

The Hadronic Gas Chiral Phase Transition within Generalized Chiral Perturbation Theory¹

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Abstract

We study the temperature evolution of the $\langle \bar{q}q \rangle$ condensate below the chiral phase transition. The hadronic gas is described using a virial expansion within Generalized Chiral Perturbation Theory. In such way, we can implement both the alternative large or small chiral condensate scenarios and analyze the condensate dependence on the values of the lightest quark masses.

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1 Introduction

The properties of QCD at finite temperature have raised considerable interest in the literature (see [1] and references therein). At low temperatures it seems that color is confined and chiral symmetry spontaneously broken. However, from asymptotic freedom, it is expected that at high temperatures, color will be liberated and chiral symmetry restored. It is a matter of intense debate whether there should be one or two phase transitions, at what temperatures they occur, and what would be their nature.

Within the standard wisdom the quark chiral condensate $\langle 0|\bar{q}q|0\rangle$ plays a central role in this problem, since it is assumed that the chiral symmetry breaking is produced by an strong condensation of quark-antiquark pairs [2]. However, in recent years, this hypothesis has been questioned, opening the possibility of small, even vanishing, $\langle 0|\bar{q}q|0\rangle$ scenarios [3, 4, 5]. (Note that we use $\langle 0|\bar{q}q|0\rangle$ for the condensate at $T = 0$ and $\langle \bar{q}q\rangle$ in general).

The evolution of the quark condensate with the temperature has indeed been addressed using several approaches. In general, the properties of $\langle \bar{q}q\rangle$ can be derived from a somewhat idealized dilute pion gas, which is commonly described using an effective lagrangian formalism [6], as we will do here, or by means of finite temperature QCD sum rules [7]. In general, all these and other approaches [8] yield a rather consistent picture, although they usually have the large condensate assumption built in.

In this work we want to know how the actual value of $\langle 0|\bar{q}q|0\rangle$, as well as the light quark masses, can modify the behavior of the chiral condensate, as for instance, with changes in the phase transition temperature. With that purpose, we will describe the pion gas by means of the virial expansion and using the interactions obtained from the Chiral Perturbation Theory (ChPT) formalism [9], although allowing for a wide range of $\langle 0|\bar{q}q|0\rangle$ values. Such a framework is usually referred as Generalized Chiral Perturbation Theory (GChPT) [3, 4]. It should be noticed that we will be dealing with two effects: First, at $T = 0$, the size of $\langle 0|\bar{q}q|0\rangle$ which may be different from the standard large value. Second, the evolution at finite temperature which is also changed through the modifications in the meson interactions due to the different scheme of the explicit chiral symmetry breaking. Our purpose is to study what is the interplay of these two effects.

The plan of the paper is as follows: In section one we describe briefly the ChPT and GChPT formalisms, focusing on the relation between the quark and meson masses with the quark condensate. The next section is devoted to the virial expansion for the pion gas, where we introduce the temperature dependence. In section three we make the study of the condensate dependence both on the temperature and the ratio of light quark masses, using the $\mathcal{O}(p^4)$ amplitudes of GChPT. Next, in section four, we estimate the contributions from heavier states, and in the conclusions we summarize our results.

2 Standard and Generalized Chiral Perturbation Theory

When considering just three massless quark flavors, the QCD Lagrangian exhibits an $SU(3)_L \times SU(3)_R$ symmetry which, even neglecting particle masses, is not present in the physical spectrum. Instead, we observe an approximate $SU_{L+R}(3)$ symmetry, which means that the $SU(3)_{L-R}$ chiral group has to be spontaneously broken. According to the Goldstone Theorem, there should be eight massless Goldstone Bosons (GB), which are identified with the pions, kaons and the eta. In a first approximation, these GB couple to the spontaneously broken currents with strength $F \sim 90$ MeV. These particles are so light compared with the typical hadronic scales, that they will dominate the hadronic dynamics at low energies or temperatures.

In order to describe the hadronic dynamics at low energies we can therefore use these fields to build an effective Lagrangian, made of the most general terms that respect the above symmetry breaking pattern. As we are interested in the low energy regime, the terms are organized according to their number of derivatives. It can be seen, by counting the powers of momenta of different diagrams, that it is possible to renormalize any calculation and obtain finite results order by order in the expansion [10]. We could also couple gauge fields, scalar and pseudoscalar sources, etc..., which would allow us to describe other processes. This whole approach is usually known as Chiral Perturbation Theory (ChPT) [9].

Explicit Chiral Symmetry Breaking

Up to the moment we have just considered the chiral limit. When quark masses are turned on, the GB become massive pseudo-GB and their masses can be obtained, generically, as

$$\begin{aligned}
 M_\pi^2 &\simeq 2B_0\hat{m} + \mathcal{O}(m_q^2) \\
 M_K^2 &\simeq (\hat{m} + m_s)B_0 + \mathcal{O}(m_q^2) \\
 M_\eta^2 &\simeq \frac{2}{3}(\hat{m} + 2m_s)B_0 + \mathcal{O}(m_q^2)
 \end{aligned}
 \tag{1}$$

where $\hat{m} = (m_u + m_d)/2$ (we will consider isospin as an exact symmetry) and the B_0, \dots coefficients are to be determined phenomenologically. Throughout this work, the first one will play a very relevant role, since it has a very physical meaning: In the chiral limit, and up to a normalization factor, it is nothing but the chiral condensate; namely

$$\langle 0|\bar{q}q|0\rangle \equiv \langle 0|\bar{u}u + \bar{d}d|0\rangle \xrightarrow{\hat{m} \rightarrow 0} -2F_0^2 B_0
 \tag{2}$$

At this point two different approaches appear in the literature. The first one, still called ChPT [9], is to assume that the mass expansions in eq.(1) are dominated by the

B_0 term. Its origin can be traced to the Gell-Mann-Okubo (GMO) and Gell-Mann-Oakes-Renner (GOR) formulae, which, within the effective formalism, are obtained at first order by eliminating B_0 in eqs.(1) and (2). This large condensate scenario usually requires $B_0 \sim \mathcal{O}(1\text{GeV})$ and, apart from the GMO and GOR formulae, it is supported by several lattice calculations [11]. Within this framework, the quark masses count as $\mathcal{O}(p^2)$. The second approach, known as Generalized Chiral Perturbation Theory (GChPT) [3, 4], is to consider also the $\mathcal{O}(m_q^2)$ terms, which can be of comparable size or even larger than the B_0 term. As a consequence, both the quark masses and B_0 count as $\mathcal{O}(p)$. This approach is supported by some deviations from the Goldberger-Treiman relation in πN , $K\Lambda$ and $K\Sigma$ [12] and some calculations using variationally improved perturbation theory or a relativistic many body approach [13].

Those two alternatives are usually compared with the spontaneous magnetization \vec{M} of spin systems: On the one hand, ferromagnets present an ordered ground state where the magnetization spontaneously acquires an $\vec{M} \neq 0$ value. That would be analogous to the standard ChPT. On the other hand, in anti-ferromagnets the magnetization remains at $\vec{M} = 0$, which would be similar to the extreme case of GChPT where $B_0 = 0$. Note that, despite their difference, in both systems the spins are oriented in one preferred spatial direction and therefore the $SO(3)$ rotational symmetry is broken.

Back to our subject, it should be noticed that both approaches have the same terms in the Lagrangian, although they are ordered differently, and their relative size is also changed. Indeed, it is possible to reobtain standard ChPT as a special case of GChPT.

At present, the experimental data does not exclude any of the two scenarios, although this question may be solved in a few years with an accurate measurement of $\pi\pi$ scattering lengths from the decay of $\pi^+\pi^-$ atoms [14].

Thus, since we are interested in high temperature differences with the standard scenario, throughout this paper we will use the GChPT formalism. As usual, the pseudo-GB fields are grouped in an $SU(3)$ matrix as follows:

$$U = \exp(i\Phi/F) \quad ; \quad \Phi = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} \quad (3)$$

And then, with the GChPT power counting described above, the $\mathcal{O}(p^2)$ Lagrangian is usually written as

$$\begin{aligned} \tilde{\mathcal{L}}^{(2)} = & \frac{4}{F^2} \left\{ \text{tr}(D_\mu U D^\mu U^\dagger) + 2B_0 \text{tr}(\mathcal{M}(U^\dagger + U)) \right. \\ & + A_0 \text{tr}(\mathcal{M}U^\dagger \mathcal{M}U^\dagger + \mathcal{M}U\mathcal{M}U) + Z_0^S \text{tr}(\mathcal{M}(U + U^\dagger))^2 \\ & \left. + Z_0^P \text{tr}(\mathcal{M}(U - U^\dagger))^2 + 2H_0 \text{tr}(\mathcal{M}^2) \right\} \quad (4) \end{aligned}$$

Where $\mathcal{M} = \text{diag}(\hat{m}, \hat{m}, m_s)$ is the quark mass matrix. In standard ChPT, only the two first terms are $\mathcal{O}(p^2)$, whereas the rest is counted as $\mathcal{O}(p^4)$. From the above

Lagrangian we obtain the following meson masses

$$\begin{aligned}
M_\pi^2 &= 2\hat{m}B_0 + 4\hat{m}^2 A_0 + 4\hat{m}(2\hat{m} + m_s)Z_0^S \\
M_K^2 &= (\hat{m} + m_s)B_0 + (\hat{m} + m_s)^2 A_0 + 2(\hat{m} + m_s)(2\hat{m} + m_s)Z_0^S \\
M_\eta^2 &= \frac{2}{3}(\hat{m} + 2m_s)B_0 + \frac{4}{3}(\hat{m}^2 + 2m_s^2)A_0 \\
&\quad + \frac{4}{3}(\hat{m} + 2m_s)(2\hat{m} + m_s)Z_0^S + \frac{8}{3}(m_s - \hat{m})^2 Z_0^P
\end{aligned} \tag{5}$$

Comparing with eq.(1), we have just added the $\mathcal{O}(m_q^2)$ terms. In the standard formalism, since only B_0 is present, it can be eliminated and one recovers, at $\mathcal{O}(p^2)$, the GMO and GOR relations. That is no longer possible in GChPT, although these relations will be recovered at higher orders. Of the three $\mathcal{O}(m_q^2)$ parameters there are two, Z_S^0 and Z_P^0 , which violate the Zweig rule. They are expected to be small from large N_c arguments and is usual to neglect their contribution, and so we will do in most of what follows.

Note that, since the pion, kaon and eta mass values are known, then, changing the value of B_0 is nothing but changing the values of the quark masses. As a matter of fact, the ChPT relations are frequently used in the literature to obtain ratios of light quark masses (for a recent update, see [15] and references therein) or even to evaluate \hat{m} itself. However, most of these works have used the standard ChPT formalism and have the large condensate assumption built in, so that their results would change if it was removed. Nevertheless, there are determinations of $m_s - m_u$, which do not rely on a large condensate value. For the sake of simplicity, and in order to facilitate the comparison with previous works [4], we will use the value $m_s - m_u = (184 \pm 32)$ MeV, given in [16]. That is,

$$\hat{m} = \frac{184 \pm 32}{r - 1} \text{MeV} \tag{6}$$

(There are other similar analysis in ref.[17], whose results are all consistent with the previous relation.) As a consequence, the parameter that determines the relative size of the $\mathcal{O}(m_q)$ and $\mathcal{O}(m_q^2)$ terms is the quark mass ratio $r = m_s/\hat{m}$, which ranges in the interval

$$r_1 \equiv 2\frac{M_K}{M_\pi} - 1 \leq r \leq 2\frac{M_K^2}{M_\pi^2} - 1 \equiv r_2 \tag{7}$$

The upper limit corresponds to the extreme case of a very large B_0 condensate, whereas the second corresponds to $B_0 = 0$. (Vacuum stability requires $B_0, A_0, Z_0^S \geq 0$).

Of course, all these formulae are valid up to $\mathcal{O}(p^2)$. For the moment, we have restricted ourselves to the $\mathcal{O}(p^2)$ case since it already displays the features of GChPT which are relevant for this work. In section four we will state our results including higher order corrections, although we will just present the GChPT formulae without such a detailed introduction.

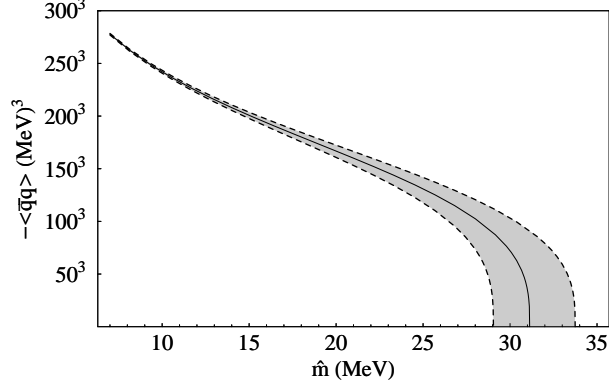


Figure 1: The chiral condensate at zero temperature as a function of the light quark mass \hat{m} , eq.(9). The continuous line corresponds to the central value of Ω . The uncertainty due to its error is covered by the shaded area.

The chiral condensate at zero temperature

Using the GChPT Lagrangian in eq.(4), the chiral condensate at $\mathcal{O}(p^2)$ is then given by

$$\langle 0|\bar{q}q|0\rangle = -2F_0^2(B + \hat{m}(A_0 + H_0) + \dots) \quad (8)$$

where $B = B_0 + 2(m_s + 2\hat{m})Z_0^S$. In practice B_0 cannot be separated from B by looking at quark masses alone, but we have already commented that the Z_0^S is expected to be very small, so that $B \sim B_0$. The parameter H_0 is associated with the contact term of two scalar sources, which does not contain meson fields. However, it is needed as a short distance counterterm, and it indeed depends on the renormalization conventions, which introduce some small ambiguity (see [9] for a discussion). Nevertheless, using QCD sum rules with a simple model for the spectral function, and *keeping fixed* $F_\pi^2 M_\pi^2$ at its physical value, it has been found [18] that the chiral condensate can be described by

$$\hat{m}\langle 0|\bar{q}q|0\rangle \simeq -F_\pi^2 [M_\pi^2 - 4\hat{m}^2\Omega] \quad (9)$$

with $\Omega = 4.7 \pm 0.7$. At $\mathcal{O}(p^2)$, the Ω parameter is nothing but $(A_0 - H_0)/2$. We will use the above equation to estimate the size of the quark condensate $T = 0$.

In Fig.1 we show the dependence of the condensate with \hat{m} , for $\Omega = 4.7$. Note that the plot starts at $\hat{m} = 7$ MeV, which is approximately the standard ChPT value. There, $\langle 0|\bar{q}q|0\rangle \simeq -(280)^3 \text{MeV}^3$, and it decreases smoothly as \hat{m} gets larger, until it vanishes around $\hat{m} \simeq 30$. The shaded area between dashed lines cover the uncertainty in Ω .

As a check of eq.(9) we can see that it is consistent with previous estimates within the framework of standard ChPT [9], where

$$\hat{m}\langle 0|\bar{q}q|0\rangle = -F_0^2 M_\pi^2 \left[1 + \frac{M_\pi^2}{32\pi^2 F_\pi^2} (4\bar{h}_1 + \bar{l}_3 - 1) \right] \equiv -F_0^2 M_\pi^2 \frac{1}{c} \quad (10)$$

and the \bar{h}_1 and \bar{l}_3 parameters play a similar role as that of the GChPT $\mathcal{O}(p^2)$ parameters, although in Standard ChPT they appear at $\mathcal{O}(p^4)$. In that case, estimates based on a simple ρ resonance model and the large N_c limit yield $c = 0.87$ and $c = 0.90 \pm 0.05$, respectively [6]. If we introduce in eq.(9) a value in the range \hat{m} from 5 to 10 MeV, and taking into account the fact that in standard ChPT $F_\pi/F_0 \simeq 1.057$, we obtain $c = 0.95 \pm 0.4$, which is a highly non-trivial check of eq.(9). (Throughout this section we have neglected higher order logarithmic contributions that would yield corrections of the order of 1%)

3 The virial expansion and temperature effects.

At low energies the free energy z is dominated by the contributions from the lightest particles. Therefore, we can use the Euclidean form of the above Lagrangian (denoted $L(x)$) within the standard finite temperature functional Euclidean formalism. Hence, in the thermodynamic limit,

$$z = -T \lim_{L \rightarrow \infty} \frac{1}{L^3} \int [dU] \exp \left(- \int_{L^3 \times [0, T]} d^4x L(x) \right) \quad (11)$$

where, as usual, the functional integration is over pion fields which are periodic in the Euclidean time, with period $\beta = 1/T$ (see ref.[6]). From the free energy we can derive any other thermodynamical property of our system, but let us first notice that since there is an spontaneously broken symmetry, even at $T = 0$ there is some non-vanishing vacuum energy density ϵ_0 . As a consequence, the pressure is defined only from the temperature dependent part of the free energy, $P \equiv \epsilon_0 - z$.

The quark condensate is now obtained as the derivative of the free energy with respect to the quark mass. That is

$$\langle \bar{q}q \rangle \equiv \frac{\partial z}{\partial \hat{m}} = \langle 0 | \bar{q}q | 0 \rangle - \frac{\partial P}{\partial \hat{m}} \quad (12)$$

where we have used that at $T = 0$ the condensate is nothing but the vacuum expectation value $\langle 0 | \bar{q}q | 0 \rangle \equiv \partial \epsilon_0 / \partial \hat{m}$.

In this section we will just concentrate on how to obtain $\partial P / \partial \hat{m}$. For that purpose, one possibility is to calculate the free energy from the effective Lagrangian, as it was done in [6] within standard ChPT. That method follows the very same philosophy of the chiral expansion, but is rather lengthy. In this paper we will make use of existing one loop calculations of elastic $\pi\pi$ scattering, together with the relativistic virial expansion of a pion gas [19, 20].

Let us then consider a gas made only of pions. This approximation seems reasonable as long as the temperatures remain sufficiently below the kaon threshold [6]. Within the virial formalism, the pressure can be expanded as follows

$$P = 3T \left(\frac{M_\pi T}{2\pi} \right)^{3/2} \sum_{k=1}^{\infty} B_k e^{-\beta M_\pi k} = 3 \frac{T}{\lambda^3} \sum_{k=1}^{\infty} B_k(T) \xi^k \quad (13)$$

The factors of three come from the fact that we are neglecting isospin breaking effects. Thus, there are effectively three different species of particles, labeled according to their third isospin component, that behave identically with respect to strong forces. We have also defined $\lambda = (2\pi/M_\pi T)^{1/2}$, which is the thermal pion wavelength. Note that the expansion parameter is the fugacity $\xi = e^{-\beta M_\pi}$. In a non-relativistic framework, the expansion is usually performed using the definition $\xi = e^{\beta\mu}$, where μ would be the chemical potential. In contrast, in the relativistic case, there is a rest energy given by M_π , whose contribution to eq.(11) is equivalent to a chemical potential $\mu = -M_\pi$ in a non-relativistic description.

There is a closed expression for the virial coefficients of the free gas, which is

$$B_n^{(0)}(T) = \frac{3}{n(M_\pi T)^{3/2}} \sqrt{\frac{2}{\pi}} \int_0^\infty dp p^2 e^{-n\beta(E(p)-M_\pi)} \quad (14)$$

In order to deal with the interacting gas, we will just consider two particle interactions, which can be justified as long as the density remains small. In [6] it was shown that this is consistent with the three loop calculation in ChPT. In such case, it is enough to keep the two first terms of the virial expansion, whose coefficients will be given by [19]

$$\begin{aligned} B_1(T) &= B_1^{(0)}(T) \\ B_2(T) &= B_2^{(0)}(T) + \frac{4e^{2M_\pi/T}}{(2\pi M_\pi T)^{3/2}} \int_{2M_\pi}^\infty dE E^2 K_1(E/T) \left(\sum_{I,J} (2I+1)(2J+1) \delta_{IJ}(E) \right) \end{aligned} \quad (15)$$

where $K_1(x)$ is the modified Bessel function which behaves as $\sqrt{\pi/2x} e^{-x}$ when $x \rightarrow \infty$. It is important to notice that the only dynamical information we need are the phase shifts δ_{IJ} . As an estimate of the applicability of this approach, it was shown in [20] that the second order virial expansion yields less than a 1% error when applied to the free gas up to $T \sim 250$ MeV.

Let us finally remark that the high temperature behavior of the chiral condensate will be then due to two different effects. First, the starting $T = 0$ value, which may differ from the standard, large condensate, value. But, second, it also depends on how the mass dependence of the phase shifts has changed with respect to standard ChPT.

In the next section the phase shifts will be obtained using the existing GChPT calculations of the $\pi\pi$ elastic scattering amplitudes [4]. In later sections we will include contributions from particles more massive than pions.

4 The General Scenario

Higher orders in GChPT

Within the standard ChPT it was shown in [6] that the $\mathcal{O}(p^4)$ contributions accelerate the melting of the chiral condensate, lowering the critical temperature. Our aim now is to include the equivalent corrections within GChPT. Unfortunately, we have already seen that the $\tilde{\mathcal{L}}^{(2)}$ Lagrangian has more terms than the standard $\mathcal{L}^{(2)}$. That means that there are many more phenomenological parameters in the Lagrangian, which in many cases are not very well known. The situation gets even worse at higher orders. In general, the GChPT Lagrangian is built of terms like [3, 4]

$$\tilde{\mathcal{L}}^{(d)} = \sum_{k+l+n} B_0^n \mathcal{L}_{(k,l)}, \quad \text{with} \quad \mathcal{L}_{(k,l)} \sim \mathcal{O}(p^k m_q^l) \quad (16)$$

Indeed, we have already given $\tilde{\mathcal{L}}^{(2)}$ in eq.(4) and we found that some of the constants are not very well determined. For the complete expression of the $\mathcal{O}(p^4)$ Lagrangian we refer to [4]. For our purposes, there are several relevant modifications to our previous discussion: First, the fact that now there are three different decay constants F_π , F_K and F_η . Second, that, neglecting Zweig rule violating parameters, the expressions for M_π and M_K in eq.(5) are now modified to

$$\begin{aligned} \frac{F_\pi^2}{F^2} M_\pi^2 &= 2\hat{m}B_0 + 4\hat{m}^2 A_0 + \frac{F_\pi^2}{F^2} \delta M_\pi^2 \\ \frac{F_K^2}{F^2} M_K^2 &= (\hat{m} + m_s)B_0 + (\hat{m} + m_s)^2 A_0 + \frac{F_K^2}{F^2} \delta M_K^2 \end{aligned} \quad (17)$$

where δM_i^2 are higher order corrections and logarithmic terms [4], whose size is $\delta M_i^2 < 0.1 M_i^2$ [21] (see below). As a consequence, the range of allowed r values is shifted upwards to

$$r_1^* \equiv 2 \frac{F_K M_K}{F_\pi M_\pi} - 1 \leq r \leq 2 \frac{(F_K M_K)^2}{(F_\pi M_\pi)^2} - 1 \equiv r_2^* \quad (18)$$

With these modifications now $r_1^* \simeq 8$ and r_2^* can be as large as 39.

Finally, there are also higher order corrections to the $T = 0$ condensate itself, which contain chiral logarithms. That means that we cannot simply say that $\Omega = (A_0 + H_0)/2$. Nevertheless we can still use the phenomenological parameter $\Omega = 4.7 \pm 0.7$ in eq.(9).

The one loop $\pi\pi$ amplitude in GChPT

Next, we need the $\pi^+\pi^- \rightarrow \pi^0\pi^0$ scattering amplitude itself. Although it has been calculated in GChPT up to two loops [3, 4], for our purposes it will be more than enough to consider the one loop result, which reads:

$$A(s, t, u) = \frac{\alpha}{3F_\pi^2} M_\pi^2 + \frac{\beta}{F_\pi^2} \left(s - \frac{4}{3} M_\pi^2 \right) \quad (19)$$

$$+ \frac{\lambda_1}{F_\pi^4} (s - 2M_\pi^2)^2 + \frac{\lambda_1}{F_\pi^4} [(t - 2M_\pi^2)^2 + (u - 2M_\pi^2)^2] + \bar{J}_{(\alpha,\beta)}(s, t, u)$$

where

$$\begin{aligned} \bar{J}_{(\alpha,\beta)}(s, t, u) = & \frac{1}{6F_\pi^4} \left\{ 4 \left[\frac{5}{6} \alpha M_\pi^2 + \beta \left(s - \frac{4}{3} M_\pi^2 \right) \right]^2 - \left[\frac{2}{3} \alpha M_\pi^2 - \beta \left(s - \frac{4}{3} M_\pi^2 \right) \right]^2 \right\} \bar{J}(s) \\ & + \frac{1}{12F_\pi^4} \left\{ 3 \left[\frac{2}{3} \alpha M_\pi^2 - \beta \left(t - \frac{4}{3} M_\pi^2 \right) \right]^2 + \beta^2 (s - u)(t - 4M_\pi^2) \right\} \bar{J}(t) \\ & + \frac{1}{12F_\pi^4} \left\{ 3 \left[\frac{2}{3} \alpha M_\pi^2 + \beta \left(u - \frac{4}{3} M_\pi^2 \right) \right]^2 + \beta^2 (s - t)(u - 4M_\pi^2) \right\} \bar{J}(u) \end{aligned} \quad (20)$$

and \bar{J} is the standard one-loop integral [9].

In the literature, the values of the α , β , λ_1 and λ_2 are fitted from experiment. However, in order to obtain the condensate dependence with the temperature, we need the derivative of the pressure with respect to M_π , and just a fitted value is not enough. Therefore, we also need to know the M_π dependence of the parameters, and, if we want to study the effects of changing the light quark masses, we also need the dependence on r .

Phenomenological parameters

The actual expressions of the α , β , parameters are rather complicated and involve many parameters from the GChPT Lagrangian, which frequently are not very well determined. In addition they contain chiral logarithms. It is therefore very convenient to expand α and β in powers of quark masses, namely

$$\alpha = \sum_{n=0}^3 \alpha^{(n)}, \quad \beta = \sum_{n=0}^3 \beta^{(n)}, \quad (21)$$

Notice that, in GChPT, since quark masses count as $\mathcal{O}(p)$, these expansions involve not only even, but also *odd* powers of momenta.

The above expansions have been worked out in [4], and they are the following:

$$\begin{aligned} \alpha(r) = & 1 + 6 \frac{r_2^* - r}{r^2 - 1} - \frac{4}{r - 1} \left(\frac{F_K^2}{F_\pi^2} - 1 \right) + 18(2 - r) \hat{\rho}_1 - 6r \hat{\rho}_2 + \alpha^{(2)}(r) \\ \beta(r) = & 1 + \frac{2}{r - 1} \left(\frac{F_K^2}{F_\pi^2} - 1 \right) + \beta^{(2)}(r) \end{aligned} \quad (22)$$

where in all the above equations we have neglected the Zweig rule violating parameters.

Let us now try to estimate the size of the different terms in the α and β expansions. Let us then look back at the allowed values of r , eq.(18). The relevant point for

our discussion is that now, even with the lowest value $r = 8$, we obtain, using eq.(6), that $\hat{m} \leq (26 \pm 4.6)$ MeV. Therefore we can estimate that the terms coming from B_0 and A_0 should be $\mathcal{O}(1)$, those from $\tilde{\mathcal{L}}^{(3)}$ should be $\mathcal{O}(10\%)$ and those from $\mathcal{L}_{(2,2)}$ and $\mathcal{L}_{(0,4)}$ should at most reach the 1% level. Consequently, we will neglect the $\alpha^{(2)}$ and $\beta^{(2)}$ effects. The only parameters that remain undetermined are the $\hat{\rho}_{1,2}$, which contribute to $\alpha^{(1)}$. However, from a dimensional analysis [4], their magnitude can be naively estimated as $|\hat{\rho}_i| \simeq (0.4 \pm 0.2)/(r-1)^3$. Their dependence on the actual value of M_π (needed for the numerical derivation) seems very weak. In our calculations we will take them first as zero and then we will include them in the uncertainty.

Concerning λ_1 and λ_2 , they come only from the terms in $\mathcal{L}_{(4,0)}$, which do not contain explicit chiral symmetry breaking. They are given by

$$\begin{aligned}\lambda_1 = \lambda_1^{(0)} &= 4(2L_1^r(\mu) + L_3) - \frac{1}{48\pi^2} \left\{ \log \frac{M_\pi^2}{\mu^2} + \frac{1}{8} \log \frac{M_K^2}{\mu^2} + \frac{35}{24} \right\} \\ \lambda_2 = \lambda_2^{(0)} &= 4L_2^r(\mu) - \frac{1}{48\pi^2} \left\{ \log \frac{M_\pi^2}{\mu^2} + \frac{1}{8} \log \frac{M_K^2}{\mu^2} + \frac{23}{24} \right\}\end{aligned}\quad (23)$$

It can be seen that these parameters do not carry any r dependence. For definiteness, we will use for them the values obtained in [4]:

$$\lambda_1 = (-5.3 \pm 2.5)10^{-3}; \quad \lambda_2 = (9.7 \pm 1.0)10^{-3}, \quad (24)$$

which are consistent with other determinations in the standard framework.

The values of α and β depend on whether there is actually a large or small condensate at $T = 0$, and we will use their r dependence to reproduce different scenarios. For illustrative purposes, let us recall that in the standard formalism both α and β are very slightly bigger than one and $r \simeq 26$. In contrast, the low condensate alternative seems to prefer $\alpha \simeq 2$ and $r \simeq 10$ [4].

Phase shifts

For the virial expansion we need the phase shifts of definite isospin and angular momentum channels. At lowest order, they are defined as follows (see [22] for a discussion on this subject)

$$\tan \delta_{IJ}(s) = \sigma(s) \text{Re}(t_{IJ}(s)), \quad (25)$$

where $\sigma(s) = \sqrt{1 - 4M_\pi^2/s}$. The partial waves $t_{I,J}$ are obtained from the isospin amplitudes

$$\begin{aligned}T_0(s, t, u) &= 3A(s, t, u) + A(t, s, u) + A(u, t, s), \\ T_1(s, t, u) &= A(t, s, u) - A(u, t, s), \\ T_2(s, t, u) &= A(t, s, u) + A(u, t, s),\end{aligned}\quad (26)$$

by means of

$$t_{IJ} = \frac{1}{64\pi} \int_{-1}^1 d(\cos \theta) P_J(\cos \theta) T_I(s, t, u). \quad (27)$$

where P_I is the corresponding Lagrange polynomial. In our calculations we have just used the lowest angular momentum for each isospin channel, namely $(I, J) = (0, 0), (1, 1)$ and $(2, 0)$. For all means and purposes, they dominate the low energy pion interactions.

The calculation of $\partial P/\partial \hat{m}$

We have then used the above phase shifts with the second order virial expansion. In order to obtain the condensate, eq.(12), we then need $\partial P/\partial \hat{m}$, which can be obtained using

$$\frac{\partial P}{\partial \hat{m}} = \frac{\partial P}{\partial M_\pi} \frac{\partial M_\pi}{\partial \hat{m}} + \frac{\partial P}{\partial M_K} \frac{\partial M_K}{\partial \hat{m}} + \frac{\partial P}{\partial F_\pi} \frac{\partial F_\pi}{\partial \hat{m}} + \frac{\partial P}{\partial F_K} \frac{\partial F_K}{\partial \hat{m}} \quad (28)$$

Naively one just expects the first term, but let us remember that M_K, F_π and F_K are \hat{m} -dependent and they appear in the amplitude either directly or indirectly through α, β, λ_1 and λ_2 . That problem was carefully avoided in [6] by using $SU(2)$ standard ChPT and only using F in the free energy expansion.

Of course, only M_π appears in the fugacity, or in the free gas virial coefficients and thus we expect the three last terms in eq.(28) to be much smaller than the first. Indeed, within the range of r and T that we are interested in, we have found that the term due to the appearance of M_K in the amplitude is smaller than 1% and we have neglected it. In contrast, F_π and F_K together generate contributions of the order of 5%, and therefore they have been included in our calculations.

The derivative of the pressure with respect to M_π, F_π and F_K have been performed numerically, with an increment of 0.1MeV. For instance, the value of the pressure is first calculated with the real M_π and then with $M_\pi - \delta M_\pi$, *including a change in the chiral parameters*, following eqs.(22) and (23). A similar procedure is followed for F_π and F_K .

In our calculations we have used

$$\begin{aligned} \frac{\partial M_\pi}{\partial \hat{m}} &\simeq \frac{M_\pi}{2\hat{m}} \left[1 + 2 \frac{r_2^* - r}{r^2 - 1} \right] \\ \frac{\partial M_K}{\partial \hat{m}} &\simeq \frac{M_\pi^2}{4\hat{m}M_K} \frac{r(2r_2^* - r) - 1}{r^2 - 1} \\ \frac{\partial F_\pi}{\partial \hat{m}} &\simeq \frac{F_\pi}{\hat{m}[(r-1) + (F_K^2/F_\pi^2 - 1)]} \left(\frac{F_K^2}{F_\pi^2} - 1 \right) \\ \frac{\partial F_\pi}{\partial \hat{m}} &\simeq \frac{r-1}{2} \frac{\partial F_\pi}{\partial \hat{m}} \end{aligned} \quad (29)$$

which we have obtained from eqs.(17) and from [4]. There are, of course, corrections, but their effects on the final results are again less than 1%.

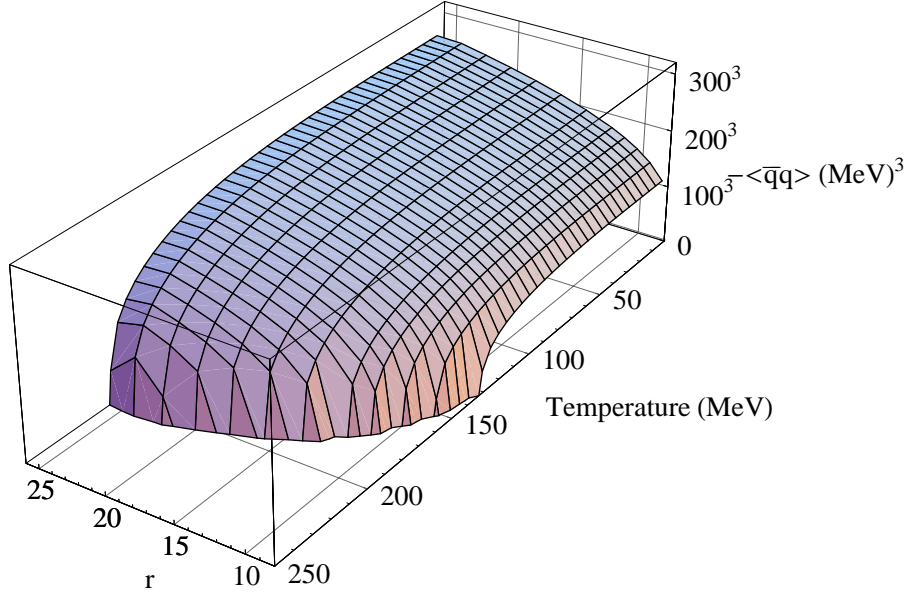


Figure 2: Quark condensate versus temperature and r for $\Omega = 4.7$.

Results

As we have already commented, the virial expansion can be trusted only at low temperatures, mostly, due to the fact that above ~ 150 MeV the contributions from other more massive particles becomes relevant. These effects will be studied in the next section and we will see that they tend to lower the critical temperature, which is therefore more favorable for our approach. For the moment, if we give in our figures results for higher temperatures, they should be interpreted with great care, as a qualitative behavior or, for instance, as a tendency towards symmetry restoration. Nevertheless, comparing between different figures could also illustrate what is the qualitative effect of a change in the parameters.

Thus, in Fig.2 we have shown the dependence of the chiral condensate with the temperature and for a light quark ratio in the range $8 \leq r \leq 26$. For Ω we have used the central value 4.7. Although the actual points at which $\langle \bar{q}q \rangle = 0$ are just gross estimates, we can see that lowering r yields a systematic decrease in the chiral phase transition temperature.

Indeed, T_C seems to be above 200 MeV for $r \geq 20$ going down to 130 MeV around $r = 8$. Note that for the latter temperature our approximations can become quite reliable. As we have already seen, smaller values of r are forbidden to ensure vacuum stability.

The previous results have been obtained using the central values of all parameters. In Fig.3a we show what happens if we take into account the uncertainty in Ω . Thus,

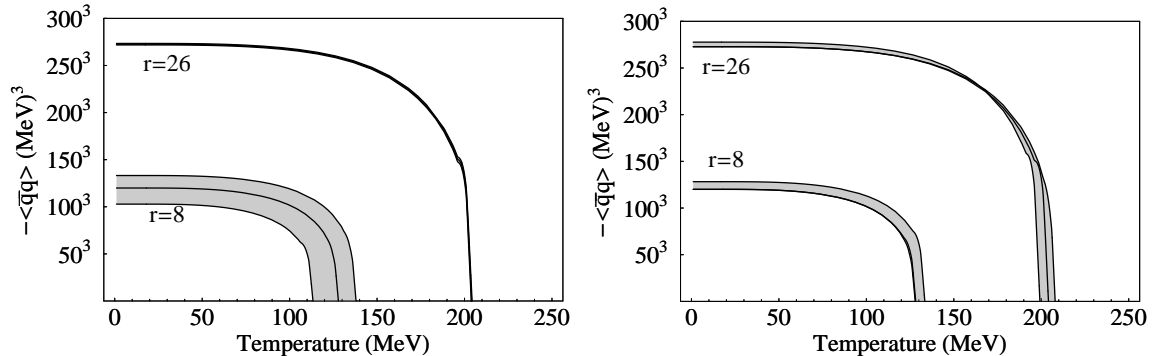


Figure 3: a) Estimate of the errors in the $\langle\bar{q}q\rangle$ evolution due to the uncertainty in Ω . b) Uncertainties in the parameters that appear in the $\pi\pi$ phase shifts, added linearly.

we are plotting the temperature dependence for the two extreme cases, $r = 26$ and $r = 8$. The former, which corresponds to the upper curve, is almost insensible to this variation. It corresponds to the standard formalism, where the value of the chiral condensate is largely dominated by the $\mathcal{O}(m)$ term and, consistently, changes in the other terms are almost negligible. On the other curve, which is associated to the lowest condensate scenario, the effect of this error is translated in a 10 to 15% change on T_C , at most.

In Fig. 3b we show the uncertainties associated to all the parameters that appear in the scattering amplitude. In the shaded areas, we have taken into account all the effects of changing λ_1 , λ_2 , $\hat{\rho}_1$ and $\hat{\rho}_1$. In addition we have also let the pion and kaon masses vary between their values for the neutral or scalar particle. Note that in the case of the pion mass, such a change also affects the coefficients of the virial expansion and the fugacity. Finally, we have included the uncertainty in $F_K/F_\pi = 1.22 \pm 0.01$ and we have let F_π change between 92.4 and 93.2 MeV which are two values currently cited in the literature. Both M_π and F_π do also appear in the expression of the $T = 0$ chiral condensate. All in all, the overall uncertainty in T_C due to these parameters seems to be of the order of ± 5 MeV at $r = 26$ and ± 3 MeV at $r = 8$. Since we have just simply *added* the different errors, we consider these numbers as a conservative estimate.

5 Other massive particles

In this section we will consider the effect of adding heavier particles to our pion gas. We will be following closely the approach of Gerber and Leutwyler [6] with slight modifications to implement also the low condensate scenario.

The density of massive states should be exponentially suppressed by Boltzmann factors $\exp(-M_i/T)$, which means that their two body interactions will carry an $\exp[-(M_i + M_j)/T]$ factor. In addition, their interactions with pions are also sup-

pressed by T^2/F^2 , due to the chiral symmetry. Hence, we can treat those heavier particles in the free gas approximation. In such case, we have an additional contribution to the pressure, which is given by

$$\Delta P = - \sum_i \frac{g_i T}{2\pi^2} \int_0^\infty dp p^2 \log \left[1 - e^{-\sqrt{p^2 + M_i^2}/T} \right] \quad (30)$$

where g_i is the state degeneracy of a state with mass M_i (that was the factor of 3 in eq(13)). Note that, since we will be dealing with temperatures much smaller than the first hadronic fermions, it makes sense to use just Bose statistics. The above formula is only meaningful at low temperatures, since as we increase the temperature, the mean distance between massive states shrinks and the dilute gas approximation is no longer valid. In ref.[9] it was estimated that the model is valid up temperatures of the order of 150 MeV, although it “rapidly deteriorates” for higher temperatures.

Back to the condensate, and in view of eq.(12), the new contributions are of the form

$$\Delta \langle \bar{q}q \rangle = - \sum_i \frac{\partial M_i}{\partial \hat{m}} \frac{\partial \Delta P}{\partial M_i} \quad (31)$$

and therefore

$$\Delta \langle \bar{q}q \rangle = \frac{1}{2\pi^2} \sum_i g_i M_i \frac{\partial M_i}{\partial \hat{m}} \int_0^\infty dp \frac{p^2}{\sqrt{p^2 + M_i^2}} \frac{1}{e^{\sqrt{p^2 + M_i^2}/T} - 1} \quad (32)$$

Thus, we only have to estimate the value of $\partial M_i/\partial \hat{m}$. Naively, one would expect that the contribution \hat{m} to a hadron mass would be roughly proportional to the number N_i of u and d quarks it contains. That estimate seemed quite appropriate in the standard framework [6]. We now have to check that it is also the case in GChPT.

Let us then go back to eqs.(5), since to get rough estimates it is enough to work at $\mathcal{O}(p^2)$. As usual, we neglect the Z_0^S and Z_0^P parameters. Then, we obtain the following derivatives

$$\begin{aligned} \frac{\partial M_K}{\partial \hat{m}} &\simeq \frac{M_\pi^2}{4\hat{m}M_K} \frac{r(2r_2 - r) - 1}{r^2 - 1} \\ \frac{\partial M_\eta}{\partial \hat{m}} &\simeq \frac{M_\pi^2}{6\hat{m}M_\eta} \left[1 + 2\frac{r_2 - r}{r^2 - 1} \right] \end{aligned} \quad (33)$$

We can reproduce the standard scenario with $r = 26$, which yields $\hat{m} \simeq 7.4 \pm 1.3$ MeV using eq.(6). In such case, we find $\partial M_K/\partial \hat{m} \simeq 1.3 \pm 0.2$, which is in very good agreement with a rough estimate of 1. We also find $\partial M_\eta/\partial \hat{m} \simeq 0.8 \pm 0.1$, again consistent with the naive estimate of 2/3. In any case, it seems that $\partial M_i/\partial \hat{m} = N_i$ is a small underestimation of the actual values of the standard scenario, as it was already pointed out in [6], where they considered that the range from N_i to $2N_i$ was a “fair representation” of the uncertainty in $\partial M_i/\partial \hat{m} = N_i$.

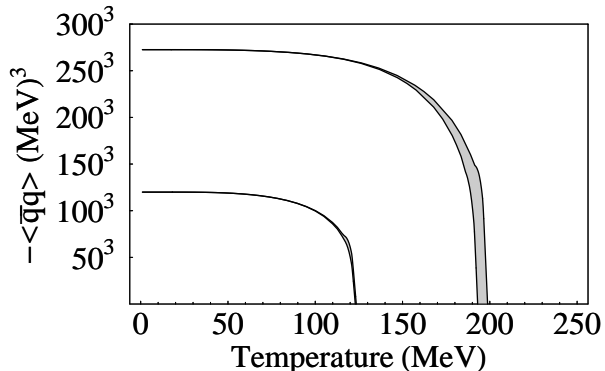


Figure 4: The evolution of the chiral condensate when we include corrections from a free gas of particles more massive than the pion. The shadowed regions cover the uncertainties in $\partial M_P/\partial\hat{m}$ described in the text. These contributions always tend to lower the critical temperature.

However, if we set $r = 8$, which is the lowest allowed $T = 0$ condensate, we find $\partial M_K/\partial\hat{m} \simeq 2.0 \pm 0.4$ and $\partial M_\eta/\partial\hat{m} \simeq 0.36 \pm 0.06$. Again, the order of magnitude is correct, although within a factor of 2, *higher or lower*. We will therefore use the estimates in eq.(33) for the kaon and the eta, instead of $\partial M_i/\partial\hat{m} = N_i$. Those are the states that will contribute more at low temperatures. For the rest, we will assume the uncertainty in $\partial M_i/\partial\hat{m} = N_i$ to be from $N_i/2$ to $2N_i$.

Thus, in Fig.4 we show the results when the massive states are taken into account. We have considered in eq.(31) all particles containing u and d quarks up to 1300 MeV and we are taking the central values of all the other parameters. The dominant contributions are, of course, those of the kaons, the eta, the rho and the omega. The shaded areas cover the uncertainty in $\partial M_i/\partial\hat{m}$ that we have just described. Obviously, the net effect is biggest for the standard scenario, since the critical temperature is higher, where T_C is decreased down to 190 to 200 MeV. This result, although it has been obtained within the generalized formalism, reproduces very nicely the standard ChPT estimate given in [6].

Indeed, the r dependence is given in Fig.5 where we plot the evolution of the chiral condensate both with T and r , for the central values of all the parameters, but also including the contributions from massive states. Note that, for the extreme case when $r = 8$, the decrease is of the order of 5 MeV, down to around 125 MeV.

6 Conclusions

In this work we have studied the generalized scenario of chiral symmetry breaking, either with a large or a small $T = 0$ condensate. For that purpose we have described a pion gas by means of the virial expansion, whose coefficients have been calculated using the amplitudes obtained within $\mathcal{O}(p^4)$ Generalized Chiral Perturbation Theory.

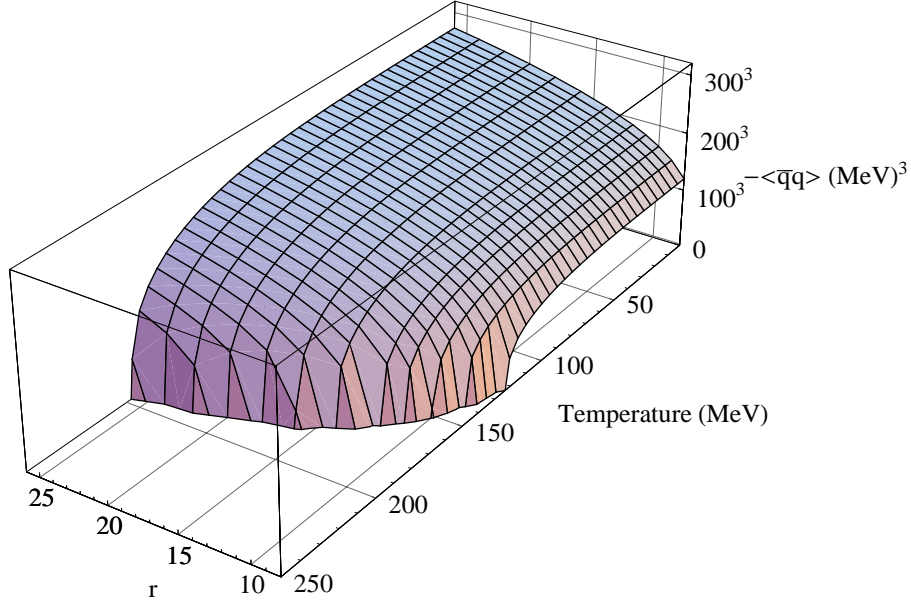


Figure 5: The evolution of the chiral condensate when we include corrections in the pion gas from heavier states, as a function of the temperature and the quark mass ratio r . (Using the central values of all parameters and estimates)

We have also added a crude estimate of contributions from particles heavier than the pion, in a free gas approximation, which can be justified at low temperatures. The effects of these particles is always to *decrease* the temperature of chiral restoration. Their net effect is to lower T_C by 10 to 20 MeV in the standard scenario, and by around 5 MeV when the $T = 0$ chiral condensate is smallest.

From our results, it seems that the chiral phase transition in a pure pionic gas may occur at energies as low as 125 MeV in the lowest possible $T = 0$ condensate scenario. The main source of uncertainty is the fact that within the Generalized approach many parameters still remain undetermined. In the worst case, which again corresponds to the lowest condensate and lowest T_C , it can be estimated at about 20%. For the standard case of a large condensate, we recover previous estimates of $T_C \simeq 190$ MeV.

In conclusion, we have found that the value of ~ 190 MeV for the critical temperature obtained from standard Chiral Perturbation theory can be seen as an upper bound if we were to include $\mathcal{O}(m_q^2)$ corrections in the mass terms, in addition to the standard condensate contribution. The effects of this corrections is always to lower the critical temperature, which, all together, could be as low as 125 with a 20% uncertainty, for the lowest condensate scenario.

Acknowledgments

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