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Energy Spectrum of Electron-Positron Pairs Produced via the Trident Process, with Application to Linear Colliders in the Deep Quantum Regime*

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Abstract

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ENERGY SPECTRUM OF ELECTRON-POSITRON PAIRS PRODUCED VIA THE TRIDENT PROCESS, WITH APPLICATION TO LINEAR COLLIDERS IN THE DEEP QUANTUM REGIME

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Explicit expressions for the energy spectrum of $e^{\pm}$ pairs produced via the trident process are derived using the quasi-classical approach of Baier, Katkov, and Strakhovenko (BKS)[4]. We examine the relevance of the trident process to the design of very high energy (E_{cm} several TeV) linear colliders having high Υ ($\Upsilon \sim 100$ to several thousand). We use our calculation of the energy spectrum of the pairs to estimate the maximum expected deflection of pair-produced particles having the same sign as the oncoming bunch in a linear collider. We also compare our results for the total pair production probability with the results obtained using formulas for very high Υ used previously in the literature on linear colliders. The agreement is very good at extremely high Υ, but in the range of Υ noted above, our result for the total trident pair production rate is about a factor of two lower than the corresponding approximate result in the literature.

1 Introduction

Consider an electron or positron of very high energy E traversing a strong electromagnetic field. Such a situation may be characterized by the Lorentz invariant parameter Υ, defined by

$$\Upsilon \equiv \frac{e\hbar}{m^3c^4}\sqrt{|F_{\mu\nu}p^{\nu}|^2} = \gamma\frac{B}{B_c} \quad . \tag{1}$$

Here $p^{\nu} = (E, \overrightarrow{p})$ is the 4-momentum of the incoming electron or positron, m is the electron mass, $\gamma \equiv E/mc^2$ is the usual Lorentz factor, $F_{\mu\nu}$ is the energy-momentum tensor of the electromagnetic field, $B = |\overrightarrow{B}| + |\overrightarrow{E}|$, and $B_c \equiv m^2c^3/\hbar e \approx 4.4 \times 10^{13}$ Gauss is the Schwinger critical field.

In linear colliders, as a tightly focused bunch consisting of $\sim 10^8$ electrons passes through a similar bunch of positrons travelling in the opposite direction, individual high energy electrons and positrons radiate photons due to their interaction with the collective electromagnetic field of the oncoming bunch. Some of these *beamstrahlung* photons convert to e^+e^- pairs as they continue moving through the collective field, which serves as the external field with corresponding Υ parameter as discussed above. The pair production may occur through a real beamstrahlung photon (we shall refer to this as the *cascade process*), or the intermediate photon may be virtual, in which case the pair production is said to occur by the *trident process*.

While pair production through a real photon has been well-studied, the virtual photon process has not been as thoroughly pursued. One motivation of this paper is to investigate further the impact of the trident process on the design of very high energy linear colliders (having center of mass energy of several TeV or more). Such collider designs, for example those using laser acceleration, typically need very short bunch lengths and thus tend to be in the strong quantum beamstrahlung regime ($\Upsilon \gg 1$). Pair production via the cascade process was first treated by Klepikov[1] and by Nikoshov and Ritus[2]. The first correct treatment of the trident process was given by Ritus[3]. Useful approximate formulas for the total pair production probability via the trident process were given by Baier, Katkov, and Strakhovenko (BKS)[4] In this paper we derive explicit expressions for the energy spectrum of pairs produced via the trident process at high Υ.

We follow the quasi-classical approach of BKS, whereby the very high energy electron can be regarded as following a classical trajectory through the magnetic field. The quantum nature of the photon emission and the corresponding recoil of the electron are, however, taken into account. Under such assumptions, BKS derive the following expression for the total pair production probability (per unit time) via a virtual intermediate photon:

$$W_{tot} = -\frac{\alpha^2 m^2}{8\pi^2 E}\frac{c^4}{\hbar}\int_0^\infty \frac{du}{(1+u)^2}\int_0^\infty \frac{d\xi}{\cosh^2\xi}\cdot I_{\sigma\tau} \quad . \tag{2}$$

Here $\alpha \equiv e^2/\hbar c \approx 1/137$ is the fine-structure constant. We denote the fractional energy of the intermediate virtual photon by $y \equiv \omega/E$. The fraction of the initial energy carried by the positron in the produced pair is denoted by $x \equiv E_+/E$, and so the electron of the pair has fractional energy $y - x$. For compactness and convenience, the above expression for the total probability was written in terms of the following variables:

$$\begin{aligned} u &\equiv y/(1-y), \\ \cosh^2\xi &\equiv \frac{y^2}{4x(y-x)}, \\ \kappa &\equiv \frac{y}{\Upsilon x(y-x)} \quad . \end{aligned} \tag{3}$$

For our purposes here it is most convenient to express $I_{\sigma\tau}$ in the following form:

$$\begin{aligned}
I_{\sigma\tau} &= \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\tau B_\sigma B_\tau \Bigg\{ \frac{u\Upsilon}{(1+u)\cosh^2\xi} \frac{\delta(\sigma-\tau)}{\sigma\tau} \\
&+ \quad \left[A_{-1,-1}\frac{1}{\tau\sigma} + A_{-1,1}\frac{\sigma}{\tau} + A_{1,-1}\frac{\tau}{\sigma} + A_{1,0}\tau + A_{1,2}\tau\sigma^2 \right. \\
&\left. + A_{2,-1}\frac{\tau^2}{\sigma} + A_{2,1}\tau^2\sigma \right] \cdot [\theta(\sigma-\tau) - \theta(\tau-\sigma)] \Bigg\} \cdot \\
&\exp\left[-i\frac{u}{\Upsilon}(\sigma + \frac{\sigma^3}{3}) - i\kappa(\tau + \frac{\tau^3}{3}) \right] \quad . \qquad (4)
\end{aligned}$$

Here $\delta(z)$ is the Dirac delta function and $\theta(z)$ the Heaviside step function. The integrals over σ and τ are regularized for $\sigma, \tau \to 0$ via the operator

$$B_\tau \tau^n e^{a\tau^3} = \begin{bmatrix} \tau^n e^{a\tau^3} & (n \geq 0) \quad , \\ \tau^n (e^{a\tau^3} - 1) & (n = -1) \quad . \end{bmatrix} \qquad (5)$$

The quantities $A_{i,j}$ depend on Υ, as well as on the fractional energies x and y (through the variables u and ξ), and are given by

$$\begin{aligned}
A_{-1,-1} &= \frac{-i}{\cosh^2\xi} \\
A_{-1,1} &= \frac{-id(u)}{(1+u)\cosh^2\xi} \\
A_{1,-1} &= \frac{ib(\xi)d(u)}{3u^2} \\
A_{1,0} = -A_{2,-1} &= \frac{2(1+u)}{3u\Upsilon} b(\xi) \\
A_{1,2} = -A_{2,1} &= \frac{2(1+u)}{3u\Upsilon} \left(\frac{b(\xi)d(u)}{1+u} - 3 \right) \qquad (6)
\end{aligned}$$

where $d(u) \equiv 1 + (1+u)^2$ and $b(\xi) \equiv 8\cosh^2\xi + 1$.

After a lengthy calculation, in which the assumption $\Upsilon \gg 1$ is used, the integrals over σ and τ in Eq. (4) may be carried out in terms of the Airy function and the related Airy function

$$\begin{aligned}
\mathrm{Ai}(z) &\equiv \frac{1}{\pi} \int_0^\infty \cos(\frac{v^3}{3} + zv) dv \quad , \\
\mathrm{Gi}(z) &\equiv \frac{1}{\pi} \int_0^\infty \sin(\frac{v^3}{3} + zv) dv \quad . \qquad (7)
\end{aligned}$$

There are three terms in $I_{\sigma\tau}$ that are significant for large Υ, of which the first is dominant. These three terms are:

$$
\begin{aligned}
I_{\sigma\tau} \;\approx\; & -\frac{8\pi}{9u^2} b(\xi) d(u) \kappa^{-2/3} \mathrm{Ai}'(\kappa^{2/3}) \ln(u/\Upsilon) \\
& + 4\pi \frac{2(1+u)}{3u} \left(\frac{b(\xi)d(u)}{1+u} - 3 \right) u^{-1} \kappa^{-2/3} \mathrm{Ai}'(\kappa^{2/3}) \\
& - \frac{4\pi}{3} (\frac{2}{3}\mathcal{C} + \frac{1}{3}\ln 3) \frac{1}{u^2} b(\xi) d(u) \kappa^{-2/3} \mathrm{Ai}'(\kappa^{2/3}) \\
= & -\frac{8\pi}{9} [(1-y)^2 + 1] \left[\frac{2y^2}{x(y-x)} + 1 \right] y^{-8/3} [x(y-x)]^{2/3} \cdot \\
& \Upsilon^{2/3} \mathrm{Ai}'(\kappa^{2/3}) \ln \left[\frac{y}{(1-y)\Upsilon} \right] \\
& + \frac{8\pi}{3} \left\{ [(1-y)^2 + 1] \left[\frac{2y^2}{x(y-x)} + 1 \right] - 3(1-y) \right\} y^{-8/3} [x(y-x)]^{2/3} \cdot \\
& \Upsilon^{2/3} \mathrm{Ai}'(\kappa^{2/3}) \\
& - \frac{4\pi}{3} \; (\frac{2}{3}\mathcal{C} + \frac{1}{3}\ln 3) [(1-y)^2 + 1] \left[\frac{2y^2}{x(y-x)} + 1 \right] y^{-8/3} [x(y-x)]^{2/3} \cdot \\
& \Upsilon^{2/3} \mathrm{Ai}'(\kappa^{2/3}) \quad . \qquad (8)
\end{aligned}
$$

Here $\mathcal{C}$ is Euler's constant (≈ 0.577). The second two terms give a correction of order 10% for parameters of interest for very high energy linear colliders. Note that all three terms depend on Υ through $\Upsilon^{2/3}\mathrm{Ai}'(\kappa^{2/3})$. The main reason for the dominance of the first term is its additional dependence on $\ln \Upsilon$. The overall dependence of the first term on Υ is roughly $\Upsilon \ln \Upsilon$.

Changing variables of integration from (u, ξ) to (x, y), we can write the total probability per unit time for producing pairs at any energy between 0 and E as follows:

$$
\begin{aligned}
W_{tot} \; &= -\frac{\alpha^2 m}{8\pi^2} \frac{c^2}{\hbar} \frac{1}{\gamma} \int_0^{1/2} dx \int_{2x}^{1} dy \frac{\frac{1}{y}(y-2x)}{[y^2/4 - x(y-x)]^{1/2}} \cdot I_{\sigma\tau} \\
&= -\frac{\alpha^2 m}{16\pi^2} \frac{c^2}{\hbar} \frac{1}{\gamma} \int_0^{1} dx \int_{x}^{1} dy \frac{\frac{1}{y}|y-2x|}{[y^2/4 - x(y-x)]^{1/2}} \cdot I_{\sigma\tau} \quad . \qquad (9)
\end{aligned}
$$

(The last equality follows from the symmetry properties of the integrand when x and $(y-x)$ are interchanged.) The total probability W_{tot} as a function of Υ is shown in Figure 1, for $E \equiv \gamma mc^2 = 2.5$ TeV. (The figure may be scaled to arbitrary γ since the vertical scale is simply proportional to $1/\gamma$.)

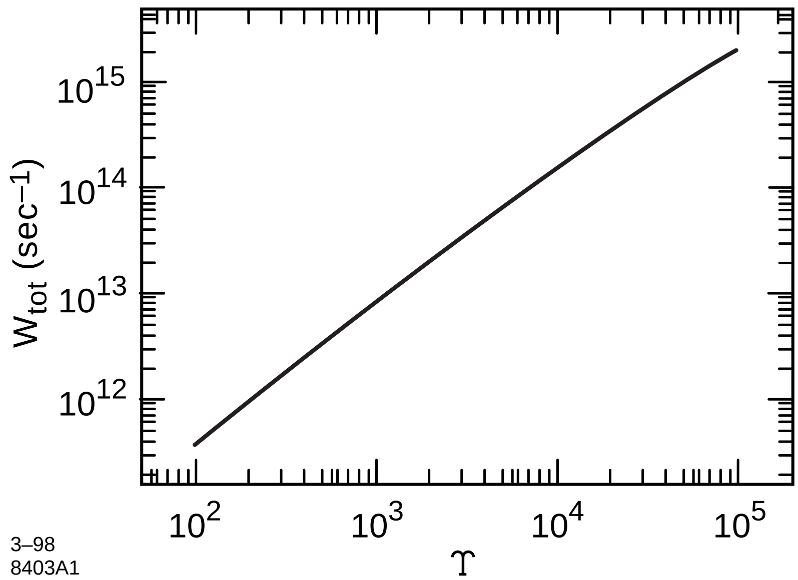

Figure 1: Total probability per unit time [$\sec^{-1}$] for pair production via the trident process, as a function of Υ, for $E \equiv \gamma mc^2 = 2.5$ TeV. This figure may be scaled to arbitrary γ since the vertical scale is proportional to $1/\gamma$.

The spectrum of produced pairs as a function of x is then

$$\frac{dW}{dx} = -\frac{\alpha^2 m}{16\pi^2}\frac{c^2}{\hbar}\frac{1}{\gamma}\int_x^1 dy \frac{\frac{1}{y}|y-2x|}{[y^2/4 - x(y-x)]^{1/2}} \cdot I_{\sigma\tau} \quad . \tag{10}$$

The spectra for $\Upsilon = 3000$ and $\Upsilon = 30000$ are shown in Figure 2. Here we have assumed $E = 2.5$ TeV, but again the particular value of the energy only affects the vertical scale through the $1/\gamma$ factor.

Using the preceding equation for dW/dx, the mean value of x as a function of Υ may also be computed, as is shown in Figure 3.

2 Total Number of Pairs in Very High Energy Linear Colliders

As a check on our results before going on to consider the spectrum, we apply our formulas for the total pair production probability to a linear collider in which the external field is that created by the oncoming bunch, and both bunches have rms length σ_z. Specifically, we compare results obtained from our formulas with those obtained using the following approximate formula

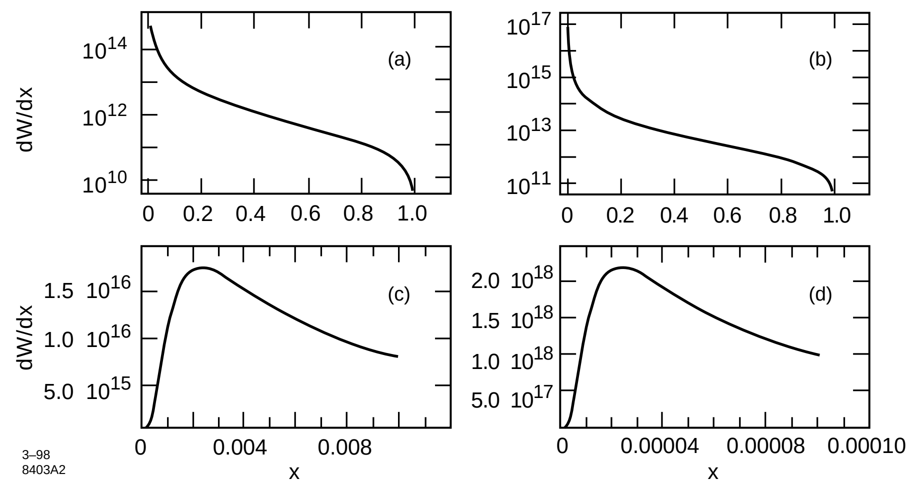

Figure 2: Spectrum of probability per unit time [sec^{-1}] for pair production via the trident process, as a function of $x \equiv E_+/E$, for (a) $\Upsilon = 3000$. (b) $\Upsilon = 30000$, (c) detailed view of $\Upsilon = 3000$ case for small x, (d) detailed view of $\Upsilon = 30000$ case for small x. The vertical axis, which scales as $1/E$, assumes $E = 2.5$ TeV.

given by BKS[4] for $\Upsilon \gg 1$:

$$W_{tot}^{BKS} = \frac{13\alpha^2 m}{9\sqrt{3}\pi}\frac{c^2}{\hbar}\frac{1}{\gamma}\Upsilon \ln \Upsilon \quad . \tag{11}$$

The total (integrated over x) number of pairs produced via the trident process, per incoming electron or positron, is $n_{tri} \approx \frac{\sqrt{3}\sigma_z}{c} W_{tot}$, where σ_z is the bunch length. We denote the BKS estimate of the pairs produced via the trident process as n_{tri}^{BKS}. We will also use the estimate[5,3]

$$n_{casc} = (0.295)\left[\frac{\alpha\sigma_z\Upsilon}{\gamma\lambda_e}\right]^2 \Upsilon^{-2/3}(\ln\Upsilon - 2.488) \qquad (\Upsilon \gg 1) \quad . \tag{12}$$

for the number of pairs per particle produced via the cascade process. Here $\lambda_e = \hbar/mc$ is the Compton wavelength of the electron. The total number of coherent pairs produced is then $n_p = n_{casc} + n_{tri}$, or, if one uses instead the BKS approximation, one has $n_p^{appx} = n_{casc} + n_{tri}^{BKS}$.

Xie, et.al.[6] give representative parameters for very high energy linear collider designs utilizing laser driven acceleration. These designs have $E_{cm} =$ 5 TeV and Υ from about 100 or so, to several thousand. The further assumption of round beams is also made. Table 1 gives results for the number of pairs

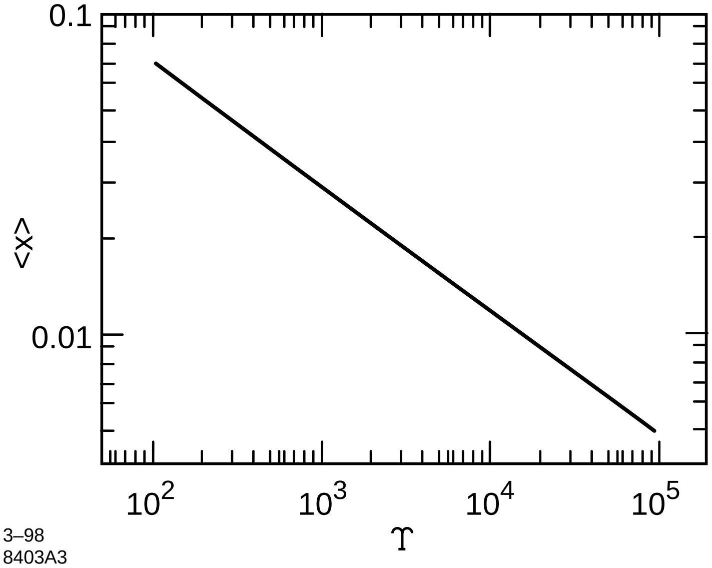

Figure 3: Mean value of $x \equiv E_+/E$ for pair production via the trident process, as a function of Υ.

per particle calculated for these three designs. The values of n_p^{appx} shown in this table differ from the values given in Xie, et.al. for the number of pairs produced, because we believe that the result[5] obtained by converting the BKS formula for W_{tot}^{BKS} to a formula for n_{tri}^{BKS} contains an error of a factor of two (making n_{tri} too low)[7], and this formula was used by Xie, et.al.

On the other hand, our values for the trident pair production (W_{tot} and thus n_{tri}) are lower than those obtained from the BKS approximation (W_{tot}^{BKS} and thus n_{tri}^{BKS}). As one goes to higher Υ, the agreement between our formulas and the BKS approximation improves, but even for a hypothetical ultra-high energy linear collider design with Υ = 30000 (case UHE shown in Table 1, for which we do not attempt to give any parameters other than to assume Υ might be roughly an order of magnitude larger for a design with E_{cm} an order of magnitude larger), there would be a discrepancy of about 25%. Note that our results for n_{tri} in the table were calculated using only the dominant term in $I_{\sigma\tau}$. The correction due to the other two significant terms increases W_{tot} (and thus n_{tri}), which (fortuitously, it seems) brings the result into closer agreement with the result obtained using the approximate result in Eqn (11).

Table 1: Number of coherent pairs per particle for representative very high energy linear collider design examples. Explanation of symbols is given in text.

Case	E $[TeV]$	Υ	σ_z $[\mu m]$	n_{casc}	n_{tri}	n_{tri}^{BKS}	$\frac{n_{tri}}{n_{tri}^{BKS}}$	n_p	n_p^{appx}
Xie III	2.5	138	2.8	0.060	0.0099	0.024	0.40	0.070	0.085
Xie II	2.5	634	1.0	0.095	0.0288	0.053	0.54	0.124	0.148
Xie I	2.5	3485	0.32	0.135	0.0774	0.118	0.66	0.212	0.253
UHE	25.0	30000					0.75		

Table 2: Calculation of maximum deflection angle of pairs for representative very high energy linear collider design examples. Explanation of symbols is given in text.

Case	Υ	σ_r $[nm]$	N	D	$< x >$	x_{cutoff}	θ_0	θ_{max}
Xie III	138	3.5	6×10^8	0.08	0.06	2×10^{-4}	9.9×10^{-5}	0.05
Xie II	634	0.56	1.6×10^8	0.29	0.03	4×10^{-5}	1.6×10^{-4}	0.11
Xie I	3485	0.1	0.5×10^8	0.92	0.02	1×10^{-5}	2.9×10^{-4}	0.25

3 Deflection Angles of Trident Pairs in Linear Colliders

In this section we use the results we have derived for the pair spectrum to examine the maximum deflection angles of pairs produced via the trident process. As examples we again use the three representative linear collider designs of Xie, et.al., discussed in the preceding section.

The maximum deflection angle for the particle of a pair that is of the same sign as the particles in the oncoming beam (and thus tends to be deflected the most) is given by[5]

$$\theta_{max} \approx \left[\frac{\ln(4\sqrt{3}D/x)}{\sqrt{3}\,xD}\right]^{1/2} \theta_0 \qquad (\frac{D}{x} \geq 1) \quad . \tag{13}$$

Here $\theta_0 \equiv D\sigma_r/\sigma_z$, where σ_r is the transverse beam size, and the disruption parameter D is given by

$$D = \frac{Nr_e}{\gamma}\frac{\sigma_z}{\sigma_r^2} \quad . \tag{14}$$

where $r_e \equiv e^2/mc^2$ is the classical electron radius.

Figure 4 shows dW/dx integrated from 0 to x and divided by the total integrated probability W_{tot}, that is, it gives (as a function of x) the fraction of pairs produced with fractional energy less than or equal to x. In Table 2 we show additional parameters and results for the deflection angle for the three designs. As a conservative value for the cutoff of each spectrum we take it to be where $W(x)/W_{tot} = 10^{-12}$, so that for a bunch with $\sim 10^8$ particles or so there is less than one trident pair per thousand bunch crossings with x less than x_{cutoff}. We see that even for this conservative value of x_{cutoff}, we have θ_{max} well under a radian.

4 Conclusions and Acknowledgments

We have derived formulas for the energy spectrum of pairs produced via the trident process. Application of our results to a linear collider with center of mass energy near 5 TeV gives a total trident pair rate in reasonable agreement with that predicted by the approximate formula of Baier, Katkov and Strakhovenko, although their approximate expression seems to overestimate the trident rate by about a factor of two, for Υ values in the range expected for a linear collider of several TeV. According to our results for the spectrum, the rate of trident-pair particles deflected to angles of a radian or more by an oncoming bunch of the same sign is negligible.

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References

1. N.P.Klepikov, Zh.Eksp.Teor.Fiz **46**, 28 (1964).
2. A.I.Nikoshov and V.I.Ritus Soviet Physics JETP **19**, 529 (1964).
3. V.I.Ritus, Nuclear Physics **B44**, 236 (1972).
4. V.N.Baier, V.M.Katkov, and V.M.Strakhovenko, Soviet Journal of Nuclear Physics **14**, 572 (1972). See also V.N.Baier and V.M.Katkov, Soviet Physics JETP **26**, 854 (1968).
5. K.Yokoya and P.Chen, "Beam-beam phenomena in linear colliders", in M.Dienes, et.al. (Eds.), Frontiers of Particle Beams: Intensity Limitations, Lecture Notes in Physics, Vol. 400 (Springer-Verlag, 1992).
6. M.Xie, T.Tajima, K.Yokoya, and S.Chattopadhyay, Proceedings of Seventh Workshop on Advanced Accelerator Concepts, Lake Tahoe, CA, October 12-18, 1996.
7. We thank John Irwin for calling our attention to this.

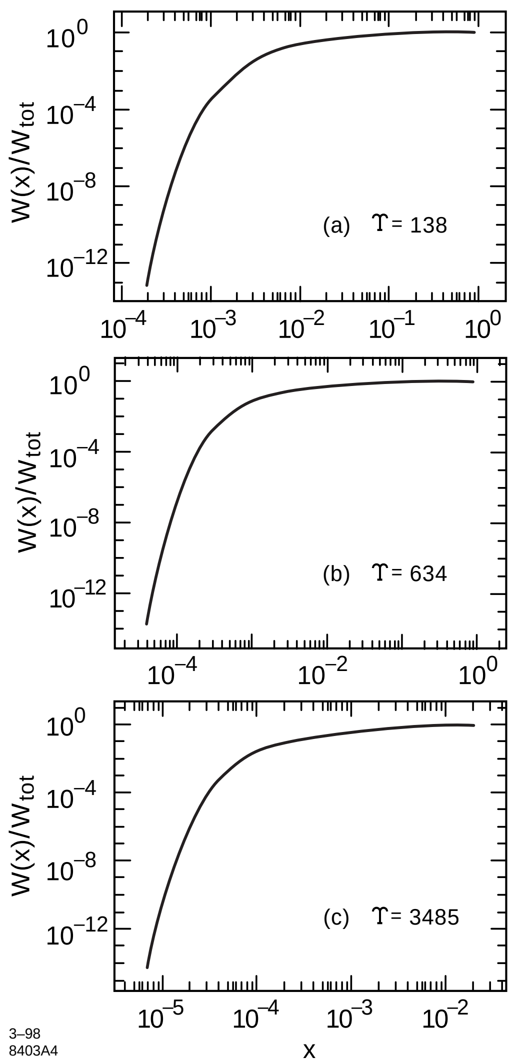

Figure 4: Probability for production via the trident process of pairs with energy less than or equal to x, as a function of x, for (a) $\Upsilon = 138$. (b) $\Upsilon = 634$, (c) detailed view of $\Upsilon = 3485$. (The vertical axis, which scales as $1/E$, is given for $E = 2.5$ TeV).