle^2)$. Notice that the effective vertex $\langle a^-|\not{K}_M\not{K}_N|a^+\rangle^2$ coincides with that given by the CS self-dual gravity Lagrangian (3), $(K_M)^\alpha{}_+(K_M)^\beta{}_+(K_N)_{\alpha+}(K_N)_{\beta+}$, when the fields ϕ carry momentum K_M and K_N, and the light-cone direction '+' corresponds to the null-vector k_a. A recursive construction of the tree-level identical-helicity off-shell gravitational current has been given by Rosly and Selivanov [30].

By combining products of two half-soft factors which share only their outside arguments, we can construct a one-loop ansatz which has simple soft properties. The ansatz is

$$\mathcal{M}_n^{\text{1-loop}}(1^+,2^+,\ldots,n^+) = -\frac{i}{(4\pi)^2\cdot 960}\left(\frac{-\kappa}{2}\right)^n \sum_{\substack{1\le a<b\le n\\ M,N}} h(a,M,b)h(b,N,a)\,{\rm tr}^3[a\,M\,b\,N]\,, \tag{16}$$

where a and b are massless legs, and M and N are two sets forming a distinct nontrivial partition of the remaining $n-2$ legs, as depicted in fig. 1. Also, $\mathrm{tr}[a\,M\,b\,N] \equiv \mathrm{tr}[\not{k}_a \not{K}_M \not{k}_b \not{K}_N]$. For $n = 4, 5, 6$, this ansatz agrees with explicit computations [25, 27], for example

$$\mathcal{M}_4^{\text{1-loop}}(1^+, 2^+, 3^+, 4^+) = -\frac{i}{(4\pi)^2 \cdot 120} \left(\frac{\kappa}{2}\right)^4 \left(\frac{s_{12}s_{23}}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,1\rangle}\right)^2 (s_{12}^2 + s_{23}^2 + s_{13}^2)\,. \qquad (17)$$

The SWI (1), applied to theories with $N = 2$ supercharges, can be used to show [25] that the contribution to the all-plus amplitude of a massless multiplet of spin s (i.e., two states, with helicity $\pm s$) circulating in the loop is simply

$$\mathcal{M}_n^{\text{1-loop, spin } s}(1^+, 2^+, \ldots, n^+) = (-1)^{2s} \mathcal{M}_n^{\text{1-loop}}(1^+, 2^+, \ldots, n^+)\,. \qquad (18)$$

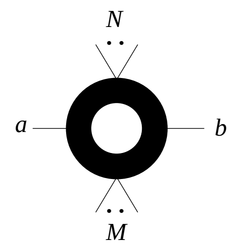

Figure 1: The configurations of external legs that are summed over in eq. (16).

The desired soft properties of $\mathcal{M}_n^{\text{1-loop}}$, eq. (9), follow from those of the half-soft functions, eq. (13). As $k_n \to 0$, the (a, M, b, N) term in $\mathcal{M}_{n-1}^{\text{1-loop}}$ gets contributions from two terms in $\mathcal{M}_n^{\text{1-loop}}$, $(a, M+n, b, N)$ and $(a, M, b, N+n)$. Each of the factors $h(a, M+n, b)$ and $h(b, N+n, a)$ in eq. (16) supplies 'half' of the soft factor in this limit, since

$$\mathcal{S}_n = \mathcal{S}_n(a, M, b) + \mathcal{S}_n(b, N, a). \qquad (19)$$

The trace factors prevent unwanted terms from occurring if a or b becomes soft, but are otherwise innocuous.

3 Comparison with All-Plus Gauge Amplitudes

The closest analog in gauge theory to the all-plus graviton amplitude is the one-loop all-plus n-gluon amplitude $\mathcal{A}_n^{\text{1-loop}}(1^+, 2^+, \ldots, n^+)$ [6, 7]. Suppressing the gauge coupling constant g and color factors (they enter the relation between $\mathcal{A}_n^{\text{1-loop}}$ and the color-ordered quantity $A_{n;1}$ [6]), we have

$$A_{n;1}(1^+, 2^+, \ldots, n^+) = -\frac{i}{48\pi^2} \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} \frac{\mathrm{tr}_-[i_1 i_2 i_3 i_4]}{\langle 1\,2\rangle \langle 2\,3\rangle \cdots \langle n\,1\rangle}\,, \qquad (20)$$

where $\mathrm{tr}_-[i_1 i_2 i_3 i_4] \equiv \frac{1}{2}\mathrm{tr}[(1-\gamma_5)\not{k}_{i_1}\not{k}_{i_2}\not{k}_{i_3}\not{k}_{i_4}]$. Define

$$g(a, \{1, 2, \ldots, n\}, b) \;\equiv\; \frac{1}{\langle a\,1\rangle \langle 1\,2\rangle \langle 2\,3\rangle \cdots \langle n-1\;\; n\rangle \langle n\,b\rangle}\,. \qquad (21)$$

Just as the identical-helicity graviton current contains $h(a,M,a)$, so does the identical-helicity gluon current [31] contain $g(a,M,a)$. The n-gluon MHV tree amplitude [32] can also be written in terms of $g(a,M,b)$, in an equation similar to eq. (12),

$$\left.\frac{g(1,\{2,3,\ldots,n-1\},n)}{\langle n\,1\rangle}\right|_{k_1+k_2+\cdots+k_n=0} = \frac{A_n^{\rm tree}(1^-,2^-,3^+,\ldots,n^+)}{i\,\langle 1\,2\rangle^4}\,. \qquad (22)$$

Note that the gauge quantities all have a definite (color) ordering of their external legs, whereas the corresponding gravitational quantities do not.

If in the all-plus gauge theory result (20) we let $i_1 \to a$, $i_3 \to b$, and rearrange the i_4 sum, then we can rewrite the non-γ_5 parts of the traces in a form very reminiscent of the gravity result (16),

$$\left.A_{n;1}(1^+,2^+,\ldots,n^+)\right|_{\text{non-}\gamma_5} = -\frac{i}{(4\pi)^2\cdot 12}\sum_{\substack{1\le a<b\le n\\ M,N}} g(a,M,b)g(b,N,a)\,{\rm tr}[a\,M\,b\,N]\,, \qquad (23)$$

where M consists of all the legs between a and b (in the cyclic sense), and N of all the legs between b and a. Unfortunately, a minus sign prevents us from writing the γ_5 terms in an analogous way. Nevertheless, the similarity between eqs. (16) and (23) is striking, and indeed eq. (23) helped to motivate the form of the gravitational ansatz.

4 Collinear Limits

As a further consistency check we examine the behavior of the ansatz (16) as any two external legs become collinear. Whereas the collinear properties of massless gauge theory amplitudes are well-known [33, 26], the corresponding behavior of gravity amplitudes has not been discussed previously in any detail.[4] The KLT relations provide a simple way to derive the gravity behavior from that of gauge theory.

In the limit that momenta k_1 and k_2 are collinear ($1 \parallel 2$), we have $k_1 \to zP$, $k_2 \to (1-z)P$, where $P = k_1 + k_2$. Color-ordered tree amplitudes in massless gauge theory satisfy

$$A_n^{\rm tree}(1,2,\ldots,n) \stackrel{1\parallel 2}{\longrightarrow} \sum_{\lambda=\pm} {\rm Split}_{-\lambda}^{\rm gauge\ tree}(z,1,2)\times A_{n-1}^{\rm tree}(P^\lambda,3,\ldots,n)\,, \qquad (24)$$

where the sum runs over the two helicities of the fused leg. The splitting amplitudes ${\rm Split}_{-\lambda}^{\rm gauge\ tree}(z,1,2)$ are universal: they depend only on the momenta and helicities of the legs becoming collinear, and not upon the specific amplitude under consideration. Here we are interested in the pure gluon splitting ampitudes with both legs of positive helicity,

$$\begin{aligned}
{\rm Split}_{+}^{\rm gauge\ tree}(z,1^+,2^+) &= 0\,,\\
{\rm Split}_{-}^{\rm gauge\ tree}(z,1^+,2^+) &= \frac{1}{\sqrt{z(1-z)}}\frac{1}{\langle 1\,2\rangle}\,.
\end{aligned} \qquad (25)$$

[4] The suggestion that collinear limits in gravity are universal was made by Chalmers and Siegel [24].

The KLT relations for $n = 4, 5$, in the limit of infinite string tension, read [3, 4]

$$\begin{aligned} \mathcal{M}_4^{\rm tree}(1,2,3,4) &= -i\left(\frac{\kappa}{2}\right)^2 s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,, \\ \mathcal{M}_5^{\rm tree}(1,2,3,4,5) &= i\left(\frac{\kappa}{2}\right)^3 s_{12}s_{34} A_5^{\rm tree}(1,2,3,4,5)\, A_5^{\rm tree}(2,1,4,3,5) \ + \ \mathcal{P}(2,3)\,. \end{aligned} \tag{26}$$

Because the gauge theory amplitudes on the right-hand side of eq. (26) (and similar equations for $n > 5$) have the universal collinear behavior (24), the gravity amplitudes must obey,

$$\mathcal{M}_n^{\rm tree}(1,2,\ldots,n) \ \stackrel{1\|2}{\longrightarrow}\ \frac{\kappa}{2}\sum_{\lambda=\pm} {\rm Split}^{\rm gravity}_{-\lambda}(z,1,2)\times \mathcal{M}_{n-1}^{\rm tree}(P^\lambda,3,\ldots,n)\,, \tag{27}$$

with splitting amplitudes

$$\begin{aligned} {\rm Split}^{\rm gravity}_{-(\lambda+\tilde\lambda)}(z,1^{\lambda_1+\tilde\lambda_1},2^{\lambda_2+\tilde\lambda_2}) \ &= \ -s_{12}\times {\rm Split}^{\rm gauge\ tree}(z,1^{\lambda_1},2^{\lambda_2}) \\ &\qquad \times {\rm Split}^{\rm gauge\ tree}(z,2^{\tilde\lambda_2},1^{\tilde\lambda_1})\,. \end{aligned} \tag{28}$$

Inserting, for example, the values of the pure gluon splitting amplitudes (25) into eq. (28), gives

$$\begin{aligned} {\rm Split}^{\rm gravity}_{+}(z,1^+,2^+) &= 0\,, \\ {\rm Split}^{\rm gravity}_{-}(z,1^+,2^+) &= \frac{-1}{z(1-z)}\frac{[1\,2]}{\langle 1\,2\rangle}\,. \end{aligned} \tag{29}$$

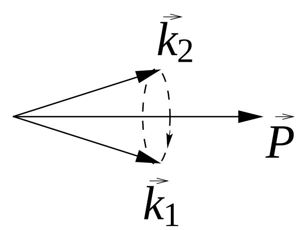

Figure 2: As two momenta become collinear the gravity S-matrix develops a phase singularity which can be detected by rotating the two momenta about the axis formed by their sum.

The meaning of eq. (27) is slightly different from the gauge theory case. In the gauge case, the leading power-law behavior of the amplitude in the collinear limit is determined by the universal terms (24); non-universal terms are suppressed by a relative power of $\sqrt{s_{12}}$. In the case of eqs. (27) and (29), there are non-universal terms of the same magnitude as $[1\,2]\,/\,\langle 1\,2\rangle$ as $s_{12}\to 0$. However, these terms do not acquire any *phase* as the nearly collinear three-vectors $\vec{k}_1$ and $\vec{k}_2$ are rotated around their sum $\vec{P}$. For example, consider the two factors,

$$({\rm a}):\quad \frac{[1\,2]}{\langle 1\,2\rangle}\,, \qquad\qquad ({\rm b}):\quad \frac{[1\,3]}{\langle 1\,3\rangle}\,.$$

If we take $\vec{k}_1$ to be nearly collinear with $\vec{k}_2$ and rotate $\vec{k}_1$ and $\vec{k}_2$ around the vector $\vec{P} = \vec{k}_1 + \vec{k}_2$ by a large angle, as depicted in fig. 2, the factor (b) undergoes only a slight numerical variation. On the other hand, from eq. (6), the factor (a) will undergo a large phase variation depending on the angle of rotation. (As $\vec{k}_1$ and $\vec{k}_2$ rotate once around their sum $\vec{P}$, the phase of ${\rm Split}^{\rm gravity}_{-}(z,1^+,2^+)$, or of the factor (a), changes by 4π, due to the angular-momentum mismatch of $2\hbar$ between the graviton P^+ and the pair of gravitons $(1^+, 2^+)$.) One may therefore numerically separate the terms with phase singularities, by evaluating the amplitude for $\vec{k}_1$ and $\vec{k}_2$ almost collinear, and subtracting the same expression but with the two collinear momenta rotated by a large angle about $\vec{P}$; the non-universal terms will then cancel.

For the one-loop all-plus amplitudes, any loop corrections to the splitting amplitudes must drop out, because they would be multiplied by vanishing tree amplitudes, according to eq. (1). Thus we require in the collinear limit,

$$\mathcal{M}_n^{\text{1-loop}}(1^+, 2^+, 3^+, \ldots, n^+) \stackrel{1\|2}{\longrightarrow} \frac{\kappa}{2} \sum_{\lambda=\pm} \text{Split}^{\text{gravity}}_{-\lambda}(z, 1^+, 2^+) \times \mathcal{M}_{n-1}^{\text{1-loop}}(P^\lambda, 3^+, \ldots, n^+) \,. \qquad (30)$$

(Actually, for any helicity configuration the gravity splitting amplitudes, like the soft factors, do not have corrections at any loop order [27].)

To verify the behavior (30), two different types of limits of the half-soft functions are needed,

$$\begin{aligned} h(a, \{1, 2, 3, \ldots, n\}, b) &\stackrel{1\|2}{\longrightarrow} \frac{1}{z(1-z)} \frac{[1\,2]}{\langle 1\,2\rangle} \times h(a, \{P, 3, \ldots, n\}, b) \,, \\ h(1, \{2, 3, \ldots, n\}, b) &\stackrel{1\|2}{\longrightarrow} \frac{1}{\langle 1\,2\rangle} \frac{\langle 1\,b\rangle \langle b^-|\not{K}_{3,n}|2^-\rangle}{\langle 2\,b\rangle^2} \times h(1, \{3, \ldots, n\}, b) \,, \end{aligned} \qquad (31)$$

where we have used the Schouten identity, and dropped terms without phase singularities. Note that the second, more singular limit reduces to the same degree of singularity in the on-shell case, $K_{1,n} + k_b = 0$ — as it must in order for eqs. (31) to give the correct collinear limits in all channels of the MHV tree amplitudes (4), via the relation (12). Using eqs. (31) it is straightforward to establish that eq. (16) has the correct phase singularities as any two momenta become collinear.

5 Comments

Although the analytic requirements that we have imposed on the amplitudes should be sufficiently stringent to uniquely fix them, it would still be useful to have a complete proof that the expression (16) is correct. The ansatz for the all-plus gauge amplitudes [6] was proven using a recursive construction of a doubly-off-shell current in gauge theory [7]. We have constructed the analogous doubly-off-shell currents in gravity, and have sewn them into the all-plus gravity amplitudes for $n = 4, 5$, obtaining numerical agreement with eq. (16) [27]. However, we have not been able to analytically simplify the resulting expressions for general n.

In this paper we have found an infinite sequence of *non*-vanishing one-loop amplitudes for identical-helicity gravitons in Minkowski space (signature $(1,3)$). (Although the $n > 6$ results are, strictly speaking, an ansatz, the structure of the soft and collinear limits certainly requires these amplitudes to be nonzero, given that the lower-point amplitudes are.) In contrast, all one-loop scattering amplitudes have been argued to vanish for the closed $N = 2$ string [15], which is supposed to describe self-dual gravity in signature $(2,2)$.[5] The same paradoxical situation has been noted for SDYM amplitudes and the open $N = 2$ string [21].

[5]However, two explicit computations have found a nonvanishing one-loop three-point amplitude [34].

It has been argued that the effective actions for both closed and open $N=2$ strings are modified upon including the effects of instantons for the $U(1)$ world-sheet current [35]. In the case of the closed $N=2$ string, originally thought [13] to be described by the 'Plebański' Lagrangian [8],

$$\mathcal{L}_{\mathrm{P}}^{\mathrm{SDG}} = \phi\Big(\frac{1}{2}\partial^2\phi + \frac{\kappa}{3}(\partial^{\alpha}{}_{+}\partial^{\beta}{}_{-}\phi)(\partial_{\alpha+}\partial_{\beta-}\phi)\Big)\,, \tag{32}$$

this Lagrangian is to be converted into the CS SDG Lagrangian (3) (or a one-field version of it) [35]. The difference between eqs. (32) and (3) might account for the different conclusions that have been reached about the vanishing of one-loop self-dual amplitudes. Clearly, a direct comparison of amplitudes obtained from the two actions in signature $(2,2)$ would be required to demonstrate this.

Another, possibly related, explanation involves a potential anomaly in the string world-sheet theory [21]. We have nothing to say about the string theory situation. However, Bardeen has suggested that the non-vanishing of the $(1,3)$ SDYM one-loop amplitudes is related to an anomaly in the SDYM conserved currents [17]. In this context, we note that the existing $(1,3)$ Minkowski-space field theory calculations used regularizations which do not respect self-duality, even though the final result is finite. The unitarity-cut calculation requires $D \neq 4$, for example, in order to detect the rational functions constituting the result. Mahlon's recursive Feynman-diagram approach to the all-plus gauge amplitudes also used dimensional regularization; without it, naive manipulations would have given zero as the answer [7].

In addition to self-dual gravity and Yang-Mills theory, it is possible to couple the two theories to each other, as in the $N=2$ open and heterotic strings [14]. Off-shell currents and MHV amplitudes for a mixture of gravitons and gauge bosons of identical helicity have been constructed at tree level [36]. The one-loop all-plus gravity/gauge amplitudes also have simple general analytic properties, and it would not be surprising if they too could be determined in closed form.

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