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RADIATION DAMPING AND QUANTUM EXCITATION IN A FOCUSING-DOMINATED STORAGE RING*

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Abstract

In this paper we calculate the effects of a linearly varying focusing field on radiation damping and quantum excitation to the transverse emittances in an electron storage ring by using a quantum mechanical perturbation approach. This method allows for arbitrarily strong focusing environment and correctly predicts the limits of both pure bending and pure focusing. We find that transverse excitation can be exponentially suppressed by the focusing field when the radiation formation length is comparable to the transverse oscillation wavelengths. Applications to the design of a focusing-dominated damping ring is also explored.

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In this paper we calculate the effects of a linearly varying focusing field on radiation damping and quantum excitation to the transverse emittances in an electron storage ring by using a quantum mechanical perturbation approach. This method allows for arbitrarily strong focusing environment and correctly predicts the limits of both pure bending and pure focusing. We find that transverse excitation can be exponentially suppressed by the focusing field when the radiation formation length is comparable to the transverse oscillation wavelengths. Applications to the design of a focusing-dominated damping ring is also explored.

1 Introduction

In an electron storage ring, synchrotron radiation created by bending magnets gives rise to the radiation damping of the beam emittances in all three degrees of freedom [1]. It is well known [1,2,3] that the damping effects are counteracted by quantum excitation due to random photon emissions, which leads to equilibrium emittances when the damping and the excitation rates balance. Electron storage rings routinely obtain such equilibrium emittances, and the art of lattice design in modern synchrotron radiation sources or damping rings is to minimize these emittances under various constraints.

On the other hand, Huang, Chen and Ruth [4] have shown that in a straight, continuous focusing channel, the transverse damping rate is independent of the particle energy, and that no quantum excitation is induced. In fact, the final normalized transverse emittance in an ideal focusing system is limited only by the uncertainty principle and is equal to one half of the Compton wavelength of the electron, which is much smaller than the equilibrium transverse (horizontal or vertical) emittance achieved in a normal damping ring.

Therefore, the radiation reaction in a focusing system is very different from that in a bending magnet. Although the transverse focusing quadrupoles are present in a storage ring to confine the beam, and they can modify the individual radiation damping rates by coupling with the bending fields in a combined-function system [3], their contributions to the overall radiation effects are usually negligible compared to the bending dipoles. The length associated with a typical photon emission (the radiation formation length) is on the order of $\rho/\gamma^{2,3}$, where ρ is the bending radius and γ is the electron energy in units of its rest energy $m_e c^2$. The standard treatment of quantum excitation can be quasiclassical because the radiation formation length is much shorter than the transverse oscillation wavelength. Thus, one can model the radiation to be instantaneous with a continuous spectrum of frequencies and treat the quantum nature of radiation as fluctuations about the average rate [1,3]. Sokolov and Ternov [2] analyzed radiation damping and quantum excitation using a rigorous quantum mechanical approach for a weak focusing synchrotron. The results agree with those of Robinson and Sands [1,3] and confirm the quasiclassical picture of quantum excitation.

However, as the strength of the transverse focusing increases or as the bending field gradually decreases, the radiation formation length and the transverse oscillation wavelengths may become comparable. The radiation in this case can not be regarded as instantaneous. Thus, it is desirable to have a general treatment of radiation effects in a storage ring with arbitrarily strong bending and focusing present. In a recent paper [6], we extend the quantum mechanical perturbation analysis [4] to include the bending case and show that quantum excitation to the horizontal emittance can be suppressed by a strong focusing environment. Both the pure bending and the pure focusing are two limiting cases of the general result. In this paper we present more detailed perturbation calculation, including the effects of focusing on the vertical emittances. We then discuss the longitudinal issues and consider some preliminary parameters for a focusing-dominated damping ring that might be useful for ultra-low emittance generation.

2 Suppression of Transverse Excitation

2.1 The Continuous Focusing Model

We consider here a simple model of storage rings with a continuous, linear focusing field around a circular electron orbit provided by a uniform magnetic field. The model for the focusing field used below is electrostatic in origin such as that created by a dilute cloud of positive ions. The more realistic magnetic focusing field will be discussed later. Suppose that a reference electron with momentum p_0 has a circular trajectory with radius ρ, the three components of the vector potential $\mathbf{A}$ for the uniform bending field in the familiar curvilinear coordinates system (x, s, y) are given by [5]:

$$A_x = A_y = 0,$$
$$A_s \equiv (\mathbf{A} \cdot \hat{\mathbf{s}}) \left(1 + \frac{x}{\rho}\right) = -\frac{p_0}{e}\left(\frac{x}{\rho} + \frac{x^2}{2\rho^2}\right). \quad (1)$$

We assume the constant focusing strengths K_x in the horizontal x direction and K_y in the vertical y direction, the total energy of the electron can be decomposed as

$$E = \sqrt{m_e^2c^4 + p_x^2c^2 + p_y^2c^2 + \frac{(p_s - eA_s)^2\, c^2}{(1 + x/\rho)^2}} + \frac{1}{2}K_x x^2 + \frac{1}{2}K_y y^2$$
$$\simeq E_s + \frac{p_x^2c^2}{2E_s} + \frac{1}{2}K_x'\,(x - x_\epsilon)^2 - \frac{1}{2}K_x' x_\epsilon^2 + \frac{p_y^2c^2}{2E_s} + \frac{1}{2}K_y y^2 \qquad (2)$$

where

$$E_s = \sqrt{m_e^2c^4 + p_s^2c^2},$$
$$K_x' = K_x + \frac{[p_0^2c^2 + 3(p_s - p_0)p_0c^2]}{(E_s\rho^2)} \simeq K_x,$$
$$x_\epsilon = \frac{(p_s - p_0)c}{(K_x\rho)} \quad \text{(equilibrium orbit)}$$
$$\omega_x = \sqrt{\frac{K_x c^2}{E_s}} \quad \text{and} \quad \omega_y = \sqrt{\frac{K_y c^2}{E_s}} \qquad (3)$$

are all functions of p_s. Thus, both transverse motions are harmonic oscillations that are coupled with the longitudinal momentum.

Since we do not care about the spin degree of freedom, we can use the Klein-Gordon equation to obtain the eigenenergies of the electron in this system [6]

$$E(n_x, n_y, p_s) = E_s + \hbar\omega_x\left(n_x + \frac{1}{2}\right) - \frac{1}{2}K_x x_\epsilon^2 + \hbar\omega_y\left(n_y + \frac{1}{2}\right). \qquad (4)$$

If we normalize the probability density of the Klein-Gordon wavefunctions to one (instead of γ) [7], then the eigenstates are found to be

$$\Psi_{n_x,n_y,p_s}(\mathbf{r}) = \sqrt{\frac{m_ec^2}{E}}\psi_{n_x,n_y,p_s}(\mathbf{r}), \quad \text{with}$$
$$\psi_{n_x,n_y,p_s}(\mathbf{r}) = \frac{1}{\sqrt{2\pi\rho}} \exp\left(i\frac{p_s}{\hbar}s\right) X_{n_x,p_s}(x) Y_{n_y,p_s}(y),$$
$$X_{n_x,p_s}(x) = \sqrt{\frac{C_{nx}}{x_0}} \exp\left[-\frac{(x - x_\epsilon)^2}{2x_0^2}\right] H_{n_x}\left(\frac{x - x_\epsilon}{x_0}\right),$$
$$Y_{n_y,p_s}(y) = \sqrt{\frac{C_{ny}}{y_0}} \exp\left(-\frac{y^2}{2y_0^2}\right) H_{n_x}\left(\frac{y}{y_0}\right), \qquad (5)$$

where n_x and n_y are the transverse quantum levels, $C_n = (2^n n! \sqrt{\pi})^{-1}$ is the normalization constant, $x_0 = \sqrt{\hbar c^2 / E_s \omega_x}$ and $y_0 = \sqrt{\hbar c^2 / E_s \omega_y}$ are the respective ground state ($n_{x,y} = 0$) oscillation amplitudes, and H_n is the n^{th} order Hermite polynomial. Both the eigenenergies and eigenstates are functions of n_x, n_y and p_s, the quantum numbers that correspond to the invariant horizontal action, the vertical action and the canonical momentum conjugate to the s variable.

2.2 The Total Transition Rate

The change of the transverse quantum levels $n_{x,y}$ due to spontaneous radiation is described by the first-order, time-dependent perturbation theory. The differential transition rate dW_{fi} from an initial state $i(n_x, n_y, p_s)$ to a final state $f(n'_x, n'_y, p'_s)$ by spontaneously emitting a photon $\mathbf{k} = k\mathbf{n}$ into the range $d\mathbf{k}$ is [7]:

$$dW_{fi} = \frac{d\mathbf{k}}{(2\pi)^3} \frac{e^2}{2\epsilon_0 \hbar \omega} \left| \int d\mathbf{r} \psi_f^*(\mathbf{r}) \sum_{\lambda=1}^{2} \mathbf{e}_\lambda \cdot \mathbf{v} e^{-i\mathbf{k}\cdot\mathbf{r}} \psi_i(\mathbf{r}) \right|^2 2\pi \delta(\omega - \omega_{fi}), \tag{6}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are two polarization vectors. The electron velocity operator, when expressed in the Cartesian coordinates, is given by [7]

$$\begin{aligned} \mathbf{v} &\simeq \frac{(\mathbf{p} - e\mathbf{A})c^2}{E_0} \\ &\simeq \left(p_x \cos\frac{s}{\rho} - \frac{p_s - eA_s}{1 + x/\rho} \sin\frac{s}{\rho}, \quad p_y, \quad p_x \sin\frac{s}{\rho} + \frac{p_s - eA_s}{1 + x/\rho} \cos\frac{s}{\rho} \right) \frac{c^2}{E_0}, \end{aligned} \tag{7}$$

where E_0 is the synchronous energy and the conjugate momentum operators are

$$p_x = -i\hbar \frac{\partial}{\partial x}, \quad p_y = -i\hbar \frac{\partial}{\partial y}, \quad \text{and} \quad p_s = -i\hbar \frac{\partial}{\partial s}. \tag{8}$$

Because of the curvilinear coordinates used for the electron, we do not have momentum conservation between the electron and the photon in the s direction even though the eigenstates of the electron have plane-wave form in s.

Since we are interested in the total radiation effects instead of the spectrum and polarization properties of the emitted photons, we can sum over the photon polarization and integrate Eq. (6) over the momentum space of the photons. First, let us expand the δ function

$$\delta(\omega - \omega_{fi}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt e^{-i(\omega - \omega_{fi})t}, \tag{9}$$

and write Eq. (6) as

$$\begin{aligned}W_{fi} =&\frac{e^2}{4\pi^2\epsilon_0\hbar c}\int_{-\infty}^{+\infty} dte^{i\omega_{fi}t}\int\int\frac{kdkd\Omega}{4\pi}e^{-ikct}\sum_{\lambda=1}^{2}\int d\mathbf{r}_1\psi_f^*(\mathbf{r}_1)e^{-i\mathbf{k}\cdot\mathbf{r}_1}\\&\times(\mathbf{v}_1\cdot\mathbf{e}_\lambda)\psi_i(\mathbf{r}_1)\int d\mathbf{r}_2\psi_f(\mathbf{r}_2)e^{i\mathbf{k}\cdot\mathbf{r}_2}(\mathbf{v}_2\cdot\mathbf{e}_\lambda)\psi_i^*(\mathbf{r}_1),\end{aligned} \tag{10}$$

where $\mathbf{v}_{1,2}$ is used to distinguish between the velocity operators that operate on $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. By applying the polarization sum

$$\sum_{\lambda=1}^{2}(\mathbf{v}_1\cdot\mathbf{e}_\lambda)(\mathbf{v}_2\cdot\mathbf{e}_\lambda)=\mathbf{v}_1\cdot\mathbf{v}_2-(\mathbf{v}_1\cdot\mathbf{n})(\mathbf{v}_2\cdot\mathbf{n}), \tag{11}$$

and introducing the Green's function [2]

$$G(t,r)=-\int\int\frac{kdkd\Omega}{4\pi}e^{-ikct+i\mathbf{k}\cdot\mathbf{r}}=\lim_{\epsilon\to+0}\frac{1}{c^2(t-i\epsilon)^2-r^2}, \tag{12}$$

we obtain

$$\begin{aligned}W_{fi} =&\frac{e^2c}{4\pi^2\epsilon_0\hbar}\int_{-\infty}^{+\infty} dte^{i\omega_{fi}t}\int\int d\mathbf{r}_1d\mathbf{r}_2\psi_f^*(\mathbf{r}_1)\psi_f(\mathbf{r}_2)\\&\times G(t,|\mathbf{r}_1-\mathbf{r}_2|)\left(1-\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right)\psi_i(\mathbf{r}_1)\psi_i^*(\mathbf{r}_2).\end{aligned} \tag{13}$$

We make the change of variables $\phi=(s_1-s_2)/\rho$ and $\phi'=(s_1+s_2)/\rho$. Insert Eq. (5) into Eq. (13) and integrate over ϕ', we arrive at

$$\begin{aligned}W_{fi} =&\frac{e^2c}{\pi\hbar}\int_{-\infty}^{+\infty} dte^{i\omega_{fi}t}\int_{-2\pi}^{2\pi}\frac{d\phi}{2\pi}\exp\left[i\frac{(p_s-p_s')\rho\phi}{\hbar}\right]\\&\times\int dx_1dx_2dy_1dy_2X_f(x_1)X_f(x_2)Y_f(y_1)Y_f(y_2)\\&\times GVX_i(x_1)X_i(x_2)Y_i(y_1)Y_i(y_2),\end{aligned} \tag{14}$$

where the Green's function

$$\begin{aligned}G =&\left[c^2(t-i\epsilon)^2-(\rho+x_1)^2-(\rho+x_2)^2\right.\\&\left.+2(\rho+x_1)(\rho+x_2)\cos\phi-(y_1-y_2)^2\right]^{-1},\end{aligned} \tag{15}$$

and the factor

$$\begin{aligned}V =&1-\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\simeq 1-\frac{(\mathbf{p}_1-e\mathbf{A}_1)\cdot(\mathbf{p}_2-e\mathbf{A}_2)}{E_0^2/c^2}\\ \simeq&1-\frac{(p_{x1}p_{x2}+p_s^2)c^2}{E_0^2}\cos\phi-\frac{(p_{x1}+p_{x2})p_sc^2}{E_0^2}\sin\phi-\frac{p_{y1}p_{y2}c^2}{E_0^2}+\mathcal{O}\left(\frac{x_{1,2}^2}{\rho^2}\right).\end{aligned} \tag{16}$$

In the last equation, p_s is the eigenvalue of both operators p_{s1} and p_{s2}. We have used the approximation that $p_s \simeq p_0$ because it is sufficient to consider the case when the electron is initially on the ideal circular orbit.

2.3 The Dipole Approximation

For simplicity, we consider small transverse oscillations in the weak undulator regime (i.e., $\gamma\theta_p^x \ll 1$ and $\gamma\theta_p^y \ll 1$)[4]. Thus, we can use the dipole approximation[4] by expanding all the operators to the first order in x or p_x, as we did in Eq. (16). We can also neglect the small betatron frequency shifts due to their p_s dependence. From Eq. (4), we obtain

$$\omega_{fi} \simeq v(p_s - p_s')/\hbar + \omega_x(n_x - n_x'), + \omega_y(n_y - n_y') \tag{17}$$

with $v = p_0 c^2/E_0$. However, the change of the equilibrium orbit must be properly taken into account. Expanding the final horizontal wavefunction in terms of the initial equilibrium orbit displacement, we have

$$\begin{aligned} X_{n_x',p_s'}(x) &= \left[1 + \frac{(p_s - p_s')c}{K_x\rho}\partial_x\right] X_{n_x',p_s}(x) + \mathcal{O}\left({\partial_x}^2\right) \\ &= \left[1 + i\frac{(p_s - p_s')c}{K_x\rho\hbar}p_x\right] X_{n_x',p_s}(x) + \mathcal{O}\left({p_x}^2\right). \end{aligned} \tag{18}$$

Introduce the notation $(p_s - p_s')\rho/\hbar = l$, $n_x' - n_x = \delta n_x$ and $n_y' - n_y = \delta n_y$, we arrive at

$$\begin{aligned} W_{fi} =& \frac{e^2 c}{4\pi^2\epsilon_0\hbar}\int_{-\infty}^{+\infty} dt \exp\left[-i(\delta n_x\omega_x + \delta n_y\omega_y)t\right]\int_{-2\pi}^{2\pi}\frac{d\phi}{2\pi}\exp\left[il\left(\phi - \frac{vt}{\rho}\right)\right] \\ &\times \int dx_1 dx_2 dy_1 dy_2 X_{n_x'}(x_1)X_{n_x'}(x_2)Y_{n_y'}(y_1)Y_{n_y'}(y_2)\left(1 + il\frac{p_{x1}c}{K_x\rho^2}\right) \\ &\times \left(1 + il\frac{p_{x2}c}{K_x\rho^2}\right) GV X_{n_x}(x_1)X_{n_x}(x_2)Y_{n_y}(y_1)Y_{n_y}(y_2), \end{aligned} \tag{19}$$

where we have dropped the subscript p_s from all the transverse wavefunctions to simplify the notations.

The expected rate of change of the horizontal quantum number is given by[4,6]

$$\left\langle\frac{dn_x}{dt}\right\rangle = \sum_{n_x',n_y',p_s'}(n_x' - n_x)W_{fi} = \sum_{\delta n_x,\delta n_y,l}\delta n_x W_{fi}. \tag{20}$$

The sum over l can be first carried out using the set of relations

$$\sum_l \exp\left[il\left(\phi - \frac{vt}{\rho}\right)\right] = 2\pi\delta_p\left(\phi - \frac{vt}{\rho}\right),$$
$$\sum_l il \exp\left[il\left(\phi - \frac{vt}{\rho}\right)\right] = 2\pi\delta_p'\left(\phi - \frac{vt}{\rho}\right), \tag{21}$$

where $\delta_p(\phi)$ is the periodic delta function with periodicity 2π, and the prime means derivative with respect to ϕ. Integration by parts over ϕ yields

$$\begin{aligned}\left\langle \frac{dn_x}{dt} \right\rangle =& \frac{e^2 c}{4\pi^2\epsilon_0\hbar} \sum_{\delta n_x} \delta n_x \sum_{\delta n_y} \int_{-\infty}^{+\infty} dt \exp\left[-i(\delta n_x\omega_x + \delta n_y\omega_y)t\right] \\ & \times \int dx_1 dx_2 dy_1 dy_2 X_{n_x'}(x_1) X_{n_x'}(x_2) Y_{n_y'}(y_1) Y_{n_y'}(y_2) \left(1 - \frac{p_{x1}c}{K_x\rho^2}\frac{\partial}{\partial\phi}\right) \\ & \times \left(1 - \frac{p_{x2}c}{K_x\rho^2}\frac{\partial}{\partial\phi}\right) GV X_{n_x}(x_1) X_{n_x}(x_2) Y_{n_y}(y_1) Y_{n_y}(y_2),\end{aligned} \tag{22}$$

where the derivative with respect to ϕ is to be evaluated at $\phi = vt/\rho$ due to the delta functions in Eq. (21).

The Green's function in Eq. (15) plays the role of determining the major contribution of the time integral. Let us define a dimensionless time variable $\tau = ct/\rho$ and expand $\cos\phi$ in the denominator of Eq. (15) to obtain

$$G \simeq \left[I(\tau,\phi) - \left(\frac{x_1 - x_2}{\rho}\right)^2 - \frac{x_1 + x_2}{\rho}\phi^2 - \left(\frac{y_1 - y_2}{\rho}\right)^2\right]^{-1} \rho^{-2}, \tag{23}$$

where $I(\tau,\phi) = (\tau - i\epsilon)^2 - \phi^2 + \phi^4/12 \simeq \tau^2(\gamma^{-2} + \tau^2/12)$ since $\phi = vt/\rho$. The time integral is significant only when $\tau \sim \phi \sim 1/\gamma$, or $ct \sim \rho/\gamma$ (the radiation formation length). Thus, we can also expand Eq. (16) for small ϕ to obtain

$$V \simeq \frac{1}{\gamma^2} + \frac{\phi^2}{2} + \frac{p_{x1}p_{x2}c^2}{E_0^2} - \frac{(p_{x1} + p_{x2})c}{E_0}\phi + \frac{p_{y1}p_{y2}c^2}{E_0^2} + \mathcal{O}\left(\frac{x_{1,2}^2}{\rho^2}\right). \tag{24}$$

We can further expand the Green's function to order x^2/ρ^2 and y^2/ρ^2:

$$\begin{aligned}G \simeq \frac{1}{\rho^2}\bigg[&\frac{1}{I} + \frac{(x_1 - x_2)^2}{\rho^2}\frac{1}{I^2} + \frac{(x_1 + x_2)}{\rho}\frac{\phi^2}{I^2} \\ &+ \frac{(x_1 + x_2)^2}{\rho^2}\frac{\phi^4}{I^3} + \frac{(y_1 - y_2)^2}{\rho^2}\frac{1}{I^2}\bigg].\end{aligned} \tag{25}$$

To evaluate the integrals of the transverse coordinates in Eq. (22), let us write the coordinates and the momenta of Eq. (24), and Eq. (15) in terms of the raising and lowering operators (a and $a^\dagger$) of the transverse harmonic oscillators:

$$p_x = -i\sqrt{\frac{E_0\omega_x\hbar}{2c^2}}(a_x - a_x^\dagger), \qquad x - x_\epsilon = \sqrt{\frac{c^2\hbar}{2E_0\omega_x}}(a_x + a_x^\dagger). \tag{26}$$

$$p_y = -i\sqrt{\frac{E_0\omega_y\hbar}{2c^2}}(a_y - a_y^\dagger), \qquad y = \sqrt{\frac{c^2\hbar}{2E_0\omega_y}}(a_y + a_y^\dagger). \tag{27}$$

Applying Eq. (26) to the horizontal wavefunctions leads to three types of selection rule for δn_x. Those generated by constant terms have the selection rule $\delta n_x = 0$, and thus have no contribution to the summation over δ due to the multiplying factor δn_x in Eq. (22). Those generated by terms proportional to $p_{x1}p_{x2}$, x_1x_2, $p_{x1}x_2$ and $p_{x2}x_1$ have the selection rule $\delta n_x = \pm 1$, and are the lowest order terms. Those generated by $x_1^2x_2^2$, $x_1^2x_2p_{x2}$ and $x_1p_{x1}x_2^2$ have the selection rule $\delta n_x = \pm 2$, but they are higher order terms in $\gamma\theta_p{}^4$, and will be ignored (consistent with the dipole approximation). Thus, the summation over δn_x can be greatly reduced by the selection rule $\delta n_x = \pm 1$. Similarly, applying Eq. (27) to the vertical wavefunctions leads to the same three types of selection rule for δn_y. The leading order terms (in $\gamma\theta_p$) are those correspond to the selection rule $\delta n_y = 0$, and the integration over y variables simply collapse the vertical wavefunctions to identity. Let us define for convenience a weighted average of an operator Q

$$\langle Q\rangle_x \equiv \sum_{\delta n_x} \delta n_x e^{-i\delta n_x \nu_x \tau} \int\int dx_1 dx_2 X_{n_x'}(x_1)X_{n_x'}(x_2)QX_{n_x}(x_1)X_{n_x}(x_2), \tag{28}$$

where $\nu_x = \rho/\beta_x$ is roughly the horizontal tune for this smooth storage ring and $\beta_x = c/\omega_x$ is the reduced betatron wavelength or the average beta function. Thus, we have

$$\begin{aligned}
\langle p_{x1}p_{x2}\rangle_x &= \frac{E_0\hbar}{2c\beta_x}\left[n_x e^{i\nu_x\tau} - (n_x+1)e^{-i\nu_x\tau}\right], \\
\langle x_1x_2\rangle_x &= -\frac{c\beta_x\hbar}{2E_0}\left[n_x e^{i\nu_x\tau} - (n_x+1)e^{-i\nu_x\tau}\right], \\
\langle p_{x1}x_2\rangle_x &= \langle p_{x2}x_1\rangle_x = i\left[n_x e^{i\nu_x\tau} + (n_x+1)e^{-i\nu_x\tau}\right].
\end{aligned} \tag{29}$$

Equation (22) can now be reduced to

$$\left\langle \frac{dn_x}{dt} \right\rangle = \frac{r_e c}{\pi\gamma\rho^2} \int_{-\infty}^{+\infty} d\tau \Bigg\{ \left[\frac{\nu_x}{2I} - \frac{2\tau^2}{\nu_x I^2} + \frac{4\tau^2 - \nu_x^2(\tau^4 - I)(\gamma^{-2} + \tau^2/2)}{\nu_x^3 I^3} \right]$$
$$\times \left[n_x e^{i\nu_x\tau} - (n_x+1) e^{-i\nu_x\tau} \right] + i \left[\frac{\tau^3}{I} - \frac{4\tau^3(\gamma^{-2} + \tau^2/2)}{\nu_x^2 I^3} \right]$$
$$\times \left[n_x e^{i\nu_x\tau} + (n_x+1) e^{-i\nu_x\tau} \right] \Bigg\} \tag{30}$$

The time integral can be performed using the residue technique in the complex τ plane. After some lengthy but straightforward algebraic manipulations, we obtain [7]

$$\left\langle \frac{dn_x}{dt} \right\rangle = -\frac{2}{3}\frac{r_e c\gamma^3}{\rho^2} \left(\varsigma_x^2 - 1\right) n_x + \frac{r_e c\gamma^3}{\rho^2} \frac{\exp(-2\sqrt{3}\varsigma_x)}{144\varsigma_x^3} F_x(\varsigma_x), \tag{31}$$

where

$$F_x(\varsigma_x) = 55\sqrt{3} + 330\varsigma_x + 262\sqrt{3}\varsigma_x^2 + 300\varsigma_x^3 + 48\sqrt{3}\varsigma_x^4, \tag{32}$$

and $\varsigma_x = (\rho/\gamma)/\beta_x$ is the ratio of the radiation formation length to the reduced horizontal betatron wavelength.

Similarly, we can find the expected rate of change for the vertical quantum number with the selection rules $\delta n_y = \pm 1$ and $\delta n_x = 0$ [7]:

$$\left\langle \frac{dn_y}{dt} \right\rangle = \sum_{\delta n_y, \delta n_y, l} \delta n_y W_{fi}$$
$$= \frac{r_e c}{\pi\gamma\rho^2} \int_{-\infty}^{+\infty} d\tau \left[\frac{\nu_y}{2I} + \left(\frac{1}{\gamma^2} + \frac{\tau^2}{2} \right) \frac{1}{\nu_y I^2} \right] \left[n_y e^{i\nu_y\tau} - (n_y+1) e^{-i\nu_y\tau} \right]$$
$$= -\frac{2}{3}\frac{r_e c\gamma^3}{\rho^2} \left(\varsigma_y^2 + 1\right) n_y + \frac{r_e c\gamma^3}{\rho^2} \frac{\exp(-2\sqrt{3}\varsigma_y)}{144\varsigma_y} F_y(\varsigma_y), \tag{33}$$

where

$$F_y(\varsigma_y) = 13\sqrt{3} + 30\varsigma_y + 12\sqrt{3}\varsigma_y^2. \tag{34}$$

and $\varsigma_y = (\rho/\gamma)/\beta_y$ is the ratio of the radiation formation length to the reduced vertical betatron wavelength $\beta_y = c/\omega_y$.

The normalized transverse emittance ε_N (ε_{Nx} or ε_{Ny}) is related to the beam average of the transverse quantum levels n (n_x or n_y) by $\varepsilon_N = \lambda\!\!\!^{-}_c\langle n+\frac{1}{2}\rangle$ [7], where $\lambda\!\!\!^{-}_c = \hbar/m_e c$ is the Compton wavelength of the electron. Thus, we have

$$\left\langle \frac{d\varepsilon_{Nx}}{dt} \right\rangle = -\Gamma_b \left[(\varsigma_x^2 - 1)\left(\varepsilon_{Nx} - \frac{\lambda\!\!\!^{-}_c}{2}\right) - \lambda\!\!\!^{-}_c \frac{\exp(-2\sqrt{3}\varsigma_x)}{96\varsigma_x^3} F_x(\varsigma_x) \right], \quad (35)$$

$$\left\langle \frac{d\varepsilon_{Ny}}{dt} \right\rangle = -\Gamma_b \left[(\varsigma_y^2 + 1)\left(\varepsilon_{Ny} - \frac{\lambda\!\!\!^{-}_c}{2}\right) - \lambda\!\!\!^{-}_c \frac{\exp(-2\sqrt{3}\varsigma_y)}{96\varsigma_y} F_y(\varsigma_y) \right], \quad (36)$$

where $\Gamma_b = 2cr_e\gamma^3/(3\rho^2)$ is the characteristic damping coefficient due to the bending field. Equations (35) and (36) describe the general results of radiation (anti-)damping (the first term) and quantum excitation (the second term) to the transverse actions in this combined-function system when the betatron oscillation amplitudes are small. The relative amount of radiation damping and quantum excitation in each transverse plane can be determined by a single dimensionless parameter ς_x or ς_y respectively, which is a measure of the radiation formation length in units of the reduced betatron wavelength.

In normal synchrotrons and storage rings, the radiation formation length is much shorter than the reduced betatron wavelength, i.e., $\rho/\gamma \ll \beta_{x,y}$ or $\varsigma_{x,y} \ll 1$, Equations (35) and (36) become

$$\left\langle \frac{d\varepsilon_{Nx}}{dt} \right\rangle = -\Gamma_b \left[-\left(\varepsilon_{Nx} - \frac{\lambda\!\!\!^{-}_c}{2}\right) - \lambda\!\!\!^{-}_c \frac{55\sqrt{3}\beta_x^3}{96(\rho/\gamma)^3} \right], \quad (37)$$

$$\left\langle \frac{d\varepsilon_{Ny}}{dt} \right\rangle = -\Gamma_b \left[\left(\varepsilon_{Ny} - \frac{\lambda\!\!\!^{-}_c}{2}\right) - \lambda\!\!\!^{-}_c \frac{13\sqrt{3}\beta_y}{96(\rho/\gamma)} \right]. \quad (38)$$

Both Eq. (37) and (38) gives the same results on radiation damping and quantum excitation to the transverse emittances as using the quasiclassical approach in a smooth storage ring [3]. Note that the first term of Eq. (37) is anti-damping instead of damping because the combined-function system studied here has a negative horizontal damping partition number ($\mathcal{J}_x = -1$).

In the opposite limit where $\rho \to \infty$ (a straight focusing channel), we have $\rho/\gamma \gg \beta_{x,y}$ or $\varsigma_{x,y} \gg 1$, both Eq. (35) and (36) reduce to

$$\left\langle \frac{d\varepsilon_{Nx}}{dt} \right\rangle = -\Gamma_b\varsigma_x^2 \left(\varepsilon_{Nx} - \frac{\lambda\!\!\!^{-}_c}{2}\right) = -\Gamma_c^x \left(\varepsilon_{Nx} - \frac{\lambda\!\!\!^{-}_c}{2}\right), \quad (39)$$

$$\left\langle \frac{d\varepsilon_{Ny}}{dt} \right\rangle = -\Gamma_b\varsigma_y^2 \left(\varepsilon_{Ny} - \frac{\lambda\!\!\!^{-}_c}{2}\right) = -\Gamma_c^y \left(\varepsilon_{Ny} - \frac{\lambda\!\!\!^{-}_c}{2}\right), \quad (40)$$

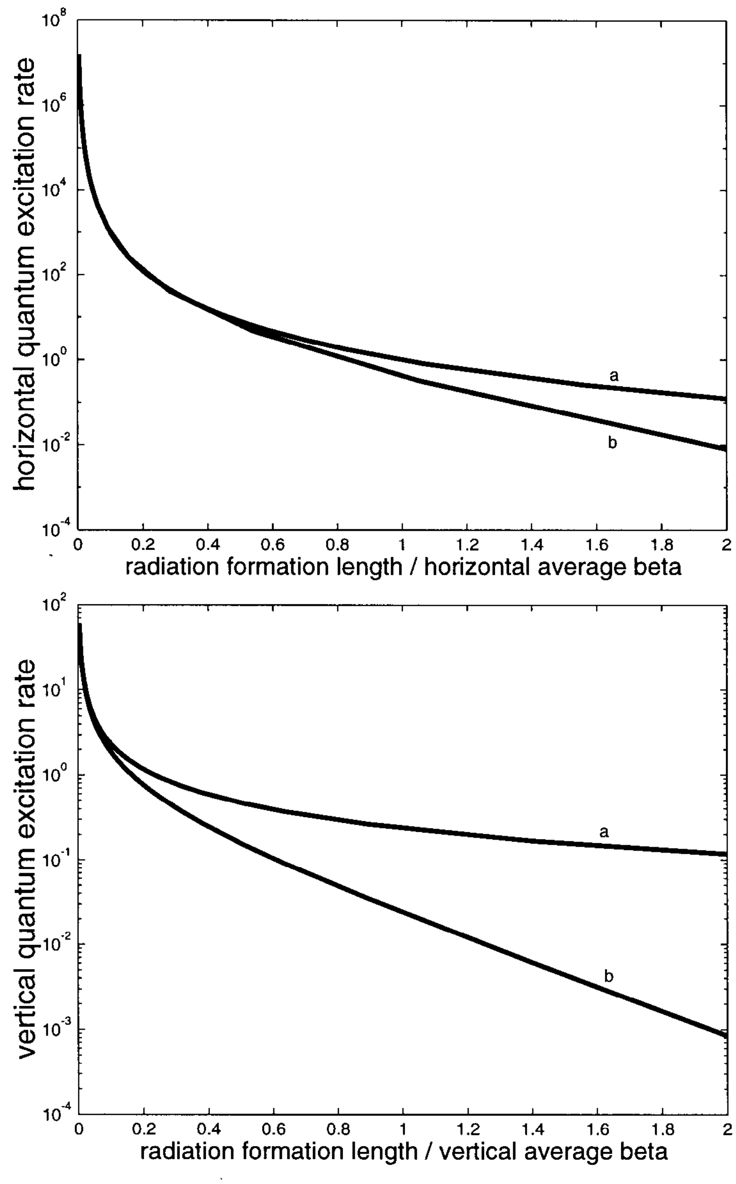

Figure 1: Horizontal and vertical quantum excitation rates in units of $\Gamma_b \lambda_c$, predicted by (a) the quasiclassical model, i.e., the second terms of Eq. (37) and (38), and (b) the quantum mechanical perturbation approach, i.e., the second terms of Eq. (35) and (36).

where $\Gamma_c^{x,y} = \Gamma_b \varsigma_{x,y}^2 = 2r_e K_{x,y}/(3mc)$ is the damping constant due to the focusing field[4]. No quantum excitation to the transverse emittances is induced, and the fundamental emittance $\lambda\!\!\!^{-}_c/2$ can in principle be reached in the ideal focusing channel in both transverse dimensions.

In the intermediate regime where the radiation formation length is on the order of the reduced betatron wavelength ($\rho/\gamma \sim \beta_{x,y}$ or $\varsigma_{x,y} \sim 1$), the horizontal action turns to damping instead of anti-damping in this combined function system, while the damping of the vertical action is enhanced by a factor of two than that from the bending alone. What's more, the rates of quantum excitation in both transverse dimensions are exponentially suppressed according to Eq. (35) or (36) and start to depart from the results based on the quasiclassical approach (see Fig. 1). Thus, the fundamental emittance can be approached very closely in such a focusing-dominated system. A physical interpretation can be given as follows: The transverse energy levels of the electron are well separated as a result of the strong focusing forces. Radiative transition to higher transverse levels becomes impossible for the electron with almost all photon emissions, and hence the quantum excitation is suppressed by the focusing environment.

3 Longitudinal Issues

We are mostly interested in the regime when the radiation formation length is on the order of the transverse betatron wavelength ($\varsigma_{x,y} \sim 1$) and when the transverse oscillation amplitudes are small ($\gamma\theta_p^{x,y} \ll 1$). In this regime, the average radiated energy loss comes predominately from the bending field because

$$\frac{(dE/dt)_{\text{bending}}}{(dE/dt)_{\text{focusing}}} = \frac{\frac{2mc^3 r_e}{3}\frac{\gamma^4}{\rho^2}}{\frac{2r_e}{3mc}\gamma^2 K_x E_x} \sim \frac{1}{\varsigma_x^2\gamma^2\theta_p^2} \gg 1, \tag{41}$$

We assume the average radiated energy loss is replenished by rf acceleration. The damping of the energy spread is achieved through the fact the higher energy electrons lose more energy than the lower energy electrons. The energy spread may be due to the initial beam preparation, may as well arise from the amplitude dependence of the radiated power in the focusing field. The equilibrium energy spread is reached when the damping effect cancels the fluctuating effect of quantum radiation. Since the rf focusing in the longitudinal direction is much weaker than the transverse focusing (i.e., $\omega_s \ll \omega_{x,y}$), the radiation formation length is always much smaller than the reduced synchrotron wavelength. Thus, the instantaneous picture of quantum emission is still valid in the longitudinal phase space and standard results on the longitudinal damping and excitation still hold. For example, the damping rate of the rms energy spread is given by [3]

$$\left\langle \frac{d(\sigma_\delta^2)}{dt} \right\rangle = -\mathcal{J}_s \Gamma_b \sigma_\delta^2, \tag{42}$$

where $\mathcal{J}_s$ is the longitudinal damping partition number, and the equilibrium energy spread is given by [3]

$$\sigma_\delta^2 = \frac{55\sqrt{3}}{96} \lambda_c \frac{\gamma^2}{\mathcal{J}_s} \frac{\langle 1/\rho^3 \rangle_s}{\langle 1/\rho^2 \rangle_s} \simeq \frac{55\sqrt{3}}{96} \lambda_c \frac{\gamma^2}{\mathcal{J}_s \rho} \quad \text{for smooth approximation.} \tag{43}$$

4 A Focusing-dominated Damping Ring

In this section, we will provide some preliminary lattice considerations on a focusing-dominated damping ring based on the results obtained in the previous sections. We note that all of the above results can be extended to alternating-gradient (AG) focusing systems when longitudinal variations of both bending and focusing fields are short compared with the radiation formation length.

Let us consider a focusing-dominated damping ring that basically consists of many repetitive FODO cells. Each cell of length $4L$ consists of four basic elements of equal length L: focusing quad, bend, defocusing quad, and another identical bend. Both quads have the same field gradient g, and both bends have the same bending radius ρ_0. Furthermore, we assume that the phase advance per cell is 60 degrees. If we treat the bending as gradual and the cell as a basic FODO cell with drift space $2L$, we obtain

$$L[\text{cm}] = \left(\frac{E_0[\text{MeV}]}{6g[\text{Tesla/cm}]} \right)^{1/2}. \tag{44}$$

The average beta function (or the reduced betatron wavelength) for the 60 degrees cell is

$$\bar{\beta} = \frac{24L}{2\pi} = \frac{12L}{2\pi} \tag{45}$$

By choosing the average bending radius $\bar{\rho} \simeq \gamma\bar{\beta}/2$, quantum excitation to the transverse emittances is kept at the very low level according to Eq. (35) and Eq. (36). Thus, the equilibrium emittances in such a ring can in principle be on the order of the Compton wavelength.

These simple lattice considerations suggest that in order to design a compact ring, it is favorable to use high-gradient focusing quads and low-energy electron beams. As an example, we assume that permanent magnet quads have a field gradient $g = 4$ Tesla/cm, and we take the electron energy to be $E_0 = 25$ MeV, we then arrive at

$$L \simeq 1.0\text{cm}, \quad \bar{\beta} \simeq 3.9\text{cm}, \quad \bar{\rho} \simeq 1.9\text{m}. \tag{46}$$

The transverse damping rate is about the same for both the focusing effect and the bending effect since $\bar{\rho}/\gamma \simeq \bar{\beta}$. The two damping constants are

$$\Gamma_b = \Gamma_c = 0.11\text{sec}^{-1}. \tag{47}$$

The longitudinal damping rate is determined by the bending effect alone.

The transverse size that corresponds to the Compton wavelength is

$$\sigma_{x,y} = \sqrt{\frac{\lambda_c \bar{\beta}}{\gamma}} = 18\text{nm}. \tag{48}$$

The energy loss per turn is mainly due to the bends, as long as the betatron amplitude is not too large. Thus, we have

$$(\Delta E) = \frac{2\pi\bar{\rho}}{c}\Gamma_b E_0 = 0.11\text{eV}. \tag{49}$$

It can be replenished by either rf or betatron-type acceleration. The equilibrium energy spread is determined by Eq. (43) with $\mathcal{J}_s = 2$:

$$\sigma_\delta = \sqrt{\frac{55\sqrt{3}}{96}\frac{\lambda_c \gamma^2}{2\bar{\rho}}} = 1.6 \times 10^{-5}. \tag{50}$$

However, at such low energy, intrabeam scattering [8] effects are significant. It might be conceivable to operate the ring below the transition energy $\gamma_t m_e c^2$ when

$$\frac{\bar{\rho}}{\bar{\beta}} \simeq \gamma_t > \gamma \tag{51}$$

is satisfied, then the Coulomb interaction between electrons, together with the external focusing environment, tend to stabilize the beam by the crystallization effect [9]. Other collective effects such as wakefields and beam-gas scattering can also influence the stability of the system and may determine the final beam emittance. These effects have yet to be studied in this new regime of operation. Generation of ultra-low emittance electron beams is an interesting subject in its own right, and the effects discussed here may have potential applications in novel accelerators or light sources.

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References

1. K. Robinson, *Phys. Rev.* **111**, 373 (1958).
2. A. A. Sokolov and I. M. Ternov, *Synchrotron Radiation*, (Pergamon Press, New York 1968).
3. M. Sands, *The Physics of Electron Storage Rings*, SLAC Report-121 (1970).
4. Z. Huang, P. Chen, and R. D. Ruth, *Phys. Rev. Lett.* **74**, 1759 (1995).
5. R. D. Ruth, *Single Particle Dynamics in Circular Accelerators*, SLAC-PUB-4103 (1986).
6. Z. Huang and R. D. Ruth, *Phys. Rev. Lett.* **80**, 2318 (1998).
7. Z. Hùang, *Radiative Cooling of Relativisitic Electron Beams*, PhD thesis, Stanford University (1998).
8. A. Piwinski, *Proceedings of the Ninth International Conference on High Energy Accelerators*, 405 (1979).
9. J. Wei and X. Li and A. Sessler, *Phys. Rev. Lett.* **73**, 3089 (1994).