

Quantum Noether Method

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Abstract. - We present a general method for constructing consistent quantum field theories with global symmetries. We start from a free non-interacting quantum field theory with given global symmetries and we determine all consistent perturbative quantum deformations assuming the construction is not obstructed by anomalies. The method is established within the causal Epstein-Glaser approach to perturbative quantum field theory (which leads directly to a finite perturbative series and does not rely on an intermediary regularization). Our construction can be regarded as a direct implementation of Noether's method at the quantum level. We illustrate the method by constructing $SU(n)$ gauge theory (where the relevant global symmetry is BRST symmetry), and the $N = 1$ supersymmetric model of Wess and Zumino. The whole construction is done before the so-called adiabatic limit is taken. Thus, all considerations regarding symmetry, unitarity and anomalies are well-defined even for massless theories.

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1 Introduction

Symmetries have always played an important and fundamental rôle in our quest of understanding nature. In classical physics they lead to conserved quantities (integrals of motion). The latter constrain the classical evolution of the system and some times they even uniquely determined it (for example, in the case of two-dimensional exactly solvable models). The rôle of symmetries in quantum physics is equally important. In quantum field theory symmetries lead to relation among the Greens' functions of the theory (Ward-Takahashi identities). The latter are instrumental in the proof of renormalizability and unitarity of the theory under question. Furthermore, recent developments in supersymmetric theories[1] show that global symmetries themselves are sometimes sufficient to determine the structure of the theory. It seems, thus, desirable to carefully understand the inter-relations between symmetries and quantum theory in a manner which is free of the technicalities inherent in the conventional Lagrangian approach (regularization-renormalization), and also in a way which is as much model independent as possible. In this article we shall undertake a first step towards this goal. We shall analyze this question within perturbative quantum field theory(QFT).

One may argue that, to a large extend, the relation between symmetries and perturbative QFT is by now well-understood. However, one would like to have an understanding at a more fundamental level. Namely, to separate the generic properties that symmetries impose from the specifics of a given model that realizes this symmetry. In addition, it would desirable to have a formulation which is mathematically as sound as possible.

A framework that encompasses most of the desired properties for this kind of questions is the causal Epstein-Glaser (EG)[2, 3, 4] approach to perturbative quantum field theory.

The explicit construction method of Epstein and Glaser rests directly on the axioms of relativistic quantum field theory. On the one hand, it clarifies how the fundamental axioms guide the perturbative construction of the S matrix, and how well-defined time-ordered products are directly constructed without the need of an intermediate regularization of the theory. On the other hand, it is an explicit construction method for the most general perturbation series compatible with causality and Poincaré invariance. The purely technical details which are essential for explicit calculations are separated from the simple physical structure of the theory. With the help of the causality condition, the well-known problem of ultraviolet (UV) divergences is reduced to a mathematically well-defined problem, namely the splitting of an operator-valued distribution with causal support into a distribution with retarded and a distribution with advanced support. Implicitly, every consistent renormalization scheme solves this problem. In this sense the explicit EG construction should not be regarded as a special renormalization scheme but as a general framework in which the conditions posed by the fundamental axioms of QFT

on any renormalization scheme are built in by construction. In the EG approach the S -matrix is directly constructed in the well-defined Fock space of free asymptotic fields in the form of a formal power series. Thus, one does not need the Haag-Ruelle (LSZ-) formalism. Interacting field operators can still be perturbatively constructed in an additional step as certain functional derivatives of the S -matrix ([2] section 8, [5], see also appendix B).

In classical physics, Noether's theorem states that there is a conserved current for every invariance of the classical action under a continuous, internal or spacetime symmetry transformation of the fields. This theorem also allows for an iterative method to construct invariant actions, called the Noether method. Noether's method has been used in the construction of theories with local symmetries starting from ones with only rigid symmetries. For example, this method was extensively used in the construction of supergravity theories [6]. In a slightly different setting, starting from a free Lagrangian one can iteratively construct consistent interactions by adding extra terms to the action and to transformation rules such that the final action is invariant under the modified transformations. One may try to elevate these results to the quantum regime by quantizing the system. To this end, one should investigate the compatibility of the classical symmetry with the quantization. The latter is reflected in the absence or presence of anomalies.

In this paper we propose a general quantum method which, as we shall see, is a direct implementation of Noether's method at the quantum level. For this reason we shall call it "Quantum Noether Method". In this new framework the conditions for constructing a consistent classical action and the conditions for absence of anomalies are associated with obstructions in the construction of the S -matrix. The classical action emerges from tree-level obstructions, whereas anomalies are associated with loop obstructions. An algebraic consistency condition for possible obstructions can be derived without using the quantum action principle[7]. Starting from a free quantum field theory, well-defined in the Fock space of free asymptotic fields, the method allows for a construction of all consistent perturbative quantum theories. Such a direct implementation of the Noether method in the quantum theory is established in the causal Epstein-Glaser approach to perturbative quantum field theory.

The strength of the EG construction lies in the operator formalism. The proof of general properties of a given quantum field theory can simply and also rigorously be reduced to the discussion of the local normalization ambiguity which is restricted by power counting. It is also the operator formalism which circumvents the classical problem of overlapping divergences - in the usual framework the latter problem is solved by the famous forest formula. Moreover, the EG formalism provides a natural framework to discuss symmetries in perturbative quantum field theories in which the regularization

and scheme independence of anomalies as well as the reduction of their discussion to local normalization ambiguities is manifest. From this it is clear that the Epstein-Glaser approach provides an ideal framework for the discussion of symmetries in quantum field theory.

In the original article [2] Epstein and Glaser applied their construction to scalar field theory. The extension of these results to the gauge theory in the four dimensional Minkowski space has been worked out some years ago in [8, 9, 10, 11]. The causal Epstein-Glaser construction of (3+1)-dimensional non-abelian gauge theory in the Feynman gauge coupled to fermionic matter fields was done in [12, 13]. There, a definition of nonabelian gauge invariance was given as an operator condition in every order of perturbation theory separately. This condition involved only the linear (abelian) BRST-transformations of the free asymptotic field operators. It was claimed that this operator condition expresses the whole content of nonabelian gauge structure in perturbation theory [12, 13]. In fact, it was proven that the operator condition directly implies the unitarity of the S -matrix in the physical subspace, i.e. decoupling of the unphysical degrees of freedom. Furthermore, it was shown that from the operator condition one can derive the Slavnov-Taylor identities for the connected Greens' functions. In [14] it was further shown that the most general couplings compatible with the asymptotic operator condition coincides with the most general BRST-invariant Lagrangian in the pure Yang-Mills case. The latter statement, namely that the concept of asymptotic BRST invariance reproduces the classical input of the usual Lagrangian theory, was recently also shown for the Higgs model [15]. A short discussion of Higgs-free non-abelian massive gauge theories in this framework was presented in [16].

Up until now, however, only the pure Yang-Mills (YM) theory was fully constructed as a quantum theory in this framework. Moreover, a deep understanding of how the asymptotic operator condition develops the full BRST symmetry [17] was missing. With this ingredient missing, it was not *a priori* clear whether the quantum theory constructed using the EG procedure actually coincides with the usual YM theory or it is some kind of "Yang-Mills-like" theory which cannot be reached from the conventional Lagrangian approach. Since the condition of asymptotic BRST invariance is a weaker condition it was not ruled out that there are new theories compatible with this symmetry - an interesting possibility because the asymptotic symmetry condition was shown to be sufficient for decoupling of the unphysical degrees of freedom. We shall argue, however, that the full BRST transformations were already present in the analysis of [12, 13], thereby establishing that the theory constructed by the EG procedure coincides with the usual YM theory. It is the auxiliary coupling which gets automatically developed in the causal construction of Yang Mills theories that provides the completion of the abelian BRST differential to the full non-abelian one. As we shall see, the auxiliary coupling is simply the new term in Noether's current that generate the new transformations in Noether's

method.

Having understood fully the case of YM theories starting with the asymptotic symmetry principle only, we are in the position to generalize the construction to any theory with any local and global symmetry. Notice that local symmetries manifest themselves through the rigid BRST symmetry. Thus, one can treat both cases in parallel. Starting with the condition of asymptotic symmetry we show explicitly how the formalism automatically develops the consistent and anomaly-free full quantum symmetry of the interacting system, provided such a consistent and anomaly-free deformation of the given free theory exists. In this paper we make this latter assumption, hence we exclude any possible obstructions of the symmetry at tree and loop level. A systematic cohomological analysis of possible obstructions will be presented in a separate paper [7]. The Epstein-Glaser framework allows for a simplified derivation of algebraic consistency conditions for obstructions. Moreover, the cohomological analysis of such obstructions is established before the adiabatic limit is taken. So the discussion is also applicable to massless theories like pure Yang-Mills theories [7].

The idea behind the construction is very simple. Given a set of free fields and a symmetry (such as supersymmetry or BRST symmetry etc.) one realizes the latter in terms of the former and computes the corresponding Noether current (which at this point only captures the linear part of the transformation rules). Then one demands that this current is conserved at the quantum level, i.e. inside correlation functions (Quantum Noether Condition). We shall show that this condition at tree-level automatically produces the most general (but polynomial in the coupling constant) non-linear completion of the transformation rules and also the corresponding Lagrangian which is invariant under these transformation rules. We shall then examine the Quantum Noether Condition at loop level. We shall show that if the anomaly consistency condition has only trivial solutions then the theory is stable, i.e. all local terms that are produced by loops are already present at tree-level. This we shall call generalized renormalizability (this corresponds to the notion of “renormalizability in the modern sense” introduced for gauge theories in [18]). If in addition the theory is also power counting renormalizable (normalizable in the EG language, see section 3) then generalized renormalizability coincides with the usual renormalizability. The only restriction needed for our construction to work is that the power counting (singular order in EG, see section 3) is bounded in every order in perturbation theory. So, in particular, our consideration also apply to effective field theories that are not power counting renormalizable.

It is rather remarkable that the only information one needs in order to construct a perturbative quantum field theory with a given global symmetry is a set of free fields linearly realizing this symmetry. Even the first term in the S -matrix, which is usually regarded as an input in the EG formalism is now derived using the Quantum Noether

Condition.

We have organized this paper as follows. In section 2 we shortly recall the Noether method. In section 3 we provide a self-contained summary of the basic ingredients of the EG construction. Section 4 is the main section where we establish the Quantum Noether Method. We illustrate the method in section 5 with two examples; the case of pure $SU(n)$ Yang-Mills theory and the $N = 1$ supersymmetric Wess-Zumino model. Special care was taken in order to illustrate every step of the general construction explicitly. In Appendix A we explain our conventions in detail. Finally, in Appendix B we discuss several issues such as the infrared problem, the construction of interacting fields, the problem of overlapping divergences that are not directly relevant for the construction of Quantum Noether method but further motivate the use of the EG formalism.

2 The Noether Method in Classical Field Theory

In this section we shortly recall the Noether method. Let us start with a classical Lagrangian density $\mathcal{L}(\phi^A, \partial_\mu \phi^A)$ that depends on a number of fields (both bosons and fermions) ϕ^A and their first derivative $\partial_\mu \phi^A$, where A is an index that distinguishes different types of fields. Suppose now that the action $S = \int \mathcal{L}$ is invariant under the symmetry transformation $s\phi^A$. This means that the Lagrangian density is either invariant under this symmetry, $s\mathcal{L} = 0$, in which case one deals with an internal symmetry or it transforms into a total derivative, $s\mathcal{L} = \partial_\mu k^\mu$, in which case we are dealing with a spacetime symmetry. A standard way to derive Noether's current is to let the parameter of the symmetry transformation ϵ become local. Then the Noether current is the expression multiplying the derivative of the local parameter.

$$\delta S = \int j^\mu (\partial_\mu \epsilon) \quad (2.1)$$

Taking the parameter ϵ rigid one sees that the variation is indeed a symmetry of the action. On the other hand, if the field equation are satisfied, $\delta S = 0$ for any ϵ and, therefore, $\partial_\mu j^\mu = 0$.

The Noether current is given by

$$j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} s\phi^A - k^\mu. \quad (2.2)$$

We shall always include the parameter of transformation in the current. In this way the current is always bosonic. Direct calculation (using $\partial_\mu k^\mu = s\mathcal{L}$) yields,

$$\partial_\mu j^\mu = \left[\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} - \frac{\partial \mathcal{L}}{\partial \phi^A} \right] s\phi^A. \quad (2.3)$$

Clearly, $\partial_\mu j^\mu = 0$ when the field equations are satisfied.

There is a natural arbitrariness in the definition of the current. One may always add terms that vanish when the field equations are satisfied as well as terms of the form $\partial_\mu b^{mn}$, where b^{mn} is antisymmetric in m, n .

The conserved charge Q is equal to

$$Q = \int d^3x (p_A s \phi^A - k^0), \quad (2.4)$$

where $p_A = \partial \mathcal{L} / \partial_0 \phi^A$ is the conjugate momentum of ϕ^A . It is straightforward to check that Q generates the corresponding variation when acting (by the Poisson bracket) to the fields. Indeed, in the case of internal symmetries, $k^0 = 0$ and $s \phi^A$ do not depend on the p_A , therefore we have

$$\{Q, \phi^A\} = \int d^3x \{p_B, \phi^A\} s \phi^B = s \phi^A. \quad (2.5)$$

When one deals with spacetime symmetries $s \phi^A$ do depend on p_A but k^0 is also not zero, and the combined effect is that Q again generates the corresponding symmetry.

As mentioned previously, Noether's theorem allows for an iterative method to construct invariant actions, called the Noether method. Starting from a free Lagrangian one can iteratively construct consistent interactions in classical field theory by adding extra terms to the action and to transformation rules such that the final action is invariant. The way the Noether method works is as follows. Start from an action $S_0 = \int \mathcal{L}_0$ invariant under transformations $s_0 \phi^A$. The goal is then to find a new action,

$$S = \int d^4x (\mathcal{L}_0 + g \mathcal{L}_1 + g^2 \mathcal{L}_2 + \dots), \quad (2.6)$$

and new transformation rules,

$$s \phi^A = s_0 \phi^A + g s_1 \phi^A + g^2 s_2 \phi^A + \dots, \quad (2.7)$$

where g is a new coupling constant (or deformation parameter), such that the new action is invariant under the new transformation rules. To first order in g the relevant equation reads

$$\frac{\delta \mathcal{L}_0}{\delta \phi^A}(s_1 \phi^A) + \frac{\delta \mathcal{L}_1}{\delta \phi^A}(s_0 \phi^A) - \partial_\mu k_1^\mu = 0 \quad (2.8)$$

Given $s_1 \phi^A$ one then tries to determine \mathcal{L}_1 such that the above equation holds or vice versa. After equation (2.8) has been solved one tries to solve the equation that appears at order g^2 , and so on. The corresponding current is given by

$$\begin{aligned} j^\mu &= [(\frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \phi^A)} s_0 \phi^A - k_0^\mu) + (g \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \phi^A)} + g^2 \frac{\partial \mathcal{L}_2}{\partial(\partial_\mu \phi^A)} + \dots) s_0 \phi^A] \\ &+ g [(\frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \phi^A)} s_1 \phi^A - k_1^\mu) + (g \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \phi^A)} + \dots) s_1 \phi^A] \dots \end{aligned} \quad (2.9)$$

where we have organized the terms in a way that it will be useful in later sections.

A systematic way to organize this procedure is to use the Batalin-Vilkovisky (or anti-field) formalism[19]. In this formalism (for a detailed exposition of the anti-field formalism see [20]) one introduces a new field, the anti-field ϕ_A^* , for each field ϕ^A . The anti-fields act as sources for symmetry variation of the corresponding field, namely one adds in the Lagrangian a term $\phi_A^* s \phi^A$ (but the solution of (2.10) may contain higher powers of anti-fields). The defining equation of the theory is the master equation

$$(S, S) = 0, \quad (2.10)$$

where (A, B) denotes the anti-bracket,

$$(A, B) = \frac{\overleftarrow{\partial} A \overrightarrow{\partial} B}{\partial \phi^A \partial \phi_A^*} - \frac{\overleftarrow{\partial} A \overrightarrow{\partial} B}{\partial \phi_A^* \partial \phi^A}, \quad (2.11)$$

where the arrow indicates from where the derivative acts. The action in this formalism generates the transformation of the fields,

$$s \phi^A = (\phi^A, S). \quad (2.12)$$

The problem is now formulated as follows[21]. Starting from an S_0 that solves the master equation $(S_0, S_0) = 0$ one seeks for a new action $S = S_0 + g S_1 + g^2 S_2 + \dots$ that solve the new master equation $(S, S) = 0$. This equation yields a tower of equations once it is expanded in g . The first few equations are the following,

$$(S_0, S_1) = 0, \quad (2.13)$$

$$2(S_0, S_2) + (S_1, S_1) = 0, \quad (2.14)$$

and so on. Solving these equations one obtains both the new terms in the action and the new transformation rules. The latter are obtained by using (2.12). Comparing with (2.7) we get the following differentials

$$s_i = (\quad , S_i). \quad (2.15)$$

A nice feature of this approach is that it allows for a systematic cohomological approach to the problem. For instance, equation (2.13) tells us that S_1 is an element of $H^0(s_0, d)$, where $H^k(s_i, d)$ denotes the cohomology group of the differential s_i relative to the differential d in ghost number k . In addition, the obstruction to the solvability of this equation lies in $H^1(s_0, d)$. Similar remarks apply for the rest of the equations.

3 The Causal Method of Epstein-Glaser.

We shall give a short but self-contained introduction into the causal Epstein-Glaser construction. This section may serve as a glossary for all quantities we are about to use in

the next section. For some of the technical details we refer to the literature [2, 3, 4, 11]. We note that we follow the original Epstein Glaser article [2] in our presentation. So we differ slightly from reference [11, 13] regarding the causality condition, and the role of the Wick submonomials in the construction.

3.1 Inductive Construction.

We recall the basic steps of the Epstein Glaser construction in the case of a massless scalar field. For concreteness, we consider the case of four dimensional spacetime. The formalism, however, is valid in any dimension. The very starting point is the Fock space \mathcal{F} of the massless scalar field (based on a representation space $H_s^{m=0}$ of the Poincaré group) with the defining equations

$$\square\varphi = 0 \quad \mathbf{(a)}, \quad [\varphi(x), \varphi(y)] = i\hbar D_{m=0}(x-y) \quad \mathbf{(b)}, \quad (3.1)$$

where $D_{m=0}(x-y) = \frac{-i}{(2\pi)^3} \int dk^4 \delta(k^2) \text{sgn}(k^0) \exp(-ikx)$ is the zero-mass Pauli-Jordan distribution (see appendix A). In contrast to the Lagrangian approach, the S -matrix is directly constructed in this Fock space in the form of a formal power series

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1^4 \cdots dx_n^4 T_n(x_1, \dots, x_n; \hbar) g(x_1) \cdots g(x_n). \quad (3.2)$$

(We do not include explicit i factors in (3.2) in order to reduce the number of i -factors in our equations.) In this approach the coupling constant g is replaced by a tempered test function $g(x) \in \mathcal{S}$ (i.e. a smooth function rapidly decreasing at infinity) which switches on the interaction.

The central objects are the n -point distributions³ T_n . They should be viewed as mathematically well-defined (renormalized) time-ordered products,

$$T_n(x_1, \dots, x_n; \hbar) = T [T_1(x_1) \cdots T_1(x_n)], \quad (3.3)$$

of a given specific coupling, say $T_1 = \frac{i}{\hbar} : \Phi^4 :$ $\mathbf{(c)}$, which is the third defining equation in order to specify the theory in this formalism.

Notice that the expansion in (3.2) is *not* a loop expansion. Each T_n in (3.2) can receive tree-graph and loop-contributions. One can distinguish the various contributions from the power of \hbar that multiplies them⁴.

Epstein and Glaser present an explicit inductive construction of the most general perturbation series in the sense of (3.2) which is compatible with the fundamental axioms

³ $T_n \in \mathcal{S}'$, where \mathcal{S}' denotes the space of functionals on \mathcal{S} .

⁴The fact that the distributions T_n are power series in \hbar follows from the way \hbar appears in defining equations (b) and (c) and the explicit construction that we describe below. Furthermore, one may deduce that \hbar is a loop counting parameter by using similar arguments as in the Lagrangian formulation.

of relativistic quantum field theory, causality and Poincaré invariance, which can be stated as follows:

- Let g_1 and g_2 be two tempered test functions. Then causality means that

$$S(g_1 + g_2) = S(g_2)S(g_1) \quad \text{if} \quad \text{supp}g_1 \preceq \text{supp}g_2 \quad (3.4)$$

the latter notion means that the support of g_1 and the support of g_2 , two closed subsets of \mathbf{R}^4 , can be separated by a space like surface; more precisely $\text{supp}g_2$ does not intersect the past causal shadow of $\text{supp}g_1$:

$$\text{supp}g_2 \cap (\text{supp}g_1 + \bar{V}^-) = 0, \quad (3.5)$$

$$(\bar{V})^- = \{x \in \mathbf{R}^4 | x^0 \leq |\vec{x}|\} \quad (3.6)$$

- Let $U(a, \mathbf{\Lambda})$ be the usual representation of the Poincaré group P_+^4 in the given Fock space \mathcal{F} . Then the condition of Poincaré invariance of the S -matrix says that

$$U(a, \mathbf{\Lambda})S(g)U(a, \mathbf{\Lambda})^{-1} = S(g_\Lambda^a) \quad \forall a \in \mathbf{R}^4, \forall \mathbf{\Lambda} \in L_+^4, g_\Lambda^a(x) = g(\mathbf{\Lambda}^{-1}(x - a)) \quad (3.7)$$

It is well-known that the heuristic solution for (3.2), namely

$$T_n(x_1, \dots, x_n; \hbar) = \sum_{\pi} T_1(x_{\pi(1)}) \dots T_1(x_{\pi(n)}) \Theta(x_{\pi(1)}^0 - x_{\pi(2)}^0) \dots \Theta(x_{\pi(n-1)}^0 - x_{\pi(n)}^0), \quad (3.8)$$

contains ultra-violet divergences (π runs over all permutations of $1, \dots, n$). The reason for this is that the product of the discontinuous Θ -step function with Wick monomials like T_1 which are operator-valued distributions is ill-defined. One can handle this problem by using the usual regularization and renormalization procedures and finally end up with the well-defined (renormalized) time-ordered products of the couplings T_1 .

Epstein and Glaser suggest another path which leads directly to the well-defined T -products without any intermediate modification of the theory using the fundamental property of causality (3.4) as a guide. They translate the condition (3.4) into an induction hypothesis for the T_n -distribution:

$$T_m(X \cup Y) = T_{m_1}(X) T_{m-m_1}(Y) \quad \text{if} \quad X \succeq Y, \quad X, Y \neq \emptyset \quad (3.9)$$

$$[T_{m_1}(X), T_{m_2}(Y)] = 0 \quad \text{if} \quad X \sim Y \quad (\Leftrightarrow X \succeq Y \wedge X \preceq Y) \quad \forall m_1, m_2 \leq m \quad (3.10)$$

Here we use the short-hand notation $T_m(x_1, \dots, x_m; \hbar) = T(X); |X| = m$.

Beside other properties they also include the Wick formula for the T_n distributions into the induction hypothesis. This is most easily done by including the so-called Wick submonomials of the specific coupling $T_1 = (i/\hbar) : \Phi^4 :$ as additional couplings in the construction $T_1^j := (i/\hbar)(4!/(4-j)!): \Phi^{4-j} ; 0 < j < 4$. Then the Wick formula for the T_n products can be written as

$$T_m[T_1^{j_1}(x_1) \dots T_1^{j_m}(x_m)] = \sum_{s_1, \dots, s_m} \langle 0 | T[T_1^{j_1+s_1}(x_1) \dots T_1^{j_m+s_m}(x_m)] | 0 \rangle : \prod_{i=1}^m \left[\frac{\Phi^{s_i}(x_i)}{s_i!} \right] : \quad (3.11)$$

In short-hand notation the formula reads

$$T_m^{\vec{j}}(X) = \sum_{\vec{s}} \langle 0 | T_m^{\vec{j}+\vec{s}}(X) | 0 \rangle \frac{\Phi^{\vec{s}} : (X)}{\vec{s}} \quad (3.12)$$

That such a quantity is a well-defined operator-valued distribution in Fock space is assured by distribution theory (see Theorem O in [2], p. 229). Note also that the coefficients in the Wick expansion are now represented as vacuum expectation values of operators.

Now let us assume that T_m distributions with all required properties are successfully constructed for all $m < n$. Epstein and Glaser introduce then the retarded and the advanced n -point distributions (from now on, in this section, we suppress the \hbar factor in our notation):

$$R_n(x_1, \dots, x_n) = T_n(x_1, \dots, x_n) + R'_n, \quad R'_n = \sum_{P_2} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X) \quad (3.13)$$

$$A_n(x_1, \dots, x_n) = T_n(x_1, \dots, x_n) + A'_n, \quad A'_n = \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n). \quad (3.14)$$

The sum runs over all partitions $P_2 : \{x_1, \dots, x_{n-1}\} = X \cup Y$, $X \neq \emptyset$ into disjoint subsets with $|X| = n_1 \geq 1$, $|Y| \leq n - 2$. The \tilde{T} are the operator-valued distributions of the inverse S-matrix:

$$S(g)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \tilde{T}_n(x_1, \dots, x_n) g(x_1) \dots g(x_n) \quad (3.15)$$

The distributions \tilde{T} can be computed by formal inversion of $S(g)$:

$$S(g)^{-1} = (\mathbf{1} + \mathbf{T})^{-1} = \mathbf{1} + \sum_{n=1}^{\infty} (-\mathbf{T})^n \quad (3.16)$$

$$\tilde{T}_n(X) = \sum_{r=1}^n (-)^r \sum_{P_r} T_{n_1}(X_1) \dots T_{n_r}(X_r), \quad (3.17)$$

where the second sum runs over all partitions P_r of X into r disjoint subsets $X = X_1 \cup \dots \cup X_r$, $X_j \neq \emptyset$, $|X_j| = n_j$.

We stress the fact that all products of distributions are well-defined because the arguments are disjoint sets of points so that the products are tensor products of distributions. We also remark that both sums, R'_n and A'_n , in contrast to T_n , contain T_j 's with $j \leq n - 1$ only and are therefore known quantities in the inductive step from $n - 1$ to n . Note that the last argument x_n is marked as the reference point for the support of R_n and A_n . The following crucial support property is a consequence of the causality conditions (3.9) and (3.10):

$$\text{supp} R_m(x_1, \dots, x_m) \subseteq \Gamma_{m-1}^+(x_m), \quad m < n \quad (3.18)$$

where Γ_{m-1}^+ is the $(m - 1)$ -dimensional closed forward cone,

$$\Gamma_{m-1}^+(x_m) = \{(x_1, \dots, x_{m-1}) \mid (x_j - x_m)^2 \geq 0, x_j^0 \geq x_m^0, \forall j\}. \quad (3.19)$$

In the difference

$$D_n(x_1, \dots, x_n) \stackrel{\text{def}}{=} R_n - A_n = R'_n - A'_n \quad (3.20)$$

the unknown n -point distribution T_n cancels. Hence this quantity is also known in the inductive step. With the help of the causality conditions (3.9) and (3.10) again, one shows that D_n has causal support

$$\text{supp} D_n \subseteq \Gamma_{n-1}^+(x_n) \cup \Gamma_{n-1}^-(x_n) \quad (3.21)$$

Thus, this crucial support property is preserved in the inductive step from $n - 1$ to n .

Given this fact, the following inductive construction of the n -point distribution T_n becomes possible: Starting off with the known $T_m(x_1, \dots, x_m)$, $m \leq n - 1$, one computes A'_n, R'_n and $D_n = R'_n - A'_n$. With regard to the supports, one can decompose D_n in the following way:

$$D_n(x_1, \dots, x_n) = R_n(x_1, \dots, x_n) - A_n(x_1, \dots, x_n) \quad (3.22)$$

$$\text{supp} R_n \subseteq \Gamma_{n-1}^+(x_n), \quad \text{supp} A_n \subseteq \Gamma_{n-1}^-(x_n) \quad (3.23)$$

Having obtained these quantities we define T'_n as

$$T'_n = R_n - R'_n = A_n - A'_n \quad (3.24)$$

Symmetrizing over the marked variable x_n , we finally obtain the desired T_n ,

$$T_n(x_1, \dots, x_n) = \sum_{\pi} \frac{1}{n!} T'_n(x_{\pi(1)}, \dots, x_{\pi(n)}) \quad (3.25)$$

One can verify that the T_n satisfy the conditions (3.9) , (3.10) and all other further properties of the induction hypothesis [2]. (The Wick formula for the T_n distributions (3.12) will be discussed below.)

3.2 Distribution Splitting.

Let us now discuss the splitting the operator-valued distribution D_n . As it follows from our discussion this is the only nontrivial step in the construction.

Let there be an operator-valued tempered distribution $D_n \in \mathcal{S}'(\mathbf{R}^{4n})$ with causal support,

$$\text{supp} D_n \subseteq \Gamma_{n-1}^+(x_n) \cup \Gamma_{n-1}^-(x_n). \quad (3.26)$$

then the question is whether it is possible to find a pair (R, A) of tempered distributions on \mathbf{R}^{4n} with the following characteristics:

$$\bullet \quad R, A \in \mathcal{S}'(\mathbf{R}^{4n}) \quad (\mathbf{A}) \quad (3.27)$$

$$\bullet \quad R \subset \Gamma^+(x_n), \quad A \subset \Gamma^-(x_n) \quad (\mathbf{B}) \quad (3.28)$$

- $R - A = D \quad (\mathbf{C}) \quad (3.29)$

The EG formalism reduces the usual renormalization program to this mathematically well-defined problem. Every renormalization scheme solves this problem implicitly. For example, the well-known BPHZ renormalization scheme which is often regarded as the most solidly founded explicit renormalization scheme also defines a splitting solution. As mentioned already in the introduction, there is an additional complication, namely the well-known problem of overlapping divergences which is solved by the famous forest formula in the BPHZ framework. The EG formalism provides a natural solution by implementing the causality condition directly on the operator level (see Appendix B).

The problem of distribution splitting has been solved in a general framework by the mathematician Malgrange in 1960 [22]. Epstein and Glaser used his general result for the special case of quantum field theory. A new formulation of the splitting problem was given recently [11].

We mention that the Wick formula for T -products, (3.12) $\forall m < n$, directly implies the corresponding Wick formula for the causal operator-valued distributions at the level n . This is easily shown by the usual Wick theorem for ordinary products of Wick monomials,

$$D_n^{\vec{j}}(X, x_n) = \sum_{\vec{s}} \langle 0 | D_n^{\vec{j}+\vec{s}}(X, x_n) | 0 \rangle \frac{\Phi^{\vec{s}}(X \cup \{x_n\})}{\vec{s}}. \quad (3.30)$$

This formula reduces the splitting problem of operator-valued distributions to the splitting of the numerical C -number distributions

$$d_n^r(x_1 - x_n, \dots, x_{n-1} - x_n) = \langle 0 | D_n^r(X, x_n) | 0 \rangle \quad (3.31)$$

The latter only depends on the relative coordinates because of translation invariance of D_n . Note that the causal support of all numerical distributions is assured by the fact that there are vacuum expectation values of operators with causal support. We can construct well-defined T_n distributions as operators by first splitting the numerical distributions d_n^r and then by defining the T_n 's as operators using the Wick formula (3.12).

The singular behavior of the distribution d_n^r for $x \rightarrow 0$ is crucial for the splitting problem because $\Gamma_{n-1}^+(0) \cap \Gamma_{n-1}^-(0) = 0$. One therefore has to classify the singularities of distributions in this region. This can be carried out with the help of the singular order ω of distributions which is a rigorous definition of the usual power-counting degree. For further details of the theory of distribution splitting we refer to the literature [2, 11] and only make the following **remarks**:

- Since we do not want to increase the singular behavior of the distribution in the splitting process, we further specify the splitting problem by requiring in addition

$$\omega(r) \leq \omega(d) \quad \wedge \quad \omega(a) \leq \omega(d). \quad (\mathbf{D}) \quad (3.32)$$

- Moreover, we have to ask whether the splitting solution of a given numerical distribution d with singular order $\omega(d)$ is unique. Let $r_1 \in \mathcal{S}'$ and $r_2 \in \mathcal{S}'$ be two splitting solutions of the given distribution $d \in \mathcal{S}'$. After construction r_1 and r_2 have their support in Γ^+ and agree with d on $\Gamma^+ \setminus \{0\}$, from which follows that $(r_1 - r_2)$ is a tempered distribution with point support and with singular order $\omega \leq \omega(d)$:

$$\text{supp}(r_1 - r_2) \subset \{0\}, \quad \omega(r_1 - r_2) = \omega(d), \quad (r_1 - r_2) \in \mathcal{S}' \quad (3.33)$$

According to a well-known theorem in the theory of distributions, we have

$$r_1 - r_2 = \sum_{|a|=0}^{\omega_0} C_a \partial^a \delta(x). \quad (3.34)$$

In the case $\omega(d) < 0$ which means that d_n^r is regular at the zero point, the splitting solution is thus unique. In the case $\omega(d) \geq 0$ the splitting solution is only determined up to a local distribution with a fixed maximal singular degree $\omega_0 \leq \omega(d)$. The demands of causality (3.4) and Poincaré invariance (3.7) leave the constants C_a in (3.34) undetermined. They have to be fixed by additional physical normalization conditions such as gauge invariance.

- We want to stress that a normalization ambiguity can also occur in tree graphs. For example the causal Pauli-Jordan distribution $d_1 := D(x - y)$ has singular order $\omega(d_1) = -2$, hence $d_2 := \partial_\mu^x \partial_\nu^y D(x - y)$ has $\omega(d_2) = 0$ (since each derivative increases the singular order by one). This implies that the splitting of d_2 is not unique according to (3.34). Because the normalization ambiguity in tree graphs will become important in our discussion in the next section let us discuss this point in more detail. Note that $[\phi(x), \phi(y)] = i\hbar D(x - y) = i\hbar(D^+ + D^-)$ where D^+ and D^- are the positive, respectively negative frequency parts of the causal Pauli-Jordan distribution D . In R' and A' of equation (3.14) the D^- and $-D^+$ occur in the case of d_1 . The so-called natural splitting of the Pauli-Jordan distribution is given by $D = D_{ret} - D_{adv}$ (see Appendix A). Here D_{red} has retarded support and D_{adv} has advanced support. So $r = D_{ret}$ and $a = D_{adv}$ and then t is defined as $t = r - r' = a - a'$ according to equation (3.24). This means in the case under consideration $t_1 = D_{red} - D^- = D_{adv} + D^+ = D_F$, so finally we end up with the Feynman propagator. Analogously, the graph with the numerical distribution d_2 leads to a t_2 -distribution $t_2 = \partial_\mu^x \partial_\nu^y D_F(x - y) + C g_{\mu\nu} \delta(x - y)$ with the Feynman propagator D_F and a free normalization constant C which has to be fixed by further physical conditions.
- It is important to note that only in tree graphs the Feynman propagator D_F occurs. In loop graphs one gets in r' and a' products of D^+ (or D^-) distributions which are well-defined as the direct product of distributions whose Fourier transform have retarded support (r') and advanced support (a'). This does *not* lead to products of Feynman propagators in the t distribution, as it would be the case if one would use the usual Feynman rules that follows from the formal solution (3.8). For details of explicit splitting solutions in loop graphs we refer to [2, 11].

- We can now discuss the ambiguities on the operator level using the *defining* Wick formula (3.12). The field itself is included in the Wick submonomials one starts with, so we have $[T_1^k(x), \Phi(y)] = 0$ if $x \sim y$ (i.e. if x and y are spacelike separated) according condition (3.10). This implies that T_1^j must be in the Borchers class of the free field $\Phi(x)$. It is well-known that the set of Wick monomials exhausts the Borchers class of a free field. Hence, the Wick monomials in formula (3.12), including the normalization ambiguity in the numerical splitting solutions mentioned above, represents the most general solution of (3.2).
- The Wick formula for time-ordered products (3.12) was used to define the T_n^j distributions (including the Wick submonomials) as well-defined operator-valued distributions. This formula makes transparent that the normalization ambiguity in T_n^j for different j is not independent. Note that the normalization ambiguity of the T_n^j is introduced in this formula through the numerical distributions in (3.12) only. Thus, normalization conditions on different T_n^j might lead to a compatibility problem. We already mention here that the symmetry conditions we analyze in the following are only include the physical $T_n^{j=0}$ distributions, so no such compatibility analysis has to be made.

3.3 Normalizability versus Renormalizability.

The question of (re-)normalizability naturally arises. In the EG approach the question of the normalizability of a quantum field theory consists of the proof of the statement that the number of the constants C_a to be fixed by physical conditions stays the same to all orders in perturbation theory. This means that finitely many normalization conditions are sufficient to determine the S -matrix completely. If the number of the normalization constants increases with the order n of perturbation theory, one usually talks of a non-normalizable theory. The non-standard terminology ‘normalizability’ is used in order to set it apart from the standard usage ‘renormalizability’. In the causal formalism, normalizability represents the quality that the theory is normalizable by a finite number of physical conditions. Normalizability, however, does not necessarily mean that this is also possible when all the symmetry properties of the classical Lagrangian are maintained, a far more reaching quality generally referred as renormalizability. As a consequence, normalizability is a quality solely determined by the scaling properties of the theory. In a second step one tries to prove that there is also a symmetric normalization of the theory. In particular this holds if the loop normalization ambiguity is fixed the same way as the tree-level normalization ambiguity, i.e. if the theory is stable under quantum corrections. By contrast, in other approaches with the help of symmetry preserving regularizations, one uses the consequences of symmetries, the Ward identities of the regularized theory, in order to prove that the theory is normalizable by a finite number of conditions in a symmetry preserving way.

Another possibility is that the theory is not normalizable, but the singular order is bounded in every order in perturbation theory. This means that although the total number of physical conditions needed to fix the S -matrix completely is infinite, nevertheless this number is finite at each order in perturbation theory and therefore the theory still has predictive power. Effective field theories belong to this class of theories. One may call these theories ‘generalized normalizable’. If in addition the theory is stable under quantum correction then we are dealing with ‘generalized renormalizable’ theory.

One may regard the clean separation of the issues of normalizability and renormalizability as one of the advantages of the EG formalism.

Several other properties of the EG formalism are discussed in appendix B.

4 Basic Construction.

4.1 The Quantum Noether Condition.

We shall now present the basic construction of theories with global(=rigid) and/or local symmetries in the EG formalism.

As explained in detail in the last section, one starts with a set of free fields in the asymptotic Fock space. These fields satisfy their (free) field equations and certain commutation relations. To define the theory one still needs to specify T_1 , the first term in the S -matrix. (Actually, as we shall see, even T_1 is not an input in our construction method but is also determined by the Quantum Noether Condition). Given T_1 one can, in a well defined manner, construct iteratively the perturbative S matrix. In this construction, a finite number of constants (in the case of a normalizable theory (see last section)) remains unspecified by the requirements of causality and Poincaré invariance.

We are interested in constructing theories where the S matrix is invariant under certain symmetry operations generated by a well defined operator Q in the asymptotic Fock space,

$$[Q, S] = 0. \tag{4.1}$$

The operator Q acting on asymptotic fields generates their asymptotic transformation rules⁵

$$[Q, \phi^A] = -i\hbar s_0 \phi^A, \tag{4.2}$$

where $[A, B]$ denotes a graded commutator. The latter are necessarily linear in the asymptotic fields. We want to carry out the construction before the adiabatic limit.

⁵The $(-i\hbar)$ in the right hand side of (4.2) as compared to (2.5) is because in (2.5) we have Poisson brackets whereas in (4.2) quantum commutators.

Thus, instead of working with (4.1), we shall require

$$[Q, T_n(x_1, \dots, x_n; \hbar)] = \sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} T_{n/l}^\mu(x_1, \dots, x_n; \hbar) \quad (4.3)$$

for $n \geq 1$ and for some $T_{n/l}^\mu$. We shall often suppress the spacetime arguments in the n -point functions. We shall also use the abbreviation $\partial/\partial x_l^\mu = \partial_\mu^l$. The meaning of the $T_{n/l}^\mu$ will be discussed in detail below. Equation (4.3) for $n=1$

$$[Q, T_1] = \partial_\mu T_{1/1}^\mu, \quad (4.4)$$

imposes restrictions on the starting point of the EG procedure, namely on the couplings T_1 . Once the couplings T_1 has been determined the rest of the equations (4.3) impose relations among the constants left unspecified by the requirement of causality and Poincaré invariance. This is analogous to the situation in the conventional Lagrangian approach where symmetry considerations restrict the possible terms in the Lagrangian and then the same symmetries at the quantum level impose certain relations among the Z factors.

Our considerations apply to the construction of theories with any global or local symmetry. In the case of linear symmetries, such as global internal symmetries or discrete C , P , T symmetries, things are much simpler and one does not need the full machinery developed in this article. This is so because linear symmetries can be directly implemented in the asymptotic Fock space by means of (anti-)unitary transformations. To achieve the invariance of the S -matrix one only needs to start from a coupling T_1 invariant under the corresponding linear symmetry. There is, of course, still the issue of compatibility of the various symmetries imposed. This question will not be analyzed in this article.

The cases of interest here are non-linear symmetries. In this case, the asymptotic transformations differ from the ones the interacting fields have. Such cases are, for example, the BRST symmetry of gauge theories and rigid spacetime symmetries such as supersymmetry. In the latter case, the transformation rules would be linear in the presence of auxiliary fields. However, apart from the fact that auxiliary field are not always known, in the EG formalism the fields are on-shell and, therefore, the auxiliary fields are necessarily absent.

Since different non-linear transformations may have the same linear limit it is not *a priori* obvious whether a theory constructed by EG and satisfies (4.1) has any underlying non-linear structure at all. To address this issue one can work out the precise consequences the operator equation (4.3) has and try to reproduce the Ward identities derived in the Lagrangian approach using the full non-linear transformation. This approach has been followed in [12, 13] for the case of $SU(n)$ gauge theory in the Feynman gauge coupled to fermions where it shown that (4.3) implies the Slavnov-Taylor identities for connected Greens' functions. An alternative and complementary approach is to try to find a direct correspondence between the Lagrangian approach and the EG formalism.

In the conventional Lagrangian approach the theory is defined by giving the Lagrangian and specifying a meaningful way to compute (regularization/renormalization). Our strategy is to identify the Lagrangian within the EG approach. If both approaches describe the same theory, then the perturbative S matrix should be identical in both. The Lagrangian always appear in the S -matrix in the tree-level graphs. We shall, therefore, identify the Lagrangian with the sum of T_1 and the local terms that arise through tree-level normalization conditions (notice that in the EG approach one performs a perturbative expansion around the free action and not around a classical solution of the full theory, so one expects to recover the classical Lagrangian through tree-level graphs). If this correspondence is correct then, for instance, one should be able to understand from the EG point of view why adding a BRST exact term in the Lagrangian does not change the theory. We will indeed see that this can be entirely understood using the EG formalism. If one, in addition, deals with a renormalizable theory then loops do not produce any further local terms from the ones already present in the Lagrangian. Therefore, the question of renormalizability in the Lagrangian approach translates, in the EG formalism, to the question of whether the local normalization ambiguity to all orders reproduces the tree-graph normalizations.

Let us further remark that these considerations also explain why the Lagrangian is such a central object in quantum field theory: according to Epstein-Glaser the perturbative S -matrix is uniquely fixed once one fixes the local ambiguity. In a renormalizable theory, the Lagrangian precisely fixes this local ambiguity.

Our proposal of the construction of theories with global and/or local symmetries in the EG formalism is rather simple. One introduces in addition to T_1 the coupling $g_\mu j_0^\mu$ in the theory, where j_0^μ is the Noether current that generates the asymptotic (linear) symmetry transformations. Actually, as we shall see, the coupling T_1 itself is determined by the construction. In addition, one imposes the condition that the Noether current is conserved at the quantum level, namely inside correlation functions

$$\partial_\mu \mathcal{J}_n^\mu(x_1, \dots, x_n; \hbar) = 0 \quad (4.5)$$

where we introduce the notation

$$\partial_\mu \mathcal{J}_n^\mu(x_1, \dots, x_n; \hbar) = \sum_{l=1}^n \partial_\mu^l \mathcal{J}_{n/l}^\mu, \quad (4.6)$$

and

$$\mathcal{J}_{n/l}^\mu = T[T_1(x_1) \cdots j_0^\mu(x_l) \cdots T_1(x_n)]. \quad (4.7)$$

(for $n = 1$, $\mathcal{J}_1^\mu(x_1) = j_0^\mu(x_1)$). In other words we consider an n -point function with one insertion of the current j_0^μ at the point x_l . Notice that since the left hand side of (4.5) is a series in \hbar , this condition is actually a set of condition; one at each loop level. This construction is so natural that one hardly has to motivate it.

We shall show in the remaining of this section that one can construct using the symmetry condition (4.5) and the free Noether current $\mathcal{J}_1^\mu(x_1) = j_0^\mu(x_1)$ as a starting point, any theory with global/local symmetry that can be viewed as deformation of a free theory. The latter is true for all perturbative QFT's. In addition, we shall establish the equivalence of any theory consistently constructed in the EG formalism with a Lagrangian theory. In the case of interest, namely theories with non-linear symmetry transformations, the corresponding Lagrangian can be assumed to be obtained by the classical Noether method. I.e. the Lagrangian \mathcal{L} and the transformation rules $s\phi^A$ under which it is invariant, are both polynomials in the coupling constant, and are obtained by solving the equations arising in Noether's method. We shall explicitly show that

1. the sum of T_1 and the tree-level normalizations that arise from the requirement (4.5) coincides with the Lagrangian that is invariant under the non-linear transformations. This shows that the full non-linear structure is present in the theory,
2. the free Noether's current j_0^μ is renormalized by the condition (4.5) in such a way that it finally generates the full non-linear transformations,
3. the loop normalization ambiguity is fixed the same way as the tree-level one provided the anomaly consistency condition has only trivial solutions. This means that the theory is stable under quantum correction,
4. the condition (4.5) is equivalent to the condition (4.3). The latter guarantees the invariance of the S -matrix under the corresponding asymptotic symmetry.

The way T_1 and j_0^μ get promoted to the full Lagrangian (point 1) and the full Noether current (point 2), respectively, is completely analogous to the classical Noether method. However, the EG method generates the full quantum theory on the way (point 3), not just a classical Lagrangian invariant under certain classical symmetry. This motivates the title of this article. In particular, the condition (4.5) (or the equivalent one (4.3)) already contains the Ward identities. Points 1 and 2 deal with the condition (4.5) at tree level and point 3 covers the loop analysis. Breaking of (4.5) by loops corresponds to anomalies.

4.2 Off-shell Formulation of the Inductive Hypothesis.

Our goal is to find which are the conditions on the T -products implied by equation (4.5). Since causality and Poincaré invariance uniquely fix the T -products up to local terms, as explained in the section 3, we only need to discuss the conditions imposed to the local normalization ambiguity (4.5). As already mentioned in the introduction, we assume in this paper that a consistent and anomaly-free deformation of the asymptotic symmetry

exists, i.e. we assume that the Quantum Noether method works successfully in the cases we consider. This assumption excludes any true obstructions of the symmetry condition (4.5) on the tree and loop level. A cohomological analysis of possible true obstructions of the condition (4.5) without using the quantum action principle will be presented in a separate paper [7].

We shall follow an iterative approach following the inductive EG construction. Namely, we shall assume that (4.5) is satisfied for all $m < n$, and then we shall examine the conditions implied by (4.5) at n th order. According to section 3 this involves three steps: We first construct the corresponding causal distribution $D_n[j_0 T_1 \cdots T_1]$, then we have to split D_n to obtain $T_{c,n}[j_0 T_1 \cdots T_1]$, and finally we impose (4.5) what leads to conditions on the normalization ambiguity of T_n . The notation T_c indicates that we use the natural splitting solution (discussed in the last section) in tree graph contributions. The latter is our reference solution. When we refer to local normalization terms in tree graph contributions in EG they are always defined with respect to T_c . Points 1 and 2 crucially depend on this choice. There is a good reason, however, why this is what one should do. Only with natural splitting in tree-level graphs the contraction between two fields becomes equal to the Feynman propagator. As we already argued, we shall identify the Lagrangian with local terms in tree-level graphs in the S -matrix. In the Lagrangian approach these graphs have been constructed using Feynman propagators. So, in order to compare the two approaches one has to use the natural splitting solution. It is only for the sake of comparison that the natural splitting solution becomes distinguished. In the analysis of (4.5) at the loop level one may likewise choose a reference splitting solution. In this case, however, there is no ‘preferred’ reference solution, but also no need to explicitly specify one. In the following the subscript c will denote natural splitting in tree-graphs and some fixed reference splitting in loops.

Let us start by noting that having satisfied our fundamental Quantum Noether condition (4.5) for all $m < n$, namely

$$\partial_\mu \mathcal{J}_m^\mu(x_1, \dots, x_m; \hbar) = \sum_{l=1}^m \partial_\mu^l \mathcal{J}_{m/l}^\mu = 0, \quad \forall m < n, \quad (4.8)$$

then the equation (4.5) at the n th order can be violated by a local distribution $A_n(\hbar)$ (which we shall call anomaly term) only:

$$\partial_\mu \mathcal{J}_n^\mu(x_1, \dots, x_n; \hbar) = \sum_{l=1}^n \partial_\mu^l \mathcal{J}_{n/l}^\mu = A_n(\hbar), \quad (4.9)$$

Let us give a short proof of the latter statement: From (4.8) one can derive the analogous condition for the causal distribution \mathcal{D} at the n th order:

$$\sum_{l=1}^n \partial_\mu^l \mathcal{D}_{n/l}^\nu = 0, \quad (4.10)$$

where $\mathcal{D}_{n/l}^\mu$ denotes $\mathcal{J}_{n/l}^\mu$ at the D -level. The latter step is somehow trivial using the basic formulae (3.13)-(3.14) which involve only tensor products of the known T_m products with $m < n$ which fulfill (4.8). Knowing (4.10), we have to split the causal distributions $\mathcal{D}_{n/l}^\nu$: Since the splitting solution $\mathcal{R}_{n/l}$ of $\mathcal{D}_{n/l}$ fulfills $\mathcal{R}_{n/l} = \mathcal{D}_{n/l}$ on $\Gamma^+ \setminus \{(x_n, \dots, x_n)\}$ and $\mathcal{R}_{n/l} = 0$ on $(\Gamma^+)^c$, the symmetry condition can be violated in this process only in the single point (x_n, \dots, x_n) , i.e. by local terms. (Note that it is exactly this point where the \mathcal{R} -distributions are not completely determined, they have some local normalization freedom here.) Therefore, the condition (4.5) at n th order can only be violated by local terms, denoted by $A_n(\hbar)$ in (4.9). The anomaly terms $A(\hbar)$ are a series in \hbar since the left hand side in (4.9) is. In addition, they are restricted by the power counting condition (3.32). Notice that we allow for theories with different (but finite) maximal singular order ω at every order in perturbation theory (i.e. we consider the class of ‘generalized normalizable’ theories, see section 3).

We shall now present an off-shell version of the inductive hypothesis. The assumption that the Quantum Noether method works successfully means that there exists local normalizations such that (4.8) is satisfied when the field equations are satisfied. This does not mean, however, that (4.8) is satisfied when any splitting is used. Actually, generically after natural splitting (this refers to tree-level graphs, for loop graphs one uses some reference splitting solution) one would end up with

$$\partial_\mu \mathcal{J}_{c,m}^\mu = A_{c,m}, \quad (4.11)$$

where the subscript c indicates that the natural splitting has been used at tree-graphs. Our assumption only means that the anomaly $A_{c,m}$ is a divergence up to terms B_m that vanish when the free field equations are used, i.e.

$$A_{c,m} = \partial_\mu A_{c,m}^\mu + B_m \quad (4.12)$$

where $A_{m,c}^\mu$ and B_m are some local distributions (since $A_{c,m}$ is local). This decomposition is not unique since one can move derivatives of field equation terms from B_m to $A_{c,m}^\mu$ ⁶. We fix this freedom by demanding that B_m does not contain any derivatives of field equations. Let us show this explicitly. To derive the general form of B_m we first note that it should have the general form of a local distribution symmetric in all variables⁷

$$B_m = O_{1,m} \delta^{(m)} + \sum_{k=1}^m \partial_\kappa^k \partial_\lambda^k (O_{2,m}^{\kappa\lambda} \delta^{(m)}) + \sum_{k=1}^m \partial_\kappa^k \partial_\lambda^k \partial_\mu^k (O_{3,m}^{\kappa\lambda\mu} \delta^{(m)}) + \dots \quad (4.13)$$

This expression is derived by moding out the relation

$$\left(\sum_{i=1}^m \partial_\mu^i \right) \delta(x_1, \dots, x_m) = 0 \quad (4.14)$$

⁶We shall see below that this arbitrariness is related to the natural arbitrariness in the definition of the classical Noether current.

⁷We use the following abbreviations for the delta function distributions $\delta^{(m)} = \delta(x_1, \dots, x_m) = \delta(x_1 - x_2) \cdots \delta(x_{m-1} - x_m)$.

from the general form of a local distribution. A detailed derivation will appear in [7]. Notice that because the power counting degree is bounded at each order in perturbation theory the series terminates after a finite number of terms. The operators $O_{i,m}$, $i = 1, 2, \dots$, are in general unrestricted, but in our case they should be such that B_m vanishes when the free field equations are satisfied. This means in particular that $O_{1,m}$ has the form⁸

$$O_{1,m} = S^{A;m} \mathcal{K}_{AB} \phi^B + \sum_p S_{\mu_1 \dots \mu_p}^{A;m} \partial^{\mu_1} \dots \partial^{\mu_p} \mathcal{K}_{AB} \phi^B \quad (4.15)$$

where

$$\mathcal{K}_{AB} \phi^B = \partial^\mu \frac{\partial \mathcal{L}_0}{\partial (\partial^\mu \phi^A)} - \frac{\partial \mathcal{L}_0}{\partial \phi^A} \quad (4.16)$$

are the free field equations. Remembering that (4.13) is a distributional relation, one may integrate by parts the derivatives from the field equations to obtain

$$O'_{1,m} = (S^{A;m} + \sum_p (-1)^p \partial^{\mu_1} \dots \partial^{\mu_p} S_{\mu_1 \dots \mu_p}^{A;m}) \mathcal{K}_{AB} \phi^B, \quad (4.17)$$

a new term of the form $\partial(O\delta)$ and appropriate modifications of the $O_{i,m}$, $i > 1$. Notice now that one may always factor out a derivative from the terms involving the $O_{i,m}$, $i > 1$. In addition, the new term is also a total derivative term. This means that these terms can be moved into $A_{c,n}^\mu$. The latter is finally removed by appropriately fixing the local normalization freedom of the left hand side. Let us also define

$$R^{A;m} = S^{A;m} + \sum_p (-1)^p \partial^{\mu_1} \dots \partial^{\mu_p} S_{\mu_1 \dots \mu_p}^{A;m} \quad (4.18)$$

An additional ambiguity is related to the global symmetries of the free action. If one makes the transformation

$$A_{c,m}^\mu \rightarrow A_{c,m}^\mu + j_a^\mu; \quad R^{A;m} \rightarrow R^{A;m} - s_a \phi^A \quad (4.19)$$

where j_a^μ is a Noether current that generates the symmetry transformations $s_a \phi^A$, then the right hand side of (4.12) remains unchanged. To fix this ambiguity we demand that $R^{A;m}$ do not contain any summand which is itself a symmetry transformation of the free action. (In practice, one would never have to deal with this problem unless one does by hand the substitutions (4.19)).

In this manner we are led to the following off-shell representation of the inductive hypothesis: for $m < n$,

$$\sum_{l=1}^m \partial_\mu^l \mathcal{J}_{m/l}^\mu = \sum_A R^{A;m}(\hbar) \mathcal{K}_{AB} \phi^B \delta(x_1, \dots, x_m). \quad (4.20)$$

⁸The polynomial character of the T_1 excludes “exotic behavior” like square root factors or terms of the form $\exp(-1/(\mathcal{K}_{AB} \phi^B)^2)$.

The coefficients $R^{A;m}(\hbar)$ may, in general, receive tree and loop contributions. We shall show below that this off-shell representation provides an alternative and simplified way of obtaining local terms arising from tree-level graphs.

We first concentrate on analyzing the condition (4.5) at tree-level. We shall consider the loop case afterwards. We therefore only need the \hbar^0 part of (4.20). Let us define

$$s_{(m-1)}\phi^A = \frac{1}{m!}R^{A;m}(\hbar^0); \quad m > 1, \quad (4.21)$$

(we shall see below that this formula also holds for $m = 1$) Depending on the theory under consideration the quantities $R^{A;m}(\hbar^0)$ may be zero after some value of m . Without loss of generality we assume that they are zero for $m > k + 1$, for some integer k (which may also tend to infinity; the same applies for k' below.). We shall show below that

$$s\phi^A = \sum_{m=0}^k g^m s_m \phi^A \quad (4.22)$$

are symmetry transformation rules that leave invariant the Lagrangian

$$\mathcal{L} = \sum_{m=0}^{k'} g^m \mathcal{L}_m, \quad (4.23)$$

where k' is also an integer (generically not equal to k). The Lagrangian \mathcal{L} will be determined from the tree-level normalization conditions as follows,

$$\mathcal{L}_m = \frac{\hbar}{i} \frac{N_m}{m!}, \quad \text{for } m > 1, \quad (4.24)$$

where N_m denotes the local normalization ambiguity of $T_m[T_1(x_1)\dots T_1(x_m)]$ in tree graphs defined with respect to the naturally splitted solution. For $m = 1$, $\mathcal{L}_1 = (\hbar/i)T_1$. The factor $m!$ reflects the fact that $T_m[\dots]$ appears in (3.2) with a combinatorial factors $m!$ while the factor \hbar/i is there to cancel the overall factor i/\hbar that multiplies the action in the tree-level S -matrix. Notice that we regard (4.24) as definition of \mathcal{L}_m .

To understand how the off-shell formulation simplifies the calculation of local terms A arising from tree-level graphs we start by first describing the traditional way of doing such a calculation. In order to obtain the local terms, one first constructs $T_{c,n}[j_0^\mu(x_1)T_1(x_2)\dots T_1(x_n)]$, differentiates with respect to the variable of the current and symmetrizes in all variables. $T_{c,n}[j_0^\mu(x_1)T_1(x_2)\dots T_1(x_n)]$ involves many terms and there will be a large number of cancellations after differentiating and symmetrizing. In particular, we already know from equation (4.9) that all non-local terms will cancel among themselves. So the idea (which gets implemented with the help of the off-shell formulation) is to only concentrate on possible local terms anticipating the cancellation of all non-local terms.

From (3.13), (3.14) we know that the causal distribution $D_n[j_0^\mu(x_1)T_1(x_2)\dots T_1(x_n)]$ is equal to the sum of terms which are products of T -products constructed at lower orders

one of which contains j_0^μ as a vertex. To calculate $T_{c,n}[j_0^\mu(x_1)T_1(x_2)\dots T_1(x_n)]$ one first does all contractions in D_n and then splits the solution. Each tree-level contraction between two fields ϕ^A and ϕ^B yields after natural splitting a factor of \mathcal{D}^{AB} , where \mathcal{D}^{AB} is the inverse of the corresponding kinetic operator \mathcal{K}_{AB} ,

$$\frac{i}{\hbar}\mathcal{K}_{AB}\mathcal{D}^{BC} = \delta_A^C \quad (4.25)$$

where δ_A^C contains a delta function, and we have included also the i and \hbar factors (see appendix A). Let us also denote by \mathcal{O}_{AB} the operator associated with the fields ϕ^A, ϕ^B that satisfies $\partial_\mu\mathcal{O}_{AB} = \mathcal{K}_{AB}$ (for bosons $\mathcal{O} \sim \partial^\mu$, for fermions $\mathcal{O} \sim \gamma^\mu$). Note that we treat a possible mass term as perturbation in order to simplify the presentation of the argument. The expression for $T_{c,n}[j_0^\mu(x_1)T_1(x_2)\dots T_1(x_n)]$ will contain, among other terms, terms of the form ⁹

$$S^1(x)S^2(y)\mathcal{O}_{AB}(x)\mathcal{D}^{BC}(x-y), \quad (4.26)$$

where x is the variable of the current and S^1, S^2 are local terms at order m ($m < n$), respectively, $n-m$. Upon differentiating with ∂_μ^x this term will bring among other terms the local term

$$S^1S^2\delta_A^C\delta(x-y). \quad (4.27)$$

(we have now explicitly written the delta function to emphasize that this is a local term). The mechanism we just described is the only one that creates local terms out of tree-level graphs. Diagrammatically, after we push the derivative in we get an inverse propagator that cancels the propagator between x and y , thus, rendering the graph local. The fact that S^1 and S^2 are local is, of course, essential. Local terms proportional to derivatives of the δ distribution are constructed in an analogous way.

The term (4.26) originated from the following T -product

$$T_{c,n}[(S^1\mathcal{O}_{AB}\phi^B\delta^{(m)})(x)(S^2\phi^C)\delta^{(n-m)}(y)], \quad (4.28)$$

upon contraction between ϕ^B and ϕ^C . We are ultimately interested in computing $\partial_\mu T_{c,n}[j_0^\mu(x_1)T_1(x_2)\dots T_1(x_n)]$, so we need to compute $\partial_\mu T_{c,n}[(S^1\mathcal{O}_{AB}\phi^B\delta^{(m)})(x)(S^2\phi^C)\delta^{(n-m)}(y)]$. Moving the derivative inside the correlation function we get (among other terms)

$$T_{c,n}[(S^1\mathcal{K}_{AB}\phi^B)(x)(S^2\phi^C)(y)] \quad (4.29)$$

Upon contracting the ϕ^B with an ϕ in $(S^2\phi^C)$ one obtains a local term. In particular, the local term in (4.27) is simply obtained from the contraction between the explicit ϕ 's

⁹In (4.26), as well as in later formulae, Wick-ordering is always understood. We also suppress two delta distributions that set n of the variables equal to x and the remaining $n-m$ equal to y . In order to keep the notation as simple as possible we shall often suppress such delta distributions. In all cases one may insert these delta distribution by simple inspection of the formulae.

(i.e. the ϕ^B and ϕ^C) in (4.29). We shall call this kind of contractions, namely the ones that involve the field ϕ^B in the field equation $\mathcal{K}_{AB}\phi^B$, “correct contractions”. These yield the local terms that one obtains by doing the calculation in the “traditional way”. All other contractions (“wrong contractions”) generically yield non-local terms proportional to field equations. These are not relevant in our case and they will be discarded.

In this way we are led to an alternative and more systematic way of obtaining all tree-level local terms. One first differentiates and then do the “correct contractions”. However, in doing the calculation in this way one should remember that one should *not* use the field equations before the end of the calculation. The terms that are proportional to the field equations are the source of the local terms.

Let us now make contact with the off-shell formulation of the induction hypothesis. After differentiation, the causal distribution $\sum_{l=1}^n \partial_\mu^l \mathcal{D}_{n/l}^\mu = 0$ at the n 'th order consists of a sum of terms each of these being a tensor product of $T_m[T_1 \dots T_1 \partial \cdot j_0 T_1 \dots T_1]$ ($m < n$) with T -products that involve only T_1 vertices according to the general formulae (3.13,3.14,3.20). By the off-shell induction hypothesis, we have for all $m < n$

$$\sum_{l=1}^m \partial_\mu^l \mathcal{J}_{m/l}^\mu = \sum_A (m! s_{m-1} \phi^A) \mathcal{K}_{AB} \phi^B \delta^{(m)}. \quad (4.30)$$

At order n the “correct contractions”, namely the contractions between the ϕ^B in the right hand side of (4.30) and ϕ in local terms, yield the seek-for local terms. This implies, in particular, that that no local term arises from terms in D_n that are products of more than two T products. This is accordance with the diagrammatic picture of creation of local terms that we mentioned above. In this manner we get the following general formula for the local term $A_{c,n}$ arising through tree-level contractions at level n ,

$$A_{c,n}(\hbar^0) = \sum_{\pi \in \Pi^n} \sum_{m=1}^{n-1} \partial_\mu \mathcal{J}_m^\mu(x_{\pi(1)}, \dots, x_{\pi(m)}) N_{n-m} \delta(x_{\pi(k+1)}, \dots, x_{\pi(n)}) \quad (4.31)$$

where it is understood that in the right hand side only “correct contractions” are made. The factors N_{n-m} are tree-level normalization terms of the T -products that contain $n-m$ T_1 vertices.

4.3 Analysis of the Quantum Noether Condition at Tree-level.

In this section we analyze the formula (4.31) for all n .

For $n = 1$, we have $\mathcal{J}_1^\mu = j_0^\mu$. Then from (2.3) follows that

$$\partial_\mu \mathcal{J}_1^\mu = s_0 \phi^A K_{AB} \phi^B. \quad (4.32)$$

Therefore, $R^{A;1} = s_0 \phi^A$ as we promised.

Let us now move to the $n = 2$ case. From our previous discussion follows immediately that

$$\partial_\mu^{x_1} T_{c,2}[j_0^\mu(x_1)\phi^A(x_2)] = \frac{\hbar}{i} s_0 \phi^A \delta(x_1 - x_2) \quad (4.33)$$

where the factor (\hbar/i) originate from the contraction (see (4.25)). Using the derivation property of single Wick contractions we immediately get that for any local function $f(\phi^A)$ of the fields that do *not* contain derivatives of the fields a similar relation holds,

$$\partial_\mu^{x_1} T_{c,2}[j_0^\mu(x_1)f(\phi^A)(x_2)] = \frac{\hbar}{i} s_0 f(\phi^A) \delta(x_1 - x_2). \quad (4.34)$$

Let us now consider derivative terms. In this case,

$$\partial_\mu^{x_1} T_{c,2}[j_0^\mu(x_1)(\partial_k \phi^A)(x_2)] = \frac{\hbar}{i} (s_0 \phi^A)(x_1) \partial_k^{x_2} \delta(x_1 - x_2) \quad (4.35)$$

Symmetrizing¹⁰ this expression with respect to x_1 and x_2 and using the distributional identity

$$a(x_1) \partial_{x_2} \delta(x_1 - x_2) + a(x_2) \partial_{x_1} \delta(x_1 - x_2) = (\partial a) \delta(x_1 - x_2) \quad (4.36)$$

we get

$$\partial_\mu^{x_1} T_c[j_0^\mu(x_1)(\partial_k \phi^A)(x_2)] + \partial_\mu^{x_2} T_c[(\partial_k \phi^A)(x_1)j_0^\mu(x_2)] = \frac{\hbar}{i} \partial_k (s_0 \phi^A) \delta(x_1 - x_2). \quad (4.37)$$

Notice that the corresponding relation with $(\partial_k \phi^A)$ replaced by ϕ^A has an extra factor of 2 in the right hand side. Combining these results we obtain¹¹ (we use the notation introduced in (4.6))

$$\partial_\mu \mathcal{J}_{c,2}^\mu(x_1, x_2) = \frac{\hbar}{i} (2s_0 T_1 - \partial^\mu (\frac{\partial T_1}{\partial (\partial_\mu \phi^A)} s_0 \phi^A)) \delta(x_1 - x_2) = A_{c,2} \quad (4.38)$$

Inserting the definition of $\mathcal{L}_1 = (\hbar/i)T_1$ the right hand side of (4.38) becomes real and independent of \hbar .

Our objective is to fix the tree-level normalization freedom in respect to $\mathcal{J}_{c,2}^\mu$ such that (4.5) holds (when the free field equations are satisfied). This is possible if and only if

$$s_0 \mathcal{L}_1 = \partial_\mu \mathcal{L}_1^\mu + \frac{1}{2} B_2 \quad (4.39)$$

where B_2 vanishes when the free field equations are satisfied (the factor 1/2 has been inserted such that we agree with (4.12)). By our assumption-the Quantum Noether Method works successfully- there is a pair $(\mathcal{L}_1, \mathcal{L}_1^\mu)$ which solves (4.39). We emphasize

¹⁰The current j_0^μ contains the parameter of the transformation rule and is therefore bosonic. If one were to use the current without the parameter then one would need to graded-symmetrize in x_1 and x_2 .

¹¹In this article we consider only theories that depend arbitrarily on ϕ^A and its first derivative $\partial_\mu \phi^A$. One could also study theories that depend on higher derivatives of ϕ^A , $\partial_{m_1} \dots \partial_{m_p} \phi^A$, for some integer p . However, it is not clear whether such theories have any relevance to the physical world, so we will not consider these theories in this article.

that also $\mathcal{L}_1 = (\hbar/i)T_1$ is not an input in our construction method but also is determined by the Quantum Noether condition (4.5). So only the free Noether current j_0^μ is used as defining equation in the Quantum Noether method. Going back to (4.38) we obtain

$$\begin{aligned}\partial_\mu \mathcal{J}_{c,2}^\mu(x_1, x_2) &= 2!(s_1 \phi^A) \mathcal{K}_{AB} \phi^B \delta(x_1 - x_2) \\ &\quad - \partial_\mu \left(-2\mathcal{L}_1^\mu + \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \phi^A)} s_0 \phi^A \right) \delta(x_1 - x_2)\end{aligned}\quad (4.40)$$

According to (4.21) the latter equation defines s_1 .

Now we consider the normalization ambiguity of $T_2[j_0 T_1]$,

$$T_2[j_0^\mu(x_1)T_1(x_2)] = T_{c,2}[j_0^\mu(x_1)T_1(x_2)] + j_1^\mu \delta(x_1 - x_2)\quad (4.41)$$

Demanding that

$$\partial_\mu \mathcal{J}_2^\mu(x_1, x_2) = 2!s_1 \phi^A \mathcal{K}_{AB} \phi^B \delta(x_1 - x_2)\quad (4.42)$$

we obtain

$$j_1^\mu = \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \phi^A)} s_0 \phi^A - 2\mathcal{L}_1^\mu.\quad (4.43)$$

The discussion for $3 \leq m \leq k+1$ goes the same way as the $n=2$ case. However, it is instructive to also directly work out the $n=3$ case as it is still relatively easier than the general case but sufficiently more complicated than the $n=2$ case. Using (3.13) and (3.14) for $n=3$ and afterwards naturally splitting one gets

$$\partial_\mu \mathcal{J}_{c,3}^\mu(x_1, x_2, x_3) = \partial_\mu \mathcal{J}_2^\mu(x_1, x_2)T_1(x_3) + \partial_\mu \mathcal{J}_1^\mu(x_1)N_2(x_2, x_3) + \text{cyclic in } x_1, x_2, x_3,\quad (4.44)$$

(this is equation (4.31) for $n=3$). In writing this expression on the right hand side we have discarded all terms that do not contribute any local terms. In particular, it is understood that only ‘‘correct contractions’’ are made. N_2 denotes the tree-normalization term of T_2 which is uniquely defined in respect to $T_{c,2}$.

Using our previous results (4.32), (4.42) and remembering that the derivative terms should be treated with care we obtain

$$\begin{aligned}\partial_\mu \mathcal{J}_{c,3}^\mu(x_1, x_2, x_3) &= [3!(s_1 \mathcal{L}_1 + s_0 \mathcal{L}_2) \\ &\quad - 2! \partial_\mu \left(2 \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \phi^A)} s_1 \phi^A + \frac{\partial \mathcal{L}_2}{\partial(\partial_\mu \phi^A)} s_0 \phi^A \right)] \delta(x_1, x_2, x_3)\end{aligned}\quad (4.45)$$

where we have used the definition in (4.24). The Quantum Noether condition (4.5) is satisfied if and only if

$$s_1 \mathcal{L}_1 + s_0 \mathcal{L}_2 = \partial_\mu \mathcal{L}_2^\mu + s_2 \phi^A \mathcal{K}_{AB} \phi^B\quad (4.46)$$

Substituting back in (4.45) we obtain

$$\begin{aligned}\partial_\mu \mathcal{J}_{c,3}^\mu(x_1, x_2, x_3) &= 3!(s_2 \phi^A) \mathcal{K}_{AB} \phi^B \delta(x_1, x_2, x_3) \\ &\quad - \partial_\mu \left[-3! \mathcal{L}_2^\mu + 2! \left(\frac{\partial \mathcal{L}_2}{\partial(\partial_\mu \phi^A)} s_0 \phi^A + 2 \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \phi^A)} s_1 \phi^A \right) \right] \delta(x_1, x_2, x_3)\end{aligned}\quad (4.47)$$

In a similar way as before we define

$$T_3[j_0^\mu(x_1)T_1(x_2)T_1(x_3)] = T_{c,3}[j_0^\mu(x_1)T_1(x_2)T_1(x_3)] + j_2^\mu\delta(x_1, x_2, x_3) \quad (4.48)$$

Then

$$\partial_\mu \mathcal{J}_3^\mu(x_1, x_2, x_3) = 3!s_2\phi^A\mathcal{K}_{AB}\phi^B\delta(x_1, x_2, x_3) \quad (4.49)$$

provided

$$j_2^\mu = -3!\mathcal{L}_2^\mu + 2!\left(\frac{\partial\mathcal{L}_2}{\partial(\partial_\mu\phi^A)}s_0\phi^A + 2\frac{\partial\mathcal{L}_1}{\partial(\partial_\mu\phi^A)}s_1\phi^A\right) \quad (4.50)$$

The general case for $1 \leq n \leq k+1$ can be worked out in a completely analogous way. The result for the current is

$$j_{n-1}^\mu = -n!\mathcal{L}_{n-1}^\mu + (n-1)!\sum_{l=0}^{n-2}(l+1)\frac{\partial\mathcal{L}_{n-1-l}}{\partial(\partial_\mu\phi^A)}s_l\phi^A \quad (4.51)$$

where we have used the definition in (4.24). To derive this result one may use following distributional identity

$$\begin{aligned} & \sum_{\pi \in \Pi^n} \delta(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(k)}) A \partial_{x_{\pi(k)}} (B \delta(x_{\pi(k)}, x_{\pi(k+1)}, \dots, x_{\pi(n)})) = \\ & \left[\binom{n}{k} A \partial B - \binom{n-1}{n-k-1} \partial (AB) \right] \delta(x_1, \dots, x_n) \end{aligned} \quad (4.52)$$

where $\binom{n}{k}$ is the binomial coefficient. In addition,

$$\partial_\mu \mathcal{J}_n^\mu = n!s_{n-1}\phi^A\mathcal{K}_{AB}\phi^A\delta(x_1, \dots, x_n) \quad (4.53)$$

Let us now move to the $n > k+1$ case. (In the case of ‘‘generalized normalizable’’ theories only the analysis of the case $n \leq k+1$ is present since k tends to infinity). One gets

$$\begin{aligned} \partial_\mu \mathcal{J}_{c,n}^\mu &= n![s_0\mathcal{L}_{n-1} + s_1\mathcal{L}_{n-2} + \dots + s_k\mathcal{L}_{n-1-k}] \\ &\quad - (n-1)!\partial^\mu \sum_{l=1}^k l \frac{\partial\mathcal{L}_{n-l}}{\partial(\partial_\mu\phi^A)} s_{l-1}\phi^A \end{aligned} \quad (4.54)$$

where again use of (4.52) has been made. This equation now implies that

$$s_0\mathcal{L}_{n-1} + s_1\mathcal{L}_{n-2} + \dots + s_k\mathcal{L}_{n-1-k} = \partial_\mu \mathcal{L}_{n-1}^\mu. \quad (4.55)$$

One achieves

$$\partial_\mu \mathcal{J}_n^\mu = 0 \quad (4.56)$$

by renormalizing the current as

$$j_{n-1}^\mu = -n! \mathcal{L}_{n-1}^\mu + (n-1)! \sum_{l=1}^k l \frac{\partial \mathcal{L}_{n-l}}{\partial (\partial_\mu \phi^A)} s_{l-1} \phi^A \quad (4.57)$$

and without the need to use the free field equations. Depending on the theory under consideration the \mathcal{L}_n will be zero for $n > k'$, for some integer k' . Given the integers k and k' , there is also an integer k'' (determined from the other two) such that $\mathcal{L}_n^\mu = 0$, for $n > k''$.

Let us recapitulate. We have calculated all local terms that arise from the tree level diagrams. Summing up the necessary and sufficient conditions (4.39), (4.46), (4.55) for the Quantum Noether method to hold at tree level we obtain,

$$s \sum_{l=1}^{k'} g^l \mathcal{L}_l = \sum_{l=1}^{k''} \partial_\mu \mathcal{L}_l^\mu + \left(\sum_{l=1}^k g^l s_l \phi^A \right) \mathcal{K}_{AB} \phi^B \quad (4.58)$$

Using $s_0 \mathcal{L}_0 = \partial_\mu k_0^\mu$ and for $l \leq k$

$$s_l \phi^A \mathcal{K}_{AB} \phi^B = \partial_\mu \left(\frac{\partial \mathcal{L}_0}{\partial (\partial_\mu \phi^A)} s_l \phi^A \right) - s_l \mathcal{L}_0 \quad (4.59)$$

we obtain,

$$s \mathcal{L} = \partial_\mu \left(\sum_{l=0}^{k''} g^l k_l^\mu \right) \quad (4.60)$$

where, for $1 < l \leq k$,

$$k_l^\mu = \mathcal{L}_l^\mu + \frac{\partial \mathcal{L}_0}{\partial (\partial_\mu \phi^A)} s_l \phi^A \quad (4.61)$$

and for $l > k$, $k_l^\mu = \mathcal{L}_l^\mu$. We therefore find that \mathcal{L} is invariant under the symmetry transformation,

$$s \phi^A = \sum_{l=0}^k g^l s_l \phi^A. \quad (4.62)$$

According to Noether's theorem there is an associated Noether current given by (2.9). Using (4.61) one may check that the current normalization terms j_m^μ ((4.43), (4.50), (4.51), (4.57)) are in one-to-one correspondence with the terms in the Noether current¹²

¹²As we discussed in section 2, there is some natural arbitrariness in the definition of the classical Noether current. The one related to the addition of field equations terms is directly related to the arbitrariness in moving (free) field equation terms from B_m to $A_{c,m}^\mu$ in (4.12). This was fixed by writing (4.20). Had we have started with (4.20) with an additional term with a derivative of the free field equation we would have ended up with an addition of a full (non-linear) equation term in the final (non-linear) Noether current. The arbitrariness of adding a divergence of an antisymmetric tensor to the current is also present (but implicitly fixed in the main text); when we fix the current normalization terms in (4.43), (4.50), (4.51), (4.57) we could also add such a term (since it lies in kernel of the derivative). Finally, the ambiguity (4.19) of (4.12) related to global symmetries of the free action reflects the fact that one could try to develop a theory with more than one global symmetry. In this paper we only discuss the case of one non-linear global symmetry. In the case of more than one symmetries one would still have to address the question of compatibility.

(compare with (2.9)); they only differ by combinatorial factors related to the perturbative expansion. Therefore the current j_0 indeed renormalizes to the full non-linear current. This finishes the proof of points 1 and 2.

4.4 Analysis of the Quantum Noether Condition at Loop-level.

We now move to point 3 and consider what happens in loops. If the local normalization ambiguity to all orders in \hbar reproduces the tree-graph normalizations then we are dealing with a (generalized) renormalizable theory. We have seen that the tree level analysis leads to the formula (4.31) for the local terms at order n . At higher loop level this formula is not correct any longer. In addition to the tree level terms present in (4.31) one also has loop graphs between the correlation function that contains the j_0^μ vertex and the rest. Furthermore, the transformation rules may receive quantum correction, i.e. $R^{A;m}$ is a series in \hbar . Symbolically one has

$$\begin{aligned} A_{c,n}(\hbar^M) &= \sum_{m_1+m_2=n} \{ (R^{A;m_1}(\hbar^0) \mathcal{K}_{AB} \phi^B \delta^{(m_1)} N_{m_2}(\hbar^M) \delta^{(m_2)}) \\ &+ \sum_{M_1+M_2+M_3=M} [R^{A;m_1}(\hbar^{M_1}) N_{m_2}(\hbar^{M_2})] (M_3 \text{ loops}) \} \end{aligned} \quad (4.63)$$

where $N_n(\hbar^M)$ denotes the local normalization freedom of $T_n[T_1(x_1)\dots T_1(x_n)]$ in the M -th loop level. In the first line in right hand side of (4.63) tree-level ‘‘correct contractions’’ are understood. These terms correspond to the right hand side of (4.31). The second line in (4.63) contains the new terms on top of the ones present in (4.31). We shall collectively denote these terms L_n .

The analysis of the first line in right hand side of (4.63) is exactly the same as the tree-level analysis. The loop terms L_n in (4.63) are also local terms. Their general form is therefore of the form (4.13),

$$L_n = n! L_{1,n} \delta^{(n)} + \sum_{k=1}^n \partial_\kappa^k \partial_\lambda^k (L_{2,n}^{\kappa\lambda} \delta^{(n)}) + \sum_{k=1}^n \partial_\kappa^k \partial_\lambda^k \partial_\mu^k (L_{3,n}^{\kappa\lambda\mu} \delta^{(n)}) + \dots \quad (4.64)$$

(the $n!$ in the first term was added for later convenience). The terms involving $L_{i,n}$, $i > 1$ are total derivative terms. We remove those by appropriately fixing the local normalization freedom of the current correlation function. Hence, we obtain

$$A_{n,c}(\hbar^M) = n! [(s_0 \mathcal{L}_{n-1}(\hbar^M) + \dots + s_k \mathcal{L}_{n-1-k}(\hbar^M)) + L_{1,n}(\hbar^M)] \delta^{(n)} \quad (4.65)$$

where we have extended the definition (4.24) to cover also the loop case. From our assumption-the Quantum Noether method works successfully-follows

$$s_0 \mathcal{L}_{n-1}(\hbar^M) + \dots + s_k \mathcal{L}_{n-1-k}(\hbar^M) + L_{1,n}(\hbar^M) = \partial_\mu C_n^\mu(\hbar^M) \quad (4.66)$$

for some C_n^μ . Summing up these relations we obtain,

$$s\mathcal{L}(\hbar^M) + L_1(\hbar^M) = \partial_\mu C^\mu(\hbar^M) \quad (4.67)$$

where we have defined $L_1 = \sum_n L_{1,n}$, $C = \sum_n C_n$ and we have extended the definition (4.23) at the loop level. Equation (4.67) constrains the local terms $N(\hbar^M)(= \sum_n N_n(\hbar^M))$. Notice that L_1 only depends on $N(\hbar^K)$ for $K < M$ (this follows from simple \hbar counting, see (4.63)). A sufficient and necessary condition for loop terms to satisfy the same condition as the tree-level terms is

$$L_1(\hbar^M) = sM(\hbar^M) + \partial_\mu M^\mu(\hbar^M) \quad (4.68)$$

If this relation is satisfied then the renormalized local terms $\mathcal{L}'(\hbar^M) = \mathcal{L}(\hbar^M) + M(\hbar^M)$ satisfy

$$s\mathcal{L}'(\hbar^M) = \partial_\mu C'_\mu \quad (4.69)$$

where $C'_\mu = C_\mu - M_\mu$, i.e. the same equation as (4.60) that the tree-level normalizations satisfy, and the theory is stable.

If we are considering a BRST-like symmetry then (4.68) will always be satisfied if $H_1(s, d) = 0$. We can rephrase this condition by saying that (4.68) will always be true if the anomaly consistency condition $sA = dB$ has only trivial solutions $A = sA_1 + dB_1$.

The general case goes along the similar lines. Consider the case where the algebra of the symmetry transformation is given by

$$[s(\epsilon), s(\eta)]\phi^A = s(\epsilon \times \eta)\phi^A \quad (4.70)$$

(for simplicity, we consider closed symmetry algebra) where $s(\epsilon)\phi^A = (s\phi^A)^a \epsilon_a$, ϵ_a and η_a are the parameters of the symmetry transformations, a is an algebra index, $(\epsilon \times \eta)^a = f^a_{bc} \epsilon^b \eta^c$, and f^a_{bc} are the structure constants of the algebra.

Wess and Zumino [23] have worked out the anomaly consistency condition for this case. For the integrated anomaly this condition reads

$$A_{int}(\epsilon \times \eta) = s(\epsilon)A_{int}(\eta) - s(\eta)A_{int}(\epsilon) \quad (4.71)$$

where $A_{int}(\epsilon) = \int A(\epsilon) = \int A^a \epsilon_a$. In terms of the local anomaly $A(\epsilon)$ the above condition reads,

$$A(\epsilon \times \eta) - \partial_\mu A^\mu(\epsilon \times \eta) = s(\epsilon)(A(\eta) - \partial_\mu A^\mu(\eta)) - s(\eta)(A(\epsilon) - \partial_\mu A^\mu(\epsilon)) \quad (4.72)$$

for some A^μ . One may easily check that

$$A(\epsilon) = s(\epsilon)K + \partial_\mu A^\mu \quad (4.73)$$

is a solution of (4.72) for any K . In the case of nilpotent symmetries, $f^{abc} = 0$, equation (4.72) implies that the anomaly is s -closed. Furthermore, the trivial solution (4.73) correspond to an exact solution.

Assuming that (4.72) has only the (trivial) solution (4.73) we now show that the theory is stable. Consider (4.67). Let us first make explicit in our notation the parameter of the transformation and suppress the \hbar^M as our considerations hold at any order in \hbar ,

$$s(\eta)\mathcal{L} + L_1(\eta) = \partial_\mu C^\mu(\eta) \quad (4.74)$$

Act in this equation with $s(\epsilon)$ and antisymmetrize in ϵ and η . Using (4.70) we get

$$s(\epsilon \times \eta)\mathcal{L} + s(\epsilon)L_1(\eta) - s(\eta)L_1(\epsilon) = \partial_\mu(s(\epsilon)C^\mu(\eta) - s(\eta)C^\mu(\epsilon)) \quad (4.75)$$

Using again (4.74) to eliminate $s(\epsilon \times \eta)\mathcal{L}$ we obtain (4.72) but with $A \rightarrow L_1, A^\mu \rightarrow C^\mu$. Therefore, since by assumption this equation has only the trivial solution (4.73) we obtain

$$L_1 = sM + \partial_\mu C^\mu \quad (4.76)$$

which just (4.68) with $M^\mu \rightarrow C^\mu$. This finishes the proof that the theory is stable if the anomaly consistency condition has only trivial solutions.

4.5 Invariance of the S -matrix.

In this subsection we analyze point 4, namely the question of equivalence between the conditions (4.5) and (4.3). Having established this equivalence the invariance of the S -matrix follows as discussed in section 4.1.

At order g the equivalence has been already established at (4.39). From (4.61) we know in addition that

$$T_{1/1}^\mu = -\frac{i}{\hbar} \left(\frac{\partial \mathcal{L}_0}{\partial (\partial_\mu \phi^A)} s_1 \phi^A - k_1^\mu \right) \quad (4.77)$$

We now define

$$T_{n/l}^\mu = T_n[T_1(x_1) \cdots T_{1/1}^\mu(x_l) \cdots T_1(x_n)]. \quad (4.78)$$

Let us also use the notation

$$\partial_\mu \mathcal{T}_n^\mu(x_1, \dots, x_n) = \sum_{l=1}^n \partial_\mu^l T_{n/l}^\mu \quad (4.79)$$

We shall now show that the condition (4.3), namely $s_0 T_n = \partial_\mu \mathcal{T}_n^\mu$, with this definition implies the same conditions on local terms as (4.5).

Because Poincaré invariance and causality already fix the time-ordered products $T_n[T_1 \dots T_1]$ up to the local normalization ambiguity N_n , we only have to show that these local normalization terms N_n are constrained in the same way by both conditions, (4.5)

and (4.3). The inductive proof of condition (4.3) proceeds along the same lines as the one of (4.5) as explained around formula (4.8): Assuming that the condition (4.3) is satisfied for all $m < n$ we can directly derive the fact that the condition at the n th order can only be violated by a local distribution (for a detailed proof see [13], section 2b). We shall discuss in some detail the local terms arising from tree level graphs. The discussion of loops goes the same way as in the previous subsection.

To find the tree-level local terms that arise in the right hand side of (4.3) we follow the same approach as before. Namely, we first differentiate, keep only the field equation terms and then do the contractions. To get the local terms at $n = 2$ we first calculate

$$\partial_\mu T_{1/1}^\mu = -\frac{i}{\hbar} \partial_\mu \left(\frac{\partial \mathcal{L}_0}{\partial (\partial_\mu \phi^A)} s_1 \phi^A \right) + \text{more} = -\frac{i}{\hbar} s_1 \phi^A \mathcal{K}_{AB} \phi^B + \text{more} \quad (4.80)$$

The terms denoted by “more” will not yield any local terms so we discard them. Then

$$\partial_\mu \mathcal{T}_{c,2}^\mu(x_1, x_2) = [Q, T_{c,2}(x_1, x_2)] - (2s_1 T_1 - \partial_\mu \left(\frac{\partial T_1}{\partial (\partial_\mu \phi^A)} s_1 \phi^A \right)) \delta(x_1, x_2) \quad (4.81)$$

where the first term in the right hand side only cancels the non-local part of the left hand side. Now we add the local normalization ambiguity N_2 , which is uniquely defined with respect to the natural splitting solution $T_{c,2}$, to the equation,

$$T_2[T_1(x_1)T_1(x_2)] = T_{c,2}[T_1(x_1)T_1(x_2)] + N_2 \delta(x_1, x_2) \quad (4.82)$$

We get

$$\begin{aligned} [Q, T_2(x_1, x_2)] - \partial_\mu \mathcal{T}_{c,2}^\mu(x_1, x_2) &= [2!(s_1 T_1 + s_0 \left(\frac{1}{2} N_2 \right)) \\ &\quad - \partial_\mu \left(\frac{\partial T_1}{\partial (\partial_\mu \phi^A)} s_1 \phi^A \right)] \delta(x_1, x_2) \end{aligned} \quad (4.83)$$

Compared with (4.45) we see that we “miss” the term $(\partial \mathcal{L}_2 / \partial (\partial_\mu \phi^A)) s_0 \phi^A$. This term “covariantizes” the current j_0^μ that generates the linear transformation $s_0 \phi^A$. In the present case, we have started with the $T_{1/1}^\mu$ coupling instead of the j_0^μ coupling, so we do not expect to find these “covariantization” terms. In addition, what matters is how the normalization freedom of the correlation functions without a current insertion are fixed. Also, now the combinatorial factor is $2!$ instead of $3!$. This is due to the fact that we are in 2nd order instead of 3rd.

Using (4.46) (and with the same identification $T_1 = (i/\hbar) \mathcal{L}_1$, $N_2/2! = (i/\hbar) \mathcal{L}_2$ as before) we obtain

$$\begin{aligned} [Q, T_2(x_1, x_2)] - \partial_\mu \mathcal{T}_{c,2}^\mu(x_1, x_2) &= \frac{i}{\hbar} [+ 2!(s_2 \phi^A) \mathcal{K}_{AB} \phi^B - 2! \partial_\mu \left(\frac{\partial \mathcal{L}_0}{\partial (\partial_\mu \phi^A)} s_2 \phi^A - k_2^\mu \right) \\ &\quad - \partial_\mu \left(\frac{\partial \mathcal{L}_1}{\partial (\partial_\mu \phi^A)} s_1 \phi^A \right)] \delta(x_1, x_2) \end{aligned} \quad (4.84)$$

We now fix the tree-level normalization freedom of $T_2[T_{1/1}^\mu T_1]$

$$T_2[T_{1/1}^\mu(x_1)T_1(x_2)] = T_{c,2}[T_{1/1}^\mu(x_1)T_1(x_2)] - j_2^\mu \delta(x_1, x_2) \quad (4.85)$$

by the condition (4.3). Then we get

$$j_2^\mu = 2! \left(\frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \phi^A)} s_2 \phi^A - k_2^\mu \right) + \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \phi^A)} s_1 \phi^A. \quad (4.86)$$

So we end up with

$$[Q, T_2(x_1, x_2)] - \partial_\mu \mathcal{T}_2^\mu(x_1, x_2) = -\frac{i}{\hbar} 2!(s_2 \phi^A) \mathcal{K}_{AB} \phi^B \quad (4.87)$$

After imposing the free field equations, we arrive at the condition (4.3).

At this point it is already clear that the condition (4.3) fixes all tree-normalization terms in the same way as our Quantum Noether condition (4.5). The derivation of the Noether consistency equations is totally analogous up to different combinatorial factors and up to covariantization terms. Moreover, the question of stability can be analyzed in exactly the same way as in section 4.4. Once the stability has been established the equivalence of (4.5) and (4.3) at loop level follows. The condition (4.3) guarantees the invariance of the S -matrix under the corresponding asymptotic symmetry. In the case of BRST symmetries the asymptotic linear part of the symmetry directly implies the unitarity of the physical S -matrix, i.e. the crucial decoupling of the unphysical degrees of freedom (for a simple proof see i.e. [13], chapter 7). So the Quantum Noether condition (4.5) also directly implies this crucial property in the case of BRST symmetries.

5 Examples.

5.1 $SU(n)$ Yang-Mills Theory.

In this section we present the construction of $SU(n)$ Yang-Mills theory. According to the discussion in section 3, to define the theory we first need to specify a set of free field and to give their commutation relations. For the $SU(n)$ YM theory, one has the YM field A_μ^a , the ghost c^a and the anti-ghost b^a , where a is an $SU(n)$ index. We shall discuss the theory in the Feynman gauge. For other gauge choices we refer to [14, 24]. The fields satisfy the free-field equations

$$\square A_\mu^a = 0; \quad \square c^a = 0; \quad \square b_a = 0, \quad (5.1)$$

and the (anti)-commutation relations,

$$\begin{aligned} [A_\mu^{(-)a}(x), A_\nu^{(+)b}(y)] &= i\hbar \delta^{ab} g_{\mu\nu} D^+(x-y) \\ \{b_a^{(-)}(x), c^{(+)b}(y)\} &= -i\hbar \delta_a^b D^+(x-y) \end{aligned} \quad (5.2)$$

where the super-index \pm designates the emission and absorption parts of the corresponding field and D^\pm is the (zero mass) Pauli-Jordan distribution (see appendix A).

The field equations in (5.1) may be derived from the following free Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + b\Box c - \frac{1}{2}(\partial \cdot A)^2 \quad (5.3)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. With these conventions, the kinetic operator $\mathcal{K}_{AB}\phi^B = \partial_\mu(\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\phi^A)}) - \frac{\partial\mathcal{L}_0}{\partial\phi^A}$, and the corresponding propagator \mathcal{D} are equal to $\mathcal{K}_{AA}A = -\Box A$, $\mathcal{K}_{bc}c = -\Box c$, $\mathcal{K}_{cb}b = \Box b$, and, $\mathcal{D}^{AA} = i\hbar g_{\mu\nu}\delta_{ab}D_F(x-y)$ and $\mathcal{D}^{bc} = -i\hbar\delta_a^b D_F(x-y)$, where $\Box D_F(x-y) = \delta(x-y)$ (see appendix A).

The free Lagrangian is invariant under the following BRST transformations

$$s_0 A_\mu^a = \partial_\mu c^a \Lambda; \quad s_0 c^a = 0; \quad s_0 b^a = -\partial \cdot A^a \Lambda \quad (5.4)$$

where Λ is a constant anticommuting variable. With this parameter present s_0 is a derivation¹³, namely one does not get any extra signs when s_0 passes through an anticommuting variable. We use the convention that Λ is removed from the right side of the equations. (This choice is co-related with the convention chosen in (3.2) to have the test functions to the right of the T -products. See the discussion in the paragraph after (5.9)).

The associated Noether current can be derived by making Λ local, varying the Lagrangian and collecting all terms that are proportional to $\partial_\mu \Lambda$ as explained in section 2. The result is

$$j_0^\mu = -\partial_\nu c F^{\mu\nu} - (\partial \cdot A)\partial^\mu c. \quad (5.5)$$

(the overall sign is fixed by our convention to always remove the anticommuting variable from the right side of the equations). The current is conserved when the free field equations are satisfied. In particular,

$$\partial_\mu j_0^\mu \equiv \partial_\mu \mathcal{J}_1^\mu = \partial_\mu c(-\Box A^\mu) + (\partial \cdot A)(-\Box c). \quad (5.6)$$

This corresponds to the general formula (4.32) in section 4:

$$\partial_\mu j_0^\mu \equiv \partial_\mu \mathcal{J}_1^\mu = (s_0 \phi^A)\mathcal{K}_{AB}\phi^B. \quad (5.7)$$

The corresponding BRST charge is given by

$$Q = \int j_0^0 d^3x = - \int (\partial \cdot A^a) \overset{\leftrightarrow}{\partial}{}^0 c^a d^3x \quad (5.8)$$

¹³Alternatively, one may treat s_0 as an anti-derivation (call it σ_0 to avoid confusion with s_0) in which case the parameter Λ should not be present in (5.4). To get the same signs in both cases one should use the Leibniz rule $\sigma_0(AB) = A\sigma_0 B + (-1)^B(\sigma_0 A)B$, where $(-1)^B$ denotes the grading of B . s_0 is more natural than σ_0 in the general case, where the parameter of the transformation may not be a scalar (for instance, in supersymmetric models the parameter is a spinor).

(with the above stated conventions, $s_0\phi^A = (i/\hbar)\{Q, \phi^A\}\Lambda$).

One may easily check that s_0 is nilpotent ($s_0^2 = 0$) when the free-field equations are satisfied. This is particular to the case of BRST symmetry¹⁴. In addition, one has the ghost charge (which can also be obtained as a Noether current),

$$Q_c = \frac{i}{\hbar} \int d^3x b_a \overset{\leftrightarrow}{\partial}^0 c^a. \quad (5.9)$$

The ghost charge introduces a gradation, the ghost number, in the algebra generated by the fundamental field operators. The ghost number of the gauge field A_μ^a is zero, of ghost field c^a is +1, and of the antighost field b_a is -1. It follows that the BRST charge has ghost number +1.

To obtain the theory in the Epstein-Glaser approach one starts with the coupling $g_\mu j_0^\mu$ and a yet unknown coupling gT_1 . Since j_0^μ is fermionic, g_μ is fermionic too, and one has to be careful with signs. A useful trick that helps keeping track of them is to write $g_\mu = \Lambda g'_\mu$, where Λ is anticommuting constant (as in (5.4)). Now, let us define $j_0'^\mu = j_0^\mu \Lambda$. Since $j_0^\mu g_\mu = j_0'^\mu g'_\mu$ the S -matrix constructed with a coupling involving the bosonic current $j_0'^\mu$ smeared out by the bosonic test function g'_μ is the same with the one constructed with a coupling involving the anticommuting current j_0^μ smeared out by the fermionic test function g_μ . However, since all vertices are now bosonic one need not worry about signs. At the end we are interested in the T -products that involve the fermionic current. To obtain those, one simply pushes Λ to the right where it recombines with g'_μ to give g_μ . This automatically produces all correct signs. (If one considers multi-current correlation functions one introduces as many anti-commuting constants as the number of current insertions.)

As we have seen the Quantum Noether condition (4.5) at second order, $\partial_\mu \mathcal{J}_2^\mu(x_1, x_2) = 0$, is equivalent to the condition

$$s_0 \mathcal{L}_1 = \partial_\mu \mathcal{L}_1^\mu \quad (5.10)$$

where $\mathcal{L}_1 = (\hbar/i)T_1$. Observe that both sides of this equation involve a nilpotent differential; the left hand side the (abelian) BRST differential and the right hand side the co-differential $\delta = *d*$, where $*$ is the Hodge operator and d the exterior derivative. Therefore, $\mathcal{L}_1 \in H_0(s_0, d)$ (the sub-index 0 denotes ghost number). Notice, however, that d is not cyclic since we are working on a space where the free field equations are satisfied. The latter introduce non-trivial cycles. An example of the latter at ghost number 2 is $C_\mu = c^a \partial_\mu c^a$. One may check that $\partial^\mu C_\mu = 0$, but $C_\mu \neq \partial_\mu B$ for any B .

¹⁴Let us mention that in the Lagrangian approach one may incorporate global symmetries into the BRST operator by introducing constant ghost fields. In other words, one formally treats the global symmetry as a local one (see, for instance, [25]). In the EG formalism, where one starts with free fields in the asymptotic Fock space, there is no natural way to incorporate constant ghost fields. We shall, therefore, not discuss further this possibility.

We shall argue below that only non-trivial elements $H_0(s_0, d)$ are important (the exact terms do not change the physics). So, one can focus only on them. The latter are equal to¹⁵

$$\mathcal{L}_1 = g f^{abc} \left(\frac{1}{2} A_{\mu a} A_{\nu b} F_c^{\nu\mu} + A_{\mu a} c_b \partial^\mu b_c \right) \quad (5.11)$$

Let us analyze further (4.5) at second order. We shall present this calculation in some detail in order to illustrate how one deals with the various subtle points discussed in section 4. We are interested in computing

$$\partial_\mu \mathcal{J}_{c,2}^\mu(x_1, x_2) = \partial_{x_1}^\mu T_{c,2}[j_\mu^0(x_1) T_1(x_2)] + \partial_{x_2}^\mu T_{c,2}[T_1(x_1) j_\mu^0(x_2)] \quad (5.12)$$

at tree level. Let us start by first computing $\partial_{x_1}^\mu T_{c,2}[j_\mu^0(x_1) T_1(x_2)]$. Since the tree-level Wick contractions satisfy the Leibniz rule, and T_1 only depends only on the fields and their first derivative, one can first compute $\partial_{x_1}^\mu T_{c,2}[j_\mu^0(x_1) \phi^A(x_2)]$ and $\partial_{x_1}^\mu T_{c,2}[j_\mu^0(x_1) \partial_\mu \phi^A(x_2)]$, where $\phi^A = A_\mu^a, c^a, b^a$, and then use the Leibniz rule. To correctly take care of the various signs we insert the anti-commuting constant Λ next to j_0^μ . As has been argued in detail in section 4, one can first move the derivative inside the T -product and then do the contractions. Let us compute $\partial_{x_1}^\mu T_{c,2}[j_\mu^0(x_1) \Lambda b^a(x_2)]$. After the first step and before the contractions one has (using (5.6))

$$[\partial_\mu c(-\square A^\mu) + (\partial \cdot A)(-\square c)](x_1) \Lambda b_a(x_2) \quad (5.13)$$

Clearly, a local term is produced when there is a contraction between the second term in the square brackets and $b_a(x_2)$. It is equal to $(\hbar/i)(-\partial^\mu A_\mu^a \Lambda)$, which up to (\hbar/i) (which is there to cancel the overall (i/\hbar)), is equal to $s_0 b_a$ as it should. One can also contract the first term with $b_a(x_2)$. This yields a non-local term proportional to the A field equation. This is an example of “wrong contraction”. As explained in section 4 these terms are irrelevant and they will be discarded. We shall, from now on, only concentrate on the “correct” contractions, namely the ones resulting from contractions involving the field ϕ^B in $\mathcal{K}_{AB} \phi^B$.

Following the procedure we have just outlined one obtains,

$$\begin{aligned} \partial_\mu \mathcal{J}_{c,2}^\mu(x, y) &= g f_{abc} [2\partial_\mu c^a A_\nu^b F^{\nu\mu c} + 2\partial_\mu c^a c^b \partial^\mu b^c - 2A_\mu^a c^b \partial^\mu \partial \cdot A^c \\ &\quad + \partial^\mu (A_\mu^a c_b \partial \cdot A^c + A_\mu^a A_\nu^b \partial^\nu c^c)] \Lambda \delta(x_1, x_2) \end{aligned}$$

¹⁵We note that the well-known Curci-Ferrari mass term [27] $\mathcal{L}'_1 = m^2(1/2 A_\mu^a A_\mu^a + c_a b_a)$ is compatible with (5.10). However, had we included this term in (5.11) the Quantum Noether condition at the next order would not be fulfilled ([14], section 3). If one starts with massive asymptotic gauge bosons the situation is slightly different. Without any further scalar field in the theory s_0 is not nilpotent any longer. One may still prove that the theory is renormalizable, but it is not unitary[28]. Including an unphysical scalar (Stüeckelberg model) nilpotency can be restored but this theory can be shown to be only ‘generalized renormalizable’ (see also [29, 16]). If one adds a physical scalar (Higgs field) to the theory one ends up with a unitary and renormalizable theory with massive gauge bosons (for a discussion in the EG framework, see [15]).

$$\begin{aligned}
&= \{ \partial^\mu [2(s_1 A_\nu^a F_{\mu\nu}^a - s_1 c^a \partial_\mu b_a) + g f_{abc} (A_\mu^a A_\nu^b \partial^\nu c^c + A_\mu^a c^b \partial^\nu A_\nu^c) \Lambda] \\
&\quad + 2(s_1 A^{\mu a} (-\square A_\mu^a) + s_1 c^a \square b^a) \} \delta(x-y)
\end{aligned} \tag{5.14}$$

where

$$s_1 A_\mu^a = g f^a_{bc} A_\mu^b c^c \Lambda; \quad s_1 c^a = \frac{g}{2} f^a_{bc} c^b c^c \Lambda \tag{5.15}$$

Notice that the field equation terms in (5.14) were created in the process of factoring out a total derivative from the rest of the terms. Let us now fix the tree-level ambiguity such that (4.5) at second order holds when the free-field equations are satisfied. To this end, we let

$$T[j_0^\mu(x_1)T_1(x_2)] = T_c[j_0^\mu(x_1)T_1(x_2)] + j_1^\mu \delta(x_1 - x_2) \tag{5.16}$$

Then (4.5) implies

$$j_\mu^1 = -2g f_{abc} (A^{\nu b} c^c F_{\mu\nu}^a + \frac{1}{2} c^b c^c \partial_\mu b^a) - g f_{abc} (A_\mu^a A_\nu^b \partial^\nu c^c + A_\mu^a c^b \partial^\nu A_\nu^c) \tag{5.17}$$

The first two terms in the Noether current are the ones that generate the s_1 transformation. The last two ‘‘covariantize’’ j_0^μ . All of them are part of the Noether current of the non-linear theory up to combinatorial factors which take care of the additional factors in the perturbative expansion,

$$j_\mu(\text{non-abelian}) = -D_\nu c^a \mathcal{F}_{\mu\nu}^a - (\partial \cdot A^a) D_\mu c^a - \frac{1}{2} g f_{abc} c^a c^b \partial_\mu b^a, \tag{5.18}$$

where

$$\mathcal{F}_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^a_{bc} A_\mu^b A_\nu^c; \quad D_\mu c^a = \partial_\mu c^a + g f^a_{bc} A_\mu^b c^c. \tag{5.19}$$

So, finally at $n = 2$ we have off-shell,

$$\partial_\mu \mathcal{J}_2^\mu(x, y) = 2[s_1 A^{\mu a} (-\square A_\mu^a) + s_1 c^a \square b^a] \delta(x_1, x_2) \tag{5.20}$$

This corresponds to the general formula

$$\partial_\mu \mathcal{J}_2^\mu(x_1, x_2) = 2! s_1 \phi^A \mathcal{K}_{AB} \phi^B. \tag{5.21}$$

One may also check that $\mathcal{L}_0 + \mathcal{L}_1$ is equal to the YM action,

$$\mathcal{L}_{YM} = -\frac{1}{4} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} + b \partial^\mu D_\mu c - \frac{1}{2} (\partial \cdot A)^2, \tag{5.22}$$

but the four-gluon term. The latter is of order g^2 and will be recovered at next order.

To examine (4.5) at third order we need the second-order result off-shell (5.20). Using

$$\partial_\mu \mathcal{J}_{c,3}^\mu(x_1, x_2, x_3) = \partial_\mu \mathcal{J}_2^\mu(x_1, x_2) T_1(x_3) + \partial_\mu \mathcal{J}_1^\mu(x_1) N_2(x_2, x_3) + \text{cyclic in } x_1, x_2, x_3, \tag{5.23}$$

one gets at $n = 3$,

$$\begin{aligned}
\partial_\mu \mathcal{J}_{c,3}^\mu(x_1, x_2, x_3) &= [3!(g^2 f_{abe} f_{ecd} A_\mu^a A_\nu^b A_\mu^c \partial_\nu c^d + s_0 \mathcal{L}_2) \Lambda \\
&\quad - 2! \partial_\mu (-2g f_{abc} A_\mu^a A_\nu^b s_1 A_\nu^c + \frac{\partial \mathcal{L}_2}{\partial (\partial_\mu \phi^A)} s_0 \phi^A)] \delta(x_1, x_2, x_3)
\end{aligned} \tag{5.24}$$

where $\mathcal{L}_2 = (\hbar/i)N_2/2!$ and N_2 denotes the unique local normalization term of T_2 with respect to $T_{c,2}$. From here we determine \mathcal{L}_2 , which is just the missing four-gluon coupling in (5.22), and also the Noether current renormalizations,

$$\begin{aligned}\mathcal{L}_2 &= -\frac{1}{4}(gf_{abc}A_\mu^b A_\nu^c)^2, \\ j_2^\mu &= -4g^2 f_{abc}f_{cde}A_\mu^a A_\nu^b A_\nu^d c^e\end{aligned}\tag{5.25}$$

where j_2 is defined as in (4.48). Notice that this precisely the missing ‘‘covariantization’’ term from (5.18). Therefore, at $n = 3$ we finally have off-shell

$$\partial_\mu \mathcal{J}_3^\mu(x_1, x_2, x_3) = 0\tag{5.26}$$

Notice the absence of field equation terms in the right hand side of (5.26). This means that no new tree-level local terms will emerge at higher orders.

One may easily check that s_0 and s_1 as defined above satisfy $\{s_i, s_j\} = 0$, where $i = 0, 1$. It follows that $s = s_0 + s_1$ squares to zero. It has been argued that only non-trivial solutions of (5.10) are physically relevant. This question will be analyzed in detail in [7]. Let us already briefly discuss this issue here. Suppose that instead of \mathcal{L}_1 in (5.11) one considers $\mathcal{L}_1 + s_0 C$. This is still a solution of (5.10). At next order, however, one obtains the equation

$$s_1(s_0 C) + s_0 \mathcal{L}'_2 = 0\tag{5.27}$$

where $N_2 = (i/\hbar)(\mathcal{L}_2 + \mathcal{L}'_2)$ and \mathcal{L}_2 is as in (5.25). It follows that $\mathcal{L}'_2 = s_1 C$. Thus, adding an exact s_0 term in \mathcal{L}_1 results in the addition of an s -exact term in the Lagrangian. It is well known that s -exact term are physically irrelevant. The corresponding statement in the EG formalism will be worked out in [7]. In our specific example one may check that adding the exact terms, $\mathcal{L}'_1 = \beta_1 s_0 (f_{abc} \partial A^a c^b c^c)$ and $\mathcal{L}''_1 = \beta_2 \partial^\mu (f_{abc} c^a b^b A_\mu^c)$ with the free constants β_1 and β_2 , results in additional terms in the transformation rules at order g and in the well-known four-ghost coupling at order g^2 .

Let us now move to loop level. It is well known[26] that in order to have a candidate anomaly (i.e. non-trivial element of $H^1(s, d)$) one needs a non-vanishing d_{abc} and an epsilon symbol $\epsilon_{\mu\nu\rho\sigma}$. For $SU(n)$ ($n > 2$) d_{abc} is non-zero. However, in a theory without chiral fermions one does not have an epsilon symbol. So, for pure Yang-Mills theory $H^1(s, d) = 0$. According to the analysis presented in section 4.4 this is sufficient to guarantee that the loop normalization ambiguity is constraint the same way as the tree-level one. We therefore conclude that the Yang-Mills theory is renormalizable.

5.2 The $N = 1$ Wess-Zumino Model.

We now turn to our supersymmetric example. The field in Wess-Zumino model[30] are a complex scalar field ϕ and its fermionic partner ψ^α . We use the two component spinor

notation of [31] (see appendix A). The fields satisfy the following field equations

$$\square\phi = 0; \quad \partial_{\alpha\dot{\alpha}}\psi^\alpha = 0. \quad (5.28)$$

The commutations relations are

$$\begin{aligned} [\phi^{(-)}(x_1), \bar{\phi}^{(+)}(x_2)] &= i\hbar 2D^+(x_1, x_2) \\ \{\psi_\alpha^{(-)}(x_1), \psi_{\dot{\alpha}}^{(+)}(x_2)\} &= i\hbar S_{\alpha\dot{\alpha}}^+(x_1, x_2) \end{aligned} \quad (5.29)$$

where $S_{\alpha\dot{\alpha}}^+ = 2i\partial^{\alpha\dot{\alpha}}D^+$.

The field equations can be derived from the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\bar{\phi}\square\phi + \psi^{\dot{\alpha}}i\partial^\alpha_{\dot{\alpha}}\psi_\alpha \quad (5.30)$$

With these conventions $\mathcal{K}_{\bar{\phi}\phi} = -\square\phi/2$, $\mathcal{K}_{\phi\bar{\phi}} = -\square\bar{\phi}/2$, $\mathcal{K}_{\dot{\alpha}\alpha}\psi^\alpha = -i\partial^\alpha_{\dot{\alpha}}\psi_\alpha$ and $\mathcal{K}_{\alpha\dot{\alpha}}\psi^{\dot{\alpha}} = -i\partial_\alpha^{\dot{\alpha}}\psi_{\dot{\alpha}}$. The corresponding Feynman propagators are given by $\mathcal{D}^{\phi\bar{\phi}} = 2i\hbar\Delta_F(x-y)$, and $\mathcal{D}^{\alpha\dot{\alpha}} = i\hbar S_F^{\alpha\dot{\alpha}}$, where $S_F^{\alpha\dot{\alpha}} = 2i\partial_x^{\alpha\dot{\alpha}}D_F(x-y)$. For a derivation of these formulae see appendix A.

This action is invariant under the linear supersymmetry transformations,

$$s_0\phi = -\epsilon^\alpha\psi_\alpha, \quad s_0\bar{\phi} = -\epsilon^{\dot{\alpha}}\psi_{\dot{\alpha}}, \quad s_0\psi^\alpha = -\epsilon^{\dot{\alpha}}i\partial^\alpha_{\dot{\alpha}}\phi, \quad s_0\psi^{\dot{\alpha}} = -\epsilon^\alpha i\partial_\alpha^{\dot{\alpha}}\phi \quad (5.31)$$

The associated Noether current is equal to

$$j_0^{\alpha\dot{\alpha}} = \epsilon^\beta\psi^\alpha\partial_\beta^{\dot{\alpha}}\bar{\phi} + \epsilon^{\dot{\beta}}\psi^{\dot{\alpha}}\partial_\beta^\alpha\phi \quad (5.32)$$

An easy calculation yields

$$\begin{aligned} \partial_{\alpha\dot{\alpha}}j_0^{\alpha\dot{\alpha}} &= (-\epsilon^\alpha\psi_\alpha)\left(-\frac{1}{2}\square\bar{\phi}\right) + (-\epsilon^{\dot{\alpha}}\psi_{\dot{\alpha}})\left(-\frac{1}{2}\square\phi\right) \\ &\quad + (-\epsilon^{\dot{\beta}}i\partial_\beta^\alpha\phi)(-i\partial_\alpha^{\dot{\alpha}}\psi_{\dot{\alpha}}) + (-\epsilon^\beta i\partial_\beta^{\dot{\alpha}}\phi)(-i\partial_\alpha^{\dot{\alpha}}\psi_\alpha) \end{aligned} \quad (5.33)$$

In a similar way as in the Yang-Mills example, one may check that this current correctly produces the supersymmetry variation inside correlation functions.

Invariance at first order requires that we find a $T_1 = (i/\hbar)\mathcal{L}_1$ such that

$$s_0\mathcal{L}_1 = \partial_\mu\mathcal{L}_1^\mu \quad (5.34)$$

holds, up to free field equations, for some \mathcal{L}_1 and \mathcal{L}_1^μ . The latter are constraint by power counting. The most general solution (for simplicity we restrict ourselves to a massless theory) is

$$\mathcal{L}_1 = (1/2)(\phi\psi^2 + \bar{\phi}\bar{\psi}^2) \quad (5.35)$$

where $\psi^2 = \psi^\alpha \psi_\alpha$ and $\bar{\psi}^2 = \psi^{\dot{\alpha}} \psi_{\dot{\alpha}}$. Then,

$$\begin{aligned} \partial_{\alpha\dot{\alpha}} \mathcal{J}_{c,2}^{\alpha\dot{\alpha}}(x_1, x_2) &= [\partial_{\alpha\dot{\alpha}} i(\epsilon^{\dot{\alpha}} \psi^\alpha \phi^2 + \epsilon^\alpha \psi^{\dot{\alpha}} \bar{\phi}^2) \\ &\quad + 2(-\frac{1}{2} \epsilon^\alpha \bar{\phi}^2)(-i \partial_\alpha^{\dot{\alpha}} \psi_{\dot{\alpha}}) + 2(-\frac{1}{2} \epsilon^{\dot{\alpha}} \phi^2)(-i \partial_{\dot{\alpha}}^\alpha \psi_\alpha)] \delta(x_1, x_2) \end{aligned} \quad (5.36)$$

The first term in the right hand side is removed by fixing the tree-level ambiguity as

$$T[j_0^{\alpha\dot{\alpha}}(x_1) T_1(x_2)] = T_c[j_0^{\alpha\dot{\alpha}}(x_1) T_1(x_2)] + j_1^{\alpha\dot{\alpha}} \delta(x_1 - x_2) \quad (5.37)$$

where

$$j_1^{\alpha\dot{\alpha}} = -i(\epsilon^{\dot{\alpha}} \psi^\alpha \phi^2 + \epsilon^\alpha \psi^{\dot{\alpha}} \bar{\phi}^2) \quad (5.38)$$

So, we end up with

$$\partial_{\alpha\dot{\alpha}} \mathcal{J}_2^{\alpha\dot{\alpha}}(x_1, x_2) = 2!(s_1 \psi^\alpha) \mathcal{K}_{\alpha\dot{\alpha}} \psi^{\dot{\alpha}} + 2!(s_1 \psi^{\dot{\alpha}}) \mathcal{K}_{\dot{\alpha}\alpha} \psi^\alpha \quad (5.39)$$

where the new symmetry variations are given by

$$s_1 \phi = 0, \quad s_1 \psi^\alpha = -\frac{1}{2} \epsilon^\alpha \bar{\phi}^2, \quad s_1 \bar{\phi} = 0, \quad s_1 \psi^{\dot{\alpha}} = -\frac{1}{2} \epsilon^{\dot{\alpha}} \phi^2 \quad (5.40)$$

We now move to the next order. We have

$$\partial_\mu \mathcal{J}_{c,3}^\mu(x_1, x_2, x_3) = \partial_\mu \mathcal{J}_2^\mu(x_1, x_2) T_1(x_3) + \partial_\mu \mathcal{J}_1^\mu(x_1) N_2(x_2, x_3) + \text{cyclic in } x_1, x_2, x_3, \quad (5.41)$$

A straightforward calculation yields

$$\partial_\mu \mathcal{J}_{c,3}^\mu(x_1, x_2, x_3) = 3![(1/2)(-\epsilon^\alpha \psi_\alpha) \bar{\phi}^2 \phi + N_2(-\epsilon^{\dot{\alpha}} \psi_{\dot{\alpha}}) \bar{\phi} \phi^2 + s_0 \mathcal{L}_2] \delta(x_1, x_2, x_3) \quad (5.42)$$

where $\mathcal{L}_2 = (\hbar/i) N_2/2$. This implies that

$$\mathcal{L}_2 = -\frac{1}{4} \phi^2 \bar{\phi}^2 \quad (5.43)$$

Hence,

$$\partial_\mu \mathcal{J}_3^\mu(x_1, x_2, x_3) = 0 \quad (5.44)$$

Therefore, no new local terms will arise through in higher orders.

The sum $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$ is equal to

$$\mathcal{L} = \frac{1}{2} \bar{\phi} \square \phi + \psi^{\dot{\alpha}} i \partial_{\dot{\alpha}}^\alpha \psi_\alpha + \frac{1}{2} g(\phi \psi^2 + \bar{\phi} \bar{\psi}^2) - \frac{1}{4} g^2 \phi^2 \bar{\phi}^2 \quad (5.45)$$

which is indeed the Wess-Zumino Lagrangian. The transformation rules which are generated by $s = s_0 + s_1$ are given by,

$$\begin{aligned} s\phi &= -\epsilon^\alpha \psi_\alpha, \quad s\bar{\phi} = -\epsilon^{\dot{\alpha}} \psi_{\dot{\alpha}}, \\ s\psi^\alpha &= -\epsilon^{\dot{\alpha}} i \partial_{\dot{\alpha}}^\alpha \phi - \frac{1}{2} g \epsilon^\alpha \bar{\phi}^2, \quad s\psi^{\dot{\alpha}} = -\epsilon^\alpha i \partial_\alpha^{\dot{\alpha}} \phi - \frac{1}{2} g \epsilon^{\dot{\alpha}} \phi^2. \end{aligned} \quad (5.46)$$

These are also the correct supersymmetry transformation rules.

Finally, we discuss the issue of stability under quantum corrections (renormalizability). It has been shown in [32] that there are no anomaly candidates in the $N = 1$ Wess-Zumino model. From our discussion ¹⁶ in section 4.4 then immediately follows that the loop normalization ambiguity is constraint the same way as the tree-level one, i.e. the theory is renormalizable.

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A Conventions.

In this appendix we explain in detail our conventions and our various sign choices. Since one of the aims of this article is to make contact between the EG and the Lagrangian formalism, it is important to have compatible conventions on both sides. Our approach will be to start from canonical quantization and the standard equal-time commutation relations and from there to derive the commutator of the fields at arbitrary spacetime points. The latter is one of the inputs in the EG formalism. Furthermore, we shall use the path integral formalism to compute the Feynman propagator which, in this formalism, is the inverse of the kinetic operator. Having fixed the conventions in this way, we shall see that after natural splitting the contraction becomes equal to the Feynman propagator exactly (i.e. no extra factors). In this way we make contact with section 4. The presentation in this appendix follows in part [33].

We start from a free massless scalar in four dimensions. The Lagrangian is given by

$$\mathcal{L}_0 = \frac{1}{2}\phi\Box\phi \tag{A.1}$$

¹⁶Strictly speaking one would have to extend the considerations of section 4.4 to theories with open symmetry algebra in order to be applicable to the present example.

The overall sign in the Lagrangian is such that the Hamiltonian is positive definite. (Our convention for metric is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and x^0 is the time variable t . For the space coordinates we use either x^i or \mathbf{x}). The canonical momentum is equal to $p = \dot{\phi}$. Then the equal-time commutation relations (ECR) read,

$$\begin{aligned} [\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] &= [\dot{\phi}(t, \mathbf{x}), \dot{\phi}(t, \mathbf{y})] = 0 \\ [\phi(t, \mathbf{x}), \dot{\phi}(t, \mathbf{y})] &= i\hbar\delta(\mathbf{x} - \mathbf{y}) \end{aligned} \quad (\text{A.2})$$

The field equation is equal to

$$\square\phi = 0 \quad (\text{A.3})$$

It follows that $\phi(x)$ may be expanded in plane waves

$$\phi(x) = \int d^3k \hbar^{1/2} (f_k(x)a(k) + f_k^*(x)a^\dagger(k)) \quad (\text{A.4})$$

where

$$f_k = \frac{1}{\sqrt{(2\pi)^3 2\omega}} e^{ikx} \quad (\text{A.5})$$

with $k^0 = \omega = \sqrt{\mathbf{k}^2}$. One may check that $\square f_k = 0$. Furthermore,

$$\int d^3x f_k^*(\mathbf{x}, t) \overleftrightarrow{\partial}_0 f_l(\mathbf{x}, t) = -i\delta(\mathbf{k} - \mathbf{l}) \quad (\text{A.6})$$

From the ECR (A.2) and with the help of (A.6) we then obtain the commutation relation that the a 's satisfy,

$$\begin{aligned} [a(\mathbf{k}), a(\mathbf{l})] &= [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{l})] = 0 \\ [a(\mathbf{k}), a^\dagger(\mathbf{l})] &= \delta(\mathbf{k} - \mathbf{l}) \end{aligned} \quad (\text{A.7})$$

We can now compute the commutator of two fields in arbitrary spacetime points. The result is

$$[\phi^{(\mp)}(x), \phi^{(\pm)}(y)] = i\hbar D^\pm(x - y) \quad (\text{A.8})$$

where $\phi^{(\pm)}(x)$ are the absorption and emission parts of $\phi(x)$, and

$$D^\pm(x - y) = \pm(-i) \int \frac{d^3k}{(2\pi)^3 2\omega} e^{\pm ik(x-y)} \quad (\text{A.9})$$

The Pauli-Jordan distribution is then equal to

$$\begin{aligned} D(x - y) &= D^+ + D^- = (-i) \int \frac{d^3k}{(2\pi)^3 \omega} \sin k(x - y) \\ &= (-i) \int \frac{d^4k}{(2\pi)^3} \delta(k^2) \text{sgn}(k^0) e^{ik(x-y)} \end{aligned} \quad (\text{A.10})$$

The Pauli-Jordan distribution has a causal decomposition into retarded and advanced part as follows,

$$D_{ret}(x-y) = \theta(x^0 - y^0)D(x-y); \quad D_{adv}(x-y) = -\theta(y^0 - x^0)D(x-y). \quad (\text{A.11})$$

We also give a couple of useful formulae

$$\partial^0 D(x-y)|_{x^0=y^0} = \delta(\mathbf{x}-\mathbf{y}); \quad D(x-y)|_{x^0=y^0} = \partial_i D(x-y)|_{x^0=y^0} = 0 \quad (\text{A.12})$$

Let us now compute the Feynman propagator $\langle 0|T(\phi(x)\phi(y))|0\rangle$. This can be done either by canonical methods (using (A.4)-(A.7) and defining the vacuum as $a|0\rangle = 0$) or by coupling the fields to sources and completing squares in the path integral. Both approaches yield the same result,

$$\mathcal{D}^{\phi\phi} \equiv \langle 0|T(\phi(x)\phi(y))|0\rangle = i\hbar D_F(x-y) \quad (\text{A.13})$$

where $\square_x D_F(x-y) = \delta(x-y)$. From the latter equation one obtains

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikx}}{-k^2 + i\epsilon} \quad (\text{A.14})$$

It follows that

$$D_F(x-y) = \theta(x^0 - y^0)D^+(x-y) - \theta(y^0 - x^0)D^-(x-y). \quad (\text{A.15})$$

It is now easy to verify (as described in section 3) that after natural splitting the commutator of two fields is replaced by the Feynman propagator exactly.

The case of gauge fields goes along similar lines. The Lagrangian in the Lorentz gauge is given by

$$\mathcal{L}_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\partial \cdot A)^2 \quad (\text{A.16})$$

The overall sign in the action is fixed by requiring positivity of the energy. The result for the commutator and the propagator are,

$$\begin{aligned} [A_\mu^{(-)a}(x), A_\nu^{(+)b}(y)] &= i\hbar\eta_{\mu\nu}\delta^{ab}D^+(x-y) \\ \mathcal{D}^{AA} \equiv \langle 0|T(A_\mu^a(x)A_\nu^b(y))|0\rangle &= i\hbar\eta_{\mu\nu}\delta^{ab}D_F(x-y) \end{aligned} \quad (\text{A.17})$$

The indices a, b indices are gauge group indices.

We now move to the ghost sector. We take for Lagrangian

$$\mathcal{L}_0 = \int b_a \square c^a. \quad (\text{A.18})$$

With this sign choice the quantum Hamiltonian acting on the vacuum (defined as the state annihilated by the annihilation operators $b_a(k)$ and $c^a(k)$ of the ghost and antighost

fields, respectively) has positive energy eigenvalues. The canonical momenta, defined by left differentiation, are given by

$$p_a(c) = -\dot{b}_a; \quad p^a(b) = \dot{c}^a \quad (\text{A.19})$$

The canonical equal time commutation relations are equal to

$$\{c^a(t, \mathbf{x}), p_b(c)(t, \mathbf{y})\} = \{b_b(t, \mathbf{x}), p^a(b)(t, \mathbf{y})\} = -i\hbar\delta_b^a\delta(\mathbf{x} - \mathbf{y}) \quad (\text{A.20})$$

and the rest of anticommutators are equal to zero. The minus sign in the right hand side of (A.20) is not a matter of convention. It is fixed once one requires the the Heiseberg equation $(i/\hbar)[H, *]$ yields the field equations for both the ghost and the antighost [33].

The field equation are $\square c^a = 0$ and $\square b_a = 0$. The mode expansion reads

$$\begin{aligned} c^a(x) &= \int d^3k \hbar^{1/2} (f_k(x)c^a(k) + f_k^*(x)c^{a\dagger}(k)) \\ b_a(x) &= \int d^3k \hbar^{1/2} (-f_k(x)b_a(k) + f_k^*(x)b_a^\dagger(k)) \end{aligned} \quad (\text{A.21})$$

Note that b_a is antihermitian whereas c^a is hermitian, so that the action is hermitian¹⁷ Using the equal time canonical commutation relation we now get

$$\{c^a(\mathbf{k}), b_b^\dagger(\mathbf{l})\} = \delta_b^a\delta(\mathbf{k} - \mathbf{l}) \quad (\text{A.22})$$

and the complex conjugate one. One may now compute the commutator of a ghost field with an antighost field at arbitrary spacetime points. The result is

$$\{b_a^{(-)}(x), c^{(+)\dagger}(y)\} = -i\hbar\delta_a^b D^+(x - y) \quad (\text{A.23})$$

To calculate the propagator we couple the fields to sources by adding to the Lagrangian the term $b j_b + j_c c$. Following the same procedure as in the case of a scalar field we obtain

$$\mathcal{D}^{bc} \equiv \langle 0|T(b_a(x)c^b(y))|0\rangle = -i\hbar\delta_a^b D_F(x - y) \quad (\text{A.24})$$

The same exercise, but considerably more tedious, can also be done for the case of fermions. It is quite clear though from the previous discussion that starting from the action and quantizing canonically one derives commutation relations that after natural splitting yield the Feynman propagator; the latter being the inverse of the kinetic operator. So, one may reverse the reasoning and derive the commutator from the Feynman propagator. The computation of the latter is far easier than the computation of the former. Before we present this computation, however, we introduce our spinor conventions. We use the two component notation of [31]. With these conventions one avoids

¹⁷Strictly speaking the ghosts and the action are pseudo-(anti-)hermitian which indicates that the actual hermiticity properties are defined in respect to a sesquilinear form (indefinite metric) and not in respect of the (positive definite) scalar product of the one-particle Hilbert space of the ghosts.

using gamma matrices, and the Fierz identities become a matter of symmetrizing and antisymmetrizing spinor indices.

The Lorentz group in four dimension is isomorphic to $SL(2, C)$. The simplest non-trivial representation of the latter is the two component complex Weyl spinor ψ^α , $\alpha = +, -$, (the $(1/2, 0)$ representation). Its complex conjugate representation (the $(0, 1/2)$) is denoted by $\psi^{\dot{\alpha}}$. Greek letters are reserved for spinor two components indices and Roman ones for vector indices. Each vector index is equivalent to one undotted and one dotted index ($\phi^a = \phi^{\alpha\dot{\alpha}}$). Indices are raised and lowered using an sl_2 invariant antisymmetric two dimensional matrix $C_{\alpha\beta}$. Since $C_{\alpha\beta}$ is antisymmetric, we have to specify how exactly we use it to raise and lower indices, and our convention is the so-called ‘down-hill’ rule from left to right for both the undotted and the dotted sector. For example,

$$\psi^\alpha C_{\alpha\beta} = \psi_\beta; \quad C^{\alpha\beta} \psi_\beta = \psi^\alpha; \quad \psi^{\dot{\alpha}} C_{\dot{\alpha}\dot{\beta}} = \psi_{\dot{\beta}}; \quad C^{\dot{\alpha}\dot{\beta}} \psi_{\dot{\beta}} = \psi^{\dot{\alpha}}. \quad (\text{A.25})$$

In addition, we have the following identity

$$C_{\alpha\beta} C^{\gamma\delta} = \delta_\alpha^\gamma \delta_\beta^\delta - \delta_\beta^\gamma \delta_\alpha^\delta, \quad (\text{A.26})$$

From (A.26) we get

$$C^{\alpha\beta} C_{\alpha\beta} = \delta_\alpha^\alpha = 2 \quad (\text{A.27})$$

$$C_{\alpha\beta} \chi_\gamma - C_{\alpha\gamma} \chi_\beta = -C_{\beta\gamma} \chi_\alpha \quad (\text{A.28})$$

$$\chi_\alpha \chi_\beta = -\frac{1}{2} C_{\alpha\beta} \chi^\gamma \chi_\gamma \quad (\text{A.29})$$

The last identity is an example of a Fierz identity.

The Lagrangian for a massless spinor field is given by

$$\mathcal{L}_0 = \psi^{\dot{\alpha}} i \partial_{\dot{\alpha}}^\alpha \psi_\alpha \quad (\text{A.30})$$

To compute the propagator one adds to the Lagrangian the source terms $j^\alpha \psi_\alpha + j^{\dot{\alpha}} \psi_{\dot{\alpha}}$. Then one may derive in a standard way the following expression for the propagator

$$\mathcal{D}^{\alpha\dot{\alpha}} \equiv \langle 0 | \psi^\alpha(x) \psi^{\dot{\alpha}}(y) | 0 \rangle = i\hbar S_F^{\alpha\dot{\alpha}} \quad (\text{A.31})$$

where

$$S_F^{\alpha\dot{\alpha}} = 2i \partial_x^{\alpha\dot{\alpha}} D_F(x - y) \quad (\text{A.32})$$

(A useful identity that one use in the derivation of the propagator is $\partial_{\dot{\alpha}}^\alpha \partial_{\alpha\dot{\beta}} = -1/2 C_{\dot{\alpha}\dot{\beta}} \square$.)

From here one can obtain the commutation relations of two fermi fields at arbitrary spacetime separation,

$$\{\psi_\alpha^{(-)}(x), \psi_{\dot{\alpha}}^{(+)}(y)\} = i\hbar S_{\alpha\dot{\alpha}}^+(x - y) \quad (\text{A.33})$$

where

$$S_{\alpha\dot{\alpha}}^+(x - y) = 2i \partial_{\alpha\dot{\alpha}}^x D^+(x - y). \quad (\text{A.34})$$

B Further Properties of the EG Formalism

In this appendix we highlight some further properties of the EG formalism. In particular we discuss the infrared problem, the construction of interacting fields, the problem of overlapping divergences and few other issues.

- The distributions T_n in (3.2) are smeared out by tempered test functions $g \in S$. This provides a natural regularization of the physical infrared problem which arises in massless theories because the tempered test function cuts off the long-distance part of the distributions. In the construction of quantum electrodynamics, for example, the infrared problem is fully separated in the causal formalism. Ultimately, one is interested in the physical (so-called adiabatic) limit $g(x) \rightarrow g \equiv const..$ Epstein and Glaser have proven that this limit exists for the vacuum expectation values of the T_n distributions (in the sense of tempered distributions) in massive theories if a suitable normalization is chosen (section 8.2 in [2]). In this limit the Greens' functions possess all the expected linear properties such as causality, Lorentz covariance and the spectral condition. The existence of Greens' functions in the adiabatic limit for the case of quantum electrodynamics was shown by Blanchard and Seneor [8]. We implement our method before the adiabatic limit. All equations are understood as distributional ones. So our examinations are also well-defined in massless theories like pure Yang-Mills theories where the adiabatic limit is related to the confinement problem which is not expected to be solved in the framework of perturbation theory.
- We have argued in the introduction that the strength of the EG construction lies in the operator formalism. However, this strength turns into a weakness of the formalism when one is interested in non-local details of the theory, for example, when one tries to translate a simple operator condition into relations of C -number distributions or if one discusses properties of a subgroup of contributions or even a single type of diagram.
- The EG formalism naturally leads to amputated connected Greens' functions and not to one-particle irreducible ones. This complicates the discussion of the multiplicative structure of the renormalization ambiguity which is the starting point to analyze the renormalization group.
- Having constructed the most general S -matrix one can construct interacting field operators (compatible with causality and Poincaré invariance) ([2] section 8, [5]).

One starts with an extended first order S -matrix

$$S(g, g_1, g_2, \dots) = \int d^4x \{T_1(x)g(x) + \Phi_1(x)g_1(x) + \Phi_2(x)g_2(x) + \dots\} \quad (\text{B.1})$$

where Φ_i represent certain Wick monomials like $(i/\hbar)\varphi$ or $(i/\hbar) : \varphi^3 :$. Following Bogoliubov and Shirkov, Epstein and Glaser defined the corresponding interacting fields Φ_i^{int} as functional derivatives of the extended S -matrix:

$$\Phi_i^{int}(g, x) = S^{-1}(g, g_1, \dots) \left. \frac{\delta S(g, g_1, \dots)}{\delta g_i} \right|_{g_i=0} \quad (\text{B.2})$$

One shows that the perturbation series for the interacting fields is given by the advanced distributions of the corresponding expansion of the S -matrix, namely

$$\Phi_i^{int}(g, x) = \Phi_i(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \frac{\hbar}{i} A_{n+1/n+1}(x_1, \dots, x_n; x), \quad (\text{B.3})$$

where $A_{n+1/n+1}$ denotes the advanced distributions with n original vertices T_n and one vertex Φ_i at the $(n+1)$ th position; symbolically we may write:

$$A_{n+1/n+1}(x_1, \dots, x_n; x) = Ad [T_1(x_1) \dots T_1(x_n); \Phi_i(x)] \quad (\text{B.4})$$

One shows that the perturbative defined object Φ_i^{int} fullfills all expected properties like locality, field equations etc. in the sense of formal power series. The definition can be regarded as a direct construction of renormalized composite operators. Epstein and Glaser showed that the adiabatic limit $g \rightarrow 1$ exists only in the weak sense of expectation values. These have the essential properties of Wightman functions. Hence one may construct the corresponding Wightman fields by using the general reconstruction theorem of Wightman.

- If the specific coupling T_1 is fermionic then the causality condition implies that the corresponding tempered test function has to be an anticommuting Grassmann variable. Also an ordering of fermionic couplings in the time-ordered products is introduced (for further details see [12, 13]).
- The polynomial character of the interactions T_1 is forced by the formalism allowing g to be any element of $\mathcal{S}(\mathbf{R}^4)$. If one restricts the choice of the test functions to the Jaffe's class one can also construct theories with nonpolynomial specific couplings (see [34]).
- Steinmann presented an approach to perturbative quantum field theory which is related to the Epstein Glaser method ([35]). He works with retarded products and his construction is done after the adiabatic limit is taken.
- A variant of the Epstein Glaser formalism has been proposed by Stora [4]. He shows that the problem of cutting a causal distribution into a retarded and advanced piece is equivalent to the problem of continuation of time-ordered products to coincident points. Actually the renormalization scheme of differential renormalization proposed by Freedman, Johnson and Latorre [36] is an operative way to perform such a continuation in configuration space (x -space). This has been illustrated in [37]. Stora's variant of the EG method allows for an extension of the EG method to theories on curved space-time. One of the main problems one encounters in trying to achieve such extension is the absence of translation invariance (which plays a crucial role in the original version of the formalism as presented by Epstein and Glaser (see section 3)) in curved space-time. For recent work see [38].
- Finally, let us shortly discuss how the problem of overlapping divergences is automatically solved by the EG formalism [39]. All T -products are defined as operators that fulfill Wick's theorem (3.12) and both conditions of causality (3.9)-(3.10). From this, we can

derive a general necessary condition for all renormalization schemes which is normally established at the level of C -number valued Greens' functions and not at the level of operators. Note that in the EG formalism a Greens' function always has a representation as a vacuum expectation value of an operator-valued distribution $\langle T(V) \rangle$.

A renormalization of a Greens' function $\langle T(V) \rangle$ has to be such that for all partition of the set of vertices $V = X \cup Y$, $X \neq \emptyset$, $Y \neq \emptyset$, it coincides in the region where $X \geq Y$ with

$$\langle T(V) \rangle = \langle T(X) \rangle \prod_{x_i \in X, y_i \in Y} D^+(x_i - y_i) \langle T(Y) \rangle \quad (\text{B.5})$$

where D^+ represents a fundamental commutation distribution and $\langle T(X) \rangle$ ($\langle T(Y) \rangle$) is a subgraph of order $|X|$ ($|Y|$). In the EG formalism the latter Greens' functions are vacuum expectation values of operator-valued distributions including Wick submonomials (see (3.12)). These are well-defined and fulfill all required conditions by the induction hypothesis.

Usual renormalization schemes are directly implemented on the level of C -number Greens' functions, so it becomes a nontrivial task to show that all subdiagrams of a given diagram can be renormalized in a consistent way, in particular such that the condition (B.5) is fulfilled. In the context of the BPHZ method this is achieved by the forest formula found by Zimmermann, which presents an algorithm that disentangles all divergences in subdiagrams. In the EG formalism it is the inductive operator formalism which disentangles the problem of the renormalization of subdiagrams.

Let us, for completeness, also mention how the renormalization using BRST techniques deals with this problem [40]. The latter also proceeds inductively. One assumes that the theory is made finite up to a given order. This means that at the next order there are no subdivergences (since any graph of lower order is finite by assumption). This leaves us with the overall overlapping divergences (these are the ones that disappear if any of the propagators is opened). One then proves (within a specific symmetric regularization scheme provided such a scheme exists) that the latter disappear if the corresponding diagrams are differentiated enough times with respect to the external momenta (see for instance [41]). Therefore, the new divergences are polynomials in the external momenta, and thus local in x -space. One then proceeds to find the structure of the latter using BRST techniques, and finally show that the divergences can be removed by appropriate rescalings.

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