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Two-Loop Large-m_t Electroweak Corrections to $K \to \pi\nu\bar{\nu}$ for Arbitrary Higgs Boson Mass *

Gerhard Buchalla[1] and Andrzej J. Buras[2]

[1]*Stanford Linear Accelerator Center*
Stanford University, Stanford, CA 94309, U.S.A.

[2]*Physik Department, Technische Universität München*
D-85748 Garching, Germany

Abstract

We consider for the first time the leading large top mass corrections, arising at higher order in electroweak interactions, to the rare decays $K \to \pi\nu\bar{\nu}$ and the related modes $B \to X_s\nu\bar{\nu}$ and $B \to l^+l^-$. Higher order effects of similar type have previously been calculated in the large-m_t limit for key observables of precision electroweak physics at Z-factories. Here we obtain the corresponding corrections of order $\mathcal{O}(G_F^2 m_t^4)$ at the amplitude level for short-distance dominated rare meson decays. This allows us to quantify the importance of higher order electroweak effects for these processes, which can be reliably computed and have very small uncertainties from strong interactions. Simultaneously it becomes possible to remove, to some extent, ambiguities in the definition of electroweak parameters describing the strength of FCNC interactions. The corrections we discuss are at the level of a few percent.

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1 Introduction

In the Standard Model flavor-changing neutral current (FCNC) interactions are generated at one-loop order. They give rise to neutral meson mixing, CP violation and rare decays, which therefore provide excellent opportunities to study flavor dynamics. A class of rare decay modes, including $K \to \pi\nu\bar{\nu}$, $B \to X_{s,d}\nu\bar{\nu}$ and $B_{s,d} \to l^+l^-$, has long been recognized to be particularly interesting in this respect. Since there are no contributions from virtual photons in these cases, the GIM cancellation pattern is powerlike ($\sim m_i^2/M_W^2$, $i = u, c, t$, for $m_i \ll M_W$), resulting in a strong suppression of potential long distance effects. The processes are dominated by short distances, related to the heavy particles (W, top, charm) in the loop, and can be reliably calculated.

The low-energy effective Hamiltonian for $K \to \pi\nu\bar{\nu}$ to lowest order in electroweak interactions can be written as

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_W} \left(\lambda_t X_0(x_t) + \lambda_c X_0(x_c)\right) (\bar{s}d)_{V-A} (\bar{\nu}_l \nu_l)_{V-A} + h.c. \tag{1}$$

where $\lambda_i = V_{is}^* V_{id}$ and $x_i = m_i^2/M_W^2$. Here the lepton mass dependence (only important for the charm contribution in the case of the τ-lepton) has been neglected for simplicity. The one-loop function is given by [1]

$$X_0(x) = \frac{x}{8} \left[\frac{x+2}{x-1} + \frac{3x-6}{(x-1)^2} \ln x \right] \tag{2}$$

Only the top quark contribution is relevant for the CP violating neutral mode $K_L \to \pi^0\nu\bar{\nu}$. For $K^+ \to \pi^+\nu\bar{\nu}$ the charm sector contributes typically 40% of the branching ratio and is therefore not negligible, though still somewhat smaller than the top contribution.

Eq. (1) provides a reasonable approximation as a basis for calculating $K \to \pi\nu\bar{\nu}$. For $K^+ \to \pi^+\nu\bar{\nu}$ the τ-lepton mass effect [1] and leading logarithmic QCD corrections [2, 3, 4] are relevant in the charm sector and have been known for some time.

Over the years important refinements have been added in the theoretical treatment of $K \to \pi\nu\bar{\nu}$. Long-distance contributions were estimated quantitatively and could be shown to be essentially negligible, as expected [5, 6, 7, 8]. The hadronic matrix elements $\langle\pi|(\bar{s}d)_V|K\rangle$ can be extracted from the leading semileptonic decay $K^+ \to \pi^0 e^+ \nu$ using isospin symmetry. Corrections due to isospin breaking from quark masses and electromagnetism have been computed in [9]. Finally, the complete next-to-leading order QCD corrections are known [10, 11, 12]. The NLO result eliminates the dominant uncertainties of the leading order predictions, improving the precision of the theoretical calculation.

All these developments have led to a fairly advanced and quantitative understanding of, and good control over theoretical uncertainties. They are at the level of 5% for $K^+ \to \pi^+\nu\bar{\nu}$, dominated by the charm sector, and even considerably smaller (below 2%) for $K_L \to \pi^0\nu\bar{\nu}$, where the charm contribution is absent. Correspondingly the prospects for precision tests of Standard Model flavor physics are quite promising [13] (for recent discussions of new physics possibilities see e.g. [14, 15]). An ongoing search for $K^+ \to \pi^+\nu\bar{\nu}$ has set a branching ratio limit of $2.4 \cdot 10^{-9}$ [16] and is approaching the Standard Model range at $\sim 10^{-10}$. The current published upper limit on $B(K_L \to \pi^0\nu\bar{\nu})$ is $5.8 \cdot 10^{-5}$ [17]. It is particularly encouraging

that various efforts are now under way to make the challenging experiments possible that are needed for precise measurements of $K^+ \to \pi^+ \nu\bar{\nu}$ [18] and $K_L \to \pi^0 \nu\bar{\nu}$ [19, 20, 21].

In this situation it is interesting to carefully consider presumably small effects that have so far always been neglected. An example are electroweak radiative corrections of higher order, which are expected to be reasonably small and are probably not the first issue one would worry about in the context of rare decays. However, given the high level of precision already obtained in the theory of $K \to \pi\nu\bar{\nu}$, a more quantitative estimate of these corrections is certainly worth pursuing. Moreover, non-decoupling effects, due to electroweak symmetry breaking, grow with m_t and could in principle be sizable. To our knowledge, these higher oder electroweak corrections have not been studied previously for FCNC rare decays. On the other hand such effects have been calculated, to leading order in large m_t, for precision electroweak physics at Z-factories [22, 23, 24]. In this context one should stress that all existing analyses of rare decays have intrinsic theoretical uncertainties related to the definition of electroweak parameters. In particular:

- There is an ambiguity in the value of $\sin^2\Theta_W$ entering the rare decay branching ratio formulas. The various possible definitions of this quantity differ by electroweak radiative corrections that amount to several percent. The related uncertainty can only be removed by considering higher order electroweak effects.

- An ambiguity further exists in whether the pole mass or the $\overline{MS}$ mass of the top quark should be used. With respect to QCD interactions this uncertainty has been eliminated through the calculation of $\mathcal{O}(\alpha_s)$ corrections. However, the ambiguity is not only due to QCD but also due to electroweak effects, which in view of large m_t are not fully negligible.

- Next there are scale ambiguities related to the top quark Yukawa coupling caused by the Higgs-top Yukawa interaction.

- Finally it is of interest to see the impact of the neutral Higgs boson on FCNC processes.

Many of these issues have been discussed in the context of electroweak precision tests, in particular in [25], but have not been considered in connection with rare decays. An exception are higher order corrections of purely electromagnetic origin and the ambiguity between the fine structure constant $\alpha = 1/137$ and $\alpha(M_Z) = 1/129$. The dominant effects of this type have already been taken into account previously. They are not related to large top quark Yukawa interactions and therefore not our major concern in the present context. We will however briefly address this topic further later on.

It is the purpose of this paper to derive the higher order electroweak effects, in the limit of large top quark mass, that correct the leading Inami-Lim function for $K \to \pi\nu\bar{\nu}$. The explicit expressions obtained will enable us to quantify the impact of these corrections. The corresponding effects will also be discussed for $B \to X_s\nu\bar{\nu}$ and $B \to l^+l^-$.

2 Leading Large-m_t Corrections to $K_L \to \pi^0\nu\bar{\nu}$

For definiteness we will focus our discussion first on $K_L \to \pi^0\nu\bar{\nu}$ and generalize to the remaining cases at the end of this section.

Large non-decoupling top quark effects from electroweak loops are of the form $G_F m_t^2$ to leading order in the large-m_t limit. In the following we shall work through order $G_F^2 m_t^4$, corresponding to two-loop electroweak effects. Such corrections modify in particular the Z-boson–fermion coupling into an effective vertex $V_{\bar{f}fZ}$ and one may write [23]

$$V_{\bar{f}fZ} = -i(\sqrt{2}\varrho G_F)^{1/2}\frac{M_Z}{2}\gamma^\mu\left((1+\tau_b)2T_{3f}(1-\gamma_5) - 4Q_f\kappa\sin^2\Theta_W\right) \tag{3}$$

Here G_F, M_Z and $\alpha = 1/137$ are taken to be the basic electroweak parameters; $\sin^2\Theta_W \equiv 1 - M_W^2/M_Z^2$ can be expressed in terms of these three quantities. T_{3f} and Q_f denote the third component of weak isospin and the charge of the fermion f, respectively. ϱ and κ are universal, propagator-type corrections. τ_b is a non-universal vertex correction, which depends on f through the top quark CKM couplings. Denoting

$$\xi_t = \frac{G_F m_t^2}{8\sqrt{2}\pi^2} \tag{4}$$

one has in the above mentioned approximation [23]

$$\varrho = 1 + \Delta\varrho = 1 + 3\xi_t + \mathcal{O}(\xi_t^2) \tag{5}$$

$$\tau_b = -2\xi_t\left(1 + \tau_b^{(2)}\xi_t + \mathcal{O}(\xi_t^2)\right) \tag{6}$$

The two-loop function $\tau_b^{(2)}$ depends on both the top quark mass m_t and the Higgs-boson mass m_H, and reads [22, 23]

$$\begin{aligned}\tau_b^{(2)} &= 9 - \frac{13}{4}a - 2a^2 - \frac{a}{4}(19+6a)\ln a - \frac{a^2}{4}(7-6a)\ln^2 a - \left(\frac{1}{4} + \frac{7}{2}a^2 - 3a^3\right)\frac{\pi^2}{6} + \\ &+ \left(\frac{a}{2} - 2\right)\sqrt{a}g(a) + (a-1)^2\left(4a - \frac{7}{4}\right)L_2(1-a) - \left(a^3 - \frac{33}{4}a^2 + 18a - 7\right)f(a)\end{aligned} \tag{7}$$

where

$$a = \frac{m_H^2}{m_t^2} \qquad L_2(1-a) = \int_1^a dt\frac{\ln t}{1-t} \tag{8}$$

$$g(a) = \begin{cases} 2\sqrt{4-a}\arccos\sqrt{a/4} & \text{for } 0 \le a \le 4 \\ \sqrt{a-4}\ln\frac{1-\sqrt{1-4/a}}{1+\sqrt{1-4/a}} & \text{for } a \ge 4 \end{cases} \tag{9}$$

$$f(a) = \int_0^1 dt\left[L_2(1-r(t,a)) + \frac{r(t,a)}{r(t,a)-1}\ln r(t,a)\right], \qquad r(t,a) = \frac{1+(a-1)t}{t(1-t)} \tag{10}$$

The expression in (7) corresponds to the pole definition of the top quark mass.
Expression (3) may be generalized to the case of the FCNC vertex $V_{\bar{s}dZ}$ by introducing $\lambda_i = V_{is}^* V_{id}$, summing over $i = u, c, t$, using CKM unitarity and noting that τ_b has to be set to zero for $i = u, c$. Additive universal contributions drop out and one obtains

$$V_{\bar{s}dZ} = -i(\sqrt{2}\varrho G_F)^{1/2}\frac{M_Z}{2}\lambda_t(-\tau_b)\gamma^\mu(1-\gamma_5) \tag{11}$$

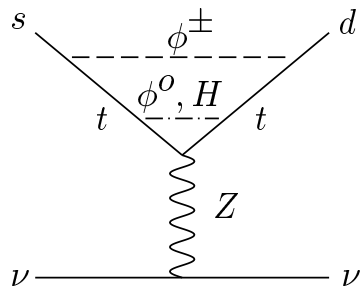

Figure 1: Typical diagram contributing at $\mathcal{O}(G_F^2 m_t^4)$ to the $K \to \pi\nu\bar{\nu}$ amplitude. Here $\phi^{\pm}$, ϕ^0 are Higgs-ghosts and H is the physical Higgs-particle. The complete set of diagrams can be found in [23].

Note in particular that the tree level part of $V_{\bar{f}fZ}$ is canceled through the GIM mechanism and $V_{\bar{s}dZ}$ is, like τ_b, a pure loop effect. The coupling of Z to neutrinos can also be read off from $V_{\bar{f}fZ}$ and is

$$V_{\bar{\nu}\nu Z} = -i(\sqrt{2}\varrho G_F)^{1/2}\frac{M_Z}{2}\gamma^{\mu}(1-\gamma_5) \tag{12}$$

Combining (11), (12) and (4)–(6), an effective Hamiltonian, valid to first and second order in $G_F m_t^2$, can be constructed for $K_L \to \pi^0\nu\bar{\nu}$

$$\mathcal{H}_{eff,FCNC} = \frac{G_F^2 m_t^2}{16\pi^2}\left(1+(3+\tau_b^{(2)})\xi_t\right)\lambda_t(\bar{s}d)_{V-A}(\bar{\nu}\nu)_{V-A} + h.c. \tag{13}$$

(13) coincides with (1) in the large top mass limit and to leading order in electroweak interactions. The corresponding effective Hamiltonian for the charged current process $K^+ \to \pi^0 e^+\nu$, useful for normalizing $K_L \to \pi^0\nu\bar{\nu}$, is given by

$$\mathcal{H}_{eff,CC} = \frac{G_F}{\sqrt{2}}V_{us}^*(\bar{s}u)_{V-A}(\bar{\nu}e)_{V-A} \tag{14}$$

From (13) and (14) it is straightforward to obtain ($\lambda \equiv V_{us}$)

$$\frac{B(K_L \to \pi^0\nu\bar{\nu})}{B(K^+ \to \pi^0 e\nu)} = 3\frac{\tau_{K_L}}{\tau_{K^+}}\frac{G_F^2 m_t^4}{64\pi^4}\left[1+2(3+\tau_b^{(2)})\xi_t\right]\left(\frac{\mathrm{Im}\lambda_t}{\lambda}\right)^2 \tag{15}$$

where we have summed over neutrino flavors.

We remark that in [22, 23] the effective vertex (3) has been derived in the limit where all external momenta are negligible in comparison with m_t and m_H. Therefore the result is applicable to both electroweak observables at the Z-boson resonance, considered in [22, 23], as well as to low energy effective Hamiltonians for rare meson decays that we are interested in here. Note also that in the large-m_t limit only Z-penguin but no box diagrams contribute to $K \to \pi\nu\bar{\nu}$. A typical two-loop electroweak diagram relevant for the $\mathcal{O}(G_F^2 m_t^4)$ correction to the decay amplitude is shown in Fig. 1.

Next we recall that, within the approximation we need for our purposes, one has [26]

$$G_F = \frac{\pi\bar{\alpha}}{\sqrt{2}M_W^2\sin^2\Theta_W}\left(1-\frac{\cos^2\Theta_W}{\sin^2\Theta_W}\Delta\varrho\right) \tag{16}$$

where $\sin^2\Theta_W \equiv 1 - M_W^2/M_Z^2$ (on-shell definition), $\bar{\alpha} \equiv \alpha(M_Z) = 1/129$ and $\Delta\varrho = 3\xi_t$. Using (16) in (13) one has

$$\mathcal{H}_{eff,FCNC} = \frac{G_F}{\sqrt{2}} \frac{\bar{\alpha}}{2\pi \sin^2\Theta_W} \lambda_t \frac{x_t}{8} \left(1 + \left(\tau_b^{(2)} + 6 - \frac{3}{\sin^2\Theta_W}\right)\xi_t\right) (\bar{s}d)_{V-A}(\bar{\nu}\nu)_{V-A} + h.c. \quad (17)$$

and (15) becomes

$$\frac{B(K_L \to \pi^0\nu\bar{\nu})}{B(K^+ \to \pi^0 e\nu)} = 3\frac{\tau_{K_L}}{\tau_{K^+}} \frac{\bar{\alpha}^2}{2\pi^2 \sin^4\Theta_W} \left[1 + \left(2\tau_b^{(2)} + 12 - \frac{6}{\sin^2\Theta_W}\right)\xi_t\right] \left(\frac{x_t}{8}\right)^2 \left(\frac{\mathrm{Im}\lambda_t}{\lambda}\right)^2 \quad (18)$$

This expression is useful since it contains the leading electroweak coupling constants in the form conventionally chosen in analyzing $K_L \to \pi^0\nu\bar{\nu}$. The various forms in which the electroweak parameters may be written are all equivalent at lowest order, where for instance $\sqrt{2}G_F M_W^2 \sin^2\Theta_W = \pi\bar{\alpha}$. These expressions differ by terms of order $\mathcal{O}(\xi_t)$ (see (16)). Consequently, the explicit $\mathcal{O}(\xi_t)$ correction will be different for different choices of electroweak couplings, while physical quantities remain unchanged (compare (15) and (18)).
From the above derivation it is clear that the appropriate QED coupling entering (17) and (18) is $\bar{\alpha} = \alpha(M_Z) = 1/129$ and not the usual fine structure constant $\alpha = 1/137$. These two quantities differ by logarithmic terms $\sim \alpha \ln M_Z/m_f$. On the other hand, the ratio in (18) does in fact receive a logarithmic QED correction $\sim \alpha \ln M_Z/m_K$ not displayed in this equation. It is due to the differences in the QED renormalization between the neutral current and the charged current transitions forming the ratio (18). This effect, which in principle is of similar nature as the difference between α and $\bar{\alpha}$, has been discussed in [9] in the context of isospin breaking corrections. We will not include this correction here, with the understanding that it is part of the known isospin breaking effects [9] to be taken into account in a complete analysis of $K_L \to \pi^0\nu\bar{\nu}$.

If we use the Hamiltonian in the form of (17), the leading large-m_t electroweak correction to $K_L \to \pi^0\nu\bar{\nu}$ may be written as a factor

$$r_{X,EW} = 1 + \frac{x_t}{4X_0(x_t)} \left(\tau_b^{(2)} + 6 - \frac{3}{\sin^2\Theta_W}\right)\xi_t \quad (19)$$

multiplying the leading order $K_L \to \pi^0\nu\bar{\nu}$ branching ratio. Here we have generalized the lowest order top-mass dependence $x_t/8$ to the complete function $X_0(x_t)$ (2). Only the leading large-m_t terms have been kept for the electroweak correction, as the full mass dependence to this order is still unknown. Equivalently we may express the electroweak effects as a correction to the lowest order Inami-Lim function $X_0(x_t)$, which then becomes

$$X_0(x_t) + \frac{x_t}{8} \left(\tau_b^{(2)} + 6 - \frac{3}{\sin^2\Theta_W}\right)\xi_t \quad (20)$$

This modification likewise affects the top contribution to $K^+ \to \pi^+\nu\bar{\nu}$. However, because of the sizable charm contribution that dominates the theoretical uncertainties in this case, electroweak corrections are less relevant here.

The same factor $r_{X,EW}$ applies also to the rare decay $B \to X_s\nu\bar{\nu}$, whose branching fraction is to lowest order given by [27]

$$\frac{B(B \to X_s\nu\bar{\nu})}{B(B \to X_c e\nu)} = \frac{3\bar{\alpha}^2}{4\pi^2 \sin^4\Theta_W} \left|\frac{V_{ts}}{V_{cb}}\right|^2 \frac{X_0^2(x_t)}{f(m_c/m_b)} \quad (21)$$

with the $b \to ce\nu$ phase space factor $f(z) = 1 - 8z^2 + 8z^6 - z^8 - 24z^4 \ln z$.

A closely related decay is $B_s \to l^+ l^-$. The effective Hamiltonian for this case is similar to (17) and can be obtained by replacing $V_{ts}^* V_{td} (\bar{s}d)_{V-A} (\bar{\nu}\nu)_{V-A} \to -V_{tb}^* V_{ts} (\bar{b}s)_{V-A} (\bar{l}l)_{V-A}$. The leading order m_t-dependence $x_t/8$ generalizes here to the function

$$Y_0(x) = \frac{x}{8} \left[\frac{x-4}{x-1} + \frac{3x \ln x}{(x-1)^2} \right] \tag{22}$$

The branching ratio [27]

$$B(B_s \to l^+ l^-) = \tau(B_s) \frac{G_F^2}{\pi} \left(\frac{\bar{\alpha}}{4\pi \sin^2 \Theta_W} \right)^2 F_{B_s}^2 m_l^2 m_{B_s} \sqrt{1 - 4\frac{m_l^2}{m_{B_s}^2}} |V_{tb}^* V_{ts}|^2 Y_0^2(x_t) \tag{23}$$

is then modified through large-m_t electroweak effects by a factor

$$r_{Y,EW} = 1 + \frac{x_t}{4Y_0(x_t)} \left(\tau_b^{(2)} + 6 - \frac{3}{\sin^2 \Theta_W} \right) \xi_t \tag{24}$$

This corresponds to a correction of the Inami-Lim function in (22), which gets replaced by

$$Y_0(x_t) + \frac{x_t}{8} \left(\tau_b^{(2)} + 6 - \frac{3}{\sin^2 \Theta_W} \right) \xi_t \tag{25}$$

The functions X_0 and Y_0 differ only by box-diagram contributions. Because these are vanishing in the large-m_t limit, the correction terms in (20) and (25) are identical.

3 Numerical Results and Discussion

In the following section we will present numerical results for the electroweak corrections and further discuss various aspects of the analysis. To this purpose we specify first the relevant input parameters. We will use

$$G_F = 1.16639 \cdot 10^{-5}\, GeV^{-2} \qquad M_W = 80.34\, GeV \tag{26}$$

$$\sin^2 \Theta_W \equiv 1 - M_W^2/M_Z^2 = 0.2238 \qquad m_t = 167\, GeV \tag{27}$$

For the W-boson mass M_W we take the central value of the Standard Model prediction [28]. The logarithmic dependence of M_W on the Higgs boson mass ($\sim \xi_t \ln(m_H/M_W) \cdot M_W^2/m_t^2$) can consistently be neglected within our approximation and is also numerically small. The value of m_t in (27) corresponds to the $\overline{MS}$ definition with respect to QCD corrections. It differs from the value of the QCD pole mass $m_{t,pole(QCD)} = 175\, GeV$ by about $8\, GeV$. The $\overline{MS}$ definition is an appropriate choice in the analysis of QCD effects, which have been discussed elsewhere [10, 11]. On the other hand, the top quark mass m_t will here be understood to refer to the pole mass definition with respect to electroweak effects. This is the choice that has been used in obtaining the electroweak corrections in the previous section.

Numerical values for the correction factors $r_{X,EW}$ and $r_{Y,EW}$ are displayed in Table 1 for various values of the Higgs-boson mass m_H. The leading large-m_t corrections shown there

m_H/GeV	60	150	300	450	600	1000
$r_{X,EW}-1$	-0.91%	-1.20%	-1.27%	-1.17%	-1.02%	-0.58%
$r_{Y,EW}-1$	-1.41%	-1.87%	-1.97%	-1.82%	-1.59%	-0.91%

Table 1: The leading large-m_t electroweak correction factors $r_{X,EW}$ and $r_{Y,EW}$, as defined in (19) and (24), respectively, for various values of the Higgs-boson mass m_H. They multiply the branching fractions of $K_L \to \pi^0\nu\bar{\nu}$, $B \to X_s\nu\bar{\nu}$ ($r_{X,EW}$) and $B \to l^+l^-$ ($r_{Y,EW}$).

are moderate and amount to typically -1%. The largest effect is obtained for m_H around $170-340\,GeV$, where the (positive-valued) function $\tau_b^{(2)}$ has a minimum. The corrections $r_{Y,EW}-1$ are larger than $r_{X,EW}-1$ by a factor of 1.56 for $m_t = 167\,GeV$ and independent of m_H. They can reach values up to -2%.

We recall that these corrections depend on the form in which the leading electroweak coupling constants are expressed. The factors $r_{X,EW}$ and $r_{Y,EW}$ refer to the choice of $\bar{\alpha}^2/\sin^4\Theta_W$ as used in (18), (21) and (23), with the on-shell Weinberg angle. If instead one were to use the coupling expressed in terms of the effective weak mixing angle $\sin^2\hat{\Theta}(M_Z) = 0.23$ [28], where†

$$\sin^2\hat{\Theta}(M_Z) = \left(1 + \frac{\cos^2\Theta_W}{\sin^2\Theta_W}\Delta\varrho\right)\sin^2\Theta_W \tag{28}$$

the correction factors would be different. $r_{X,EW}$, for instance, would become

$$\hat{r}_{X,EW} = 1 + \frac{x_t}{4X_0(x_t)}\left(\tau_b^{(2)} + 3\right)\xi_t \tag{29}$$

This change compensates for the corresponding change in the coupling constants, which also differ by terms of $\mathcal{O}(\xi_t)$ (see (28)). The compensation is not exact in our approximation where we use $X_0(x_t)$ instead of $x_t/8$ as the leading m_t-dependent function. It holds strictly only in the large-m_t limit. Numerically the different choices for $\sin^2\Theta$ lead to a difference in the branching ratio by a factor of $\sin^4\hat{\Theta}(M_Z)/\sin^4\Theta_W = 1.056$ if no higher order electroweak corrections are applied. After inclusion of $\mathcal{O}(\xi_t)$ corrections this discrepancy is reduced to

$$\frac{\sin^4\hat{\Theta}(M_Z)}{\sin^4\Theta_W}\cdot\frac{r_{X,EW}}{\hat{r}_{X,EW}} = 1.034 \tag{30}$$

This indicates a reduction of the uncertainty from about $\pm 2.8\%$ to $\pm 1.7\%$, where these numbers are independent of the Higgs boson mass. While $r_{X,EW}$ is smaller than unity by about 1% (see Table 1) and reduces the larger parameter choice of $1/\sin^4\Theta_W$, the smaller normalization using $1/\sin^4\hat{\Theta}(M_Z)$ is enhanced by roughly the same amount through $\hat{r}_{X,EW}$. Previous analyses usually employed the latter choice of $\sin^2\hat{\Theta}(M_Z) = 0.23$, in which case $\hat{r}_{X,EW}$ is the appropriate correction factor.

We remark that the ambiguities discussed here in connection with the weak mixing angle are particularly large since they are reinforced by a factor of $\cos^2\Theta_W/\sin^2\Theta_W \approx 3.5$ as seen in

†Note that the approximate relations in (16) and (28) hold quite accurately for realistic values of the parameters.

(28). The above estimate of about $\pm 2\%$ for the uncertainty due to as yet unknown subleading electroweak contributions should therefore be quite conservative.

We turn next to a discussion of scheme and scale dependence, which is useful to further investigate the structure of the electroweak corrections.
Instead of using the on-shell (pole) definition of m_t (with respect to electroweak interactions), which we have employed so far, one may adopt the $\overline{MS}$ scheme for the top quark mass. These two definitions differ by terms of $\mathcal{O}(\xi_t)$ and are related by [25]

$$\bar{x}_t = x_t(1 + \Delta_t(\mu, a)\xi_t) \tag{31}$$

where $x_t = m_t^2/M_W^2$, $\bar{x}_t = \bar{m}_t^2/M_W^2$ and $\bar{m}_t$ is the $\overline{MS}$-mass. The function Δ_t reads [25][‡]

$$\Delta_t(\mu, a) = 18 \ln \frac{\mu}{m_t} + 11 - \frac{a}{2} + \frac{a(a-6)}{2} \ln a + \frac{a-4}{2}\sqrt{a} g(a) \tag{32}$$

The corrected Inami-Lim function has been given in (20) for the on-shell scheme. Alternatively we may use the $\overline{MS}$-scheme (31), in which case (20) is replaced by

$$X_0(\bar{x}_t) + \frac{x_t}{8}\left(\tau_b^{(2)} - \Delta_t + 6 - \frac{3}{\sin^2\Theta_W}\right)\xi_t \tag{33}$$

The ratio of (33) to (20), to be denoted by R, provides a measure of scheme dependence (we will put $\mu = m_t$ for the moment). To linear order in ξ_t we have

$$R = 1 + s_R\ \Delta_t\ \xi_t \qquad s_R = \frac{1}{X_0(x_t)}\left(x_t \frac{\partial X_0}{\partial x_t} - \frac{x_t}{8}\right) \tag{34}$$

In the large-m_t limit $X_0 \to x_t/8$ and $s_R \equiv 0$, ensuring the scheme independence of the corrected Inami-Lim function to first order in ξ_t. If we use the full leading order function $X_0(x_t)$, a residual scheme dependence persists, since the corrections are only known in the large-m_t limit. Numerically $R = 1.002$ for $m_H = 300\,GeV$. This is to be compared with $X_0(\bar{x}_t)/X_0(x_t) = 1.006$, indicating the scheme dependence when the corrections are altogether omitted. The scheme dependence is thus reduced from 0.6% to 0.2% in the decay amplitudes. The effects are twice as big for the branching fractions. The scheme ambiguities can be somewhat larger for other values of the Higgs-boson mass, but the reduction by a factor of three observed above is independent of m_H.
Note that the reduction in scheme dependence is quite sizable, although the asymptotic limit is not a good approximation for realistic values of m_t as $X_0(x_t) \approx 2.8 \cdot x_t/8$. This can be understood by considering the large-x expansion of $X_0(x)$

$$X_0(x) = \frac{x}{8} + \frac{3\ln x + 3}{8} + \frac{3}{8x} + \mathcal{O}\left(\frac{1}{x^2}\right) \tag{35}$$

which shows that the dominant x-dependence stems from the leading term $x/8$.

[‡] This expression relates, strictly speaking, the $\overline{MS}$ and the pole definition of the top quark Yukawa coupling, which we then write in terms of the top quark mass.

Related to the scheme ambiguity is the issue of scale dependence, resulting from the running of the top quark Yukawa coupling. This question can be studied using the $\overline{MS}$ formulation in equation (31), which exhibits explicitly the μ-dependence of $\bar{m}_t$ due to the Higgs-top Yukawa interaction. Changing μ between $100\,GeV$ and $300\,GeV$ results in a variation of $X_0(\bar{x}_t(\mu))$ by $\pm 1.6\%$. This sensitivity is reduced to $\pm 0.6\%$ when the leading large-m_t corrections from (33) are included. Unfortunately the residual μ-dependence in the branching ratios is then still $\pm 1.2\%$, of the same order of magnitude as the electroweak corrections in Table 1 themselves. This indicates again that subleading m_t-terms in the electroweak corrections are important and have to be taken into account if a higher precision is required.
We finally note that the difference between the two definitions of m_t in (31) amount to typically $1-2\,GeV$. This is still smaller than the current experimental uncertainty in the top quark mass of $\pm 5.5\,GeV$ [29], but will become relevant if future measurements reduce this error to $\pm 1\,GeV$ or below. The top quark pole- and $\overline{MS}$-mass in QCD, by contrast, differ by about $8GeV$ as already mentioned before. The situation is similar with respect to the scale dependence. Here the electroweak scale ambiguity of $\pm 1.6\%$ in the uncorrected lowest order term $X_0(\bar{x}_t(\mu))$ may be compared with the corresponding QCD effect of $\pm 5\%$ ($100\,GeV \leq \mu \leq 300\,GeV$). The latter is reduced to $\pm 0.5\%$ when the full $\mathcal{O}(\alpha_s)$ corrections are included [11].

For definiteness we have restricted our discussion to the function $X_0(x_t)$, relevant for $K_L \to \pi^0 \nu\bar{\nu}$ and $B \to X_s \nu\bar{\nu}$. Similar observations hold for the decays $B \to l^+ l^-$ governed by $Y_0(x_t)$. Here the situation is generally somewhat more favorable, since the function $Y_0(x_t)$ is closer to its asymptotic limit $x_t/8$ ($Y_0(x_t) = 1.8 \cdot x_t/8$) than it is the case for $X_0(x_t)$.

4 Conclusions

In this paper we have investigated the electroweak radiative corrections of $\mathcal{O}(G_F^2 m_t^4)$ to the decay amplitudes of $K_L \to \pi^0 \nu\bar{\nu}$, $K^+ \to \pi^+ \nu\bar{\nu}$, $B \to X_s \nu\bar{\nu}$ and $B \to l^+ l^-$. These corrections arise at the two-loop level and are the formally leading electroweak corrections to the one-loop induced FCNC in the limit of large top Yukawa coupling. Our analysis was motivated by the theoretically clean nature of the rare decay processes under consideration. The main benefits of this investigation may be summarized as follows.

- It serves to illustrate the general issues involved in the calculation of higher order electroweak corrections to rare decays.
- It provides a quantitative order of magnitude estimate of these effects.
- It helps to reduce the impact of ambiguities in the definition of electroweak parameters on observable quantities.

In the large-m_t limit the lowest oder amplitudes are of $\mathcal{O}(G_F m_t^2)$. The inclusion of the $\mathcal{O}(G_F^2 m_t^4)$ correction eliminates various ambiguities, of order several percent, that are related to the definition of electroweak parameters in the lowest order expressions. Such ambiguities exist for instance between $\sqrt{2} G_F M_W^2$ and $\pi\bar{\alpha}/\sin^2\Theta_W$ or between $\sin^2\Theta_W$ and $\sin^2\hat{\Theta}(M_Z)$, which differ due to higher order electroweak corrections. Another example is the uncertainty due to (electroweak) scheme- and scale dependence in the top quark Yukawa coupling.

Unfortunately the asymptotic, large-m_t limit is not fully realistic in the cases at hand. Since only the formally leading corrections are known, the above ambiguities can at present not be removed completely. They become however smaller when the large-m_t corrections are applied. Scheme- and scale dependence are reduced by a factor of three to typically $\pm 1\%$ in the branching ratios. The presumably largest uncertainty is due to the difference between $\sin^2\Theta_W = 0.224$ and $\sin^2\hat{\Theta}(M_Z) = 0.23$ that leads to a change in the lowest order branching fractions by 5.6%. At order $\mathcal{O}(G_F^2 m_t^4)$ this is reduced to a total variation of 3.4%. We estimate the uncertainty due to presently unknown subleading (in m_t) electroweak corrections for the top quark dominated decays $K_L \to \pi^0\nu\bar{\nu}$, $B \to X_s\nu\bar{\nu}$ and $B \to l^+l^-$ to be about $\pm 2\%$. In comparison to previous calculations of these rare decays, which employed $\sin^2\hat{\Theta}(M_Z) = 0.23$ in the overall normalization, the $\mathcal{O}(G_F^2 m_t^4)$ effects lead to a slight enhancement of about $1-2\%$ in the central value of the branching ratio.

We remark that the corrections discussed in this paper have no impact on the extraction of the CKM parameter $\sin 2\beta$ from $K_L \to \pi^0\nu\bar{\nu}$ and $K^+ \to \pi^+\nu\bar{\nu}$ [13], as the top contribution essentially cancels out in this case.

An improvement of the uncertainties in the decay rates beyond the $\pm 2\%$ quoted above would require the explicit calculation of at least the first subleading two-loop contributions of $\mathcal{O}(G_F^2 m_t^2 M_W^2)$. Such corrections are yet unknown for rare decays, but have been calculated for the ϱ-parameter, relevant for electroweak precision observables at the Z resonance [30].

In any case our work confirms the expectation that higher order electroweak effects are well below the experimental sensitivity in the forseeable future. A further, systematic improvement over the present situation is however still possible, if it should indeed appear necessary.

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