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SOLUTION OF A RELATIVISTIC THREE BODY PROBLEM

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Abstract

Starting from a relativistic s-wave scattering length model for the two particle input we construct an unambiguous, unitary solution of the relativistic three body problem given only the masses m_a, m_b, m_c and the masses of the two body bound states $\mu_{bc}, \mu_{ca}, \mu_{ab}$.

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1 INTRODUCTION

Some time ago one of us made an analysis of what was then needed to "solve" the three nucleon problem[17]. The conclusions were rather discouraging. In particular, it appeared to us that the enormous effort that Tjon and his collaborators, and a number of other few body nuclear physicists, were making to obtain and use "realistic" two-nucleon potential models to calculate the triton wave function and related problems were necessarily plagued by ambiguities which it would be difficult to remove. We eventually came to the conclusion that a "zero-range" or "on-shell" treatment of the problem might aid in simplifying the contact between these efforts and experiment. Unfortunately our efforts did not lead to the desired result in the form of a unique theoretical analysis for reasons we discuss to some extent in our paper reporting this body of research[21]. The reasons are rather complicated, but can be briefly summarized by the statement that *only* a finite particle number *relativistic* scattering theory can provide the needed framework. This paper presents a relativistic scattering length model which we believe will be a practical first step toward creating such a fundamental theory.

The key to understanding how we can create a simple, soluble and unitary model for a relativistic three body system is to realize that we embed the two body subsystems in the three body space from the start. We consider two particles with invariant masses m_a, m_b which scatter elastically when Mandelstam variable s (square of the invariant four momentum) lies in the range $(m_a + m_b)^2 \leq s \leq M_{th}^2$; here M_{th} is the total energy of the first inelastic (particle creation) threshold in the zero 3-momentum reference frame and $k(s)$ is the magnitude of the momentum of either particle in this frame. On 3-momentum and energy shell, we write $s = M^2$ with the conventional algebraic connections[2] to the energies of the free particles outside the elastic scattering volume:

$$\begin{aligned} e_a^2 - m_a^2 &= p^2 = e_b^2 - m_b^2 \\ M^2 &= (e_a + e_b)^2 - |\vec{p}_a + \vec{p}_b|^2 \\ |\vec{p}|(M; m_a, m_b) &= \frac{[(M^2 - (m_a + m_b)^2)(M^2 - (m_a - m_b)^2)]^{\frac{1}{2}}}{2M} \end{aligned} \tag{1}$$

$$(m_a + m_b)^2 \leq s \;\; = \;\; M^2 \leq M_{th}^2$$

As in our non-relativistic treatment[21], we consider only an interaction in which no angular momentum is transferred. We also analytically continue to values of $s \neq M^2$, keeping 3-momentum conserved as in the non-relativistic case, but using this kinematics and two-particle amplitudes embedded in a multi-particle space. As was pointed out to one of us (HPN) by J.V.Lindesay [14] in commenting on a preliminary version of this paper, the correct way to carry out this analytic continuation is to take

$$k_{ab}(s; m_a + m_b) = +\sqrt{\frac{[s - (m_a + m_b)^2][s - (m_a - m_b)^2]}{4s}} \tag{2}$$

This has the effect of embedding our two-particle interaction in a multi-particle space. We were earlier led to a version of this "off shell extension" by other considerations[16]. We are much indebted to Castillejo[6] for pointing out to us in a discussion of [16] that we had, in fact, implicitly assumed we were in a multi-particle space.

Whatever the off-shell extension, we can insure on-shell unitarity for the scattering amplitude $T(s)$ with the normalization $Im\ T(s) = k(s)|T|^2$ in the elastic scattering region by using the scattering length formula

$$T(s) = \frac{1}{\gamma - ik(s)} = \frac{e^{i\delta(s)} sin\ \delta(s)}{k(s)} \tag{3}$$

where γ is any finite constant. To generate a bound state pole at $s = \mu_{ab}^2$ we can (for $\mu_{ab}^2 > (m_a - m_b)^2$) take

$$4\mu_{ab}^2\gamma^2 = [(m_b + m_c)^2 - \mu_{ab}^2][\mu_{ab}^2 - (m_a - m_b)^2] > 0 \tag{4}$$

and pick the branch in the square root defining $k(s)$ by analytic continuation below elastic scattering threshold to insure that a pole occurs [25]. As we discuss to some extent in [21], p.1869 and as Weinberg explored extensively in his quasi-particle approach to the three body problem[29, 30, 31], this prescription amounts to assuming that the two particles composing the bound state are "structureless". This completes our specification of the two particle input to our three body problem.

In the next section we show that in our context this *algebraic* construction of on-shell two body unitarity plus the algebraic form of the Faddeev equations guarantees the three body unitarity of their solution. This was not obvious to some people in the context of our previous non-relativistic analysis, which led to considerable formal complexity in the final presentation[21]. In the current context the triviality becomes manifest for the 2-2 channels below three body breakup threshold where the solution is algebraic, as we show in Sec. 3.4. The 3 free to 3 free, coalesence and breakup amplitudes require a more detailed analysis of the zero angular momentum equations than we initially intended to make here. However, in order to meet a referee comment on an earlier version of this paper[23], we have extended our discussion in Sec. 3 and there provide algebraic solutions of the separable integral equations which follow from our assumptions when the three free particle channels are explicitly included. Examples of these solutions will be explored elsewhere [15]. Our concluding section contains speculations as to where this simple result might be applied.

2 RELATIVISTIC THREE BODY UNITARITY

2.1 Relativistic Faddeev 2-2 channel kinematics

Our model contains three structureless particles labeled a, b, c, which taken pairwise can form three bound states $(bc), (ca), (ab)$ whose only structure arises from these simple constituents. Although we are dealing with a relativistic system, we restrict our energies to the range which allows no particle creation; in this respect the situation is the same as in the non-relativistic three body problem, and the Faddeev[8, 9] channel decomposition can be employed. Because of this simple structure, if we start in a state with zero total 3-momentum and angular momentum, this situation will persist so long as our interactions have no internal degrees of freedom. This is true by hypothesis for the model we described in the introduction. We further assume that we start the system out with a scattering between one of the particles (which we can pick to be a) and the implied bound state pair (which will then be bc). Following Faddeev, we drop the redundant index. We distinguish the bare particles from the bound states by using the symbols m and μ for their respective masses. Then a specific scattering problem is completely specified by supplying numerical values for the following eight parameters:

$$\begin{aligned}
Particle\ masses &: m_a,\ m_b,\ m_c \\
Bound\ state\ masses &: \mu_a, \mu_b, \mu_c \\
Invariant\ 4-momentum &: M = \sqrt{S} \\
Input\ momentum &: k_a
\end{aligned} \tag{5}$$

The input momentum is the magnitude of the 3-momentum of particle a of mass m_a in the zero momentum frame with the bound state of particles b and c of mass μ_a having 3-momentum of equal magnitude but opposite direction. We call this momentum κ_a Calling the corresponding energy ϵ_a and the energy of the bare particle e_a *fixes* the exterior kinematics as follows:

$$\begin{aligned}
e_x^2 - k_x^2 &= m_x^2; \quad x \in a, b, c \\
\epsilon_x^2 - \kappa_x^2 &= \mu_x^2; \quad x \in a, b, c \\
\kappa_x &= -k_x \\
M_{abc} &= e_x + \epsilon_x = M\ independent\ of\ x
\end{aligned} \tag{6}$$

As can be seen from Eq.7 below, the constraint $M > m_x + \mu_x$ is required, introducing the θ-functions explicitly noted in Eq.52 below which mark the opening of the 2+1 channels at their energetic thresholds; these θ-functions are required for consistency with the orthogonality (Eq.45) and completeness (Eq.46) relations needed in the most general case we discuss.

We will not need to leave the three body zero momentum frame in this paper. Given M, we can in this restricted environment immediately compute the input momenta because

$$\begin{aligned}
4M^2k_x^2(\mu_x, M) = \ &[M^2 - (m_x + \mu_x)^2][M^2 - (m_x - \mu_x)^2] \ = 4M^2\kappa_x^2 \\
x \ &\in \ a, b, c
\end{aligned} \tag{7}$$

Thus the input momentum is not a parameter, but we still have to specify whether the input channel is a or b or c. Above three body breakup threshold ($M > m_a + m_b + m_c$) we must specify both the input and the output momenta for the "spectator" (see Sec.

3.1). Because we can have one or two rearrangement output channels in addition to elastic scattering in the entrance channel, the entrance channel has to be open for *anything* to happen, but how many output channels will then be reached introduces an asymmetry into the predictions. The next problem is to insure that these various possibilities are described by a unitary formalism.

2.2 On Shell Faddeev Equations

If we start in the a channel, the simplest thing that can happen is that the bc pair scatter without the initial spectator a being affected. These are the "disconnected diagrams" which initially caused so much trouble in analyzing the three body scattering problem, until Faddeev realized that that they could be subtracted out of the amplitude, leaving only connected diagrams. Here we call the full amplitude

$$ {}^{(3)}T(M) \equiv \Sigma_{x,y\in a,b,c}\ {}^{(3)}T_{xy}(M)\theta(M-m_x-\mu_x)\theta(M-m_y-\mu_y) \tag{8}$$

where ${}^{(3)}T_{xy}(M)$ are the unique solutions of Eq. 10 below.

We will return to the $\theta-$functions in Sec. 3, where we compute only those processes allowed by the conservation laws, except when a bound state vertex opens or closes in a single channel. We call the disconnected amplitude ${}^{(3)}T_a(M)$ and define it by

$$ {}^{(3)}T_a(M) \equiv\ {}^{(2)}T_{bc}([M-e_a(M)]^2) \tag{9}$$

where $e_a(M)$ is to be computed from Eq. 6. Then the relativistic version of the "on-shell Faddeev Equations" [20] for our simple pole model become

$$\begin{aligned} T_{ab}-T_a\delta_{ab} &= T_aRT_{bb}+T_aRT_{cb} = T_a\Sigma_x\bar{\delta}_{ax}RT_{xb} \\ &= T_{aa}RT_b+T_{ac}RT_b = [\Sigma_yT_{ay}R\bar{\delta}_{yb}]T_b \\ cyclic \quad on & \quad a,b,c \\ \bar{\delta}_{ab} &\equiv 1-\delta_{ab} \end{aligned} \tag{10}$$

Here R is the three free particle propagator, which because of our on-shell kinematics is simply a constant whose value we will determine from unitarity in the next two sections. The fact that the two alternative forms of Eq. 10 define the same function is critical for time reversal invariance. That requirement was what led us to conclude[21] that our non-relativistic zero range model could not be consistently defined when the two body amplitudes contain "left hand cuts" and not just bound state or CDD[7] poles. Algebraically the fact that summing the multiple scattering series starting from either the first or the last scattering — which is what the two alternative forms of Eq.10 express — follows from the fact that the two forms define the same multiple scattering series. Convergence follows if we use two-body input amplitudes which are always less than unity in absolute magnitude[14].

In our theory the convergence of the two forms of the multiple scattering series given in Eq. 10 to the same function suffices to prove the uniqueness of the solution. In a normal Hamiltonian theory, one would require that the homogeneous solutions of the two equations also must be proved to be identical. A related question is to ask "what operator must be diagonalized in order to make the spectral expansion (46)." Both questions are discussed more fully in the non-relativistic context when taking the "zero range limit" (see Ref. 21). Note that the same dispersion relation used to establish the unitarity of the two-body input in the non-relativistic case also applies to the relativistic scattering length model used here.

There is no known unambiguous way to go from *any* field-theoretic description of strongly interacting particles to a non-relativistic potential model of the class used *phenomenologically* in nuclear physics[17]. Ref.21 tried to remedy this defect by starting from a dispersion- theoretic representation of the two body amplitudes which *is* the non-relativistic limit of dispersion-theoretic relativistic S-Matrix amplitudes (cf. Ref.25), leading to a "left hand cut" representing virtual relativistic particle exchanges (eg pions). The difficulty which showed up in implementing this idea was that this approach cannot be made general enough. In particular, the Low equation (which, superficially, would seem to allow us to construct the non-relativistic potential from knowledge of the two-body phase shifts *and* the left hand cuts in the corresponding amplitudes) cannot be constructed. Specifically, the requirement of time reversal invariance for the two-body potential model which results from the attempted construction cannot be satisfied when the dispersion-theoretic input has left hand cuts.

Consequently any bound state pole in the two body-input will not vanish in the zero range limit, and corresponds to a CDD pole (Ref.7) as noted above. Consequently, some relativistic theory must be invoked to provide the parameters of this pole. As mentioned in Sec.3.4 we use here the "handy-dandy formula" connecting masses to coupling constants. We conclude that there is no direct way to relate our model to a non-relativistic Hamiltonian model, and hence that our completeness relation is part our model; it need not be derived.

The second point to emphasize is that our two-body bound states are *structureless*. They only appear in our three body system above energetic threshold, when the third particle can play the kinematic role of a "spectator". For the same reason, the equations in this paper cannot lead directly to three body bound states, which is why we do not need the (non-existent) "homogeneous solutions" to establish the equivalence of the two orders of the operators in Eq.10. The convergence of the summation of the two multiple scattering series to the same result suffices. Consistent treatment of three body bound states requires us to embed this model in a four body space in order to provide a kinematic "spectator", and is not attempted in this paper.

2.3 Three Particle Unitarity from Two Particle Unitarity

That the unitarity of the *off-shell* Faddeev equations follows from the unitarity of the two body input amplitudes was shown by Freedman, Lovelace and Namyslowski[11] and independently by Kowalski[13], who taught HPN the simple algebraic proof given below. The key is to write the unitarity condition on the two-body amplitude t_a *in the three body space* as

$$t_a(M) - t_a^*(M) = t_a(M)(R - R^*)t_a^*(M) \tag{11}$$

which determines the normalization of R in terms of the normalization of t_a. In order to avoid kinematic factors, it is convenient to use

$$t_a(M) = [e^{i\delta_s(s)} sin\ \delta_a(s)]_{s=[M-e_a(M)]^2} = [k_{bc}(s)\ {}^{(2)}T_{bc}(s)]_{s=[M-e_a(M)]^2} \tag{12}$$

so that $t - t^* = 2i\ tt^*$. Then $R = +i$ provides a channel independent propagator in the three particle space. The proper thresholds for the opening of the various channels are provided by the $\theta-$functions in Eq. 8. Using the same normalization, the unitarity condition on the three body channel amplitudes is then simply

$$T(M) - T^*(M) = T(M)(R - R^*)T^*(M) \tag{13}$$

and below breakup threshold the on-shell Faddeev equations become the algebraic equations

$$\begin{aligned} T_{ax} - t_a\delta_{ax} &= +it_a(T_{bx} + T_{cx}) = +it_a\Sigma_y\bar{\delta}_{ay}T_{yx} \\ &= +i((T_{ay} + T_{az})t_x = +i[\Sigma_y T_{ay}\bar{\delta}_{yx}]t_x \end{aligned} \tag{14}$$

The proof of unitarity follows the same steps taken in[20], Eq.'s (2)-(5), but below breakup threshold these are now actually algebraic rather than symbolic. If in T_{ab} we call $L_a = \Sigma_b T_{ab}$ and $F_b = \Sigma_a T_{ab}$, we clearly have that

$$T(M) = \Sigma_a L_a = \Sigma_b F_b \tag{15}$$

Then the unitarity condition we wish to prove becomes

$$\begin{aligned} T - T^* &\stackrel{?}{=} \\ \Sigma_x F_x(R - R^*)L^*_x &+ \Sigma_{x,y}\bar{\delta}_{xy}F_x(R - R^*)L^*_y \end{aligned} \tag{16}$$

where we have separated out the term where the indices differ so that we can make use of the two body unitarity condition. This is possible because, in the current notation, we can rewrite the Faddeev equations (Eq.10) as

$$\begin{aligned} F_x &= (1 - \Sigma_w F_w\bar{\delta}_{wx}R)t_x \\ L^*_x &= t^*_x(1 - \Sigma_z R^*\bar{\delta}_{zx}L^*_z) \end{aligned} \tag{17}$$

Consequently the equation to be proved becomes

$$\begin{aligned} T - T^* & & \stackrel{?}{=} & \\ \Sigma_x(1 - \Sigma_w F_w\bar{\delta}_{wx}R) & \quad t_x(R - R^*)t^*_x & & (1 - \Sigma_z R^*\bar{\delta}_{zx}L^*_z) \\ & & + & \Sigma_{x,y}\bar{\delta}_{xy}F_x(R^* - R)L^*_y \end{aligned} \tag{18}$$

We can now take the critical step of substituting $t_x - t_x^*$ for $t_x(R - R^*)t_x^*$ and find that

$$
\begin{aligned}
& T - T^* \stackrel{?}{=} \\
& \Sigma_x(1 - \Sigma_w F_w \bar{\delta}_{wx} R)t_x - t_x^*(1 - \Sigma_z R^* \bar{\delta}_{zx} L_z^*) \\
- \quad & \Sigma_{x,y}[t_x - \Sigma_w F_w \bar{\delta}_{wx} R t_x - F_x] R^* \bar{\delta}_{xy} L_y^* \\
+ \quad & \Sigma_{x,y} F_x R \bar{\delta}_{xy}[t_y^* - t_y^* \Sigma_z R^* \bar{\delta}_{zx} L_z^* - L_y^*] \\
& = \qquad T - T^* \quad Q.E.D.
\end{aligned}
\tag{19}
$$

where the unwanted terms vanish because F_x and L_y^* are solutions of the Faddeev equations.

3 SOLUTION OF THE ON-SHELL FADDEEV EQUATIONS

3.1 3-3 Kinematics

In order to properly relate the amplitudes for elastic and rearrangement collisions below and above breakup threshold it is useful to first express the amplitudes as operators in an orthonormal and complete space and then reduce them to integral or algebraic equations. In our non-relativistic treatment[21] this happened automatically because there we took the "zero range limit" of equations originally written in a larger space. Here we make a new approach by formulating the operators directly in the on-shell space of empirically observable particle and bound state momenta. For 3-3 collisions, under the restriction to s-channel driving terms and total angular momentum zero (and ignoring for the moment the two-body bound states and channels associated with them), this is simply the space of the relativistic Dalitz plot. We modify the standard notation[2] to accommodate the Faddeev channel decomposition (a the spectator of a bc pair interaction or scattering) as follows. For single free particles $p_i = (E_i, \vec{p}_i)$, $i \in a, b, c$, and $p_i^2 = E_i^2 - \vec{p}_i \cdot \vec{p}_i = m_i^2$. Then, for i, j, k cyclic or anti-cyclic on a, b, c, if p_i is the four-momentum of the spectator, we define the

Mandelstam invariant s_i for this channel by

$$s_i \equiv m_{jk}^2 = (p_j + p_k)^2 = (E_j + E_k)^2 - (\vec{p}_j + \vec{p}_k) \cdot (\vec{p}_j + \vec{p}_k) \tag{20}$$

In an arbitrary coordinate system

$$P \equiv p_a + p_b + p_c; P^2 \equiv M^2 = (E_a + E_b + E_c)^2 - (\vec{p}_a + \vec{p}_b + \vec{p}_c) \cdot (\vec{p}_a + \vec{p}_b + \vec{p}_c) \tag{21}$$

It follows immediately that in the coordinate system where $\vec{P} \equiv (\vec{p}_a + \vec{p}_b + \vec{p}_c) = 0$, that

$$\begin{aligned} s_i &= M^2 + m_i^2 - 2ME_i \\ s_a + s_b + s_c &= M^2 + m_a^2 + m_b^2 + m_c^2 \equiv \Sigma^2 \end{aligned} \tag{22}$$

and that, cf (34.20b), with $|\vec{p}_a|^2 \to k_a^2$,

$$\begin{aligned} 4M^2 k_a^2(s_a) &= [M^2 - (s_a^{\frac{1}{2}} + m_a)^2][M^2 - (s_a^{\frac{1}{2}} - m_a)^2] \\ &= (M^2 - m_a^2 - s_a)^2 - 4m_a^2 s_a \\ &= (M^2 + m_a^2 - s_a)^2 - 4M^2 m_a^2 \end{aligned} \tag{23}$$

allowing us to define $k_a(s_a) \equiv |\vec{p}_a|$ in this zero 3-momentum frame.

The magnitudes of the three momenta k_a, k_b, k_c, or equivalently (cf. Eq.'s 22 and 23) the three invariants s_a, s_b, s_c , specify a rigid triangle. The orientation of this triangle relative to some space fixed system of axes in which the system as a whole has zero 3-momentum can be specified by three Euler angles, (α, β, γ), cf. Eq. 34.18. In, for example, Osborn's treatment[27], the three degrees of freedom connecting the initial to the final state are discretized as a rotation matrix $d^J_{\lambda,\lambda'}(\theta)$ and $cos\ \theta$ is expressed in terms of the scalar momenta k, k'. Under our s-channel, zero total angular momentum assumption, this rotation matrix is simply a constant, which is a way of seeing why we have only three degrees of freedom even before we go on shell (restrict ourselves to the interior of the Dalitz plot by fixing M in Eq.22).

3.2 Separable integral equations for 3 free- 3 free scattering

Define the operators $\mathcal{M}_{ax} \equiv \mathcal{T}_a \delta_{ax} + \mathcal{W}_{ax}$, making the Faddeev equations starting from entrance channel a read

$$\begin{aligned} \mathcal{W}_{aa} - \mathcal{T}_a \mathcal{R}(\mathcal{W}_{ba} + \mathcal{W}_{ca}) &= 0 \\ -\mathcal{T}_b \mathcal{R} \mathcal{W}_{aa} + \mathcal{W}_{ba} - \mathcal{T}_b \mathcal{R} \mathcal{W}_{ca} &= \mathcal{T}_b \mathcal{R} \mathcal{T}_a \\ -\mathcal{T}_c \mathcal{R}(\mathcal{W}_{aa} + \mathcal{W}_{ba}) + \mathcal{W}_{ca} &= \mathcal{T}_c \mathcal{R} \mathcal{T}_a \end{aligned} \tag{24}$$

Separate $\mathcal{W}_{ax}$ into the four amplitude operators

$$\begin{aligned} \mathcal{W}_{ax} &= \mathcal{A}_{ax} + \mathcal{B}_{ax} + \mathcal{C}_{ax} + \mathcal{D}_{ax} \\ \mathcal{A}_{ax} &: \textit{Anelastic scattering} \\ \mathcal{B}_{ax} &: \textit{Breakup} \\ \mathcal{C}_{ax} &: \textit{Coalesence} \\ \mathcal{D}_{ax} &: 3\ free - 3\ free\ scattering \end{aligned} \tag{25}$$

Assume that the first three are zero. Then the operator Faddeev equations for the 3-free to 3-free amplitude starting in entrance channel a are

$$\begin{aligned} \mathcal{D}_{aa} - \mathcal{T}_a \mathcal{R}(\mathcal{D}_{ba} + \mathcal{D}_{ca}) &= 0 \\ -\mathcal{T}_b \mathcal{R} \mathcal{D}_{aa} + \mathcal{D}_{ba} - \mathcal{T}_b \mathcal{R} \mathcal{D}_{ca} &= \mathcal{T}_b \mathcal{R} \mathcal{T}_a \\ -\mathcal{T}_c \mathcal{R}(\mathcal{D}_{aa} + \mathcal{D}_{ba}) + \mathcal{D}_{ca} &= \mathcal{T}_c \mathcal{R} \mathcal{T}_a \end{aligned} \tag{26}$$

and the two obvious cyclic permutations for entrance channels b and c. The alternative form, with exit channel (rather than entrance channel) specified as a is

$$\begin{aligned} \mathcal{D}_{aa} - (\mathcal{D}_{ab} + \mathcal{D}_{ac}) \mathcal{R} \mathcal{T}_a &= 0 \\ -\mathcal{D}_{aa} \mathcal{R} \mathcal{T}_b + \mathcal{D}_{ab} - \mathcal{D}_{ac} \mathcal{R} \mathcal{T}_b &= \mathcal{T}_a \mathcal{R} \mathcal{T}_b \\ -(\mathcal{D}_{aa} + \mathcal{D}_{ab}) \mathcal{R} \mathcal{T}_c + \mathcal{D}_{ac} &= \mathcal{T}_a \mathcal{R} \mathcal{T}_c \end{aligned} \tag{27}$$

To convert these into one-variable integral equations, we assume the orthogonality and completeness relations

$$\begin{aligned} < s_x | s'_y > &= \delta_{xy} \delta(s_x - s'_y) \\ \int_{(m_y+m_z)^2}^{(M-m_x)^2} ds_x | s_x >< s_x | &= \mathbf{1} \end{aligned} \tag{28}$$

for the states onto which we project. Here, and from now on, the kinematic constraint (cf. Eq.22) on the variables

$$s_a + s_b + s_c = M^2 + m_a^2 + m_b^2 + m_c^2 \equiv \Sigma^2 = s'_a + s'_b + s'_c \tag{29}$$

is to be understood. The matrix elements of the operators are taken to be

$$\begin{aligned} < s_x|\mathcal{R}|s'_y > &= R\bar{\delta}_{xy}\theta(M - m_a - m_b - m_c) \\ < s_x|\mathcal{T}_x|s'_y > &= \delta_{xy}\delta(s_x - s'_y)e^{i\delta_x} sin\ \delta_x(s_x, M)\theta(s_x - (m_y + m_z)^2) \\ < s_x|\mathcal{D}_{xy}|s'_y > &= D_{xy}(s_x; M; s'_y)\theta(M - m_a - m_b - m_c) \end{aligned} \tag{30}$$

where R is a constant to be fixed by requiring on-shell unitarity and $\bar{\delta}_{xy} = 1 - \delta_{xy}$. It is convenient to also define $t_x(s_x, M) \equiv e^{i\delta_x} sin\ \delta_x(s_x, M)$. Then, by taking matrix elements of the operator equations between the appropriate bras and kets, and invoking the completeness relation to bring in the appropriate integrals, we find that, for a the entrance channel,

$$\begin{aligned} &D_{aa}(s_a; s'_a) - t_a(s_a)R[\textstyle\int_{(m_a+m_c)2}^{(M-m_b)^2} ds''_b D_{ba}(s''_b; s'_a) + \int_{(m_a+m_b)2}^{(M-m_c)^2} ds''_c D_{ca}(s''_c; s'_a)] \\ &= 0 \\ &-t_b(s_b)R\textstyle\int_{(m_b+m_c)2}^{(M-m_a)^2} ds''_a \mathcal{D}_{aa}(s''_a; s'_a) + D_{ba}(s_b; s'_a) - t_b(s_b)R\int_{(m_a+m_b)2}^{(M-m_c)^2} ds''_c D_{ca}(s''_c; s'_a) \\ &= t_b(s_b)Rt_a(s'_a) \\ &-t_c(s_c)R[\textstyle\int_{(m_b+m_c)2}^{(M-m_a)^2} ds''_a D_{aa}(s''_a; s'_a) + \int_{(m_a+m_c)2}^{(M-m_b)^2} ds''_b D_{ba}(s''_b; s'_a)] + \mathcal{D}_{ca}(s_c; s'_a) \\ &= t_c(s_c)Rt_a(s'_a) \end{aligned} \tag{31}$$

Here we have suppressed the fixed argument M in the functions and retained only the variables s_x, s'_x, s''_x of the integral equation and the fixed parameter s'_a describing the initial state. The alternative form of the equations (Eq. 27) reverses the role of variables and parameters, leading to

$$\begin{aligned} &D_{aa}(s'_a; s_a) - [\textstyle\int_{(m_a+m_c)2}^{(M-m_b)^2} ds''_b D_{ab}(s'_a; s''_b) + \int_{(m_a+m_b)2}^{(M-m_c)^2} ds''_c D_{ac}(s'_a; s''_c)]Rt_a(s_a) \\ &= 0 \\ &-\textstyle\int_{(m_b+m_c)2}^{(M-m_a)^2} ds''_a \mathcal{D}_{aa}(s'_a; s''_a)Rt_b(s_b) + D_{ab}(s'_a; s_b) - \int_{(m_a+m_b)2}^{(M-m_c)^2} ds''_c D_{ac}(s'_a; s''_c)Rt_b(s_b) \\ &= t_a(s'_a)Rt_b(s_b) \end{aligned} \tag{32}$$

$$-[\int_{(m_b+m_c)^2}^{(M-m_a)^2} ds''_a D_{aa}(s'_a;s''_a) + \int_{(m_a+m_c)^2}^{(M-m_b)^2} ds''_b D_{ba}(s'_a;s''_b)]Rt_c(s_c) + \mathcal{D}_{ca}(s_c;s'_a)$$
$$= t_a(s'_a)Rt_c(s_c)$$

We see that Eq.31 and Eq.32 are, as the sub-section title claims, separable integral equations for one-variable, one-parameter functions. But the two alternative forms have to be solved for all nine functions and not just for three of them before we can even ask the question as to whether they both define the *same* nine functions $D_{xy}(s_x;M;s_x)$. Until we have made this proof, it will be safer to call the solutions of Eq. 31 (and their two cyclic permutations) $L_{xy}(s_x;M;s'_y)$ with the understanding that s_x is the variable and s'_y the parameter while the solutions of Eq. 32 will be designated by $R_{xy}(s'_x;M;s_y)$.

3.3 Solution of the 3-3 problem

To solve these equations we first define

$$\begin{aligned} \bar{t}_x &\equiv \int_{(m_y+m_z)^2}^{(M-m_x)^2} ds_x t_x(s_x;M) \\ l_{xy}(s_y) &\equiv \int_{(m_y+m_z)^2}^{(M-m_x)^2} ds_x L_{xy}(s_x;M;s_y) \\ r_{xy}(s_x) &\equiv \int_{(m_z+m_x)^2}^{(M-m_y)^2} ds_y R_{xy}(s_x;M;s_y) \end{aligned} \tag{33}$$

Then, integrating Eq. 31 over the variables yields

$$\begin{pmatrix} 1 & -\bar{t}_a R & -\bar{t}_a R \\ -\bar{t}_b R & 1 & -\bar{t}_b R \\ -\bar{t}_c R & -\bar{t}_c R & 1 \end{pmatrix} \begin{pmatrix} l_{aa}(s'_a) \\ l_{ba}(s'_a) \\ l_{ca}(s'_a) \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{t}_b R t_a(s'_a) \\ \bar{t}_c R t_a(s'_a) \end{pmatrix} \tag{34}$$

While integrating Eq. 32 over the variables yields

$$\begin{pmatrix} 1 & -\bar{t}_a R & -\bar{t}_a R \\ -\bar{t}_b R & 1 & -\bar{t}_b R \\ -\bar{t}_c R & -\bar{t}_c R & 1 \end{pmatrix} \begin{pmatrix} r_{aa}(s'_a) \\ r_{ab}(s'_a) \\ r_{ac}(s'_a) \end{pmatrix} = \begin{pmatrix} 0 \\ t_a(s'_a) R \bar{t}_b \\ t_a(s'_a) R \bar{t}_c \end{pmatrix} \tag{35}$$

These equations are clearly algebraic and, together with the two cyclic permutations, define the functions $l_{xy}(s'_y)$ and $r_{xy}(s'_x)$ for any value of the entrance or exit parameters allowed by the fixed value of M. Assuming that Eq. 34 has been inverted (see below), we then can insert the solution in Eq. 31 to specify the full solution

$$\begin{aligned} L_{aa}(s_a;M;s'_a) &= t_a(s_a;M)R[l_{ba}(s'_a)+l_{ca}(s'_a)] \\ L_{ba}(s_b;M;s'_a) &= t_b(s_b;M)R[t_a(s'_a;M)+l_{aa}(s'_a)+l_{ca}(s'_a)] \\ L_{ca}(s_c;M;s'_a) &= t_c(s_c;M)R[t_a(s'_a;M)+l_{aa}(s'_a)+l_{ba}(s'_a)] \end{aligned} \tag{36}$$

while if Eq. 35 has been inverted we find that

$$\begin{aligned} R_{aa}(s'_a;M;s_a) &= [r_{ab}(s'_a)+r_{ac}(s'_a)]Rt_a(s_a;M) \\ R_{ab}(s'_a;M;s_b) &= [t_a(s'_a;M)+r_{aa}(s'_a)+r_{ac}(s'_a)]Rt_b(s_b;M) \\ R_{ac}(s'_a;M;s_c) &= [t_a(s'_a;M)+r_{aa}(s'_a)+r_{ab}(s'_a)]Rt_c(s_c;M)R \end{aligned} \tag{37}$$

It still remains to prove that, when we complete the system by supplying the remaining permutations, the two *different* routes by which we arrive at the solutions define the *same* nine functions.

One way to do this is to define

$$\begin{aligned} t_{xy} &\equiv \bar{\delta}_{xy}\bar{t}_x R\bar{t}_y = t_{yx} \qquad (38) \\ l_{xy} &\equiv \int_{(m_y+m_z)^2}^{(M-m_x)^2} ds_x \int_{(m_z+m_x)^2}^{(M-m_y)^2} ds_y L_{xy}(s_x;M;s_y) \\ r_{xy} &\equiv \int_{(m_y+m_z)^2}^{(M-m_x)^2} ds_x \int_{(m_z+m_x)^2}^{(M-m_y)^2} ds_y R_{xy}(s_x;M;s_y) \end{aligned}$$

Then, integrating Eq. 34 over the remaining parameter yields

$$\begin{pmatrix} 1 & -\bar{t}_a R & -\bar{t}_a R \\ -\bar{t}_b R & 1 & -\bar{t}_b R \\ -\bar{t}_c R & -\bar{t}_c R & 1 \end{pmatrix} \begin{pmatrix} l_{aa} \\ l_{ba} \\ l_{ca} \end{pmatrix} = \begin{pmatrix} 0 \\ t_{ba}(=t_{ab}) \\ t_{ca}(=t_{ac}) \end{pmatrix} \tag{39}$$

while integrating Eq. 35 over the remaining parameter yields

$$\begin{pmatrix} 1 & -\bar{t}_a R & -\bar{t}_a R \\ -\bar{t}_b R & 1 & -\bar{t}_b R \\ -\bar{t}_c R & -\bar{t}_c R & 1 \end{pmatrix} \begin{pmatrix} r_{aa} \\ r_{ab} \\ r_{ac} \end{pmatrix} = \begin{pmatrix} 0 \\ t_{ab}(=t_{ba}) \\ t_{ac}(=t_{ab}) \end{pmatrix} \tag{40}$$

We see immediately that the two alternative forms define the same algebraic matrix and hence that $l_{xy} = r_{yx}$. Because of the symmetry of the driving terms $l_{xy} = l_{yx}$, $r_{xy} = r_{yx}$ and we are free to define

$$z_{xy} \equiv l_{xy} = r_{xy} = l_{yx} = r_{yx} \tag{41}$$

The inversion is algebraic and straightforward[12]. Explicitly, with

$$R = i \tag{42}$$

we find that

$$\begin{pmatrix} z_{aa} \\ z_{ba} \\ z_{ca} \end{pmatrix} = \begin{pmatrix} -\bar{t}_a\bar{t}_b - \bar{t}_a\bar{t}_c - 2i\bar{t}_a\bar{t}_b\bar{t}_c \\ i\bar{t}_b - \bar{t}_b\bar{t}_c \\ i\bar{t}_c - \bar{t}_c\bar{t}_b \end{pmatrix} \frac{\bar{t}_a}{1 + \bar{t}_a\bar{t}_b + \bar{t}_b\bar{t}_c + \bar{t}_c\bar{t}_a + 2i\bar{t}_a\bar{t}_b\bar{t}_c} \tag{43}$$

We have now solved the 3-3 problem (with no bound states) because we can reconstruct the variable and parameter content of the solution as

$$\begin{aligned} & For\ \mathcal{A} = \mathcal{B} = \mathcal{C} = 0 \\ M_{aa}(s_a; M; s'_a) &= t_a(s_a, M)\delta(s_a - s'_a) + t_a(s_a, M) z_{aa} t_a(s'_a, M) \\ M_{ba}(s_b; M; s'_a) &= t_b(s_b, M) z_{ba} t_a(s'_a, M) \\ M_{ca}(s_c; M; s'_a) &= t_c(s_c, M) z_{ca} t_a(s'_a, M) \end{aligned} \tag{44}$$

Clearly, the remaining six amplitudes can be written in the same way.

The Kowalski version of the unitarity proof is now algebraic and trivial, provided we use the state normalization which makes two-particle unitarity require that $t - t^* = 2itt^*$ and $R - R^* = 2iRR^*$, which forces us to choose the constant $R = i$.

3.4 Coalesence, Breakup, Anelastic Scattering coupled to 3 free-3 free Scattering

If we now include $N_a + N_b + N_c$ two-body bound states $|\mu_x^{n_x} >$ with $x \in a, b, c$ and $n_x \in 1, 2, .., N_x$ the orthonormality condition must be extended to include

$$\begin{aligned} < \mu_x^{n_x} | \mu_{x'}^{n'_{x'}} > &= \delta_{xx'} \delta_{n_x n'_{x'}} \\ < s_x | \mu_x^n > &= 0 \end{aligned} \tag{45}$$

and the completeness equation becomes

$$\int_{(m_y+m_z)^2}^{(M-m_x)^2} ds_x|s_x><s_x| + \Sigma_{n_x=1}^{N_x}|\mu_x^{n_x}><\mu_x^{n_x}| = \mathbf{1} \tag{46}$$

Although the propagator $\mathcal{R}\bar{\delta}_{xy}$ remains constant, using our normalizations in this extended space, we can add a new dynamical element to the system if we allow (as we can, conserving on-shell energy and 3-momentum) the 2-2 scattering operators $\mathcal{T}_a(M)$, whose matrix elements in the 3-3 part of the space are given above, to couple 1,2 states (anelastic scattering) to 3 free particle states (breakup) or visa versa (coalesence) via the three free particle propagator $\mathcal{R}$ preserving probability conservation (unitarity). In constructing our specific way of providing this coupling, we have been guided by (a) the success of the dispersion-theoretic non-relativistic approximation of single pion exchange in predicting the ${}^1S_0$ shape parameter in the archetypal strong interaction problem (nucleon-nucleon scattering [24, 19]), (b) Faddeev's analysis of the role of the "essential singularities" in the non-relativistic case and (c) the success of the "handy dandy formula" connecting masses to coupling constants[16, 22].

We note first that the thresholds for the opening of the elastic and anelastic channels can be uniquely specified by the thresholds:

$$M_{n_x} \equiv m_x + \mu_x^{n_x} \tag{47}$$

and (assuming no degeneracies and at least one bound state) uniquely ordered by some integer parameter $1 \leq n \leq N_a + N_b + N_c$. So we can always specify and order uniquely the parameters

$$M_{n_x}^1 < M_{n_{x'}}^n < M_{n_{x''}}^{n+1} < m_a + m_b + m_c; \quad x, x', x'' \in a, b, c; \quad n \leq N_a + N_b + N_c \tag{48}$$

Consider first only anelastic scattering ($M_{n_x}^1 < M < m_a+m_b+m_c$). Then, relying on Faddeev's insight (b), we will assume that the scattering operator has the matrix element

$$[\lim_{s_a\to(\mu_x^{n_x})^2}]((s_a - (\mu_x^{n_x})^2) < \mu_x^{n_x}|\mathcal{T}_x|\mu_y^{n'_y} >) = \tag{49}$$
$$\delta_{xy}\delta_{n_x n'_x}\ i\Gamma_x^{n_x}(k_x^{n_x}, M)\theta(M - m_x - \mu_x^{n_x})$$

Here $\Gamma_x^{n_x}(k_x^{n_x}, M)$ is the residue at the pole of the two-body scattering amplitude at $s_x = (\mu_x^{n_x})^2$ and for our scattering length model is given by (cf.Eq.'s 2,3,4,8)

$$\Gamma_x^{n_x}(k_x^{n_x}, M) = \frac{\gamma_x^{n_x}}{M}\sqrt{[M^2 - (m_x + \mu_x^{n_x})^2][M^2 - (m_x - \mu_x^{n_x})^2]} \tag{50}$$

This is a simple algebraic consequence of the algebraic structure of our two particle input model; The θ-function in Eq.49 makes this limit consistent with the orthogonality (45) and completeness (46) relations. Both below and above breakup threshold we model the opening and closing of the bound state vertex by assumung that

$$\begin{aligned} < \mu_x^{n_x}|\mathcal{R}|\mu_y^{n'_y} > &= \bar{\delta}_{xy}R \\ < \mu_x^{n_x}|\mathcal{R}|s_y > &= \bar{\delta}_{xy}R \\ < s_x|\mathcal{R}|\mu_y^{n'_y} > &= \bar{\delta}_{xy}R \\ < s_x|\mathcal{R}|s_y > &= \bar{\delta}_{xy}R \end{aligned} \tag{51}$$

For $M < m_a + m_b + m_c$, taking matrix elements of the Faddeev equations between the bound states and inserting the completeness relations as we did in the continuum case, we clearly obtain N algebraic equations for N amplitudes where N is the number of open channels allowed by the θ-fuctions using the fixed value of M. For one bound state in each channel, we obtain a triple of three equations in three unknowns, whose solution is of the same algebraic form as those for the z_{xy} exhibited above, with $\bar{t}_x$ replaced by Γ_x^1. These were the equations we had in mind in the first version of this paper[23]; the intent of the original paper was to explore breakup and related problems in subsequent work[15]. As already noted, the more complete treatment here was needed to meet referee comments, for which we are grateful.

The postulate of a "channel independent" (indeed constant) "three free particle propagator" below breakup threshold is our way of taking over the Faddeev idea that "once a scattering occurs, we must allow one of the pair to interact with a third particle before anything more can happen" into our on-shell theory. Another way to put it is that "once a bound state vertex opens up, another degree of freedom must intervene before that pair can interact again." This is familiar in other Faddeev contexts; only this particular articulation of the idea is novel. This analysis also shows us how the coalesence and breakup amplitudes couple into the system (cf. Eq.'s 49,50,51).

It is now straightforward to write down the full coupled equations for $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ above breakup threshold for any arbitrary finite number of bound states and reduce them to algebraic equations for the constants $z_{xy}^{n_x n'_y}$. Once solved, we can reconstruct the full variable content of the amplitudes by writing

$$\begin{aligned}
\mathcal{A}_{xy}^{n_x n'_y}(M) &= \Gamma_x^{n_x} z_{xy}^{n_x n'_y} \Gamma_y^{n'_y} \\
&\quad \theta(M - (m_x + \mu_x^{n_x}))\theta(M - (m_y + \mu_y^{n'_y})) \qquad (52)\\
\mathcal{B}_{xy}^{n'_y}(s_x; M) &= t_x(s_x, M) z_{xy}^{n'_y} \Gamma_y^{n'_y} \\
&\quad \theta(s_x - (m_a + m_b + m_c)^2) \qquad (53)\\
\mathcal{C}_{xy}^{n_x}(M; s'_y) &= \Gamma_x^{n_x} z_{xy}^{n_x} t_y(s'_y, M) \\
&\quad \theta(s'_y - (m_a + m_b + m_c)^2) \qquad (54)\\
\mathcal{D}_{xy}(s_x; M; s'_y) &= t_x(s_x, M) z_{xy} t_y(s'_y, M) \\
&\quad \theta(s_x - (m_a + m_b + m_c)^2)\theta(s'_y - (m_a + m_b + m_c)^2) \qquad (55)
\end{aligned}$$

Knowing $z_{xy}^{n_x n'_y}$, $z_{xy}^{n'_y}$, $z_{xy}^{n_x}$ and z_{xy} we can now provide predictions for all elastic, rearrangement, three body breakup and 3-3 scattering cross sections for all energy-momentum-$J = 0$ conserving processes over the entire kinematic region $M < M_{th}$.

The two body model used assumes that we know *either* the binding energy of a (single) two body bound state *or* the scattering length in the isolated two-particle system *but not both*. Our treatment up to this point has implicitly assumed that our six parameters m_x, μ_x, $x \in a, b, c$, are all consistent with known two body data over the range of interest. The failure of this connection, either by the two parameters (which can be determined by different types of experiment) being inconsistent with it, or by the departure of the predicted elastic scattering cross section in the physical region from experiment suffices to show that we must enrich the parameter content of the model. This would be analagous to what happened historically[4] in the study of neutron-proton scattering and established the spin-dependence of nuclear forces. Precise experiments analyzed using related ideas have even demonstrated the existence of pions using nucleon-nucleon s-wave experiments below 10 Mev[24, 19].

When it comes to the unique predictions of our three body model, the comparison with experiment becomes richer than for the two-body scattering length model. In particular we can now predict *three* scattering lengths or their equivalent and compare these with low energy s-wave experiments. If any one of these fails to agree with experiment, one place to look for an explanation is to postulate a single three-body bound state. If our matrix formulation of the Faddeev equations, when analytically continued below the threshold for the lowest kinematically allowed 2-2 elastic scattering, possesses a homogeneous solution at some value of M, this value of M predicts the existence of a 3-body bound state at that invariant energy. As already noted, this analytic continuation in our theory is only possible after we embedded our model in an appropriate 4-body space. If the state with the right quantum numbers is found, but the value of M differs experimentally from that predicted, we can extend our model phenomenologically by explicitly introducing an $(abc) \leftrightarrow (abc)$ channel in addition to the three channels we started with. Then we must solve four equations for four unknowns, but no new conceptual problems arise. Since direct scattering of three free particles to three free particles is usually too difficult to measure, particularly at three body breakup threshold, we do not count this as a new piece of available experimental information. However, we now have one parameter to explain (if the bound state exists at a known mass m_{abc}) four experimental numbers, giving a stringent consistency check even using threshold data. We intend to explore consistency conditions on these parameters elsewhere.

As a significant example, particularly relevant to Tjon's *oevre*, the n, n and n, p singlet scattering lengths and the deuteron binding energy can be used as the three 2-2 channel parameters for the n, n, p system. Only two scattering lengths (n, d doublet and n, d quartet) are measured in addition to the binding energy of the triton. ${}^4a_{nd}$ is well predicted while ${}^2a_{nd}$ and ϵ_t are highly correlated ("Phillips line") for reasons that can be understood from a dispersion-theoretic point of view[3]. Of course Barton and Phillips' explanation is consistent with the physics underlying our on-shell approach, so we expect our model to achieve a comparable result. In historical fact, it was their work which helped us start thinking about the usefulness of a more general approach in the first place.

It should be obvious from the treatment of the four body problem in our non-relativistic paper[21] that the current approach can be readily generalized to relativistic four particle systems, as we intend to do on another occasion.

4 SPECULATIONS AND CONCLUSIONS

The example of the applicability of this model to nuclear physics given in the last section hardly begins to suggest the range of problems which we believe can profitably be explored using the approach presented here. For instance, the equivalent of the relativistic scattering length formula used here was first written down by Bohr in 1915 [5, 16]. Viewed from the current point of view, this makes the hydrogen atom a relativistic bound state of a proton and an electron. This suggests looking at the three body systems e, e, p (H^-), e, p, p (H_2^+) and similar atomic systems using the explicit model presented in this paper.

For strong interactions, we suggest treating the deuteron as a neutron-proton-pion bound state. If we include crossing and the Fermi-Yang model for the pion[10] as a bound state of a nucleon and an anti-nucleon the usefulness of the approach for deeply bound states should become manifest. We suspect that relativistic models of quarks and quark confinement could also be attacked using these methods.

The relativistic scattering length model employed here actually arose in a study of the fine structure spectrum of hydrogen [16]. In conjunction with combinatorial arguments, this model leads to the Sommerfeld formula *without* any specific use of the concept of "spin", and to an improvement of the lowest order combinatorial calculation of the low energy fine structure constant ($(\alpha_e(m_e^2)^{-1} = 137$) by four significant figures (to 137.03596...). Similar improvement of our understanding of "bit-string-physics"[22] can be anticipated when we make use of the three and four body dynamics adumbered here. We hope that others may be induced to try these simple methods and see how far they might lead.

Because of the occasion to which this issue of Few Body Physics is dedicated, one of us (HPN) thinks it appropriate to reiterate here, as was stated long ago[18], that the reduction of the three body problem from three to two continuous variables presented by Osborn and Noyes[28] was first, independently, developed by Ahmadzadeh and Tjon[1]. This is yet another reminder, of which there will be many in this issue, of how continuously useful and important Prof. Tjon's dedication to our field has been.

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