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Electron-Positron Pair Production in the Deep Quantum Regime*

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Abstract. Electron-positron pair production via real and virtual photons is significant to the design of linear colliders, especially in the deep quantum regime (i.e., beamstrahlung parameter $\Upsilon >> 1$). In this regime, pair production via a virtual photon (the trident process) can become comparable in rate to pair production via a real beamstrahlung photon. We derive characteristics of the e+e- pairs produced via the trident process, using the quasi-classical approach of Baier, Katkov, and Strakhovenko [1]. We have also examined some of the implications of e+e- pair production for the design of very high energy (several TeV in the center of mass) linear colliders in the deep quantum regime, both in this paper and elsewhere [2].

INTRODUCTION

In extremely high energy linear collider designs (several TeV in the center of mass), a tightly focused bunch consisting of $\sim 10^8$ electrons is to pass through a similar bunch of positrons travelling in the opposite direction. Individual high energy electrons and positrons will radiate photons due to their interaction with the collective electromagnetic field of the oncoming bunch. Some of these beamstrahlung photons convert to e^+e^- pairs as they continue moving through the collective field. The strength of the interaction with the external field is characterized by the usual Υ parameter (see next section). Very high energy collider designs, for example those using laser acceleration [3], typically need very short bunch lengths and thus tend to be in the deep quantum beamstrahlung regime ($\Upsilon \gg 1$). Coherent pair production (i.e. that due to the interaction with the strong collective field produced by the other bunch) may occur through a real beamstrahlung photon (we shall refer to this as the *cascade process*), or the intermediate photon may be virtual, in which case the pair production is said to occur by the *trident process.* The virtual photon process becomes comparable to the cascade process in the deep quantum regime. Thus the main motivation of this paper is to investigate further the possible impact of the trident process on the design of very high energy linear colliders.

Pair production via the cascade process was first treated by Klepikov [4] and by Nikishov and Ritus [5]. The first correct treatment of the trident process was given by Ritus [6]. Useful approximate formulas for the total pair production probability via the trident process were given by Baier, Katkov, and Strakhovenko (BKS) [1] including one for the high Υ limit that is of interest to us, but almost no details of the derivation that would enable one to easily obtain an expression for the energy spectrum of the pairs were given. Thus the main work of our paper is to reconstruct a derivation (which is presumably along the lines of one already carried out by BKS) that will allow us to obtain an explicit expression for the energy spectrum of pairs produced via the trident process at high Υ. We shall also compare our result for the total trident pair production rate with that obtained by BKS, as well as with that for the rate via the cascade process; both of which have already been widely used in the literature on linear colliders.

CALCULATION OF PAIR PRODUCTION RATE

Consider an electron or positron of very high energy E traversing a strong electromagnetic field. Such a situation may be characterized by the Lorentz invariant parameter Υ, defined by

$$\Upsilon \equiv \frac{e\hbar}{m^3c^4}\sqrt{|F_{\mu\nu}p^\nu|^2} = \gamma\frac{B}{B_c} \quad . \tag{1}$$

Here $p^\nu = (E, \overrightarrow{p})$ is the 4-momentum of the incoming electron or positron, m is the electron mass, $\gamma \equiv E/mc^2$ is the usual Lorentz factor, $F_{\mu\nu}$ is the energy-momentum tensor of the electromagnetic field, $B = |\overrightarrow{B}| + |\overrightarrow{E}|$, and $B_c \equiv m^2c^3/\hbar e \approx 4.4 \times 10^{13}$ Gauss is the Schwinger critical field.

We follow the quasi-classical approach of BKS, whereby the very high energy electron can be regarded as following a classical trajectory through the magnetic field. The quantum nature of the photon emission and the corresponding recoil of the electron are, however, taken into account. Under such assumptions, BKS derive the following expression for the total pair production probability (per unit time) via a virtual intermediate photon:

$$W_{tot} = -\frac{\alpha^2 m^2}{8\pi^2 E}\frac{c^4}{\hbar}\int_0^\infty \frac{du}{(1+u)^2}\int_0^\infty \frac{d\xi}{\cosh^2\xi}\cdot I_{\sigma\tau} \quad . \tag{2}$$

Here $\alpha \equiv e^2/\hbar c \approx 1/137$ is the fine-structure constant. The variables u and ξ are defined by

$$\begin{aligned} u \quad &\equiv \frac{\hbar\omega}{E-\hbar\omega} = \frac{y}{1-y}, \\ \cosh^2\xi &\equiv \frac{(\hbar\omega)^2}{4E_+E_-} = \frac{y^2}{4x(y-x)}. \end{aligned} \tag{3}$$

where $\hbar\omega$ is the energy of the intermediate virtual photon, E_+ is the energy of the positron of the produced pair, $E_- = \hbar\omega - E_+$ is the energy of the produced electron, $y \equiv \hbar\omega/E$ is the fractional energy of the intermediate virtual photon, and $x \equiv E_+/E$ is the fraction of the initial energy carried by the positron in the produced pair.

For the moment, it is most convenient to express $I_{\sigma\tau}$ in the following form:

$$I_{\sigma\tau}= \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\tau B_\sigma B_\tau \Big\{ \frac{u\Upsilon}{(1+u)\cosh^2\xi} \frac{\delta(\sigma-\tau)}{\sigma\tau}$$
$$+ \quad \Big[A_{-1,-1}\frac{1}{\tau\sigma} + A_{-1,1}\frac{\sigma}{\tau} + A_{1,-1}\frac{\tau}{\sigma} + A_{1,0}\tau + A_{1,2}\tau\sigma^2$$
$$+A_{2,-1}\frac{\tau^2}{\sigma} + A_{2,1}\tau^2\sigma\Big] \cdot [\theta(\sigma-\tau) - \theta(\tau-\sigma)]\Big\} \cdot$$
$$\exp\Big[-i\frac{u}{\Upsilon}(\sigma + \frac{\sigma^3}{3}) - i\kappa(\tau + \frac{\tau^3}{3})\Big] \quad . \tag{4}$$

where the τ and σ dependences are shown explicitly, and the coefficients $A_{i,j}$ still depend on the remaining integration variables u, ξ (or x, y) and are defined below. Here $\delta(z)$ is the Dirac delta function, $\theta(z)$ is the Heaviside step function, and we have defined

$$\kappa \equiv \frac{4\cosh^2\xi\,(1+u)}{u\Upsilon} = \frac{y}{\Upsilon x(y-x)} \quad . \tag{5}$$

The integrals over σ and τ are regularized for $\sigma, \tau \to 0$ via the operator B_τ, which is defined by:

$$B_\tau \tau^n e^{a\tau^3} = \begin{bmatrix} \tau^n e^{a\tau^3} & (n \geq 0) \quad , \\ \tau^n(e^{a\tau^3} - 1) & (n = -1) \quad . \end{bmatrix} \tag{6}$$

The quantities $A_{i,j}$ depend on Υ, as well as on the fractional energies x and y (through the variables u and ξ), and are given by

$$A_{-1,-1} = \frac{-i}{\cosh^2\xi}$$
$$A_{-1,1} = \frac{-id(u)}{(1+u)\cosh^2\xi}$$
$$A_{1,-1} = \frac{ib(\xi)d(u)}{3u^2}$$
$$A_{1,0} = -A_{2,-1} = \frac{2(1+u)}{3u\Upsilon}b(\xi)$$
$$A_{1,2} = -A_{2,1} = \frac{2(1+u)}{3u\Upsilon}\Big(\frac{b(\xi)d(u)}{1+u} - 3\Big) \tag{7}$$

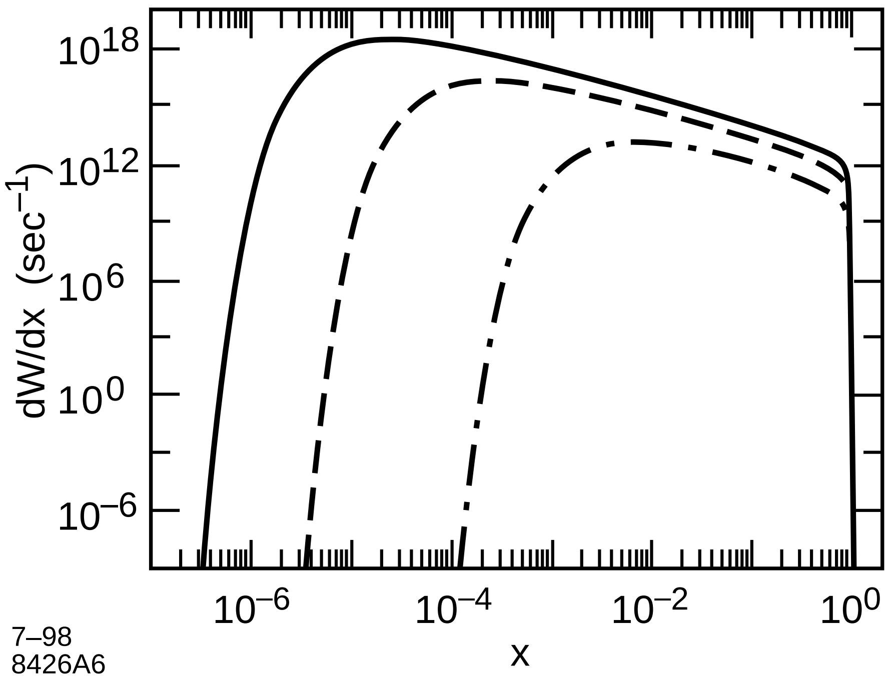

FIGURE 1. Spectrum of probability per unit time [sec^{-1}] for pair production via the trident process, as a function of $x \equiv E_+/E$, for $\Upsilon = 100$ (dot-dashed curve), $\Upsilon = 3000$ (dashed curve), and $\Upsilon = 30000$ (solid curve). The vertical axis, which scales as $1/E$, assumes $E = 2.5$ TeV.

where $d(u) \equiv 1 + (1+u)^2$ and $b(\xi) \equiv 8\cosh^2\xi + 1$.

After a lengthy calculation, in which the assumption $\Upsilon \gg 1$ is used, the integrals over σ and τ in Eq. (4) may be carried out in terms of the Airy function Ai(z) and the related Airy function Gi(z):

$$\mathrm{Ai}(z) \equiv \frac{1}{\pi}\int_0^\infty \cos(\frac{v^3}{3} + zv)dv \quad ,$$

$$\mathrm{Gi}(z) \equiv \frac{1}{\pi}\int_0^\infty \sin(\frac{v^3}{3} + zv)dv \quad . \tag{8}$$

The full result for $I_{\sigma\tau}$ is given in the Appendix. There are three terms in $I_{\sigma\tau}$ that are significant for large Υ:

$$\begin{aligned} I_{\sigma\tau} \approx & -\frac{8\pi}{9u^2} b(\xi) d(u) \kappa^{-2/3} \mathrm{Ai}'(\kappa^{2/3}) \ln(u/\Upsilon) \\ & + 8\pi \frac{(1+u)}{3u} \left(\frac{b(\xi)d(u)}{1+u} - 3 \right) u^{-1} \kappa^{-2/3} \mathrm{Ai}'(\kappa^{2/3}) \\ & - \frac{4\pi}{3} (\frac{2}{3}\mathcal{C} + \frac{1}{3}\ln 3) \frac{1}{u^2} b(\xi) d(u) \kappa^{-2/3} \mathrm{Ai}'(\kappa^{2/3}) \end{aligned}$$

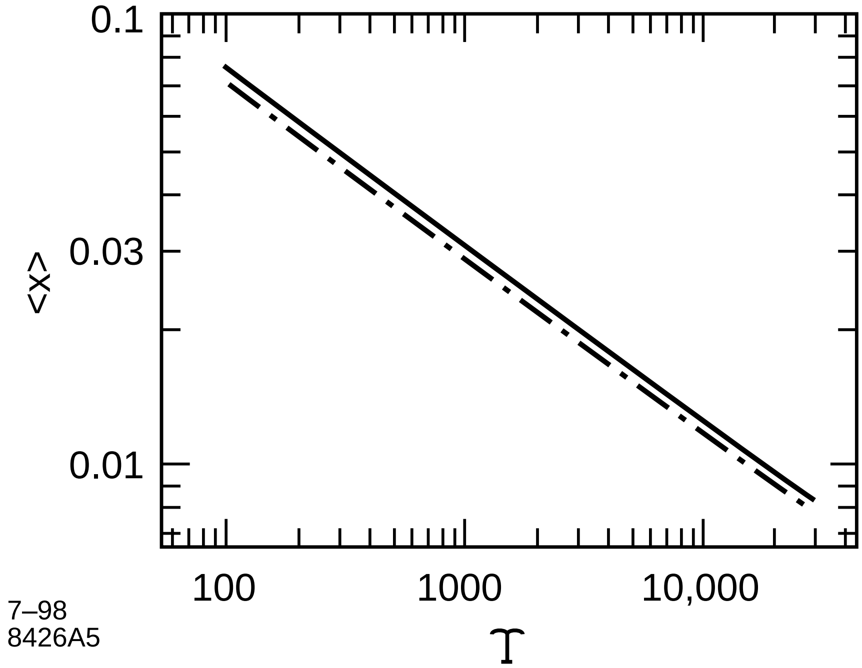

FIGURE 2. Mean value of $x \equiv E_+/E$ for pair production via the trident process, as a function of Υ. The lower curve includes only the dominant term in Eq. (9), while the solid curve includes all three terms in Eq. (9).

$$
\begin{aligned}
&= -\frac{8\pi}{9}[(1-y)^2+1]\left[\frac{2y^2}{x(y-x)}+1\right]y^{-8/3}[x(y-x)]^{2/3}\cdot \\
&\quad \Upsilon^{2/3}\mathrm{Ai}'(\kappa^{2/3})\ln\left[\frac{y}{(1-y)\Upsilon}\right] \\
&\quad + \frac{8\pi}{3}\left\{[(1-y)^2+1]\left[\frac{2y^2}{x(y-x)}+1\right]-3(1-y)\right\}y^{-8/3}[x(y-x)]^{2/3}\cdot \\
&\quad \Upsilon^{2/3}\mathrm{Ai}'(\kappa^{2/3}) \\
&\quad - \frac{4\pi}{3}\,(\frac{2}{3}\mathcal{C}+\frac{1}{3}\ln 3)[(1-y)^2+1]\left[\frac{2y^2}{x(y-x)}+1\right]y^{-8/3}[x(y-x)]^{2/3}\cdot \\
&\quad \Upsilon^{2/3}\mathrm{Ai}'(\kappa^{2/3}) \quad . \qquad (9)
\end{aligned}
$$

Here $\mathcal{C}$ is Euler's constant (≈ 0.577). The first term shown dominates. The second two terms give a correction of order 10% for parameters of interest for very high energy linear colliders. Note that all three terms depend on Υ through $\Upsilon^{2/3}\mathrm{Ai}'(\kappa^{2/3})$. The main reason for the dominance of the first term is its additional dependence on $\ln\Upsilon$.

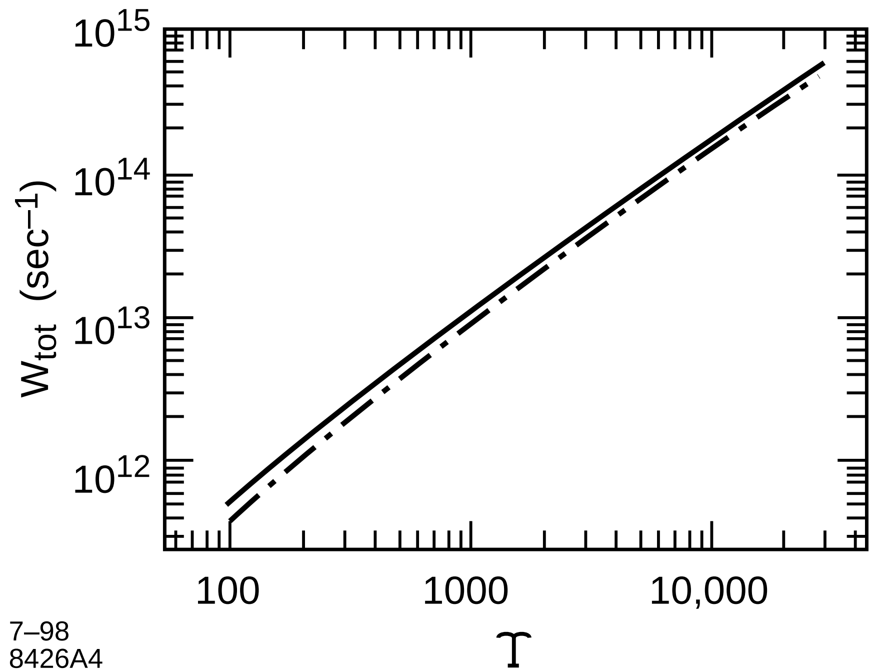

FIGURE 3. Total probability per unit time [sec^{-1}] for pair production via the trident process, as a function of Υ, for $E \equiv \gamma mc^2 = 2.5$ TeV. The lower curve includes only the dominant term in Eq. (9), while the solid curve includes all three terms in Eq. (9). [This figure may of course be scaled to arbitrary energy since the vertical scale is proportional to $1/\gamma$.]

Energy spectrum of pairs

In order to obtain the energy spectrum, we express the remaining integrations in terms of x and y rather than u and ξ. In terms of x and y, the total probability per unit time for producing pairs at any energy between 0 and E is:

$$\begin{aligned} W_{tot} &= -\frac{\alpha^2 m}{8\pi^2}\frac{c^2}{\hbar}\frac{1}{\gamma}\int_0^{1/2} dx \int_{2x}^{1} dy \frac{\frac{1}{y}(y-2x)}{[y^2/4 - x(y-x)]^{1/2}} \cdot I_{\sigma\tau} \\ &= -\frac{\alpha^2 m}{16\pi^2}\frac{c^2}{\hbar}\frac{1}{\gamma}\int_0^{1} dx \int_{x}^{1} dy \frac{\frac{1}{y}|y-2x|}{[y^2/4 - x(y-x)]^{1/2}} \cdot I_{\sigma\tau} \quad . \end{aligned} \tag{10}$$

(The last equality follows from the symmetry properties of the integrand when x and $(y-x)$ are interchanged.) Given Eq.(10), it is straightforward to do the integration over y numerically to get the spectrum dW/dx, and to do both integrations numerically to get W_{tot}. The spectra for $\Upsilon = 100$, $\Upsilon = 3000$, and $\Upsilon = 30000$ are shown in Figure 1. Here we have assumed $E = 2.5$ TeV, but again the particular value of the energy only affects the vertical scale through the $1/\gamma$ factor. Using the result for dW/dx, the mean value of x as a function of Υ may also be computed, as is shown in Figure 2.

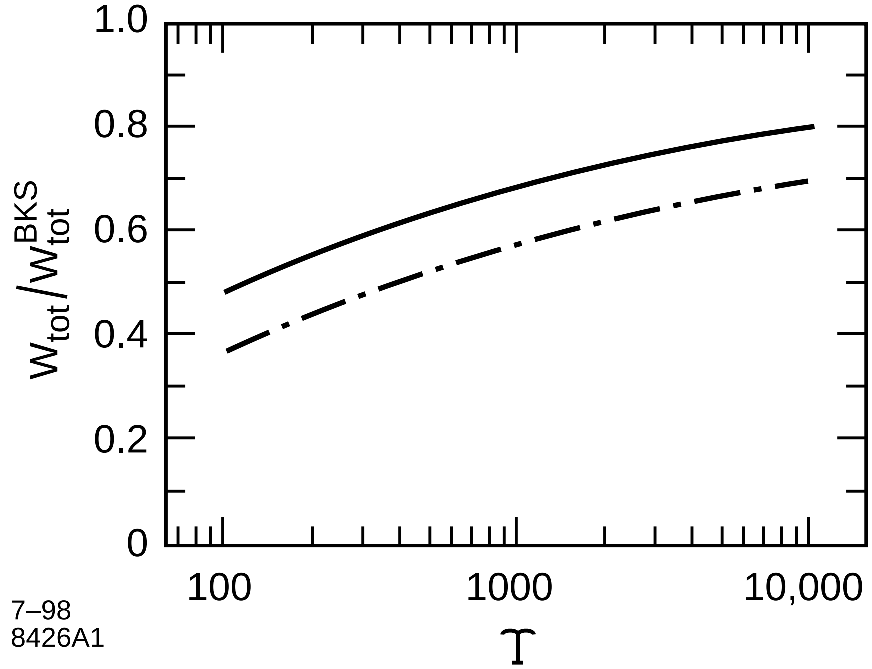

FIGURE 4. Ratio of W_{tot}, the total probability per unit time for pair production via the trident process to W_{tot}^{BKS}, the approximation of Baier, Katkov, and Strakhovenko, as a function of Υ. The lower curve includes only the dominant term in Eq. (9), while the solid curve includes all three terms in Eq. (9).

Total rates of pairs

The total probability W_{tot} as a function of Υ is shown in Figure 3, for $E \equiv \gamma mc^2 = 2.5$ TeV. (The figure may be scaled to arbitrary γ since the vertical scale is simply proportional to $1/\gamma$.) The upper curve includes all three terms in Eq. (9), while the lower curve includes only the first term.

Next we compare our result for W_{tot} with the result given by BKS [1]. These authors appear to have made an approximation to the first term of Eq. (9) which allows them to carry out the integrations over u and ξ, obtaining the following convenient analytic expression:

$$W_{tot}^{BKS} = \frac{13\alpha^2 m}{9\sqrt{3}\pi}\frac{c^2}{\hbar}\frac{1}{\gamma}\Upsilon\ln\Upsilon \quad . \tag{11}$$

In Figure 4 we show the ratio of W_{tot} to W_{tot}^{BKS}, as a function of Υ. The lower curve includes only the dominant term in Eq. (9), while the solid curve includes all three terms in Eq. (9). The overall Υ dependence of our result is in fact quite close to $\Upsilon\ln\Upsilon$ for very large Υ, but we do not know what approximations BKS made to obtain precisely their result (or indeed whether or not our derivation agrees with theirs up to the point where such an approximation would be made).

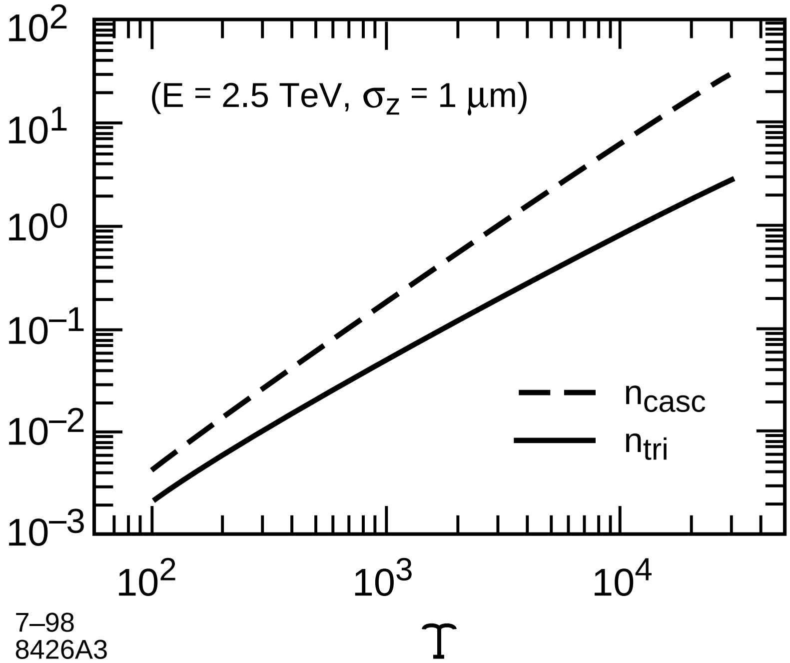

FIGURE 5. Number of pairs per particle produced at an energy $E = 2.5$ TeV and for a bunch length $\sigma_z = 1\mu$m, via the cascade process (dashed curve) and the trident process (solid curve).

Consider now a linear collider in which the external field is that created by the oncoming bunch, and in which the bunch lengths are σ_z. The bunches are assumed to be Gaussian in all three space dimensions, and the charge and transverse size of the bunch are then the remaining determinants of the effective Υ. The total (integrated over x) number of pairs produced via the trident process, per incoming electron or positron, is $n_{tri} \approx \frac{\sqrt{3}\sigma_z}{c} W_{tot}$. For comparison, an estimate [6,7] of the number of pairs per particle produced via the cascade process is

$$n_{casc} = (0.295)\left[\frac{\alpha\sigma_z\Upsilon}{\gamma ƛ_e}\right]^2 \Upsilon^{-2/3}(\ln\Upsilon - 2.488) \qquad (\Upsilon \gg 1) \quad . \tag{12}$$

Here $ƛ_e = \hbar/mc$ is the Compton wavelength of the electron.

In Figure 5 we show the number of pairs per particle produced via the cascade process (dashed curve) and the trident process (solid curve) as a function of Υ, for energy $E = 2.5$ TeV and bunch length $\sigma_z = 1$ μm.

In Figure 6 we show the ratio of the number of pairs per particle produced via the trident process to the number of pairs produced via the cascade process, normalized to an energy $E = 2.5$ TeV and for a bunch length $\sigma_z = 1$ μm. To obtain the ratio n_{tri}/n_{casc} for arbitrary energy and bunch length, one would multiply by energy in TeV and divide by the bunch length in microns. Keep in mind, of course, that changing the bunch length will change the effective Υ, unless the bunch charge and transverse size are adjusted to compensate.

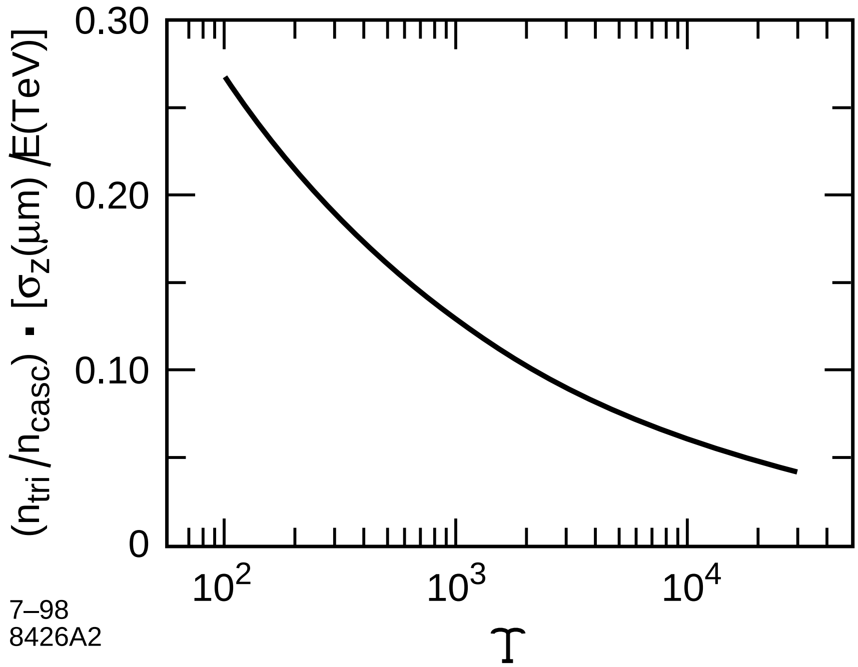

FIGURE 6. Ratio of number of pairs per particle produced via the trident process to the number of pairs produced via the cascade process, normalized to an energy $E = 2.5$ TeV and for a bunch length $\sigma_z = 1$ μm. (To obtain n_{tri}/n_{casc} for arbitrary energy and bunch length, multiply by E in TeV and divide by σ_z in microns.)

CONCLUSIONS AND ACKNOWLEDGMENTS

Our results for the total trident pair production rate W_{tot} are in reasonable agreement with that predicted by the approximate formula of Baier, Katkov and Strakhovenko, although their approximate expression is somewhat larger than ours. The agreement is within 20% at extremely high Υ ($\Upsilon > 10000$ up to the limit at which the assumptions in our calculations break down), but there is a discrepancy of about a factor of two for Υ values of a few hundred.

For reasonable linear collider parameters, it does not appear that the total number of trident pairs would exceed the total number of cascade-process pairs, although the trident pairs can can comprise a significant fraction of the total. Another possible issue, which we have addressed elsewhere, is whether there are a significant number of trident pairs with sufficiently low energy that the pair particle with the same sign as the oncoming beam can be deflected to large angles in the beam-beam field. Our conclusion [2] is that there is much less than one trident pair per bunch crossing that would reach an outgoing angle of a radian or more, assuming very high energy (several TeV) linear collider parameters similar to those proposed in Ref. [3].

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APPENDIX

We give here the full expression for $I_{\sigma\tau}$:

$$\begin{aligned}
I_{\sigma\tau} \quad = \quad & -\frac{8\pi}{9u^2} b(\xi) d(u) \kappa^{-2/3} \mathrm{Ai}'(\kappa^{2/3}) \ln\left(\frac{u}{\Upsilon}\right) \\
& +8\pi \frac{(1+u)}{3u^2} \left(\frac{b(\xi)d(u)}{1+u} - 3\right) \kappa^{-2/3} \mathrm{Ai}'(\kappa^{2/3}) \\
& -\frac{4\pi}{3u^2} (\frac{2}{3}\mathcal{C} + \frac{1}{3}\ln 3) b(\xi) d(u) \kappa^{-2/3} \mathrm{Ai}'(\kappa^{2/3}) \\
& +\frac{4\pi^2}{3u^2} b(\xi) d(u) \kappa^{-2/3} \mathrm{Ai}'(\kappa^{2/3}) \int_0^{(\frac{u}{\Upsilon})^{2/3}} \mathrm{Gi}(v) dv \\
& +8\pi^2 \frac{(1+u)}{3u^{5/3}\Upsilon^{1/3}} \left(\frac{b(\xi)d(u)}{1+u} - 3\right) \kappa^{-1/3} \mathrm{Ai}(\kappa^{2/3}) \mathrm{Gi}'((\frac{u}{\Upsilon})^{2/3}) \\
& +8\pi^2 \frac{(1+u)}{3u^{4/3}\Upsilon^{2/3}} b(\xi) \kappa^{-1/3} \mathrm{Ai}'(\kappa^{2/3}) \mathrm{Gi}((\frac{u}{\Upsilon})^{2/3}) \\
& -8\pi^2 \frac{(1+u)}{3u^{4/3}\Upsilon^{2/3}} \left(\frac{b(\xi)d(u)}{1+u} - 3\right) \kappa^{-2/3} \mathrm{Ai}'(\kappa^{2/3}) \mathrm{Gi}((\frac{u}{\Upsilon})^{2/3}) \\
& -8\pi^2 \frac{(1+u)}{3u\Upsilon} b(\xi) \kappa^{-1/3} \mathrm{Ai}(\kappa^{2/3}) \int_0^{(\frac{u}{\Upsilon})^{2/3}} \mathrm{Gi}(v) dv \\
& +8\pi (\frac{2}{3}\mathcal{C} + \frac{1}{3}\ln 3) \frac{(1+u)}{3u\Upsilon} b(\xi) \kappa^{-1/3} \mathrm{Ai}(\kappa^{2/3}) \\
& +\frac{16\pi}{3} \frac{(1+u)}{3u\Upsilon} b(\xi) \kappa^{-1/3} \mathrm{Ai}(\kappa^{2/3}) \ln\left(\frac{u}{\Upsilon}\right) \quad . \qquad (13)
\end{aligned}$$

The Υ dependence of the terms beyond the first three is through non-positive powers of Υ and through the Airy and related Airy functions.