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SIMULATION OF SYNCHROTRON RADIATION IN AN ELECTRON STORAGE RING[a]

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We studied a method of using symplectic integrator to simulate incoherent synchrotron radiation in an electron storage ring. The simulation includes both classical radiation and its quantum fluctuations. We found that the emission process of photon using a compound Poisson process cannot be applied to the fourth-order integrator because it has negative path length which implies a negative number of emitted photons. The simulations are carried out using only the second-order integrator. The result of equilibrium beam size agrees with the analytical calculation well. The method can be applied for all types of magnets.

1 Introduction

The subject of synchrotron radiation in an electron storage ring has been studied by many authors: the classical radiation by Schwinger [1], the effects of quantized photons by Sands [2], and recently by Ohmi [3] working with the second moment of beam distribution.

Motivated by a need of understanding beam halo distribution, we try to develop a method to simulate the synchrotron radiation in a same way for all kinds of magnets including their magnetic imperfections. We choose the method of symplectic integrator because it has been successfully used to handle the simulation in storage rings when synchrotron radiation is ignored.

In this paper, we will first introduce symplectic integration as a technique of tracking through all type of magnets when the synchrotron radiation is not included. Then we continue to discuss how to incorporate the classical radiation and quantum fluctuations into the tracking. Finally we discuss how to compute beam size using the second moment.

2 Single Particle Dynamics

2.1 Hamiltonian

Let's consider a sector bending magnet combined with multipole fields. The Hamiltonian that describes single particle dynamics in a curved coordinate system with curvature h can be expressed as [4]

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$$H = -(1+hx)\sqrt{(1+\delta)^2 - P_x^2 - P_y^2} + hx + \frac{1}{2}h^2x^2 - \frac{e}{p_0}A_s \tag{1}$$

where $\delta = (p - p_0)/p_0$, P_x and P_y are momentum normalized by the design momentum p_0. The s is the designed path length. It is used as an independent variables. For simplicity, we assume that the electron is an ultra-relativistic particle. The difference between the path length and time of flight is ignored. Our canonical coordinates are

$$\vec{z} = \begin{pmatrix} x \\ P_x \\ y \\ P_y \\ \delta \\ \tau \end{pmatrix}. \tag{2}$$

Please note that we use δ as the fifth component because in many situations we need to treat it as a static variable. We have also assumed that the bending angle matches the curvature h. A_s is the longitudinal component of vector potential which can be written as multipole expansion

$$A_s = -Re(\sum_{n=1} \frac{1}{n}(b_n + ia_n)(x+iy)^n), \tag{3}$$

where b_n and a_n are normal and skew components of multipoles respectively. In our convention, b_3 is a normal sextupole. The magnetic field can be computed from the vector potential using $\vec{B} = \nabla \times \vec{A}$. The result is

$$B_y + iB_x = \sum_{n=1}(b_n + ia_n)(x+iy)^{n-1}. \tag{4}$$

When a machine is large: $x << 1/h$ and the energy of the electron is high: $P_x << 1, P_y << 1$, we can simplify the Hamiltonian in Eq. 1 by expanding the square root and keeping only the quadratic part. The simplified Hamiltonian is

$$H = \frac{1}{2}\frac{P_x^2 + P_y^2}{(1+\delta)} - (1+hx)\delta + \frac{1}{2}h^2x^2 - \frac{e}{p_0}A_s, \tag{5}$$

where a constant of -1 has been dropped since it will not affect any dynamics. This Hamiltonian describes most elements in a storage accelerators since by selecting different parameters, it can describe drift, dipole, quadrupole and sextupoles. It also contains a dispersion, $-hx\delta$, and weak focusing, $(1/2)h^2x^2$, generated by the bending magnet.

2.2 Symplectic Integrator

The simplified Hamiltonian cannot be solved in its general forms without further approximation because its non-linearity. We choose symplectic integrators as the technique to solve it at least approximately. One of the advantages of symplectic integrator is that symplecticity is preserved in the process of its integration. This property is very important when the long-term stability of particles is the issue of concern. Another advantage is that one can easily obtain a transfer map to an arbitrary order by integrating a truncated power series [5] through the element that contains very high-order multipoles.

The idea of symplectic integrator is very simple. It is based on the observation that although the Hamiltonian as a whole can not be solved but if we separate it into two parts [6]

$$H = H_0 + H_1, \tag{6}$$

where

$$H_0 = \frac{1}{2}\frac{P_x^2 + P_y^2}{(1+\delta)}, \tag{7}$$

and

$$H_1 = -(1+hx)\delta + \frac{1}{2}h^2x^2 - \frac{e}{p_0}A_s, \tag{8}$$

then each of them can be solved exactly. H_0 is a "drift" since it depends only on the momentum and H_1 is a "kick" that depends only the coordinates.

To see how these exact solvable solutions can be used to approximate the integration of the whole Hamiltonian we write the integration process as a Lie operator: [7]

$$\vec{z}_{out} = e^{-s:H:}\vec{z}_{in}, \tag{9}$$

where $:H: f = \{H, f\}_{poisson}$ is a short notation of the Lie operation on a function f using the Poisson bracket.

It can be shown by applying the Cambell-Bake-Hausdorf theorem that

$$e^{:-lH:} = e^{:-\frac{l}{2}H_0:}e^{:-lH_1:}e^{:-\frac{l}{2}H_0:} + O(l^3). \tag{10}$$

The result can be seen as simply placing the integrated kick at the middle of the drift. It does not depend on the specific form of H_0 or H_1. This integrator is called second-order symplectic integrator since its residual is third order in the length of the integration.

In fact, we can make a fourth-order integrator [8,9] by using three kicks and four drifts symmetrically as given by

$$
\begin{aligned}
e^{:-lH:} = {} & e^{:-c_1 l H_0:} e^{:-d_1 l H_1:} e^{:-c_2 l H_0:} e^{:-d_2 l H_1:} \\
& e^{:-c_2 l H_0:} e^{:-d_1 l H_1:} e^{:-c_1 l H_0:} + O(l^5),
\end{aligned}
\tag{11}
$$

where l is the length of integration and

$$
\begin{aligned}
c_1 &= \frac{1}{2(2-2^{\frac{1}{3}})}, c_2 = \frac{1-2^{\frac{1}{3}}}{2(2-2^{\frac{1}{3}})}, \\
d_1 &= \frac{1}{2-2^{\frac{1}{3}}}, d_2 = \frac{-2^{\frac{1}{3}}}{2-2^{\frac{1}{3}}}.
\end{aligned}
\tag{12}
$$

Please note there are two negative drifts and one negative kick used in the formula. In most cases, if the information of where and how the kicks occurred in the physical space is not needed, it is an excellent approximation. It is often used for strong quadrupoles in interaction regions where strong focusing is required. Also $2c_1 + 2c_2 = 1$ and $2d_1 + d_2 = 1$ ensure that the total path length and integrated magnetic strength are kept the same as in the total Hamiltonian.

This process can be continued to construct higher order symplectic integrators [10]. In practice, we slice evenly a magnet into a few segments and then for each segment we select a proper symplectic integrator.

2.3 Solvable Solutions

To make this paper self-contained, we list the solutions of some useful integrators explicitly.

It is trivial to solve the Hamiltonian equation of a drift. The change of phase vector $\vec{z}$ after the drift described by the Lie operator $e^{:-lH_0:}$ is

$$
\Delta\vec{z} = \begin{pmatrix} \frac{lP_x}{1+\delta} \\ 0 \\ \frac{lP_y}{1+\delta} \\ 0 \\ 0 \\ \frac{l}{2(1+\delta)^2}(P_x^2 + P_y^2) \end{pmatrix}.
\tag{13}
$$

The solution of a kick is also well known. The change of phase vector after passing $e^{:-lH_1:}$ is

$$\Delta \vec{z} = \begin{pmatrix} 0 \\ -lb_y + lh\delta - lxh^2 \\ 0 \\ lb_x \\ 0 \\ l(1+hx) \end{pmatrix}, \tag{14}$$

where $\vec{b} = (e/p_0)\vec{B}$ is the normalized magnetic fields.

3 Synchrotron Radiation

3.1 Radiation Force

In order to simulate the synchrotron radiation, we need to know the velocity of electron since the emission of photons by an accelerated electron is mostly in its forward direction. Its velocity can be computed using Hamiltonian equations given by

$$\begin{aligned} x' &= \frac{\partial H}{\partial P_x} = \frac{P_x}{(1+\delta)}, \\ y' &= \frac{\partial H}{\partial P_y} = \frac{P_y}{(1+\delta)}, \\ \tau' &= -\frac{\partial H}{\partial \delta} = (1+hx) + \frac{1}{2}\frac{P_x^2 + P_y^2}{(1+\delta)^2}, \end{aligned} \tag{15}$$

where the minus sign in the third equations is due to the use of the path length τ as the sixth variable of the canonical coordinates in our convention instead the fifth in the standard convention.

From the velocity, we can define an unit vector, $\vec{n} = n_x\vec{e}_x + n_y\vec{e}_y + n_\tau\vec{e}_\tau$, along the forward direction of the electron where

$$\begin{aligned} n_x &= \frac{x'}{\sqrt{x'^2 + y'^2 + \tau'^2}}, \\ n_y &= \frac{y'}{\sqrt{x'^2 + y'^2 + \tau'^2}}, \\ n_\tau &= \frac{\tau'}{\sqrt{x'^2 + y'^2 + \tau'^2}}. \end{aligned} \tag{16}$$

When a photon is emitted in the forward direction, the change of canonical momenta of the electron can be evaluated by applying the law of momentum conservation including the emitted photon and the position of the electron remains unchanged [11]

$$\Delta\vec{z} = \frac{U}{E_0}\begin{pmatrix} 0 \\ n_x \\ 0 \\ n_y \\ 1 \\ 0 \end{pmatrix}, \tag{17}$$

where U is the energy loss by the electron and p_0 is the design momentum. Again we assume that the electron is an ultra-relativistic particle and its designed energy $E_0 = cp_0$. In this paper, we will assume also that all photons are emitted at the places where the kicks applied inside the symplectic integrators.

3.2 Classical Radiation

The integrated radiation power lost by an accelerated electron is given by [1]

$$P = \frac{2e^4}{3m^4c^3}|\vec{p}\times\vec{B}|^2 = \frac{2r_e p_0^4}{3m^3c}(1+\delta)^2|\vec{n}\times\vec{b}|^2, \tag{18}$$

where $r_e = e^2/(cm^2)$ is the classic radius of electron and $\vec{b}$ is the total normalized magnetic field that includes the main bending field h in the vertical direction as an additional component to the multipole fields Eq. 4.

The relative energy loss in distance ds inside a segment of magnet can be derived from the integrated power:

$$\frac{U_0}{E_0} = -\frac{P}{E_0}dt = -\frac{2}{3}r_e\gamma^3(1+\delta)^2|\vec{n}\times\vec{b}|^2(-\frac{\partial H}{\partial\delta})ds, \tag{19}$$

where γ is the energy of electron in the unit of its rest energy and $-\partial H/\partial\delta$ is given in Eq. 15. Please note that ds is negative in the middle of the fourth-order integrator. We can imagine it as a virtual process in which the electron gains energy and it occurs only inside the integrator. Still, the electron loses its energy after it passes the whole integrator.

To compensate the loss of the energy, we often set cavities in their synchronous phases to provide the energy to electrons. This combined process-namely the electron loses its energy in the arcs then gains the energy back at cavities-results the radiation damping which forces all electrons to a closed orbit.

3.3 *Quantum Fluctuations*

As shown first by Sands [2], the quantum fluctuations due to the quantization of the synchrotron radiation cause the growth of emittance. The balance between the emittance growth and the radiation damping results an equilibrium of the beam size in electron storage accelerators.

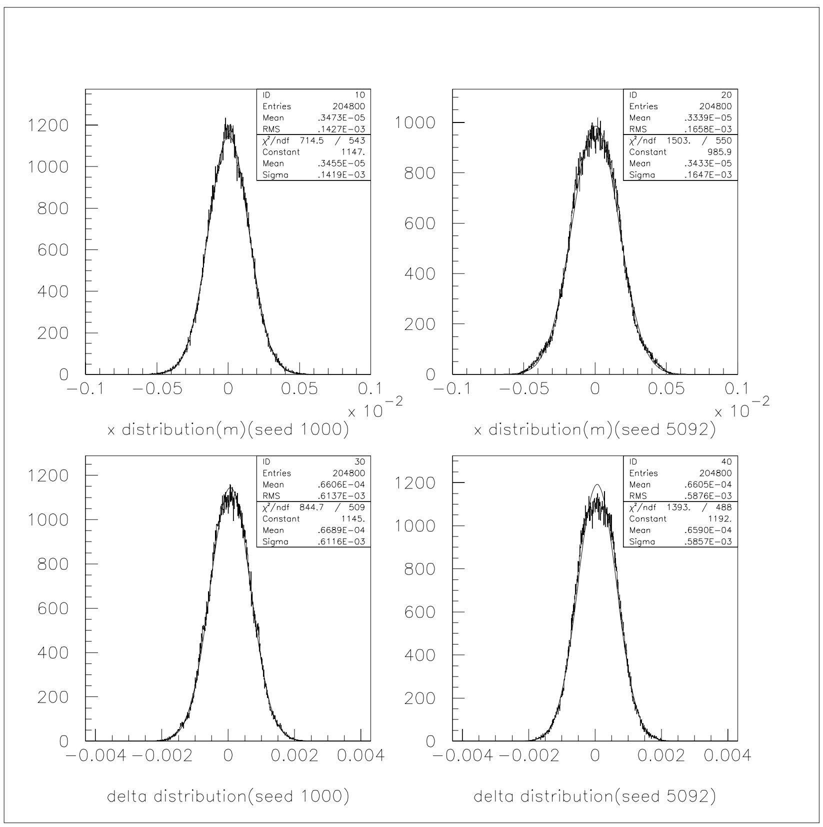

Figure 1: *Tracking a particle about 20 transverse damping time with two different seeds. The distribution is at the interaction point of the high energy ring of PEP-II. The expected beam size is 0.155mm and the energy spread 0.61E-03.*

We can simulate the emission process of photons by a compound Poisson process in which the actual number of the emitted photons is generated by the Poisson distribution with a mean of

$$< N >= \frac{5\sqrt{3}}{6}\alpha\gamma|\vec{n} \times \vec{b}|(-\frac{\partial H}{\partial \delta})ds, \tag{20}$$

where $\alpha = e^2/(\hbar c)$ and ds is the integration length inside the magnet. The energy loss due to the emission of N photons is given by

$$U = -\mu_c \sum_{j=1}^{N} \xi_j, \tag{21}$$

where μ_c is the critical energy of the emitted photons. It can be expressed as

$$\mu_c = \frac{3}{2}\gamma^3\hbar c(1+\delta)^2|\vec{n} \times \vec{b}| \tag{22}$$

and ξ is a random variable that has following the probability density function:

$$\rho(\xi) = \frac{3}{5\pi}\int_{\xi}^{\infty} K_{\frac{5}{3}}(\eta)d\eta \tag{23}$$

where $K_{\frac{5}{3}}$ is a modified Bessel function. This energy spectrum is valid only for a bending magnet. For a given segment we always calculate its local magnetic field. Hence we can use the spectrum as a good approximation for a small segment of any type of magnets.

However, this method of simulation cannot be applied to the fourth-order integrator because the number of the emitted photons is negative at the middle of the integrator due to its negative ds. This observation requires us to replace the fourth-order integrators in the strong focusing regions with thin lens in the physical space [12] if we want to know the radiation effects due to those quadrupoles. Fortunately, we still can use the second-order integrator for all dipole magnets because of its weak focusing strength. And they are the dominant source of the synchrotron radiation.

Simulation results of the high energy ring of PEP-II using an object-oriented code [13] is shown in figure 1. We see a variation of the beam size at a few percentage from its expected equilibrium beam size after 20 transverse damping time.

4 Beam Size

In the last section we show how to simulate the process of synchrotron radiation in a circular electron accelerator. We see that simulation is not an efficient way to compute the equilibrium beam size since many random seeds need to be simulated to reduce the statistical error.

There are many ways to compute equilibrium emittance such as the synchrotron radiation integrals by Helm[15] and the beam-envelope matrix formulation by Ohmi[3]. We will give a brief summary in a discrete version of the second-moment approach.

We can evaluate the mean and variance of the energy loss from the compound Poisson process by using the moment of the random sum and the fact of both the mean and variance of the Poisson distribution are $< N >$[14]. The results are

$$
\begin{aligned}
< U > &= - < N >< \xi > \mu_c = -\frac{8}{15\sqrt{3}} < N > \mu_c, \\
< (U - < U >)^2 > &= < N >< \xi^2 > \mu_c^2 = \frac{11}{27} < N > \mu_c^2.
\end{aligned}
\tag{24}
$$

Although both $< N >$ and μ_c depend upon the Plank constant $\hbar$ separately, the average of the energy $< U >$ does not depend on that constant. In fact, it is the same as the classical radiation U_0 in Eq. 19. Therefore, the motion of the beam centroid follows the same orbit as in the case of the classical radiation. And the quantum effects contribute through only the variance of the energy fluctuation.

Statistically, we can replace the compound Poisson process by a Gaussian distribution that has the same mean and variance,

$$
U = U_0 + \sigma_U \zeta, \tag{25}
$$

where ζ a stochastic variable that has a Gaussian distribution normalized to one and

$$
\sigma_U = \mu_c \sqrt{\frac{11}{27} < N >}. \tag{26}
$$

It has been shown[3] that the second-moment of the beam distribution is evolved through a segment of magnet according to

$$
\Sigma_{out} = M_{(in,out)} \Sigma_{in} M^T_{(in,out)} + M_{(q,out)} \Sigma_q M^T_{(q,out)}, \tag{27}
$$

where $M_{(in,out)}$ is the linear transport matrix of the segment and $M_{(q,out)}$ from where the emission occurred to the exit of the segment. These matrices can be extracted easily with the differential algebra method[5] in a tracking code. T is the notation of transport of a matrix.

The second-moment is define as

$$\Sigma = < \vec{z} \otimes \vec{z}^T >, \Sigma_q = < \vec{\sigma}_q \otimes \vec{\sigma}_q^T > . \tag{28}$$

Here we use $\otimes$ for the notation of the direct product of matrices; $\vec{z}$ is the orbit deviation from the beam centroid. The radiation vector can be read directly from Eq. 17 and 25 and given by

$$\vec{\sigma}_q = -\frac{\sigma_U}{E_0} \begin{pmatrix} 0 \\ \frac{P_x^c}{(1+\delta)} \\ 0 \\ \frac{P_y^c}{(1+\delta)} \\ 1 \\ 0 \end{pmatrix} \tag{29}$$

Please note that it depends on the beam centroid $\vec{z}^c$. The negative path length in the fourth-order integrator does not cause any problems since σ_U^2 is used in the formula of advancing the moment.

A one-turn moment can be calculated from segment-to-segment based on the equation of the moment. The equilibrium emittance and beam size can be computed by using eigen values and eigen vectors of the matrix M [16].

5 Conclusion

We have studied how to simulate incoherent synchrotron radiation in a tracking code based on symplectic integration. The simulation results agree reasonably well with the analytical analysis.

When generating the photons, we found a limitation of the fourth-order integrator because it contains a negative path length making the number of the photon negative. This problem limits us to simulate the quantum fluctuations with only the second-order integrators. Fortunately, the problem does not cause any troubles mathematically when we propagate the beam centroid or second-order moment.

We have limited the discussion to electron storage ring in this paper. However, many results can be applied to transport lines as well since the process of synchrotron radiation is treated locally for each element. They can also be applied to any other codes based on thin-element.

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