hep-ph/9708239
August, 1997

SLAC–PUB–7529
Saclay/SPhT-T97/090
UCLA/97/TEP/10

One-Loop Amplitudes for $e^+ e^-$ to Four Partons

Zvi Bern[♯]
Department of Physics
University of California, Los Angeles
Los Angeles, CA 90024
bern@physics.ucla.edu

Lance Dixon[⋆]
Stanford Linear Accelerator Center
Stanford University
Stanford, CA 94309
lance@slac.stanford.edu

and

David A. Kosower
Service de Physique Théorique[†]
Centre d'Etudes de Saclay
F-91191 Gif-sur-Yvette cedex, France
kosower@spht.saclay.cea.fr

Abstract

We present the first explicit formulæ for the complete set of one-loop helicity amplitudes necessary for computing next-to-leading order corrections for $e^+ e^-$ annihilation into four jets, for W, Z or Drell-Yan production in association with two jets at hadron colliders, and for three-jet production in deeply inelastic scattering experiments. We include a simpler form of the previously published amplitudes for $e^+ e^-$ to four quarks. We obtain the amplitudes using their analytic properties to constrain their form. Systematically eliminating spurious poles from the amplitudes leads to relatively compact results.

Submitted to Nuclear Physics B

[♯]Research supported in part by the US Department of Energy under grant DE-FG03-91ER40662 and in part by the Alfred P. Sloan Foundation under grant BR-3222.
[⋆]Research supported by the US Department of Energy under grant DE-AC03-76SF00515.
[†]Laboratory of the *Direction des Sciences de la Matière* of the *Commissariat à l'Energie Atomique* of France.

1. Introduction

The discovery of new physics at colliders relies to a large extent on our ability to understand the known physics producing the bulk of the data. For processes involving hadronic jets, perturbative QCD predictions are required. Leading-order calculations often reproduce the shapes of distributions well but suffer from practical and conceptual problems whose resolution requires the use of next-to-leading order (NLO) calculations. In many processes at modern colliders, the dominant theoretical uncertainties are due to unknown higher-order perturbative corrections. These corrections can be enhanced by various logarithms. For some processes, NLO corrections are known, and programs implementing them have already played an important role in analyzing data from a variety of high-energy collider experiments. Other processes have awaited the computation of the required one-loop matrix elements.

The first type of logarithm contributing to theoretical uncertainties is 'ultraviolet' in nature. Such logarithms are connected with the renormalization scale μ, which we are forced to introduce in order to define the running coupling, $\alpha_s(\mu)$. Physical quantities, such as cross-sections or differential cross-sections, should be independent of μ. When we compute such a quantity in perturbation theory, however, we necessarily truncate its expansion in α_s, and this introduces a spurious dependence on μ. Together, these effects can lead to anywhere from a 30% to a factor of 2–3 normalization uncertainty in predictions of experimentally-measured distributions. In general, the inclusion of NLO corrections significantly reduces the over-all sensitivity of a prediction to variations in μ.

The other type of logarithm is 'infrared' in nature. Such logarithms are connected with the presence of soft and collinear radiation. Jets in a detector consist of a spray of hadrons spread over a finite region of phase space. Experimental measurements of jet distributions depend on resolution parameters, such as the jet cone size and minimum transverse energy. In a leading-order calculation, jets are modeled by lone partons. As a result, these predictions either lack a dependence on these parameters or have an incorrect dependence on them. In addition, the internal structure of a jet cannot be predicted at all.

In the case of $e^+ e^-$ annihilation into jets, leading-order predictions for the production of up to five jets have been available for quite some time [1,2,3,4,5]. The NLO matrix elements for three-jet production and other $\mathcal{O}(\alpha_s)$ observables are also known [3], and numerical programs implementing these corrections [6,7] have been widely used to extract a precise value of α_s from hadronic event shapes at the Z pole [8].

Next-to-leading order corrections for more complicated processes are important, however, if we wish to use QCD to probe for new physics in other standard model processes. In $e^+ e^-$ annihilation, for example, four-jet production is the lowest-order process in which the quark and gluon color charges can be measured independently. Four-jet production is thus sensitive to the presence of light colored fermions such as gluinos [9]. At LEP 2 the process $e^+ e^- \to (\gamma^*, Z) \to 4$ jets is a background to threshold production of W pairs, when both Ws decay hadronically.

The calculation of $e^+ e^- \to 4$ jets at NLO requires the tree-level amplitudes for $e^+ e^- \to 5$ partons [4,5] (at NLO two of the partons may appear inside a single jet), and the one-loop amplitudes for $e^+ e^- \to 4$ partons. In a previous paper [10] we presented the one-loop helicity amplitudes for electron-positron annihilation into four massless quarks, $e^+ e^- \to (\gamma^*, Z) \to \bar{q}q\bar{Q}Q$ (q, Q may have the same or different flavors).

In this paper, we present the $e^+ e^- \to (\gamma^*, Z) \to \bar{q}qgg$ one-loop helicity amplitudes, as well as simplified versions of the $e^+ e^- \to (\gamma^*, Z) \to \bar{q}q\bar{Q}Q$ amplitudes. We give all contributions at order g^4 in the strong coupling, including those where the vector boson couples directly to a quark loop via a vector or axial-vector coupling. We take all quarks to be massless, except for the top quark, whose virtual effects we include through order $1/m_t^2$. Thus the list of helicity amplitudes required to construct a numerical program for $e^+ e^- \to 4$ jets is now complete. Indeed, the amplitudes presented here and in refs. [11,10] have already been incorporated into the first NLO program for $e^+ e^- \to 4$ jets [12]. Crossing symmetry and simple coupling constant modifications allow the same amplitudes to be used in NLO computations of the production of a vector boson (W, Z, or Drell-Yan pair) in association with two jets at hadron colliders, and three-jet production at ep colliders. Finally, these amplitudes will also enter the next-to-next-to-leading (NNLO) study of three-jet production at the Z pole. Such a study (which awaits the computation of appropriate two-loop matrix elements as well) would be desirable in order to reduce the theoretical uncertainties in determining α_s via this process.

Glover and Miller [13] have reported on a calculation of the helicity-summed interference term between four-quark one-loop matrix elements and the appropriate tree-level matrix elements. Recently, Campbell, Glover and Miller [14] have also calculated the analogous 'squared' matrix elements for the two-quark two-gluon final state. In neither of these papers did the authors provide any explicit formulæ. In order to compare with the results described in refs. [13,14], we considered the case of virtual photon exchange, dropped the contributions where the photon couples directly to a quark loop, constructed the unpolarized (helicity-summed) cross-section, and then performed an integration over the orientation angles of the lepton pair (in the virtual-photon rest frame). After accounting for the different versions of dimensional regularization used [15], we have verified numerically that the two sets of results agree, for both the four-quark and two-quark-two-gluon final states.† Also, the squared matrix elements in refs. [13,14] have been incorporated into a NLO program for four-jet fractions and shape variables by Nagy and Trócsányi [16], and their numerical results for the four-jet fractions confirm the corresponding results of ref. [12].

The amplitudes we present retain all correlations between the daughter or parent leptons of the vector boson, and the colored partons in the process. Such correlations are important for computations that take into account experimental constraints. For example, in the production of a W along with jets at a hadron collider, followed by the decay $W \to \ell\bar{\nu}_\ell$, the longitudinal component of the W momentum cannot be observed, only that of the decay lepton, and the latter should be isolated from the hadronic jets in order for the event to pass detector cuts. In the case of jet production in deep-inelastic scattering, the orientation of the entire event with respect to the detector is dictated by the lepton scattering angle as well as the square of the virtual photon four-momentum.

Recent years have seen a number of technical advances in the computation of one-loop amplitudes. These advances have made possible the calculation of all one-loop five-parton processes [17,18,19], as well as a number of infinite sequences of loop amplitudes [20,21,22,23,24]. Many of the techniques used in the present calculation have been reviewed in ref. [25]. The rather complicated six-body kinematics encountered here necessitate further techniques for removing certain spurious singularities from the amplitudes. The removal of spurious singularities is essential in order to find (relatively) compact final expressions, and also plays a role in improving the numerical stability of the results.

The general strategy employed in this paper is to obtain amplitudes directly from their analytic properties instead of from Feynman diagrams. In particular, we use the constraints of unitarity [26,22,23,27] and factorization [28,29,20,30], as summarized in ref. [25]. We construct the amplitudes by finding functions that have the correct poles and cuts in the various channels. The required pole and cut information is extracted from previously obtained amplitudes (tree amplitudes or lower-point loop amplitudes); manifest gauge invariance is therefore maintained at each step. This approach leads to compact expressions, especially when compared with those obtained from a traditional diagrammatic computation. In a Feynman diagram approach each diagram alone is not gauge invariant, and is often much more complicated than the sum over all diagrams. As a check, we performed a numerical evaluation of the Feynman diagrams at one kinematic point, and verified that their sum agrees with a numerical evaluation of our analytic results.

The spinor helicity method [31] and color decompositions [32] are crucial to the success of this approach, because they simplify the analytic structures that must be computed. Because

† We thank J.M. Campbell and E.W.N. Glover for assistance in comparing the results.

of the intricate analytic structure of the amplitudes it is rather non-trivial to remove spurious singularities that can appear in the amplitudes. (By a spurious singularity we mean a kinematic pole or singularity whose residue vanishes.) By evaluating the cuts appropriately we can prevent the worst of the spurious singularities from appearing. However, some of the spurious singularities are inherent in the amplitudes when they are expressed in terms of logarithms and dilogarithms. As we shall see, the spinor helicity method is quite useful for simplifying the terms containing spurious singularities.

This paper is organized as follows. In section 2, we briefly review spinor helicity and color decompositions and provide formulæ relating the full amplitudes to the 'primitive' amplitudes in terms of which the results are expressed. In section 3, we outline the construction of amplitudes from their analytic properties. Procedures for eliminating or simplifying spurious singularities are given in section 4. Sample calculations are given in section 5; in particular, examples are provided for cut constructions, rational function reconstructions, and simplifications via numerical analysis. The general structure of the primitive amplitudes including regularization issues is given in section 6. The results for the primitive amplitudes are collected in sections 7–12. In section 7 the 'master functions', which are a set of functions which appear in multiple amplitudes, are given. Section 8 contains the amplitudes which are leading in the number of colors and flavors. Subleading-in-color contributions are contained in sections 9 and 10. Contributions with the vector boson coupled to a closed fermion loop are give in section 11; this includes both vector and axial-vector contributions. Simplified versions of the four quark amplitudes (which have been previously presented in ref. [10]) are given in section 12. Some concluding remarks in are given in section 13. There are a total of four appendices. Appendices I and II concern the evaluation of loop integrals and their associated spurious singularities. Appendix III lists some spinor-product identities that are useful for simplifying the spurious pole structure of kinematic coefficients. Appendix IV records the helicity amplitudes for $e^+ e^- \to \bar{q}qg$ [7] in the same notation and conventions used in the paper; these amplitudes are useful because they appear in many collinear limits of the $e^+ e^- \to \bar{q}qgg$ amplitudes.

2. Brief Review of Basic Tools

We shall present our results in terms of the spinor helicity method and $SU(N_c)$ color decompositions. The reader is referred to review articles [29] and references therein for details beyond our following brief review.

2.1 Spinor Helicity

We represent the gluon polarization vectors in terms of Weyl spinors $|\,k^\pm\rangle$ [31],

$$\varepsilon_\mu^+(k;q) = \frac{\langle q^-|\,\gamma_\mu\,|k^-\rangle}{\sqrt{2}\,\langle qk\rangle}\,, \qquad \varepsilon_\mu^-(k;q) = \frac{\langle q^+|\,\gamma_\mu\,|k^+\rangle}{\sqrt{2}\,[kq]}\,, \tag{2.1}$$

where k is the gluon momentum and q is an arbitrary null 'reference momentum' which drops out of final gauge-invariant amplitudes. The plus and minus labels on the polarization vectors refer to the gluon helicities. Our (crossing-symmetric) convention takes all particles to be outgoing, and labels the helicity and particle vs. antiparticle assignment accordingly. For incoming (negative energy) momenta the helicity and particle vs. antiparticle assignment are reversed. It is convenient to define the following *spinor strings*,

$$\begin{aligned}
&\langle ij\rangle \equiv \langle k_i^-|k_j^+\rangle\,, \qquad\qquad [ij] \equiv \langle k_i^+|k_j^-\rangle\,,\\
&\langle i|l|j\rangle \equiv \langle k_i^-\,|\,\not{k}_l\,|\,k_j^-\rangle\,, \qquad \langle i|(l+m)|j\rangle \equiv \langle k_i^-\,|\,(\not{k}_l+\not{k}_m)\,|\,k_j^-\rangle\,,\\
&[i|lm\cdots|j] \equiv \langle k_i^+\,|\,\not{k}_l\not{k}_m\cdots\,|\,k_j^-\rangle\,,\\
&\langle i|lm\cdots|j\rangle \equiv \langle k_i^-\,|\,\not{k}_l\not{k}_m\cdots\,|\,k_j^\pm\rangle\,,\\
&[i|(l+m)(n+r)\cdots|j] \equiv \langle k_i^+\,|\,(\not{k}_l+\not{k}_m)(\not{k}_n+\not{k}_r)\cdots\,|\,k_j^-\rangle\,,\\
&\langle i|(l+m)(n+r)\cdots|j\rangle \equiv \langle k_i^-\,|\,(\not{k}_l+\not{k}_m)(\not{k}_n+\not{k}_r)\cdots\,|\,k_j^\pm\rangle\,,
\end{aligned} \tag{2.2}$$

which is the notation we shall use to quote the results. In the last definition we take the $|\,k_j^\pm\rangle$ to mean $|\,k_j^-\rangle$ for an odd number of gamma-matrices in the string and $|\,k_j^+\rangle$ when there are an even number. All the momenta k_i are massless, $k_i^2 = 0$. Sometimes we will also use the notation

$$\langle i|\ell_m|j\rangle \equiv \langle k_i^-\,|\,\not{\ell}_m\,|\,k_j^-\rangle\,, \tag{2.3}$$

etc., where ℓ_m is a loop momentum. The spinor inner products $\langle ij\rangle$, $[ij]$ are antisymmetric and satisfy $\langle ij\rangle\,[ji] = 2k_i\cdot k_j \equiv s_{ij}$. In addition to $s_{ij} \equiv (k_i+k_j)^2$ we also define the three-particle invariants $t_{ijl} \equiv (k_i+k_j+k_l)^2$.

2.2 Color Decomposition

It is convenient to decompose one-loop amplitudes in terms of group-theoretic factors (color structures) multiplied by kinematic functions called 'partial amplitudes' [32]. We present results for the general gauge group $SU(N_c)$ ($N_c = 3$ for QCD), and normalize the group generators in the fundamental representation, T^a, so that $\mathrm{Tr}(T^aT^b) = \delta^{ab}$. Color decompositions are obtained by rewriting the structure constants f^{abc} as

$$f^{abc} \;=\; -\frac{i}{\sqrt{2}}\,\mathrm{Tr}\left(\left[T^a,T^b\right]T^c\right). \tag{2.4}$$

Then one applies the $SU(N_c)$ Fierz identity

$$(X_1\, T^a\, X_2)\ (Y_1\, T^a\, Y_2) \ = \ (X_1\, Y_2)\ (Y_1\, X_2) - \frac{1}{N_c}(X_1\, X_2)\ (Y_1\, Y_2)\ , \tag{2.5}$$

where X_i, Y_i are strings of generator matrices T^{a_i}, in order to remove contracted color indices.

Here we are interested in the amplitude $\mathcal{A}_6(1_q, 2, 3, 4_{\bar{q}}; 5_{\bar{e}}, 6_e)$, where legs $1, 4$ are the quark-anti-quark pair, legs $2, 3$ are the gluon legs, and legs 5,6 are the lepton pair. We label the (outgoing) quark, anti-quark, electron and positron lines with subscripts q, $\bar{q}$, e, and $\bar{e}$, while the gluon lines do not have additional labels. The color decomposition of the tree-level contribution to $\mathcal{A}_6$ is

$$\mathcal{A}_6^{\rm tree}(1_q, 2, 3, 4_{\bar{q}}) = 2e^2 g^2 \big(-Q^q + v^e_{L,R} v^q_{L,R}\, \mathcal{P}_Z(s_{56})\big) \sum_{\sigma\in S_2} (T^{a_{\sigma(2)}} T^{a_{\sigma(3)}})_{i_1}^{\ \ \bar{\imath}_4}\ A_6^{\rm tree}(1_q, \sigma(2), \sigma(3), 4_{\bar{q}})\,. \tag{2.6}$$

Here we have suppressed the $5, 6$ labels of the electron pair, e is the QED coupling, g the QCD coupling, Q^q is the charge of quark q in units of e, and the ratio of Z and photon propagators is given by

$$\mathcal{P}_Z(s) = \frac{s}{s - M_Z^2 + i\ ,\ _Z\, M_Z}\,, \tag{2.7}$$

where M_Z and $,\ _Z$ are the mass and width of the Z.

The left- and right-handed couplings of fermions to the Z boson are

$$\begin{aligned} v_L^e &= \frac{-1 + 2\sin^2\theta_W}{\sin 2\theta_W}\,, & \qquad v_R^e &= \frac{2\sin^2\theta_W}{\sin 2\theta_W}\,, \\ v_L^q &= \frac{\pm 1 - 2Q^q\sin^2\theta_W}{\sin 2\theta_W}\,, & \qquad v_R^q &= -\frac{2Q^q\sin^2\theta_W}{\sin 2\theta_W}\,, \end{aligned} \tag{2.8}$$

where θ_W is the Weinberg angle. The two signs in $v^q_{L,R}$ correspond to up $(+)$ and down $(-)$ type quarks. The subscripts L and R refer to whether the particle to which the Z couples is left- or right-handed. That is, v_R^q is to be used for the configuration where the quark (leg 1) has plus helicity and the anti-quark (leg 4) has minus helicity, which we denote by the shorthand $(1_q^+, 4_{\bar{q}}^-)$. Similarly, v_L^q corresponds to the configuration $(1_q^-, 4_{\bar{q}}^+)$. Because the electron and positron are incoming in $e^+\, e^-$ annihilation, our outgoing-momenta notation reverses their helicities and particle vs. anti-particle assignment. Thus, v_R^e corresponds to the helicity configuration $(5_{\bar{e}}^-, 6_e^+)$ whereas v_L^e corresponds to the configuration $(5_{\bar{e}}^+, 6_e^-)$.

The one-loop color decomposition is given by

$$
\begin{aligned}
\mathcal{A}_6^{1-\mathrm{loop}}(1_q,2,3,4_{\bar q}) = 2e^2 g^4 \Big\{ & \big(-Q^q + v^e_{L,R} v^q_{L,R}\, \mathcal{P}_Z(s_{56})\big) \\
& \times \Big[N_c \sum_{\sigma\in S_2} (T^{a_{\sigma(2)}} T^{a_{\sigma(3)}})_{i_1}^{\;\bar\imath_4}\, A_{6;1}(1_q,\sigma(2),\sigma(3),4_{\bar q}) + \delta^{a_2 a_3}\, \delta_{i_1}^{\;\bar\imath_4}\, A_{6;3}(1_q,4_{\bar q};2,3) \Big] \\
& + \sum_{i=1}^{n_f} \Big(-Q^i + \frac{1}{2} v^e_{L,R} (v^i_L + v^i_R) \mathcal{P}_Z(s_{56}) \Big) \\
& \qquad \times \Big[(T^{a_2} T^{a_3})_{i_1}^{\;\bar\imath_4} + (T^{a_3} T^{a_2})_{i_1}^{\;\bar\imath_4} - \frac{2}{N_c}\, \delta^{a_2 a_3}\, \delta_{i_1}^{\;\bar\imath_4} \Big] A^{\mathrm{v}}_{6;4}(1_q,4_{\bar q};2,3) \\
& + \frac{v^e_{L,R}}{\sin 2\theta_W}\, \mathcal{P}_Z(s_{56}) \Big[\sum_{\sigma\in S_2} \Big((T^{a_{\sigma(2)}} T^{a_{\sigma(3)}})_{i_1}^{\;\bar\imath_4} - \frac{1}{N_c} \delta^{a_2 a_3}\, \delta_{i_1}^{\;\bar\imath_4} \Big) A^{\mathrm{ax}}_{6;4}(1_q,4_{\bar q};\sigma(2),\sigma(3)) \\
& \qquad + \frac{1}{N_c} \delta^{a_2 a_3}\, \delta_{i_1}^{\;\bar\imath_4}\, A^{\mathrm{ax}}_{6;5}(1_q,4_{\bar q};2,3) \Big] \Big\},
\end{aligned}
\tag{2.9}
$$

where Q^i is the electric charge (in units of e) of the ith quark and n_f is the number of light quark flavors. The partial amplitudes $A^{\mathrm{v}}_{6;4}$, $A^{\mathrm{ax}}_{6;4}$ and $A^{\mathrm{ax}}_{6;5}$ represent the contributions from a photon or Z coupling to a fermion loop through a vector or axial-vector coupling. We take all quarks to be massless except the top quark. We assume that the top quark mass squared, m_t^2, is larger than the other kinematic invariants in the process, and expand the fermion loop contributions in $1/m_t^2$, keeping terms of order $1/m_t^2$, but dropping those of order $1/m_t^4$. In this approximation the top quark contribution to $A^{\mathrm{v}}_{6;4}$ vanishes (see section 11). On the other hand, in the axial vector channel isodoublet cancellations for massless quarks ensure that only the t,b isodoublet contributes to $A^{\mathrm{ax}}_{6;4}$ and $A^{\mathrm{ax}}_{6;5}$. There are also order $1/m_t^2$ contributions to $A_{6;1}$.

For convenience we also quote the color decomposition for the four-quark amplitudes. At tree level, we have

$$
\begin{aligned}
\mathcal{A}_6^{\mathrm{tree}}(1_q,2_{\bar Q},3_Q,4_{\bar q}) = 2e^2 g^2 \Big[& \Big(-Q^q + v^e_{L,R} v^q_{L,R}\, \mathcal{P}_Z(s_{56}) \Big) A_6^{\mathrm{tree}}(1_q,2_{\bar Q},3_Q,4_{\bar q}) \\
& + \Big(-Q^Q + v^e_{L,R} v^Q_{L,R}\, \mathcal{P}_Z(s_{56}) \Big) A_6^{\mathrm{tree}}(3_Q,4_{\bar q},1_q,2_{\bar Q}) \Big] \\
& \times \Big(\delta_{i_1}^{\;\bar\imath_2}\, \delta_{i_3}^{\;\bar\imath_4} - \frac{1}{N_c} \delta_{i_1}^{\;\bar\imath_4}\, \delta_{i_3}^{\;\bar\imath_2} \Big),
\end{aligned}
\tag{2.10}
$$

while the one-loop decomposition [10] is

$$\mathcal{A}_6^{1-\text{loop}}(1_q,2_{\bar{Q}},3_Q,4_{\bar{q}}) =$$
$$2e^2g^4\Bigg[\Big(-Q^q+v^e_{L,R}v^q_{L,R}\mathcal{P}_Z(s_{56})\Big)\Big[N_c\,\delta_{i_1}^{\,\bar\imath_2}\,\delta_{i_3}^{\,\bar\imath_4}\,A_{6;1}(1_q,2_{\bar{Q}},3_Q,4_{\bar{q}})+\delta_{i_1}^{\,\bar\imath_4}\,\delta_{i_3}^{\,\bar\imath_2}\,A_{6;2}(1_q,2_{\bar{Q}},3_Q,4_{\bar{q}})\Big]$$
$$+\Big(-Q^Q+v^e_{L,R}v^Q_{L,R}\,\mathcal{P}_Z(s_{56})\Big)\Big[N_c\,\delta_{i_1}^{\,\bar\imath_2}\,\delta_{i_3}^{\,\bar\imath_4}\,A_{6;1}(3_Q,4_{\bar{q}},1_q,2_{\bar{Q}})+\delta_{i_1}^{\,\bar\imath_4}\,\delta_{i_3}^{\,\bar\imath_2}\,A_{6;2}(3_Q,4_{\bar{q}},1_q,2_{\bar{Q}})\Big]$$
$$+\frac{v^e_{L,R}}{\sin 2\theta_W}\mathcal{P}_Z(s_{56})\left(\delta_{i_1}^{\,\bar\imath_2}\,\delta_{i_3}^{\,\bar\imath_4}-\frac{1}{N_c}\delta_{i_1}^{\,\bar\imath_4}\,\delta_{i_3}^{\,\bar\imath_2}\right)A_{6;3}(1_q,2_{\bar{Q}},3_Q,4_{\bar{q}})\Bigg]\,. \tag{2.11}$$

For the case of identical quark flavors ($q = Q$) see ref. [10]. We also include here contributions of order $1/m_t^2$ from vacuum polarization loops to $A_{6;1}$ and $A_{6;2}$ (which are very small at present e^+e^- machines). The partial amplitudes $A_{6;1}$ and $A_{6;2}$ also appear in $W + 2$ jet production at hadron colliders; we leave the coupling constant and mass conversions as an exercise.

The virtual part of the next-to-leading order correction to the parton-level cross-section is given by the sum over colors of the interference between the tree amplitude $\mathcal{A}_6^{\text{tree}}$ and the one-loop amplitude $\mathcal{A}_6^{1-\text{loop}}$. Using the color decompositions (2.6) and (2.9), and the Fierz rules (2.5) the color-sum for $e^+e^-\to\bar{q}qgg$ in terms of partial amplitudes is,

$$\sum_{\text{colors}}[\mathcal{A}_6^*\mathcal{A}_6]_{\text{NLO}} = 8e^4\,g^6\,(N_c^2-1)\text{Re}\Bigg\{\Big(-Q^q+v^e_{L,R}v^q_{L,R}\,\mathcal{P}_Z^*(s_{56})\Big)A_6^{\text{tree}*}(1_q,2,3,4_{\bar{q}})$$
$$\times\Bigg[\Big(-Q^q+v^e_{L,R}v^q_{L,R}\,\mathcal{P}_Z(s_{56})\Big)\big[(N_c^2-1)A_{6;1}(1_q,2,3,4_{\bar{q}})$$
$$-A_{6;1}(1_q,3,2,4_{\bar{q}})+A_{6;3}(1_q,4_{\bar{q}};2,3)\big]$$
$$+\sum_{i=1}^{n_f}\Big(-Q^i+\tfrac{1}{2}v^e_{L,R}(v^i_L+v^i_R)\mathcal{P}_Z(s_{56})\Big)\left(N_c-\frac{4}{N_c}\right)A_{6;4}^{\text{v}}(1_q,4_{\bar{q}};2,3)$$
$$+\frac{v^e_{L,R}}{\sin 2\theta_W}\,\mathcal{P}_Z(s_{56})\Bigg[\left(N_c-\frac{2}{N_c}\right)A_{6;4}^{\text{ax}}(1_q,4_{\bar{q}};2,3)$$
$$-\frac{2}{N_c}A_{6;4}^{\text{ax}}(1_q,4_{\bar{q}};3,2)+\frac{1}{N_c}A_{6;5}^{\text{ax}}(1_q,4_{\bar{q}};2,3)\Bigg]\Bigg]\Bigg\}\;+\;\{2\leftrightarrow 3\}\,. \tag{2.12}$$

The corresponding formula for $e^+e^-\to\bar{q}q\bar{Q}Q$ is straightforward to obtain, but lengthier, and so we omit it here.

2.3 Primitive Amplitudes

One may perform a further decomposition of the partial amplitudes in terms of gauge invariant *primitive amplitudes* [19,10]. Primitive amplitudes are gauge-invariant objects from which we can build partial amplitudes. They can be defined as the sum of all Feynman diagrams with a fixed cyclic ordering of the colored lines, with a definite routing of the fermion lines, and with vertices that are given by color-ordered Feynman rules [29]. These differ from ordinary Feynman rules in that they have been stripped of the usual color factors. Here we choose *not* to fix the cyclic ordering of the colorless lepton pair with respect to the colored partons. In this case the lepton pair plays no role in the color structure, and so the equations expressing the partial amplitudes $A_{6;1}$ and $A_{6;3}$ as sums of primitive amplitudes are identical to those derived in ref. [19] for one-loop two-quark two-gluon amplitudes — see eqs. (2.13) below.

The main utility of primitive amplitudes as compared to partial amplitudes (i.e. the coefficients of a particular color structure) is that their analytic structures are generally simpler because fewer orderings of external legs appear. They also contain somewhat more color information, making it more straightforward to convert the amplitudes to those for other processes (for example, to replace gluons by photons, or quarks by gluinos).

For the $e^+ e^- \to \bar{q}qgg$ amplitudes, 'parent diagrams' for each gauge invariant class are depicted in fig. 1. By a 'parent' diagram we mean a diagram from which all other diagrams in the set can be obtained via a continuous 'pinching' process, in which two lines attached to the loop are brought together to a four-point interaction — if such an interaction exists — or further pulled out from the loop, and left as the branches of a tree attached to the loop. The cyclic ordering of external legs is always preserved by pinching. Because we do not fix the ordering of the lepton pair with respect to the partons, primitive amplitudes with an external gluon cyclicly adjacent to the vector boson (γ^*, Z) have more than one parent diagram. For example, $A_6(1_q, 2, 3, 4_{\bar{q}})$ (fig. 1b) has only one parent diagram, while $A_6(1_q, 2, 3_{\bar{q}}, 4)$ (fig. 1c) has two parent diagrams, and $A_6(1_q, 2_{\bar{q}}, 3, 4)$ (fig. 1d) has three. Note that these primitive amplitudes are different functions, not the same function with a different permutation of the arguments: $A_6(1_q, 2_{\bar{q}}, 3, 4) \neq A_6(1_q, 2, 3, 4_{\bar{q}})$.

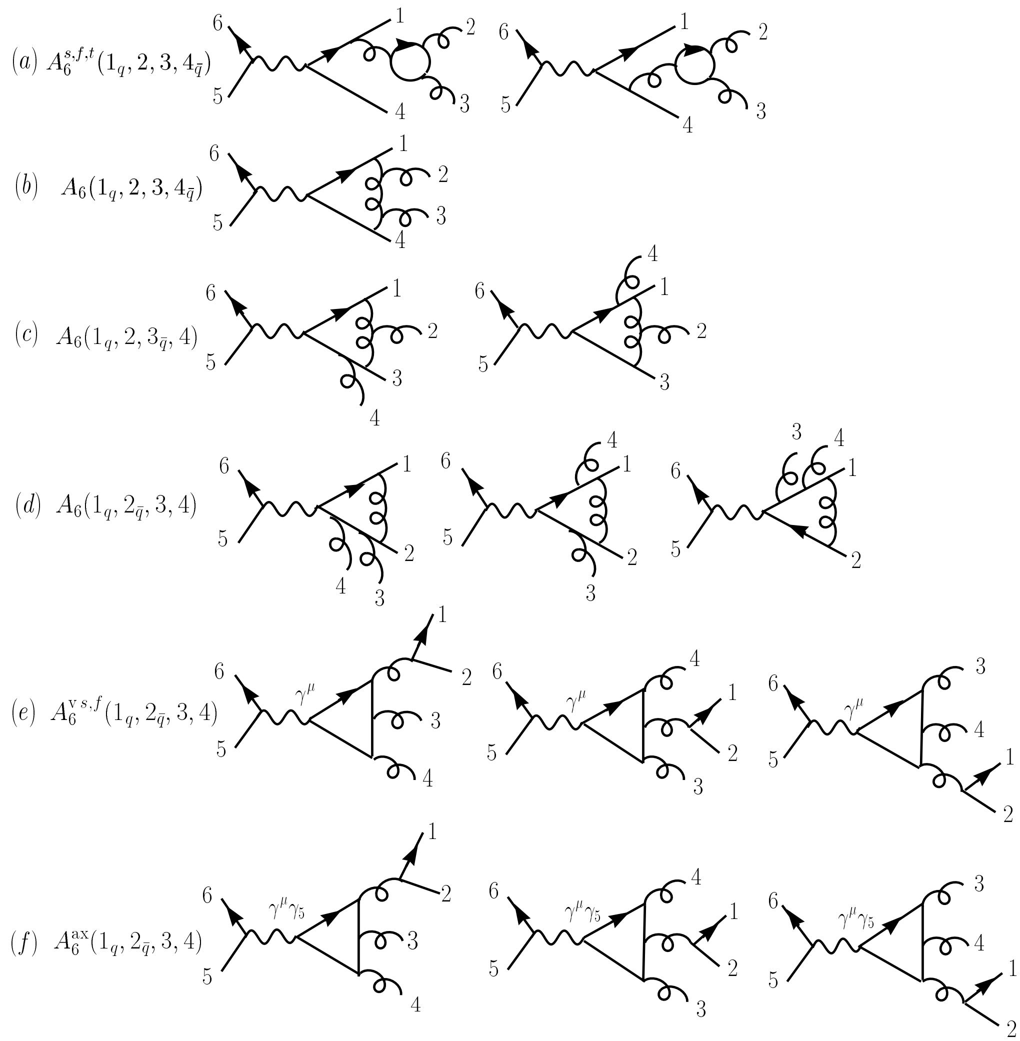

Figure 1. Parent diagrams for the various two quark and two gluon primitive amplitudes. Straight lines represent fermions, curly lines gluons, and wavy lines a vector boson (γ^* or Z).

It turns out to be useful, in diagrams of the type shown in figs. 1b, c and d, to replace a gluon loop contribution with two separate contributions, that of a scalar and that of the difference of a gluon and scalar, as shown in fig. 2. This separates the gluon loop contribution into two gauge-invariant pieces. As we shall discuss in section 3 this separation is advantageous because of differing analytic properties of the two pieces. Terms where a scalar replaces a gluon are labeled with a superscript '*sc*', while those with the difference of gluon and scalar are labeled with a superscript '*cc*'. (As we shall discuss in section 3.1 the '*cc*' terms are 'cut-constructible' meaning that they can be constructed solely from four-dimensional cuts.) Furthermore, terms proportional either to the number of scalars* n_s (which vanishes in QCD), or fermions n_f, are separately gauge invariant so we

* As in refs. [17,19], each scalar here contains four states (to match the four states of Dirac fermions) so that n_s must be divided by two for comparisons to conventional normalizations of scalars.

also separate these out explicitly. (Due to Furry's theorem [33], or charge conjugation invariance, these terms appear only in $A_{6;1}$.)

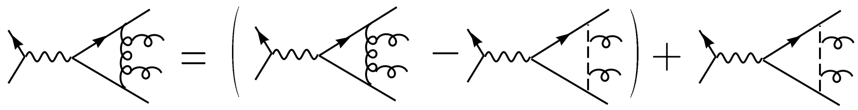

Figure 2. The contribution from a gluon in the loop is separated into the difference of a gluon and scalar, plus a scalar.

Thus we take the decomposition of the partial amplitudes in terms of the primitive amplitudes to be

$$
\begin{aligned}
A_{6;1}(1_q,2,3,4_{\bar q}) &= A_6(1_q,2,3,4_{\bar q}) - \frac{1}{N_c^2} A_6(1_q,4_{\bar q},3,2) \\
&\quad + \frac{n_s-n_f}{N_c} A_6^s(1_q,2,3,4_{\bar q}) - \frac{n_f}{N_c} A_6^f(1_q,2,3,4_{\bar q}) + \frac{1}{N_c} A_6^t(1_q,2,3,4_{\bar q})\,, \\
A_{6;3}(1_q,4_{\bar q};2,3) &= A_6(1_q,2,3,4_{\bar q}) + A_6(1_q,3,2,4_{\bar q}) + A_6(1_q,2,4_{\bar q},3) + A_6(1_q,3,4_{\bar q},2) \\
&\quad + A_6(1_q,4_{\bar q},2,3) + A_6(1_q,4_{\bar q},3,2)\,, \\
A_{6;4}^{\rm v}(1_q,4_{\bar q};2,3) &= -A_6^{{\rm v}s}(1_q,4_{\bar q},2,3) - A_6^{{\rm v}f}(1_q,4_{\bar q},2,3)\,, \\
A_{6;4}^{\rm ax}(1_q,4_{\bar q};2,3) &= A_6^{\rm ax}(1_q,4_{\bar q},2,3)\,, \\
A_{6;5}^{\rm ax}(1_q,4_{\bar q};2,3) &= A_6^{\rm ax,sl}(1_q,4_{\bar q},2,3)\,.
\end{aligned}
\tag{2.13}
$$

We have decomposed the fermion loop in $A_{6;4}^{\rm v}$ into a scalar loop piece $A_6^{{\rm v}s}$ and an additional piece $A_6^{{\rm v}f}$. As mentioned above, we perform a further decomposition of the A_6 into *cc* and *sc* pieces, as depicted in fig. 2,

$$
\begin{aligned}
A_6(1_q,2,3,4_{\bar q}) &= A_6^{cc}(1_q,2,3,4_{\bar q}) + A_6^{sc}(1_q,2,3,4_{\bar q})\,, \\
A_6(1_q,2,3_{\bar q},4) &= A_6^{cc}(1_q,2,3_{\bar q},4) + A_6^{sc}(1_q,2,3_{\bar q},4)\,, \\
A_6(1_q,2_{\bar q},3,4) &= A_6^{cc}(1_q,2_{\bar q},3,4) + A_6^{sc}(1_q,2_{\bar q},3,4)\,.
\end{aligned}
\tag{2.14}
$$

Finally, A_6^t gives the top quark contribution to $A_{6;1}$, through order $1/m_t^2$.

We choose a set of helicity amplitudes from which all others may be obtained by applying the discrete symmetries of parity and charge conjugation. Parity reverses all external helicities in a partial amplitude; it is implemented by the "complex conjugation" operation, which substitutes $\langle jl\rangle \to [lj]$, $[jl] \to \langle lj\rangle$. For the axial-vector fermion loop partial amplitudes one must multiply by an additional overall minus sign. Charge conjugation changes the identity of a fermion to an anti-fermion and vice-versa. These operations allow us to fix the helicity of the electron and the quark to be positive, and the positron and anti-quark to be negative. In addition, if the two gluons have the same helicity, we can fix that common helicity to be positive.

For the four-quark amplitudes, a similar use of charge conjugation and parity reduces the independent partial amplitudes to $A_{6;i}(1_q^+, 2_{\bar{Q}}^+, 3_Q^-, 4_{\bar{q}}^-, 5_{\bar{e}}^-, 6_e^+)$ and $A_{6;i}(1_q^+, 2_{\bar{Q}}^-, 3_Q^+, 4_{\bar{q}}^-, 5_{\bar{e}}^-, 6_e^+)$. The formulæ for the four-quark partial amplitudes, $A_{6;i}(1_q^+, 2_{\bar{Q}}^\pm, 3_Q^\mp, 4_{\bar{q}}^-)$, in terms of primitive amplitudes are [10]

$$\begin{aligned}
A_{6;1}(1_q^+, 2_{\bar{Q}}^+, 3_Q^-, 4_{\bar{q}}^-) &= A_6^{++}(1,2,3,4) - \frac{2}{N_c^2}\big(A_6^{++}(1,2,3,4) + A_6^{+-}(1,3,2,4)\big) + \frac{1}{N_c^2} A_6^{\rm sl}(2,3,1,4) \\
&\quad + \frac{n_s - n_f}{N_c} A_6^{s,++}(1,2,3,4) - \frac{n_f}{N_c} A_6^{f,++}(1,2,3,4) + \frac{1}{N_c} A_6^{t,++}(1,2,3,4)\,, \\
A_{6;2}(1_q^+, 2_{\bar{Q}}^+, 3_Q^-, 4_{\bar{q}}^-) &= A_6^{+-}(1,3,2,4) + \frac{1}{N_c^2}\big(A_6^{+-}(1,3,2,4) + A_6^{++}(1,2,3,4)\big) - \frac{1}{N_c^2} A_6^{\rm sl}(2,3,1,4) \\
&\quad - \frac{n_s - n_f}{N_c} A_6^{s,++}(1,2,3,4) + \frac{n_f}{N_c} A_6^{f,++}(1,2,3,4) - \frac{1}{N_c} A_6^{t,++}(1,2,3,4)\,, \\
A_{6;3}(1_q^+, 2_{\bar{Q}}^+, 3_Q^-, 4_{\bar{q}}^-) &= A_6^{\rm ax}(1,4,2,3)\,,
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
A_{6;1}(1_q^+, 2_{\bar{Q}}^-, 3_Q^+, 4_{\bar{q}}^-) &= A_6^{+-}(1,2,3,4) - \frac{2}{N_c^2}\big(A_6^{+-}(1,2,3,4) + A_6^{++}(1,3,2,4)\big) - \frac{1}{N_c^2} A_6^{\rm sl}(3,2,1,4) \\
&\quad + \frac{n_s - n_f}{N_c} A_6^{s,+-}(1,2,3,4) - \frac{n_f}{N_c} A_6^{f,+-}(1,2,3,4) + \frac{1}{N_c} A_6^{t,+-}(1,2,3,4)\,, \\
A_{6;2}(1_q^+, 2_{\bar{Q}}^-, 3_Q^+, 4_{\bar{q}}^-) &= A_6^{++}(1,3,2,4) + \frac{1}{N_c^2}\big(A_6^{++}(1,3,2,4) + A_6^{+-}(1,2,3,4)\big) + \frac{1}{N_c^2} A_6^{\rm sl}(3,2,1,4) \\
&\quad - \frac{n_s - n_f}{N_c} A_6^{s,+-}(1,2,3,4) + \frac{n_f}{N_c} A_6^{f,+-}(1,2,3,4) - \frac{1}{N_c} A_6^{t,+-}(1,2,3,4)\,, \\
A_{6;3}(1_q^+, 2_{\bar{Q}}^-, 3_Q^+, 4_{\bar{q}}^-) &= -A_6^{\rm ax}(1,4,3,2)\,.
\end{aligned} \tag{2.16}$$

Although color decompositions do not depend on the helicity choices, the sign differences in these equations appear because we have used symmetries of the four-quark primitive amplitudes to reduce the number of independent ones required. In fig. 3 we display the parent diagrams associated with each of the primitive amplitudes

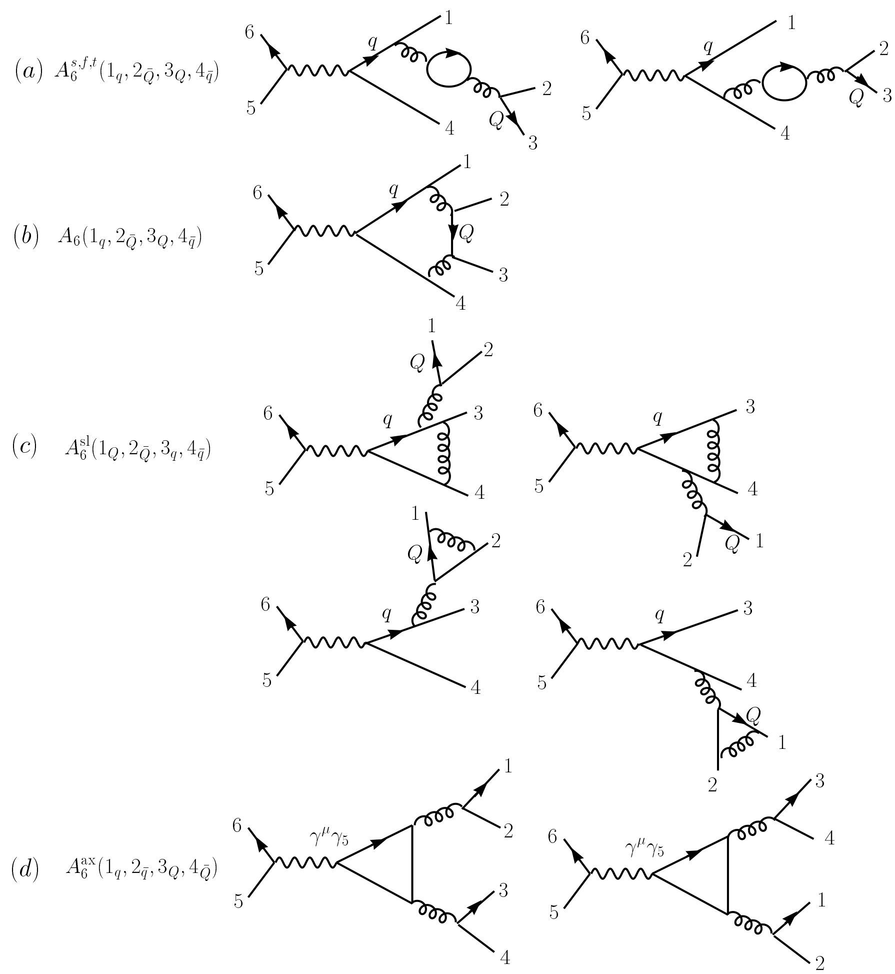

Figure 3. Parent diagrams for the four-quark primitive amplitudes. In each case the vector boson can appear on either side of a gluon line that is attached to the same quark line.

3. Analytic Construction of Amplitudes

In this section, we review the construction of one-loop amplitudes starting from their analytic properties. The two analytic properties we shall use are the determination of imaginary parts by Cutkosky rules, and factorization on particle poles. These properties of amplitudes have, of course, played an important role in field theory for many decades; the recent development which we focus on here is the ability to obtain, efficiently, complete amplitudes with no subtractions or ambiguities. These techniques, reviewed in ref. [25], have been applied to obtain results for both nonsupersymmetric and supersymmetric maximally helicity violating n-point amplitudes [20,22,23], and more recently for multi-loop $N = 4$ supersymmetric four-point amplitudes [24].

3.1 Cutting Rules

Cutkosky rules [26,34] allow one to obtain the imaginary parts of one-loop amplitudes directly from products of tree amplitudes. (By imaginary parts we mean absorptive parts, that is discontinuities across branch cuts.) We apply Cutkosky rules to amplitudes instead of diagrams because amplitudes, being gauge invariant, are simpler. In the channel with momentum squared $K^2 \equiv (k_{m_1} + k_{m_1+1} + \cdots + k_{m_2})^2$ the cut of an amplitude is (see fig. 4)

$$A_n^{1-\text{loop}}(1,2,\ldots,n)\Big|_{K^2\ \text{cut}} \\ = i \int d^D\text{LIPS}(-\ell_1,\ell_2)\ A^{\text{tree}}(-\ell_1,m_1,\ldots,m_2,\ell_2) \times A^{\text{tree}}(-\ell_2,m_2+1,\ldots,m_1-1,\ell_1)\,, \tag{3.1}$$

where the integration is over D-dimensional Lorentz-invariant phase space with with momenta $-\ell_1$ and ℓ_2 for the intermediate states. For this channel K^2 is taken positive and all other kinematic invariants are taken negative.

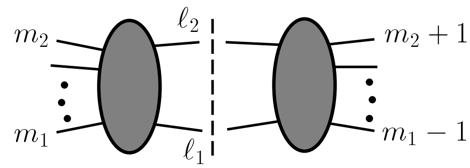

Figure 4. A cut amplitude, with momentum K^2 flowing across the cut. The lines represent gluons, scalars or fermions.

The Cutkosky rules determine imaginary parts of the amplitudes. Dispersion relations are conventionally used to reconstruct real parts from imaginary parts. Here we bypass dispersion relations; instead we replace phase-space integrals with cuts of unrestricted loop momentum integrals,

$$A_n^{1-\text{loop}}(1,2,\cdots n)\Big|_{K^2\ \text{cut}} \\ = \int \frac{d^D\ell_1}{(2\pi)^D}\ A^{\text{tree}}(-\ell_1,m_1,\ldots,m_2,\ell_2)\,\frac{1}{\ell_2^2}\,A^{\text{tree}}(-\ell_2,m_2+1,\ldots,m_1-1,\ell_1)\,\frac{1}{\ell_1^2}\Bigg|_{K^2\ \text{cut}}\,. \tag{3.2}$$

Whereas eq. (3.1) includes only imaginary parts, eq. (3.2) contains both real and imaginary parts. As indicated, eq. (3.2) is valid only for those terms with a K^2-channel branch cut; terms without such a branch cut may not be correct. A useful property of this formula is that one may continue to use on-shell conditions for the cut intermediate legs inside the tree amplitudes without affecting the result. Only terms containing no cut in this channel would change. We are able to avoid the use of dispersion relations because we have additional information, namely that the reconstructed analytic functions are given by Feynman loop integrals.

A similar equation holds for every branch cut. If one now combines all cuts into a single function having the correct cut in each channel, one obtains the full amplitude — up to the possible addition of a *rational function*, i.e. a function having no cuts at all, if one approximates the $(4-2\epsilon)$-

dimensional cuts by their four-dimensional limits (see discussion below). The full amplitude can also be expressed as a linear combination of various types of basic one-loop integral functions, multiplied by rational function coefficients. Many of the integral functions, for example scalar box integrals (which depend on the momenta of four external legs), have cuts in more than one channel. The coefficients of those integral functions as extracted from cuts in different channels must agree, and this provides a strong consistency check on the reconstructed amplitude.

In general, it is convenient to take the tree amplitudes on either sides of the cuts to be four-dimensional. This is natural in the helicity formalism, which implicitly assumes that momenta are four-dimensional. On the other hand, we wish to regulate the ultraviolet and infrared divergences by letting $D = 4 - 2\epsilon$ in the loop integral (3.2). When the $(4 - 2\epsilon)$-dimensional cut momenta are replaced by four-dimensional momenta an error may occur. Although the (-2ϵ)-dimensional parts are implicitly of $\mathcal{O}(\epsilon)$, if the associated integral has an ultraviolet pole in ϵ an $\mathcal{O}(\epsilon^0)$ rational function may remain. (As discussed in refs. [22,23], infrared poles never give rise to such rational contributions.) Despite this seeming error, a complete reconstruction of an amplitude is possible even when the $\mathcal{O}(\epsilon)$ parts of the cut momenta are dropped, *if* the amplitude satisfies a certain power-counting criterion; we call such amplitudes *cut-constructible* [23,25]. The power-counting criterion is that the n-point integrals appearing in the amplitude should have (for $n > 2$) at most $n - 2$ powers of the loop momentum in the numerator of the integrand; two-point integrals should have at most one power of the loop momentum. (By an n-point integral we mean an integral with n propagator factors in the denominator, as in equation (I.1).) Cut-constructible amplitudes are composed of a restricted set of integral functions, and sufficient information exists from the four-dimensional cuts to determine the coefficients of each such function. These integral functions automatically include the cut-free rational functions in the amplitude [23].

The full $e^+ e^- \to \bar{q}qgg$ and $e^+ e^- \to \bar{q}q\bar{Q}Q$ amplitudes are not cut-constructible. Diagrams containing a closed fermion, scalar or gluon loop can have up to n powers of the loop-momenta in the numerator of an n-point integral; other diagrams typically have up to $n-1$ powers. However, one can split the amplitudes into 'scalar' contributions, plus additional terms which are cut-constructible; in fact, we have already performed such a decomposition in eqs. (2.13) and (2.14). In the case of a closed fermion loop, we wrote the fermion loop as the negative of a scalar loop, plus a second term (which is the contribution of an $N = 1$ supersymmetric chiral multiplet). As is apparent in a second-order formalism for fermions, as reviewed in ref. [25], the latter contribution is cut-constructible since the leading two powers of loop momentum cancel.

In the case where an external quark line attaches directly to the loop, for example the leftmost diagram in fig. 2, an n-point integral has a maximum of $n - 1$ powers of the loop momentum. If one subtracts from this diagram an identical diagram but with the gluon replaced by a scalar (also shown in fig. 2), suitably adjusts the scalar-fermion Yukawa coupling, and works in background-field Feynman gauge [35], then the leading loop-momentum terms cancel. Thus the difference is cut-constructible [23].

In both cases, although the scalar contributions are not cut-constructible, they are simpler in some respects than the full amplitudes; for example, certain cuts vanish in the scalar contribution, but not in the full amplitude. They also have spurious singularities of different degree from the cut-constructible terms. To determine the rational functions for such amplitudes, which do not satisfy the power-counting criterion (but contain only massless particles), one may compute to one higher power in the dimensional regularization parameter ϵ [36,27,25]. However, a more convenient approach here is to ignore all $\mathcal{O}(\epsilon)$ contributions, and instead use the amplitudes' factorization properties to reconstruct their rational functions.

3.2 Factorization

The properties of tree-level QCD amplitudes as kinematic invariants vanish have been presented in various reviews [29]. The corresponding one-loop factorization properties have also been extensively discussed [20,30,25], so here we will briefly review only the salient features necessary for obtaining the rational function parts of amplitudes.

For amplitudes with six- and higher-point kinematics, the properties of amplitudes under factorization when a kinematic invariant vanishes are in general sufficiently powerful to obtain the rational function parts of amplitudes. Although factorization is complicated by the appearance of infrared divergences, nevertheless, as any kinematic variable vanishes, one-loop amplitudes have a universal behavior quite similar to that of tree-level amplitudes [22,30]. If one finds a function which obeys the proper factorization equations in all channels, one has an ansatz for the rational function part of an amplitude. Although no proof of the uniqueness of such a construction has been presented, for six- and higher-point amplitudes no counterexample is known. (For a five-point amplitude counterexample see ref. [37].) Factorization can be a particularly efficient way to obtain the rational function terms; it avoids the need to perform loop integrals. This complements the efficiency of the cut-construction technique for obtaining the logarithms and dilogarithms. (Resorting to Feynman diagrams to obtain analytic expressions for the rational function parts is not satisfactory because such pieces tend to have the most complicated diagrammatic representation: they are associated with the maximal power of loop momenta and the largest number of diagrams.)

Of particular utility are the two-particle factorization properties, depicted in fig. 5. As two momenta become collinear the amplitudes behave as [20,22]

$$A_n^{1-\text{loop}} \stackrel{a\|b}{\longrightarrow} \sum_{\lambda=\pm}\Bigg(\text{Split}^{\text{tree}}_{-\lambda}(a^{\lambda_a},b^{\lambda_b})\,A_{n-1}^{1-\text{loop}}(\ldots(a+b)^\lambda\ldots) \\ + \text{Split}^{1-\text{loop}}_{-\lambda}(a^{\lambda_a},b^{\lambda_b})\,A_{n-1}^{\text{tree}}(\ldots(a+b)^\lambda\ldots)\Bigg)\,, \tag{3.3}$$

where $k_a \to zP$ and $k_b \to (1-z)P$, with $P = k_a+k_b$, $P^2 = s_{ab} \to 0$. The helicity of the intermediate parton P is labeled by λ. The tree and loop splitting amplitudes, $\text{Split}^{\text{tree}}_{-\lambda}$ and $\text{Split}^{1-\text{loop}}_{-\lambda}$, behave as $1/\sqrt{s_{ab}}$ in this limit. A complete tabulation of the splitting amplitudes appearing in one-loop computations in massless QCD has been given in refs. [22,19,30]. Given the $n-1$ point amplitude and splitting amplitudes (or 'factorization functions' in multi-particle channels), eq. (3.3) provides an extremely stringent check since one must obtain the correct limits in all channels. A sign or labeling error, for example, will invariably be detected in some limits.

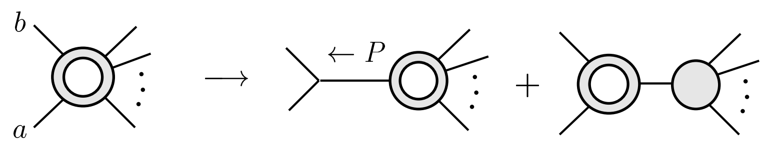

Figure 5. A schematic representation of the behavior of one-loop amplitudes as the momenta of two legs become collinear.

The physical poles that can appear in any massless amplitudes are square-root singularities in two-particle channels, or single poles in multi-particle channels [29,25],

$$\frac{1}{\langle ij\rangle} \sim \frac{1}{\sqrt{s_{ij}}}\,, \qquad \frac{1}{[ij]} \sim \frac{1}{\sqrt{s_{ij}}}\,, \qquad \frac{1}{t_{ijk}}\,. \tag{3.4}$$

Any other kinematic pole-type singularity that appears in individual terms of an amplitude must be spurious; that is, the residue of the pole must vanish for the full amplitude.

In this paper we use factorization to construct ansätze for the rational function parts of the amplitudes. Although the construction of such an ansatz involves a certain amount of guesswork, the procedure can be systematized somewhat. In general, the rational functions contain non-removable spurious singularities (see section 4). In the full amplitude, this singular behavior cancels against singular behavior in terms that have logarithms and dilogarithms, and so it can be inferred from the information provided by the cuts. Therefore one can readily introduce rational function terms that account for all, or at least most, of the spurious singularities. Subtracting these terms from the full rational function ansatz leads to an ansatz for the remainder which is (largely) free of spurious singularities. At this stage it is relatively straightforward to proceed channel by channel, adding terms to the ansatz that correctly reproduce the desired singular behavior in each channel, using the factorization information provided by lower-point amplitudes. The last few channels are simplest, because by then the remaining terms have very few singularities left. For six-point kinematics this procedure invariably gives the correct result, as we have verified by numerical comparisons to Feynman diagram computations. We determine the rational function terms for a simple example in section 5.2.

4. Spurious Singularities.

In this section we discuss the procedure for removing or at least greatly simplifying spurious kinematic singularities appearing the amplitudes. This is essential in order to obtain compact results. The presence of large numbers of such singularities leads to unwieldy results which tend to be unsuitable for use in jet programs since they are numerically unstable. Although the expressions encountered in the calculation of the cuts via eq. (3.2) are rather compact when compared to a direct Feynman diagram calculation, the amplitudes are sufficiently intricate that even a small number of spurious singularities can seriously hinder attempts to obtain compact results.

Spurious singularities fall into two categories, removable and non-removable: After expressing the amplitudes in terms of logarithms and dilogarithms there are spurious singularities that can be removed from the amplitudes by algebraic simplification, and ones that cannot be removed. As a trivial example, consider the functions

$$f_1 = \frac{1-x^2}{1-x}\,; \qquad f_2 = \frac{\ln x}{1-x}\,, \qquad f_3 = \frac{\ln x + 1 - x}{(1-x)^2}\,, \tag{4.1}$$

where the poles at $x = 1$ have vanishing residues. The first function can be re-expressed as $f_1 = (1+x)$ so the spurious pole at $x = 1$ is removable. However, the spurious poles in f_2 and f_3 are not removable if we wish to express the function in terms of a logarithm. One may, of course, define a new set of functions, of which f_2 and f_3 are examples, which absorb the spurious poles. Indeed, this is the role of the L_i and Ls_i functions [17] defined in appendix II. (Generalizations of such functions have been presented in ref. [38].) In the L_i functions x is a ratio of kinematic invariants; for example, $x = s_{23}/t_{123}$ arises from the Gram determinant $\Delta_3^{(2,5)}$ in eq. (I.12). In Feynman diagram calculations both removable and non-removable singularities occur. The removable singularities are an artifact of the integral reduction techniques employed, but the non-removable ones are an inherent part of the amplitudes when they are expressed in terms of logarithms and dilogarithms.

The complications arising from spurious denominators follow largely from dimensional analysis and the fact that they carry positive dimensions. Their appearance implies that the numerators must have compensating powers of momenta. In a six-point amplitude there are five independent momenta, leading to a substantial proliferation in the number of possible numerators. As an example, consider a five-point tensor integral with four powers of loop momentum in the numerator encountered in a Feynman diagram evaluation of $e^+e^- \to \bar{q}qgg$. If this integral were evaluated by conventional means using a Passarino-Veltman reduction [39], summarized in appendix I.1, one would have up to four inverse powers of the pentagon Gram determinant Δ_5 defined in eq. (I.10). As we shall show below, these spurious singularities can always be removed from the amplitudes of this paper, but if they appear in intermediate expressions, their removal is an arduous task. The appearance of a spurious Δ_5^{-4} denominator implies that the numerators must contain an appropriate polynomial to cancel the poles. Since the terms in the numerator of a brute force calculation appear in a seemingly haphazard pattern, one must deal with the order of $22^4 \sim 10^5$ terms to remove the spurious singularities in Δ_5. Moreover, the various spurious singularities can get tangled together in rather intricate ways. Our goal is to eliminate those spurious poles which do not belong in the amplitudes and to simplify those which are are inherently associated with the logarithms and dilogarithms.

4.1 Types of Spurious Singularities Appearing in the Amplitudes

The simplest type of spurious singularities are unphysical poles in the kinematic variables s_{ij} and t_{ijk}. Even at tree level the kinematic singularities in the amplitude can be non-trivial. For example, consider the tree amplitude $A_6^{\rm tree}(1_q^+, 2^+, 3^-, 4_{\bar{q}}^-)$ given in eqs. (8.8) and (8.9). In the first form (8.8) the amplitude exhibits the proper poles in the t_{123} and t_{234} three-particle channels, but the behavior as s_{23} or s_{56} vanish is not manifest — the apparent full poles in these variables actually cancel down to square-root singularities. The second form (8.9) of the amplitude exhibits the proper square-root singularities in all two-particle channels, at the expense of more obscure behavior in three-particle channels.

At one loop, the situation is greatly complicated by the large variety of spurious singularities that arise from loop integrals. The parent diagrams for the A_6 primitive amplitudes in figs. 1 and 3 require the evaluation of pentagon integrals where all internal lines are massless, and all external legs are massless except for the leg composed of the lepton pair 5–6, which has invariant mass s_{56}. Besides this one-mass pentagon integral, there are a number of different types of box integrals, where either one or two external legs are massive. These integrals may appear directly in a diagram or a term in a cut evaluation, or they may be generated in the reduction of pentagon integrals, as reviewed in appendix I. Similarly, we find triangle integrals with one, two or three external massive legs, as well as bubble (two-point) integrals. All these integrals have associated with them different kinematic factors, which can appear in the denominators of coefficients of logarithms and dilogarithms, and whose vanishings correspond to separate spurious singularities.

In appendix I we summarize some of the standard integral reduction methods [39,40,41,42], and their associated spurious singularities. As discussed in the appendix, when evaluating tensor n-point loop integrals one obtains denominators containing (minus) the Gram determinants

$$\Delta_n = -\det(2K_i \cdot K_j)\,, \qquad\qquad i,j = 1,2,\ldots,n-1, \tag{4.2}$$

where the K_i are external momenta or sums of external momenta. Other kinematic denominators which appear are the determinants

$$\det(S_{ij})\,, \qquad\qquad i,j = 1,2,\ldots,n, \tag{4.3}$$

where the symmetric matrix S_{ij} is given by

$$S_{ii} = 0\,, \qquad\qquad S_{ij} = -\tfrac{1}{2}(K_i + K_{i+1} + \cdots + K_{j-1})^2\,, \quad \text{for } i < j\,. \tag{4.4}$$

In appendix I we collect the explicit forms of the various determinants that can appear as poles in the amplitudes. Particularly important are poles in the three-mass triangle Gram determinant

$$\Delta_3 \equiv \Delta_3^{(2,4)} = s_{12}^2 + s_{34}^2 + s_{56}^2 - 2\,s_{12}\,s_{34} - 2\,s_{34}\,s_{56} - 2\,s_{56}\,s_{12}\,, \tag{4.5}$$

in the spinor strings

$$\langle 1|(2+3)|4\rangle\,, \qquad\qquad \langle 4|(2+3)|1\rangle\,, \tag{4.6}$$

and in objects related to these by permutations of the external legs $1,2,3,4$. The spinor strings (4.6) vanish in the kinematic configuration where $k_2+k_3 = ak_1+bk_4$ where a and b are arbitrary constants. We call this configuration 'back-to-back' because in the center-of-mass frame for particles 1 and 4, viewed as incoming, the outgoing three-momenta $\vec{k}_2+\vec{k}_3$ and $\vec{k}_5+\vec{k}_6$ must be parallel to $\vec{k}_1$ and $\vec{k}_4$. In many cases, these singularities cancel only after taking into account the behavior of the dilogarithms and logarithms that appear, in analogy to the behavior of f_2 and f_3 in eq. (4.1). The appearance of the three-external-mass triangle Gram determinants (4.5) and the back-to-back singularities (4.6) explains to a large extent the significant increase in complexity of the amplitudes presented in this paper, as compared to the massless five-parton amplitudes [17,18,19]. It is essential to simplify terms containing these singularities if we wish to obtain (relatively) compact expressions.

4.2 Integral Reductions

We now describe techniques we used to help minimize the algebraic complexity of intermediate expressions when evaluating a cut amplitude (3.2). In particular, these techniques prevent the appearance of the pentagon Gram determinants, which are by far the most noxious of the unwanted determinantal denominators. The same techniques apply just as well to Feynman diagrams, although in evaluating a cut one generally begins with a much more compact expression, making it simpler to keep its size small.

An important aspect to the calculations performed in this paper is the use of a helicity basis for both quark and gluon external states. This basis simplifies general gauge theory amplitudes, as reviewed in refs. [29,25]. (To make effective use of the helicity formalism at the loop level it is important to use a compatible regularization scheme, such as the FDH scheme [43], which at one-loop has been shown to be equivalent to a helicity form of dimensional reduction [44,15].) The spinor helicity method also leads to a useful procedure for evaluating tensor integrals. The basic observation is that certain combinations of Lorentz invariant products, such as $s_{56}s_{23} - t_{123}t_{234}$, which are destined to appear in the denominators of amplitudes for reasons discussed in the previous subsection, and which cannot be factored in terms of Lorentz invariants, *can* be factored in terms of spinor strings such as $\langle 1|(2+3)|4\rangle$, as shown in eq. (I.16). By multiplying and dividing by such spinor square roots, and by further manipulating the spinor strings in the numerator of the loop momentum integral for a cut, one can extract inverse propagators. These factors cancel propagators in the denominator, leaving behind much simpler lower-point integrals to evaluate. This general strategy is similar to the more conventional Passarino-Veltman reduction in that it expresses tensor integrals as linear combinations of simpler integrals, but it differs in its economical use of expressions which already appear in the loop momentum integrands. The strategy has already been applied to a number of cases [22,23,18,27,45,24]. The amplitudes of this paper have rather complicated kinematics, and so it is very important to exploit factorization relations such as (I.16), in order to appropriately arrange the spinor strings and thus maximize the simplifications.

When evaluating a cut (3.2) one typically encounters a pentagon loop momentum integral of the form

$$\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{\langle a|\ell_i|b\rangle\langle c|\ell_j|d\rangle\cdots}{\ell_1^2\,\ell_2^2\,\ell_3^2\,\ell_4^2\,\ell_5^2}\,, \tag{4.7}$$

where $\ell_i = p - k_1 - \cdots - k_{i-1}$ is the momentum of the ith loop propagator. The kinematic configuration of this pentagon integral is shown in fig. 6. By multiplying and dividing by $\langle b|d|c\rangle$, we may convert the numerator factors into a single spinor string containing both $\not\ell_i$ and $\not\ell_j$,

$$\langle a|\ell_i|b\rangle\langle c|\ell_j|d\rangle = \frac{1}{\langle b|d|c\rangle}\,\langle a|\ell_i b d c \ell_j|b\rangle\,. \tag{4.8}$$

We may then extract inverse propagators by commuting $\not\ell_i$ and $\not\ell_j$ towards each other. In this way we generate terms of the form

$$2\,k_j\cdot\ell_m\,, \qquad\qquad 2\,\ell_i\cdot\ell_j\,. \tag{4.9}$$

For this to be a useful rearrangement, these dot products must be expressible as sums of inverse propagators and external kinematic variables. This requirement dictates some care in deciding which pairs of strings $\langle a|\ell_i|b\rangle$, etc., to work with, and which momentum d to use in the string $\langle b|d|c\rangle$

that one multiplies and divides by. An inappropriate choice can lead to a large algebraic expression without any reduction in powers of loop momenta in the numerators of the integrals.

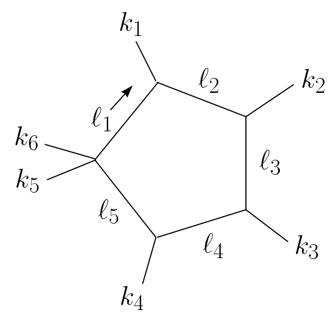

Figure 6. The momenta associated with the pentagon integral appearing in the example.

For example, consider a numerator containing the product

$$\langle 5|\ell_1|1\rangle\langle 4|\ell_5|6\rangle\cdots\,, \tag{4.10}$$

which occurs in the evaluation of the cut in the s_{56} channel for the amplitude $A_6^{sc}(1_q^+,2^-,3^+,4_{\bar q}^-,5_{\bar e}^-,6_e^+)$. In this channel we may use the on-shell conditions $\ell_5^2=0$ and $\ell_1^2=0$. Before explaining a good choice for forming a spinor string we mention first some choices which are *not* very helpful. For example, we might combine the spinors via

$$\langle 5|\ell_1|1\rangle\langle 4|\ell_5|6\rangle = \frac{1}{\langle 16\rangle}\langle 5|\ell_1|1\rangle\,\langle 16\rangle\,\langle 6^+|\ell_5|4^+\rangle = \frac{1}{\langle 16\rangle}\langle 5|\ell_1 16\ell_5|4\rangle\,. \tag{4.11}$$

In this case, if we commute $\not{\ell}_1$ or $\not{\ell}_5$ past $\not{k}_6$ we obtain a term containing $k_6\cdot\ell_1$ or $k_6\cdot\ell_5$, neither of which can be expressed in terms of inverse propagators, since only k_5+k_6 appears in the loop propagators. Other choices, such as multiplying and dividing by $\langle 1|3|4\rangle$, are better but still introduce unwanted spurious singularities in the amplitudes, which must be removed at later stages in the calculation.

A much better choice is to multiply and divide by $\langle 1|(2+3)|4\rangle = -\langle 1|(5+6)|4\rangle$, so that

$$\langle 5|\ell_1|1\rangle\langle 4|\ell_5|6\rangle = -\frac{1}{\langle 1|(2+3)|4\rangle}\langle 5|\ell_1 1(5+6)4\ell_5|6\rangle\,, \tag{4.12}$$

which will ensure that we commute $\not{\ell}_1$ and $\not{\ell}_5$ only with neighboring momenta. This choice is motivated by the appearance of this kinematic singularity in the scalar pentagon integral reduction formula (I.7) (with $n=5$), after expressing the determinantal denominator (I.13) in the factored form (I.16).

Once a numerator term is in a form where at least two $\not{\ell}_i$ are in the same inner product we can commute these terms toward each other. In particular, for the spinor string appearing in eq. (4.12) we have

$$\begin{aligned}\langle 5|\ell_1 1(5+6)4\ell_5|6\rangle &= \ell_4^2\langle 5|\ell_1 15|6\rangle + \ell_2^2\langle 5|6\ell_5 4|6\rangle + \langle 5|1\ell_1(5+6)\ell_5 4|6\rangle \\ &= \ell_4^2\,[65]\,\langle 51\rangle\,\langle 5|\ell_1|1\rangle - \ell_2^2\,\langle 56\rangle\,[64]\,\langle 4|\ell_5|6\rangle\,,\end{aligned} \tag{4.13}$$

where we used

$$\langle 5|1\ell_1(5+6)\ell_5 4|6\rangle = \langle 5|1\ell_1(\ell_5-\ell_1)\ell_5 4|6\rangle = 0\,, \tag{4.14}$$

and

$$2\ell_5\cdot k_4 = (\ell_5+k_4)^2 = \ell_4^2\,, \qquad\qquad 2\ell_1\cdot k_1 = -\ell_2^2\,, \tag{4.15}$$

which follow from the s_{56}-cut conditions $\ell_1^2 = \ell_5^2 = 0$. Since both terms in eq. (4.13) contain inverse propagators we have succeeded in reducing the pentagon integral to a sum of two box integrals. The rather clean simplification is due to our choice of multiplying and dividing by $\langle 1|(2+3)|4\rangle$. Of course, not all cases are reduced as easily, but this example does illustrate the importance of choosing appropriate factors to multiply and divide by.

Each inverse propagator appearing in a numerator cancels a propagator, leaving a lower-point integral with one less power of loop momentum; if there are any terms without an inverse propagator then they are also down by one power of loop momentum. Thus, by combining spinor strings and commuting pairs of $\not{\ell}_i$ toward each other, we can always reduce a pentagon integral with $m>1$ powers of loop momentum to a linear combination of pentagon and box integrals with at most $m-1$ powers of loop momentum. When one or no powers of loop momentum are obtained we may use the reduction formulæ (I.7) and (I.9) (with $n=5$), which are free of pentagon Gram determinants. In this way, we avoid encountering any Δ_5s in the calculation.

Integral	**Back-to-Back Singularities**
$I_5\,,\ I_4^{(3)}\,,\ \tilde{I}_4^{(3)}$	$\langle 1|(2+3)|4\rangle\,,\ \langle 4|(2+3)|1\rangle$
$\tilde{I}_5\,,\ I_4^{(2)}\,,\ \tilde{I}_4^{(2)}\,,\ \hat{I}_4^{(2)}$	$\langle 4|(1+2)|3\rangle\,,\ \langle 3|(1+2)|4\rangle$
$\hat{I}_5\,,\ \tilde{I}_4^{(1)}\,,\ \hat{I}_4^{(1)}$	$\langle 3|(4+1)|2\rangle\,,\ \langle 2|(4+1)|3\rangle$
$I_4^{(4)}\,,\ \hat{I}_4^{(4)}$	$\langle 1|(3+4)|2\rangle\,,\ \langle 2|(3+4)|1\rangle$

Table 1: The back-to-back singularities associated with reductions of each type of integral function.

The best quantities to multiply and divide by when forming a spinor string are usually the back-to-back spinor products which would occur in the Passarino-Veltman integral reductions. In many cases, these singularities are not removable and appear in our final expressions. Using the results summarized in appendix I, we have collected in table 1 the back-to-back singularities associated with the reduction of each type of integral. This table provides guidance in actual calculations as to which spinor products to introduce, although in some cases simpler alternatives are available (see below). Since the (γ^*, Z) is not colored, for a given color ordering it may appear with either cyclic ordering with respect to a cyclicly adjacent external gluon. This means that when the first four legs are ordered 1234 by the color flow, one may still have integral functions with three possible orderings for the six legs: 123456, 123564 and 125634. We denote the pentagon integrals corresponding to these three orderings by I_5, $\tilde{I}_5$, and $\hat{I}_5$. The box integrals are similarly denoted by $I_4^{(i)}$, $\tilde{I}_4^{(i)}$ and $\hat{I}_5^{(i)}$ where the label (i) indicates that the box is obtained from the pentagon by removing the propagator prior (in the clockwise ordering of legs) to the ith external leg. For convenience we have collected the integrals appearing in Table 1 in fig. 7.

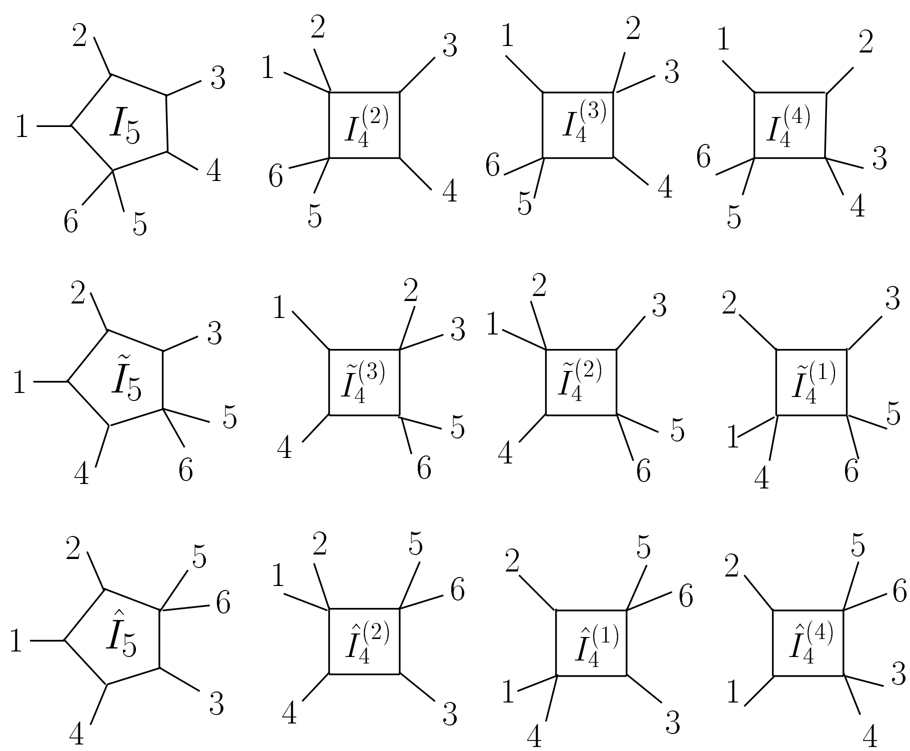

Figure 7. The integrals that can lead to spurious back-to-back singularities.

Sometimes the pentagon integral I_5 appearing in a cut can be reduced to boxes without introducing the 'back-to-back' factor $\langle 1|(2+3)|4\rangle$ or its complex conjugate $\langle 4|(2+3)|1\rangle$. In such cases, the required factor one should multiply and divide by is either $\langle 23\rangle$, or else its complex conjugate $[23]$. For example, the cut in the s_{56} channel for the amplitude $A_6^{sc}(1_q^+, 2_{\bar{q}}^-, 3^-, 4^+, 5_{\bar{e}}^-, 6_e^+)$ contains a term with the factor

$$\langle 4^+|\ell_3|2^+\rangle\langle 3|\ell_3|4\rangle = \frac{1}{[23]}[4|\ell_3 23\ell_3|4]\,, \qquad (4.16)$$

which is easily reduced to

$$\frac{1}{[23]}\Big[(\ell_2^2-\ell_3^2)\ [43]\,\langle 3|\ell_3|4\rangle + (\ell_4^2-\ell_3^2)\ [42]\,\langle 2|\ell_3|4\rangle + \ell_3^2\,[4|23|4]\Big]\,. \tag{4.17}$$

After having reduced pentagon integrals to box integrals, the next step is to reduce the box integrals. Boxes with two adjacent massive legs, depicted in columns two and four of fig. 7, have the same kind of back-to-back singularities associated with them as does the pentagon integral (see table 1); therefore they can usually be reduced by multiplying numerator and denominator by the string $\langle i|(k+l)|j\rangle$ (or its complex conjugate), along the lines of eq. (4.12). Table 1 and fig. 7 show that for this box reduction i and j should be the two adjacent massless legs. In contrast, for the other two types of box integrals that appear, boxes with only one massive leg and boxes with two diagonally opposite massive legs, the appropriate factor to multiply and divide by turns out to be a single spinor product, $\langle ij\rangle$ (or its complex conjugate), where now i and j represent the two diagonally opposite massless legs. For a box integral with one massive leg, such as $I_4^{(5)}$, this is suggested by the factor of $s_{13}=-\langle 13\rangle\ [13]$ in the Gram determinant $\Delta_4^{(5)}$ in eq. (I.11). For a box integral with diagonally opposite massive legs, such as $I_4^{(3)}$, the corresponding factor in $\Delta_4^{(3)}$ is $s_{14}=-\langle 14\rangle\ [14]$. Even though the spinor string $\langle 1|(2+3)|4\rangle$ also appears as a factor in $\Delta_4^{(3)}$, it is never required for the reduction of this box integral.

The appropriateness of these factors for simplifying numerators is due to the fact that they terminate with spinors corresponding to massless legs of the relevant *integrals*, not just the amplitude. The key to the previous pentagon reductions was having only massless legs of the pentagon integral interposed between two loop momenta in a single spinor string. At the level of box and lower-point integrals, fewer of the external legs of the integral are massless, because they can instead be sums of massless legs of the amplitude. For example, in the box integral $I_4^{(2)}$ in fig. 7, legs 3 and 4 are massless legs of the integral, but legs 1 and 2 only appear as constituents of a massive leg. If we had a string of the form $\cdots 1\ell_1\cdots$, and we tried to commute $\not\ell_1$ past $\not k_1$, we would generate $2\ell_1\cdot k_1=\ell_1^2-\ell_2^2$. But $1/\ell_2^2$ is not a propagator for $I_4^{(2)}$, hence the commutation procedure fails to reduce the integral. (This is the same problem encountered when the momentum k_6 was introduced in the pentagon numerator (4.11).)

In some cases, not enough of the massless spinors terminating the spinor strings in the numerator of an integral correspond to massless legs of the integral. For example, if the string

$$\langle a|\ell_3|b\rangle\langle 3|\ell_3|c\rangle \tag{4.18}$$

appears in the numerator of the adjacent two-mass box integral $I_4^{(2)}$, and none of a,b and c is equal to 3 or 4, then we are seemingly blocked from using the above procedures. One way to handle this situation is to use the Schouten identities,

$$\begin{aligned} \langle ij\rangle\,\langle kl\rangle &= \langle il\rangle\,\langle kj\rangle + \langle ik\rangle\,\langle jl\rangle\ , \\ [ij]\,[kl] &= [il]\,[kj] + [ik]\,[jl]\ , \end{aligned} \tag{4.19}$$

to put more 'useful' momenta at the ends of strings. In the present case, we multiply and divide by [34]. Then we use

$$\langle a|\ell_3|b\rangle\,[34] = \langle a|\ell_3|3\rangle\,[b4] + \langle a|\ell_3|4\rangle\,[3b]\ . \tag{4.20}$$

The string becomes

$$\frac{1}{[34]}\Big[(\ell_3^2-\ell_4^2)\,[b4]\,\langle a|\ell_3|c\rangle - \ell_3^2\,[b4]\,\langle a|3|c\rangle + [3b]\,\langle a|\ell_3|4\rangle\langle 3|\ell_3|c\rangle\Big]\,, \tag{4.21}$$

and the last term can now be reduced further, after multiplying and dividing by $\langle 4|(1+2)|3\rangle$.

The trick of multiplying and dividing by spinor product factors stops working when one has too few massless legs, which here basically happens at the level of triangle integrals. Fortunately, the triangle integrals with one and two external masses do not generate terribly complicated expressions even when reduced via a general ('brute force') formula, for example using Feynman parametrization. On the other hand, three-mass triangle integrals with two or three powers of the loop momenta inserted can generate quite lengthy formulæ. Indeed such terms — coefficients of $I_3^{3\mathrm{m}}(s_{12},s_{34},s_{56})$, $\ln(\frac{-s_{12}}{-s_{56}})$, etc. — account for much of the length of our final expressions. Part of the problem is that the three-mass triangle Gram determinant $\Delta_3^{(2,4)}$ in eq. (I.12) — which 'belongs' in the various coefficients in some form — has mass dimension 4, yet apparently cannot be factored at all, even employing spinor strings.

The cuts for one-loop six-point amplitudes can be divided into s_{ij} and t_{ijk} cuts, according to whether the momentum flowing across the cut is the sum of two or three external momenta. In general, the t_{ijk} cuts are much simpler to evaluate, largely because the three-mass triangle cannot appear in such a cut — it has cuts only in three s_{ij} channels. One way to handle the more intricate s_{ij} cuts is to first evaluate a 'triple cut', where three loop propagators are required to be on-shell. For example, the s_{12}-s_{34}-s_{56} triple cut, depicted in fig. 8, can be defined by the conditions $\ell_1^2=\ell_3^2=\ell_5^2=0$. Such triple cuts pick out those integral functions containing cuts in all three channels, in particular the three-mass triangle functions. The utility of considering such cuts is that instead of having to evaluate the phase-space integral of a six-point tree amplitude with a four-point tree amplitude (as one would for an s_{ij} cut), one gets an expression where the six-point amplitude is replaced by the product of the two four-point amplitudes that it factorizes on. The full s_{56} cut (say) can then be written as a sum of the triple cut and a residual term which has no $1/\ell_3^2$ propagator, and therefore no three-mass triangle integral; that is, the three-mass triangle complications can be confined to the triple cuts.

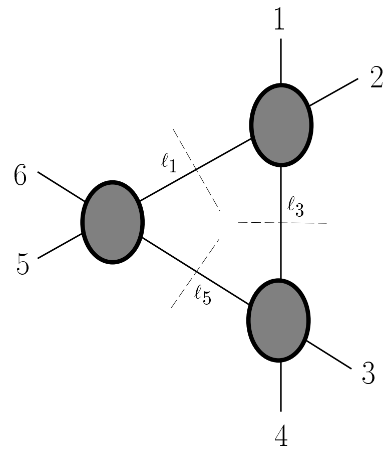

Figure 8. The kinematics of the triple cut. The cut lines are all on-shell.

4.3 Numerical simplifications

Even after employing the above spinor-product manipulations (among others) to help reduce the size of the expressions for cuts, one may still generate in intermediate steps complicated analytical expressions for the coefficients of $I_3^{3\mathrm{m}}(s_{12}, s_{34}, s_{56})$, $\ln(-s_{12})$, and so on. It is possible to use numerical techniques to simplify such an expression, which is some rational function of the spinor products, *if* one has enough information about the analytic behavior of the coefficient. (The analytic information does not have to be manifest in the complicated expression.) The basic idea is to write down an ansatz for the complicated expression, as a linear combination of all possible kinematic terms that have the proper analytic behavior, where each term is multiplied by an as-yet unknown numerical coefficient. The more one knows about the analytic behavior of the coefficient, the fewer factors one has to put in the denominator of the ansatz, and (by dimensional analysis and combinatorics) the fewer the possible terms. Then one evaluates both the complicated expression and the ansatz at a number of random kinematic points, which should exceed the number of unknown coefficients. This gives an over-determined set of linear numerical equations which can be solved for the unknown coefficients, which are required to be simple rational numbers. The solution can be checked by numerical evaluation at further kinematic points. An example of this procedure is provided in section 5.3.

In practice we have been able to carry out this 'numerical simplification' procedure efficiently once the number of linearly independent terms in the ansatz for a given coefficient is reduced to about a hundred. To get a manageable number of terms like that, one generally needs to know about not just the physical factorization limits discussed in section 3.2, but also the leading spurious singularities of the coefficient. However, this latter information can be typically be inferred from simpler cuts which have already been performed.

For example, the coefficients of both $I_3^{3\mathrm{m}}(s_{12}, s_{34}, s_{56})$ and $\ln(-s_{12})$ receive contributions from the adjacent two-mass box integral $I_4^{(2)}$, and thus they will typically have denominators containing $\langle 3|(1+2)|4\rangle$. In the full amplitude the singularities in these terms as $\langle 3|(1+2)|4\rangle \to 0$ cancel against singularities in the terms containing $\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{34}, t_{123}; s_{12}, s_{56})$. The precise way this cancellation happens is known from the structure of the tensor box integrals [38]. In the present example we have, in the limit as $\langle 3|(1+2)|4\rangle \to 0$,

$$
\begin{aligned}
&\big[\text{coefficient of } I_3^{3\mathrm{m}}(s_{12}, s_{34}, s_{56})\big] \\
&\to \big[\text{coefficient of } \mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{34}, t_{123}; s_{12}, s_{56})\big] \times (-1)\, \frac{\langle 3|(1+2)|4\rangle\langle 4|(1+2)|3\rangle(t_{123}-t_{124})s_{12}s_{56}}{t_{123}^2\Delta_3}\,, \\
&\big[\text{coefficient of } \ln(-s_{12})\big] \\
&\to \big[\text{coefficient of } \mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{34}, t_{123}; s_{12}, s_{56})\big] \times (-1)\, \frac{\langle 3|(1+2)|4\rangle\langle 4|(1+2)|3\rangle(t_{123}-t_{124})s_{12}\delta_{12}}{2s_{34}t_{123}^2\Delta_3}\,.
\end{aligned}
\tag{4.22}
$$

On the other hand, the coefficient of $\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{34}, t_{123}; s_{12}, s_{56})$ can be determined from the cut in the t_{123} channel, which as we have mentioned is much simpler, typically only one or two terms in length. The known analytic behavior of the coefficients of $I_3^{3\mathrm{m}}(s_{12}, s_{34}, s_{56})$ and $\ln(-s_{12})$ as $\langle 3|(1+2)|4\rangle \to 0$ can also be verified numerically by choosing kinematics close to this 'back-to-back' singularity.

Another denominator factor usually present in the coefficient of $\ln(-s_{12})$ is $(s_{12} - t_{123})$. Again these terms can be inferred from the t_{123}-cut; this time the relation is through the simpler functions $\mathrm{L}_0(\frac{-s_{12}}{-t_{123}})$ and $\mathrm{L}_1(\frac{-s_{12}}{-t_{123}})$.

The singularity of $\ln(-s_{12})$ as $\Delta_3(s_{12}, s_{34}, s_{56}) \to 0$ is related to that of $I_3^{3\mathrm{m}}(s_{12}, s_{34}, s_{56})$, but it cannot be related to a t_{ijk} cut, and has to be extracted from the leading loop-momentum behavior of the s_{12}-s_{34}-s_{56} triple cut. The numerical study of the $\Delta_3 \to 0$ limit is also subtler: In this limit the three vectors $k_1 + k_2$, $k_3 + k_4$ and $k_5 + k_6$ all become proportional to each other; thus there is a simultaneous 'back-to-back' vanishing of $\langle 1|(3+4)|2\rangle$, $\langle 3|(1+2)|4\rangle$ and $\langle 5|(1+2)|6\rangle$. Terms with Δ_3 in the denominator often have such vanishing factors in the numerator (or other more complicated ones — see eq. (I.17)), which mask the presence of $1/\Delta_3$. On the bright side, these numerator factors imply that individual kinematic coefficients are less singular as $\Delta_3 \to 0$ than a simple counting of denominator Δ_3s would suggest, improving their numerical stability near the pole.

5. Sample Calculations

5.1 Evaluation of Cuts

Here we describe the initial stages of evaluation of two different cuts, in order to illustrate some of the features that are involved.

The first cut we consider is the cut in the t_{234} channel of the amplitude $A_6^{sc}(1_q^+, 2_{\bar{q}}^-, 3^-, 4^+, 5_{\bar{e}}^-, 6_e^+)$. This amplitude has a scalar replacing the gluon in the loop, and so the cut propagator ℓ_2 is that of a scalar. (The configuration of external legs and loop momenta is shown in fig. 6.) The emission of the scalar from the quark line is via a Yukawa coupling that reverses the helicity of the quark line. Thus the required product of tree amplitudes, to be integrated over the two-body phase space for $\ell_2 + (-\ell_5)$, is

$$P_{234} = A_5^{\rm tree}(1_q^+, (\ell_2)_s, (-\ell_5)_{\bar{q}}^+, 5_{\bar{e}}^-, 6_e^+) \times A_5^{\rm tree}((-\ell_2)_s, 2_{\bar{q}}^-, 3^-, 4^+, (\ell_5)_q^-)\,. \tag{5.1}$$

These two five-point tree amplitudes are easily evaluated, up to overall signs (which can always be fixed at the end of the calculation, using e.g. a factorization limit),

$$\begin{aligned} P_{234} &= \pm \frac{\langle \ell_2 5\rangle^2}{\langle 1\ell_2\rangle \langle \ell_2\ell_5\rangle \langle 56\rangle} \times \frac{[24]\,[\ell_2 4]^2}{[23]\,[34]\,[\ell_2\ell_5]\,[\ell_2 2]} \\ &= \pm \frac{[24]}{[23]\,[34]\,t_{234}\,\langle 56\rangle} \frac{\langle 5|\ell_2|4\rangle\langle 5|\ell_2|1\rangle\langle 2|\ell_2|4\rangle}{\ell_1^2\ell_3^2}\,. \end{aligned} \tag{5.2}$$

In the second step we used the on-shell conditions $\ell_2^2 = \ell_5^2 = 0$ to replace $1/\langle 1\ell_2\rangle$ with $[\ell_2 1]/(2\ell_2\cdot k_1) = [\ell_2 1]/\ell_1^2$, and a similar replacement for $1/[\ell_2 2]$. Notice that the propagator ℓ_4^2 is 'missing'. Hence no pentagon reduction is necessary in this example; the integral is already the box integral $I_4^{(4)}$ with two adjacent masses. (The missing propagator can be attributed to a supersymmetry Ward identity (SWI) [46]: The limit $\ell_4^2 \to 0$ is also the collinear limit $\ell_5 \parallel k_4$ for the tree amplitude $A_5^{\rm tree}((-\ell_2)_s, 2_{\bar{q}}^-, 3^-, 4^+, (\ell_5)_q^-)$, in which it factorizes onto $A_4^{\rm tree}((-\ell_2)_s, 2_{\bar{q}}^-, 3^-, P_q^-)$, which vanishes by a SWI.)

Table 1 and the structure of the numerator in eq. (5.2) suggest that we multiply and divide this expression by $\langle 1|(3+4)|2\rangle$, and then commute the pair of ℓ_2s toward each other in the following string:

$$\begin{aligned} &\langle 5|\ell_2|1\rangle\langle 1|(3+4)|2\rangle\langle 2|\ell_2|4\rangle \\ &= \langle 5|\ell_2 1(2+3+4)2\ell_2|4\rangle \\ &= \ell_1^2\,\langle 5|(3+4)|2\rangle\langle 2|\ell_2|4\rangle + \ell_3^2\,\langle 51\rangle\,[1|\ell_2(2+3)|4] + \langle 51\rangle\,[24]\,\langle 1^+|\ell_2(2+3+4)\ell_2|2^+\rangle \\ &= \ell_1^2\,\langle 5|(3+4)|2\rangle\langle 2|\ell_2|4\rangle + \ell_3^2\,\langle 51\rangle\,[1|\ell_2(2+3)|4] + \langle 51\rangle\,[24]\,t_{234}\,\langle 2|\ell_2|1\rangle\,. \end{aligned} \tag{5.3}$$

The first two terms in eq. (5.3), after multiplication by $\langle 5|\ell_2|4\rangle$, are triangle integrals with two external masses and two powers of the loop momenta in the numerator. They can be handled straightforwardly by Feynman parametrization. The third term is still a quadratic box integral (i.e. two loop momenta in the numerator), but it can be reduced further, using a second factor of $\langle 1|(3+4)|2\rangle$ and a Fierz rearrangement (since $\ell_2^2 = 0$),

$$\langle 1|(3+4)|2\rangle\langle 5|\ell_2|4\rangle\langle 2|\ell_2|1\rangle = \langle 5|\ell_2|1\rangle\langle 1|(3+4)|2\rangle\langle 2|\ell_2|4\rangle\,, \tag{5.4}$$

and then following exactly the same steps as in eq. (5.3).

Now we have five terms, only one of which is a box integral, and that one is merely linear in the loop momenta,

$$I_4^{(4)}[\langle 2|\ell_2|1\rangle] \; = \; -\frac{\mathrm{Ls}_{-1}^{2\mathrm{m}h}\big(s_{12}, t_{234}; s_{34}, s_{56}\big)}{\langle 1|(3+4)|2\rangle} \; + \; (\text{terms lacking a } t_{234} \text{ cut}). \tag{5.5}$$

Finally, we perform the four triangle integrals, dropping terms with no t_{234} branch cut, and assemble the pieces, thus obtaining all terms in eqs. (10.10) and (10.11) for $A_6^{sc}(1_q^+, 2_{\bar{q}}^-, 3^-, 4^+, 5_{\bar{e}}^-, 6_e^+)$ that have logarithms or dilogarithms with t_{234} in the argument. (Terms representing possible scalar one- and two-mass triangle contributions are proportional to $(-t_{234})^{-\epsilon}/\epsilon^2$ and therefore can be inferred from the known structure of the poles in ϵ [7,47]. This can be used as a check on the cut calculation, or to save labor.)

The second cut we consider is the cut in the t_{412} channel of the amplitude $A_6^{sc}(1_q^+, 2^+, 3_{\bar{q}}^-, 4^-, 5_{\bar{e}}^-, 6_e^+)$. In this case leg 4 has to be adjacent to leg 1, as in the integral $\tilde{I}_5$ in fig. 7. For this configuration we label the propagator momenta around the loop, starting just clockwise of the 5–6 lepton pair, by $\ell_5', \ell_1, \ell_2, \ell_3, \ell_4$. Now the required product of tree amplitudes, to be integrated over the two-body phase space for $\ell_5' + (-\ell_3)$, is

$$\begin{aligned} P_{412} &= A_5^{\text{tree}}((\ell_5')_q^-, (-\ell_3)_s, 3_{\bar{q}}^-, 5_{\bar{e}}^-, 6_e^+) \times A_5^{\text{tree}}((-\ell_5')_{\bar{q}}^+, 4^-, 1_q^+, 2^+, (\ell_3)_s) \\ &= \pm \frac{[\ell_3 6]^2}{[\ell_5' \ell_3]\,[\ell_3 3]\,[56]} \times \frac{\langle \ell_3 4\rangle^2}{\langle 12\rangle\,\langle 2\ell_3\rangle\,\langle \ell_3 \ell_5'\rangle} \,. \end{aligned} \tag{5.6}$$

Fortunately, this cut can be obtained from the first one we evaluated. If we multiply eq. (5.2) by $[34]\,/\,[24]$, and then perform the following 'flip' (pairwise permutation plus complex conjugation of all spinor products)

$$\ell_1 \leftrightarrow \ell_4, \quad \ell_2 \leftrightarrow \ell_3, \quad \ell_5 \leftrightarrow \ell_5', \quad 1 \leftrightarrow 3, \quad 5 \leftrightarrow 6, \quad \langle ab\rangle \leftrightarrow [ab] \,, \tag{5.7}$$

we recover eq. (5.6).

This second example illustrates the principle of *recycling* cuts. One can save a lot of effort by identifying different cuts that are actually the same up to permutations and trivial overall factors. In many cases, it may not be possible to obtain an entire cut in this way, but portions of it may be recyclable. The 'master functions' defined in section 7, which enter three different amplitudes, are the most complicated contributions we have been able to recycle, but there are several other instances as well.

5.2 Rational Function Reconstruction

In Section 3.2 we described the general factorization properties of amplitudes, and how that information can be used to determine the rational function parts of amplitudes. As an example, we explicitly construct the rational function terms proportional to n_f in the leading-color helicity amplitude, which are given by $A_6^{s,f}(1_q^+,2^+,3^+,4_{\bar q}^-)$ in eq. (8.2), and outline the construction of the scalar (non-cut-constructible) piece (V^{sc}, F^{sc}) for the same helicity configuration, eq. (8.7).

The first step is to account for possible spurious singularities in the rational function terms. These singularities can always be identified after all cuts have been calculated, because they have to cancel against terms containing logarithms and dilogarithms. For $A_6^{s,f}(1_q^+,2^+,3^+,4_{\bar q}^-)$, the only possible cut, that in the s_{23} channel, vanishes identically, because the tree-level $\bar q^- q^+ g^+ g^+$ and $s s g^+ g^+$ amplitudes vanish for massless quarks and scalars. So this term is purely a rational function, and therefore can have no spurious singularities, and we can immediately focus on the physical (multi-particle and collinear) singularities.

It is usually convenient to match the multi-particle behavior — here, the limits $t_{123}, t_{234} \to 0$ — before attacking the collinear poles. The residue of a t_{ijk} pole in a one-loop six-point amplitude receives three contributions:† C_1 from a one-loop four-point amplitude multiplying a tree-level four-point amplitude; C_2 where the loop amplitude is replaced by the corresponding tree amplitude, and the tree by the loop; and C_3 is associated with a loop correction to the intermediate propagator, multiplied by the two tree amplitudes. In the case of $e^+ e^- \to$ four partons, two of the three contributions are easy to describe in general, because one of the two four-point amplitudes is the relatively simple $e^+ e^- \to q\bar q$ process. Let us define the C_2 contribution to be where the $e^+ e^- \to q\bar q$ amplitude is a loop amplitude. Then for the n_f-dependent term, C_2 and C_3 both vanish. Also, for the scalar (sc) part of an amplitude, $C_2 = \frac{1}{2} \times \lim A_6^{\rm tree}$, and $C_3 = (-1) \times \lim A_6^{\rm tree}$, where $\lim A_6^{\rm tree}$ is the appropriate $t_{ijk} \to 0$ limit of the six-point tree amplitude.

For the present $(1_q^+,2^+,3^+,4_{\bar q}^-)$ helicity configuration, the tree amplitude (8.4) has no t_{ijk} poles, hence only the C_1 contribution survives. Using a calculation of the n_f terms in the $\bar q q g g$ one-loop four-point amplitudes [15,19], we have

$$\begin{aligned}
\frac{A_6^s(1_q^+,2^+,3^+,4_{\bar q}^-)}{i\,c_\Gamma} &\sim \frac{\langle 45\rangle\,[6P]}{s_{56}} \times \frac{1}{t_{123}} \times \frac{1}{3}\frac{s_{12}}{s_{23}}\frac{\langle P|1|3\rangle}{\langle 12\rangle\,\langle 23\rangle} \qquad (P \equiv 1+2+3), \qquad \text{as } t_{123}\to 0,\\
&\sim \frac{-[16]\,\langle 5P\rangle}{s_{56}} \times \frac{1}{t_{234}} \times \frac{1}{3}\frac{s_{34}}{s_{23}}\frac{\langle 4|P|3\rangle}{\langle P2\rangle\,\langle 23\rangle} \qquad (P \equiv 2+3+4), \quad \text{as } t_{234}\to 0.
\end{aligned} \tag{5.8}$$

Similarly, the limits of the scalar pieces are found to be

$$\begin{aligned}
\frac{A_6^{sc}(1_q^+,2^+,3^+,4_{\bar q}^-)}{i\,c_\Gamma} &\sim \frac{\langle 45\rangle\,[6P]}{s_{56}} \times \frac{1}{t_{123}} \times \left[\frac{1}{2} + \frac{1}{3}\frac{s_{12}}{s_{23}}\right]\frac{\langle P|1|3\rangle}{\langle 12\rangle\,\langle 23\rangle} \qquad (P \equiv 1+2+3), \qquad \text{as } t_{123}\to 0,\\
&\sim \frac{-[16]\,\langle 5P\rangle}{s_{56}} \times \frac{1}{t_{234}} \times \left[\frac{1}{2} + \frac{1}{3}\frac{s_{34}}{s_{23}}\right]\frac{\langle 4|P|3\rangle}{\langle P2\rangle\,\langle 23\rangle} \qquad (P \equiv 2+3+4), \quad \text{as } t_{234}\to 0.
\end{aligned} \tag{5.9}$$

† In the presence of infrared divergences, such as those due to virtual gluons, the one-loop factorization is a bit more subtle, but still has a universal form [30].

Next we need to modify the terms in eq. (5.8) to improve their collinear limits, while preserving their t_{ijk} limits. For example, we can improve the $k_2 \parallel k_3$ limit of the t_{123} limit in (5.8) as follows:

$$\begin{aligned}&\frac{1}{3}\frac{\langle 45\rangle\,[6|(1+2+3)1|3]s_{12}}{\langle 12\rangle\,\langle 23\rangle\;s_{23}\,s_{56}\,t_{123}} = \frac{1}{3}\frac{\langle 45\rangle\,[6|(1+2+3)1|3]\,[12]}{\langle 23\rangle^2\,[23]\;s_{56}\,t_{123}}\\ &= -\frac{1}{3}\frac{\langle 45\rangle\,[6|(1+2+3)2|3]\,[12]}{\langle 23\rangle^2\,[23]\;s_{56}\,t_{123}} + \cdots = \frac{1}{3}\frac{\langle 45\rangle\,[6|(1+2+3)2|1]}{\langle 23\rangle^2\;s_{56}\,t_{123}} + \cdots\,,\end{aligned} \tag{5.10}$$

where nonsingular terms in the $t_{123} \to 0$ limit are represented by '$\cdots$'. A similar manipulation of the t_{234} limit, using also $s_{2P} = -s_{34}$ in that limit, gives

$$-\frac{1}{3}\frac{[16]\,\langle 5|(2+3+4)|2\rangle\langle 4|2|3\rangle}{\langle 23\rangle\;s_{23}\,s_{56}\,t_{234}} + \cdots = -\frac{1}{3}\frac{[16]\,\langle 5|(2+3+4)2|4\rangle}{\langle 23\rangle^2\;s_{56}\,t_{234}} + \cdots\,. \tag{5.11}$$

Thus a first guess for $A_6^s(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-)/(i\,c_\Gamma)$ would be

$$F^s_{\rm guess} = \frac{1}{3}\frac{1}{\langle 23\rangle^2\;s_{56}}\left[\frac{\langle 45\rangle\,[6|(1+2+3)2|1]}{t_{123}} - \frac{[16]\,\langle 5|(2+3+4)2|4\rangle}{t_{234}}\right]\,. \tag{5.12}$$

It might appear that the $k_2 \parallel k_3$ and $k_5 \parallel k_6$ collinear limits of (5.12) both need to be improved, since the expected limits are $\sim 1/\sqrt{s_{ij}}$, not $1/s_{ij}$. But in fact there is a cancellation between the two terms in eq. (5.12) in both limits. In the $k_2 \parallel k_3$ limit, using the identity

$$\begin{aligned}s_{12}\,t_{234} - s_{24}\,t_{123} &= s_{12}\,s_{34} - s_{24}\,s_{13} + s_{23}\,s_{14} + \mathcal{O}(s_{23}) = {\rm tr}[2341] + \mathcal{O}(s_{23})\\ &= \langle 23\rangle\,[34]\,\langle 41\rangle\,[12] + [23]\,\langle 34\rangle\,[41]\,\langle 12\rangle + \mathcal{O}(s_{23}),\end{aligned} \tag{5.13}$$

and letting $P \equiv 2+3$, we have

$$\begin{aligned}F^s_{\rm guess} &\sim \frac{1}{3}\frac{1}{\langle 23\rangle^2\;s_{56}}\left[\frac{\langle 45\rangle\,[61]}{t_{123}t_{234}}(\langle 23\rangle\,[34]\,\langle 41\rangle\,[12] + [23]\,\langle 34\rangle\,[41]\,\langle 12\rangle)\right.\\ &\qquad\left. - \langle 23\rangle\,\frac{\langle 45\rangle\,[63]\,[21]}{t_{123}} + [23]\,\frac{[16]\,\langle 53\rangle\,\langle 24\rangle}{t_{234}}\right]\\ &\sim \frac{1}{3}\frac{\sqrt{z(1-z)}}{\langle 23\rangle^2\;s_{56}}\left[-\langle 23\rangle\,\langle 45\rangle\,\frac{\langle 4|1|6\rangle + \langle 4|P|6\rangle}{\langle 1P\rangle\,\langle P4\rangle} + [23]\,[16]\,\frac{\langle 5|4|1\rangle + \langle 5|P|1\rangle}{[1P]\,[P4]}\right]\\ &\sim \frac{1}{3}\frac{\sqrt{z(1-z)}}{\langle 23\rangle^2\;s_{56}}\left[\langle 23\rangle\,\frac{\langle 45\rangle^2\,[56]}{\langle 1P\rangle\,\langle P4\rangle} + [23]\,\frac{[16]^2\,\langle 56\rangle}{[1P]\,[P4]}\right]\\ &\sim \frac{1}{3}\frac{\sqrt{z(1-z)}}{\langle 23\rangle} \times \frac{-\langle 45\rangle^2}{\langle 1P\rangle\,\langle P4\rangle\,\langle 56\rangle} + \frac{-1}{3}\frac{\sqrt{z(1-z)}\,[23]}{\langle 23\rangle^2} \times \frac{[16]^2}{[1P]\,[P4]\,[56]}\,, \qquad k_2 \parallel k_3.\end{aligned} \tag{5.14}$$

Not only does the leading $1/|s_{23}|$ term cancel in eq. (5.14), but the next terms in the expansion give precisely the desired limit, corresponding to the n_f-dependent piece of the

$${\rm Split}^{1-{\rm loop}}_{\mp}(2^+, 3^+) \times A_5^{\rm tree}(1_q^+, P^\pm, 4_{\bar{q}}^-, 5_{\bar{e}}^-, 6_e^+) \tag{5.15}$$

terms in eq. (3.3). The loop splitting functions are given in ref. [22] and the five-point $e^+ e^- \to \bar{q}qg$ amplitudes in appendix IV. Notice that there are no n_f terms in $A_5^{1-\text{loop}}$, hence no $\text{Split}^{\text{tree}} \times A_5^{1-\text{loop}}$ terms contribute here. Similar manipulations show that the $k_5 \parallel k_6$ limit of F^s_{guess} is also precisely correct,

$$F^s_{\text{guess}} \sim \frac{1-z}{[56]} \times \frac{1}{3} \frac{[23]\,\langle 24\rangle\,\langle 34\rangle}{\langle 23\rangle^2\,\langle 4P\rangle\,\langle P1\rangle} + \frac{z}{\langle 56\rangle} \times \frac{1}{3} \frac{[12]\,[13]}{[4P]\,[P1]\,\langle 23\rangle}\,, \qquad k_5 \parallel k_6. \tag{5.16}$$

Here there are only $\text{Split}^{\text{tree}} \times A_5^{1-\text{loop}}$ terms, with the relevant loop amplitudes obtainable from ref. [19]. All singular collinear limits of F^s_{guess} have now been verified, so we expect the result to be correct as is. A simple identity shows that indeed $F^s_{\text{guess}} = A_6^s(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-)/(i\,c_\Gamma)$, the result given in eq. (8.2).

A slightly more complicated example, which we only summarize here, is the rational function terms in the scalar piece (V^{sc}, F^{sc}) for the same helicity configuration, $A_6^{sc}(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-)$, as given in eq. (8.7). By comparing the t_{ijk} limits (5.8) and (5.9), and similarly the collinear limits, it is easy to see that there must be a term of the form A_6^s in A_6^{sc}. As for the remaining terms, the first step is again to account for possible spurious singularities. Assuming that all the cuts have previously been calculated, the $\ln(\frac{-s_{ij}}{-t_{234}})/(s_{ij} - t_{234})^2$ terms contained in the L_1 functions in eq. (8.7) are already known. But the L_1 functions are designed to cancel the spurious $(s_{ij} - t_{234})$ behavior between logarithms and rational functions. Hence by completing the $\ln(\frac{-s_{ij}}{-t_{234}})/(s_{ij} - t_{234})^2$ terms into $\text{L}_1(\frac{-s_{ij}}{-t_{234}})/t_{234}^2$ functions, we ensure that the remaining rational function terms are free of $1/(s_{ij} - t_{234})$ poles. Since there are no other non-rational-function terms in V^{sc} or F^{sc}, there are no other sources of spurious poles, and we next turn to the physical singularities.

Using the t-channel limits (5.9), and manipulations similar to those leading to eq. (5.12), we arrive at an ansatz for what has to be added to the L_1 terms,

$$F^{sc}_{\text{guess}} = \frac{1}{2}\left[\frac{\langle 45\rangle\,[6|(2+3)1|3]}{\langle 12\rangle\,\langle 23\rangle\; t_{123}\, s_{56}} + \frac{[23]\,[16]\,\langle 5|(2+4)|3\rangle}{\langle 23\rangle\,[34]\; t_{234}\, s_{56}} - \frac{\langle 5|2|3\rangle^2}{\langle 12\rangle\,\langle 23\rangle\,[34]\,\langle 56\rangle\; t_{234}} + \frac{\langle 4|(2+3)|6\rangle^2}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,[56]\; t_{234}}\right]. \tag{5.17}$$

The first two terms reproduce (5.9), while the last two terms are needed to cancel off t_{234} poles in the L_1 terms.

This time, however, the $k_5 \parallel k_6$ limit,

$$\begin{aligned} F^{sc}_{\text{guess}} &\sim -\frac{1}{2}\left[\frac{\langle 45\rangle\,[64]\,\langle 41\rangle\,[13]}{\langle 12\rangle\,\langle 23\rangle\; t_{123}\, s_{56}} + \frac{[23]\,[16]\,\langle 51\rangle\,[13]}{\langle 23\rangle\,[34]\; t_{234}\, s_{56}}\right] \\ &\sim -\frac{\sqrt{z(1-z)}}{2} \frac{[13]}{\langle 23\rangle\; s_{56}}\left[\frac{\langle 4P\rangle\,[P4]\,\langle 41\rangle}{\langle 12\rangle\; s_{4P}} + \frac{[23]\,[1P]\,\langle P1\rangle}{[34]\; s_{1P}}\right] \\ &\sim \frac{\sqrt{z(1-z)}}{2} \frac{[3P]\,\langle P1\rangle\,[13]}{\langle 12\rangle\,\langle 23\rangle\,[34]\; s_{56}}\,, \end{aligned} \tag{5.18}$$

still has to be improved, for example by adding

$$\delta F_1^{sc} \equiv \frac{1}{2} \frac{[36]\,\langle 5|(2+4)|3\rangle}{\langle 12\rangle\,\langle 23\rangle\,[34]\;s_{56}} \tag{5.19}$$

to $F^{sc}_{\rm guess}$. At this stage, the polynomial ansatz formed by the $\rm L_1$ terms, $F^{sc}_{\rm guess}$ and δF_1^{sc} has only $1/\sqrt{s_{ij}}$ collinear singularities. One can now systematically add additional terms to match the known $k_1 \parallel k_2$, $k_2 \parallel k_3$, $k_3 \parallel k_4$, and $k_5 \parallel k_6$ limits. (One can also rewrite the answer in a form where the $1/\sqrt{s_{56}}$ behavior is manifest, as in eq. (8.7).)

Other helicity amplitudes may possess more spurious singularities and/or physical singularities. For these cases the determination of the rational function terms becomes somewhat more involved, but the basic principles remain as illustrated above.

5.3 Numerical Simplification

In order to illustrate the numerical simplification technique outlined in section 4.3, we consider the particular example of the coefficient $c^{\rm 3m}$ of the three-mass triangle integral $I_3^{\rm 3m}\,(s_{12}, s_{34}, s_{56})$ in the axial-vector fermion-loop contribution $A_6^{\rm ax}(1_q^+, 2_{\bar q}^-, 3^+, 4^-)$, eq. (11.9). Note that the three-mass triangle contribution is contained in the common function $C^{\rm ax}$ defined in eq. (11.8), plus its image under the symmetry operation $\rm flip_2$ defined in eq. (6.8). Thus we determine simultaneously the corresponding contribution to $A_6^{\rm ax}(1_q^+, 2_{\bar q}^-, 3^-, 4^+)$, eq. (11.10).

The first step in the technique is to write down terms reproducing all of the spurious and physical singularities in the various channels. One subtlety is that a term reproducing a singularity in one channel may be too singular, or otherwise have the wrong type of singularity, in another channel. In this case the term will have to be 'improved'.

In the present example of $c^{\rm 3m}$, we first write down a term $c_1^{\rm 3m}$ which reproduces the known $\langle 3|(1+2)|4\rangle \to 0$ behavior of this coefficient. To do this we use eq. (4.22) and the coefficient of the hard-two-mass box function in $C^{\rm ax}$ (which we assume has already been obtained via the t_{123} cut), to get

$$\begin{aligned} c^{\rm 3m}_{1,\rm guess} &= \frac{1}{2} \frac{\langle 2|(1+3)|4\rangle^2 \langle 3|(1+2)|6\rangle^2 - \langle 23\rangle^2\,[46]^2\; t_{123}^2}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle^4} \\ &\qquad\qquad \times \frac{\langle 3|(1+2)|4\rangle \langle 4|(1+2)|3\rangle (t_{123}-t_{124}) s_{12} s_{56}}{t_{123}^2 \Delta_3} + \cdots \\ &= -\frac{\langle 2|(1+3)|6\rangle\,[12]\,\langle 23\rangle\,[46]\,\langle 56\rangle\;(t_{123}-t_{124})\,\langle 4|(1+2)|3\rangle}{t_{123}\,\langle 3|(1+2)|4\rangle^2 \Delta_3} + \cdots\,. \end{aligned} \tag{5.20}$$

In the second line of eq. (5.20) we used the spinor identity

$$\langle 2|(1+3)|4\rangle \langle 3|(1+2)|6\rangle = \langle 3|(1+2)|4\rangle \langle 2|(1+3)|6\rangle - \langle 23\rangle\,[46]\;t_{123}\,, \tag{5.21}$$

and dropped all but the leading terms as $\langle 3|(1+2)|4\rangle \to 0$. As it stands, $c^{3\mathrm{m}}_{1,\mathrm{guess}}$ contains a pole in t_{123}, but an identity similar to (5.21) removes the pole while preserving the leading $\langle 3|(1+2)|4\rangle \to 0$ behavior, and so we take the first singular term to be

$$c_1^{3\mathrm{m}} = \frac{[14]\,\langle 35\rangle\;(t_{123}-t_{124})\,\langle 4|(1+2)|3\rangle\langle 2|(1+3)|6\rangle}{\langle 3|(1+2)|4\rangle^2\,\Delta_3}\,. \tag{5.22}$$

Actually $c^{3\mathrm{m}}$ should contain a pole in t_{123} (but with a different structure than that found in eq. (5.20)); it has to cancel the pole in the explicit formula for the hard two-mass box function, eq. (II.3). Thus a term reproducing that pole is given by

$$c^{3\mathrm{m}}_{2,\mathrm{guess}} = \frac{1}{2}\frac{s_{12}s_{56}}{t_{123}} \times \frac{\langle 2|(1+3)|6\rangle^2}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle^2} + \cdots\,, \tag{5.23}$$

where we used identity (5.21) again and dropped nonsingular terms as $t_{123} \to 0$. We still have to remove the leading singularity as $\langle 3|(1+2)|4\rangle \to 0$ in this term, which we can do using

$$\langle 1^+|2(1+3)6|5^+\rangle = -\langle 1^+|3(1+2+3)6|5^+\rangle + \cdots = \langle 1^+|3(1+2)4|5^+\rangle + \cdots\,, \tag{5.24}$$

as $t_{123} \to 0$, thus obtaining

$$c_2^{3\mathrm{m}} = -\frac{1}{2}\frac{[13]\,\langle 45\rangle\,\langle 2|(1+3)|6\rangle}{t_{123}\,\langle 3|(1+2)|4\rangle} \tag{5.25}$$

as our second inferred singular term in $c^{3\mathrm{m}}$.

As mentioned above, the leading singular terms as $\Delta_3 \to 0$ require an explicit calculation of the s_{12}-s_{34}-s_{56} triple cut, but fortunately only the leading loop-momentum terms in this cut have to be retained, yielding

$$c_3^{3\mathrm{m}} = -3\,\frac{\delta_{34}\left(\langle 5|2|1\rangle\,\delta_{12} - \langle 5|6|1\rangle\,\delta_{56}\right)\langle 4|(1+2)|3\rangle\langle 2|(1+3)|6\rangle}{\langle 3|(1+2)|4\rangle\,\Delta_3^2}\,. \tag{5.26}$$

The remaining terms can have at most a $1/\Delta_3$ singularity. Furthermore, there are no collinear singularities ($k_i \parallel k_j$) in the three-mass triangle coefficient. (This feature is general, as long as the adjacent two-mass box function $\mathrm{Ls}_{-1}^{2\mathrm{m}h}$ is used, and not $\widetilde{\mathrm{Ls}}_{-1}^{2\mathrm{m}h}$.)

At this stage we have identified enough of the singularities of $c^{3\mathrm{m}}$ to write an ansatz for the remaining, less singular terms. The unknown coefficients in the ansatz can then be determined by comparing it to a numerical evaluation of $c^{3\mathrm{m}}$ at a number of phase-space points, equal to the number of unknown coefficients. The full three-mass triangle coefficient is symmetric under both the operations flip_2 and flip_3, defined in eqs. (6.8) and (6.9), and the full singular terms include also the images under flip_2 of the above terms. (They cancel against terms containing the other box function, $\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{34}, t_{124}; s_{12}, s_{56})$.) Thus we may write

$$c^{3\mathrm{m}} = c_1^{3\mathrm{m}} + c_2^{3\mathrm{m}} + c_3^{3\mathrm{m}} + \frac{p_1}{\Delta_3} + p_2\,\frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle\,\Delta_3} \;+\; \mathrm{flip}_2\,, \tag{5.27}$$

where flip_2 is to be applied to all preceding terms. In eq. (5.27) we have explicitly identified all denominator factors; i.e., p_1 and p_2 can only be linear combinations of products of spinor strings. They are further restricted by the observation that any helicity amplitude can be assigned a definite *phase weight* — namely, an integer n_i for each external leg i, which is equal to twice the helicity of that leg. For $A_6^{\text{ax}}(1_q^+, 2_{\bar{q}}^-, 3^+, 4^-, 5_{\bar{e}}^-, 6_e^+)$, and hence for C^{ax}, the phase weight is $\{1, -1, 2, -2, -1, 1\}$, the six entries corresponding to each of the six legs. The phase weight of an expression A may also be calculated by making the substitution $\langle ij\rangle \to \lambda_i^{-1}\lambda_j^{-1}\langle ij\rangle$, $[ij] \to \lambda_i\lambda_j\,[ij]$, and reading the n_i off of the resulting transformation $A \to (\prod_i \lambda_i^{n_i}) \times A$.

Since Δ_3 carries no phase weight, the phase weight of p_1 must match that of C^{ax}, namely $\{1, -1, 2, -2, -1, 1\}$. In terms of spinor string terminations, it must have the form

$$[1,\ \langle 2,\ \big([3\big)^2,\ \big(\langle 4\big)^2,\ \langle 5,\ [6\ . \tag{5.28}$$

On the other hand, dimensional analysis implies that the mass dimension of p_1 is 4. (The mass dimension of C^{ax}, like that of any six-point amplitude, is -2; but $I_3^{3\text{m}}$ also has dimension -2, and Δ_3 has dimension 4.) Thus there are exactly four spinor products $(\langle ij\rangle, [kl])$ in p_1, all of whose arguments are accounted for in eq. (5.28). Since $\langle ii\rangle = [ii] = 0$, we see that p_1 has just one independent term,

$$p_1 = a_0\ [13]\,[36]\,\langle 24\rangle\,\langle 45\rangle\ , \tag{5.29}$$

where a_0 is a constant.

Similarly, p_2 also has dimension 4, but its phase weight is $\{1, -1, 0, 0, -1, 1\}$, corresponding to the spinor string terminations

$$[1,\ \langle 2,\ \langle 5,\ [6\ . \tag{5.30}$$

Equation (5.30) allows terms of the form $\langle 5|i|6\rangle\langle 2|j|1\rangle$, $\langle 5|i|1\rangle\langle 2|j|6\rangle$, $\langle 52\rangle\,[1|ij|6]$, $\langle 5|ij|2\rangle\,[16]$ and $\langle 52\rangle\,[16]\,s_{ij}$, where i and j are arbitrary legs. Not all such terms are independent. A Fierz identity lets us eliminate all terms of the first type in favor of terms of the second and fourth types. The symmetry flip_2 relates terms of the third and fourth types, while combinations of these terms that are symmetrized in $i \leftrightarrow j$ are already included in the fifth type of terms. Using momentum conservation and the two flip symmetries, we can reduce the number of independent terms in p_2 to just seven,

$$\begin{aligned} p_2 = a_1\,\langle 5|3|1\rangle\langle 2|3|6\rangle + a_2\,\langle 5|3|1\rangle\langle 2|4|6\rangle + a_3\,\langle 5|31|2\rangle\,[16] \\ + \langle 52\rangle\,[16]\,(a_4\,s_{12} + a_5\,s_{23} + a_6\,s_{24} + a_7\,s_{25})\ +\ \text{flip}_3\,. \end{aligned} \tag{5.31}$$

Thus it suffices to evaluate $c^{3\text{m}}$ numerically at eight phase-space points, and solve the resulting linear equations for $a_0, a_1, \ldots, a_7$, which should be simple rational numbers. We find

$$\begin{aligned} p_1 &= -\,[13]\,\langle 45\rangle\,\langle 24\rangle\,[36]\ , \\ p_2 &= -\frac{3}{2}\big(\langle 5|2|1\rangle\langle 2|1|6\rangle + \langle 5|6|1\rangle\langle 2|5|6\rangle - \langle 5|3|1\rangle\langle 2|4|6\rangle - \langle 5|4|1\rangle\langle 2|3|6\rangle\big)\ . \end{aligned} \tag{5.32}$$

(Evaluation at additional phase-space points may be used to confirm this answer.) If we combine these terms with the singular terms $c_i^{3\mathrm{m}}$ according to eq. (5.27), we obtain the coefficient of the three-mass triangle given in eq. (11.9), via the function C^{ax} defined in eq. (11.8).

This numerical simplification procedure may be somewhat more involved for other coefficient functions, if there are more spurious and physical singularities to account for, but the principle is the same. In practice we found it convenient to obtain an analytic, though often very complicated, representation of the desired answer, before attempting to simplify it numerically; of course, an analytic representation is not strictly necessary. It is also not necessary to eliminate all the linear dependences between terms in the ansatz before trying to solve for the unknowns.

6. General Form of Primitive Amplitudes

The simple structure of the poles in ϵ of the primitive amplitudes [7,47] permits us to decompose them further into divergent (V) and finite (F) pieces,

$$A_6^{1-\mathrm{loop}} = c_\Gamma \left[A_6^{\mathrm{tree}} V + i\, F \right] , \tag{6.1}$$

where the $A_6^{1-\mathrm{loop}}$ are any of the primitive amplitudes and the prefactor is

$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} . \tag{6.2}$$

The tree amplitudes [5] are denoted by A_6^{tree}. (The $A_6^{s,f,t}$ amplitudes are rather simple so we will not bother with this additional decomposition for them.)

The amplitudes we present are bare ones, i.e., no ultraviolet subtraction has been performed. To obtain the renormalized amplitudes in an $\overline{\mathrm{MS}}$-type subtraction scheme, one should subtract the quantity

$$c_\Gamma N_c\, g^2 \left[\frac{1}{\epsilon} \left(\frac{11}{3} - \frac{2}{3}\frac{n_f}{N_c} - \frac{1}{3}\frac{n_s}{N_c} \right) \right] \mathcal{A}_6^{\mathrm{tree}} , \tag{6.3}$$

from the amplitudes (2.9) and (2.11).

We quote the results in the four-dimensional helicity (FDH) scheme [43,44], since this scheme is convenient for performing computations in the helicity formulation. The conversion between the various schemes is discussed in refs. [43,15]. (The more conventional regularization schemes alter the number of gluon polarizations and are therefore not natural when using a helicity basis.) The $e^+ e^- \to \bar{q}qgg$ amplitudes may be converted to the 't Hooft-Veltman (HV) scheme by adding the quantity

$$-\frac{1}{2} c_\Gamma N_c\, g^2 \left(1 - \frac{1}{N_c^2} \right) \mathcal{A}_6^{\mathrm{tree}} \tag{6.4}$$

to the amplitude (2.9) and changing the coupling constant from the non-standard $\alpha_{\overline{DR}}$ to the standard $\alpha_{\overline{MS}}$. Similarly, the $e^+ e^- \to \bar{q}q\bar{Q}Q$ amplitudes may be converted by adding the quantity

$$-c_\Gamma N_c\, g^2 \left(\frac{2}{3} - \frac{1}{N_c^2}\right) \mathcal{A}_6^{\rm tree} \tag{6.5}$$

to the amplitude (2.11) and making the same coupling constant conversion. The conversion of the HV scheme to the conventional dimensional regularization (CDR) scheme is accomplished by accounting for the fact that in the HV scheme observed gluons (at the partonic level) are in four-dimensions but in the CDR scheme they are in $(4-2\epsilon)$ dimensions. This conversion is rather simple as it involves only the coefficients of the poles in ϵ which are proportional to the tree amplitudes. In the final matrix elements squared (e.g., eq. (2.12)) one simply replaces all terms originating from the singular parts of eqs. (6.1) and (6.3) with their values in the CDR scheme. (Further details may be found in ref. [15].) Using this conversion recipe, our results (integrated over lepton orientation) agree numerically [48] with those reported in refs. [13,14].

In order to present the amplitudes compactly, we define, in addition to the previously-defined spinor-strings, the following combinations of kinematic variables, related to the three-mass triangle integrals that appear,

$$\begin{gathered}
\delta_{12} = s_{12} - s_{34} - s_{56}\,, \qquad \delta_{34} = s_{34} - s_{56} - s_{12}\,, \qquad \delta_{56} = s_{56} - s_{12} - s_{34}\,, \\
\Delta_3 = s_{12}^2 + s_{34}^2 + s_{56}^2 - 2s_{12}s_{34} - 2s_{34}s_{56} - 2s_{56}s_{12}\,, \\
\tilde\delta_{14} = s_{14} - s_{23} - s_{56}\,, \qquad \tilde\delta_{23} = s_{23} - s_{56} - s_{14}\,, \qquad \tilde\delta_{56} = s_{56} - s_{14} - s_{23}\,, \\
\tilde\Delta_3 = s_{14}^2 + s_{23}^2 + s_{56}^2 - 2s_{14}s_{23} - 2s_{23}s_{56} - 2s_{56}s_{14}\,.
\end{gathered} \tag{6.6}$$

Certain 'flip' symmetries relate either various terms, or various amplitudes. For later convenience, we collect the definitions of these symmetry operations here:

$$\text{flip}_1:\ 1 \leftrightarrow 4\,, \quad 2 \leftrightarrow 3\,, \quad 5 \leftrightarrow 6\,, \quad \langle ab\rangle \leftrightarrow [ab]\,, \tag{6.7}$$

$$\text{flip}_2:\ 1 \leftrightarrow 2\,, \quad 3 \leftrightarrow 4\,, \quad 5 \leftrightarrow 6\,, \quad \langle ab\rangle \leftrightarrow [ab]\,, \tag{6.8}$$

$$\text{flip}_3:\ 1 \leftrightarrow 5\,, \quad 2 \leftrightarrow 6\,, \quad 3 \leftrightarrow 4\,, \quad \langle ab\rangle \leftrightarrow [ab]\,, \tag{6.9}$$

$$\text{flip}_4:\ 1 \leftrightarrow 3\,, \quad 5 \leftrightarrow 6\,, \quad \langle ab\rangle \leftrightarrow [ab]\,, \tag{6.10}$$

In general, $\langle ab\rangle \leftrightarrow [ab]$ denotes complex conjugation of *all* spinor products, including the various strings defined in eq. (2.2), $\langle i|j|l\rangle \leftrightarrow \langle l|j|i\rangle$, $\langle i|(l+m)|j\rangle \leftrightarrow [i|(l+m)|j]$, and so forth. We also define the exchange operations

$$\begin{aligned}
&\text{exch}_{16,25}:\ 1 \leftrightarrow 6\,, \quad 2 \leftrightarrow 5\,, \\
&\text{exch}_{34}:\ 3 \leftrightarrow 4\,.
\end{aligned} \tag{6.11}$$

7. Master Functions

As mentioned in section 5.1, many of the cuts (or portions of cuts) for different amplitudes are related to each other by simple permutations and overall spinor-product prefactors. In particular, one 'master function' $M_1(1,2,3,4)$ enters the the cut-constructible part of one of the leading-color primitive amplitudes, as well as two of the subleading-color ones. Two additional 'master functions', $M_2(1,2,3,4)$ and $M_3(1,2,3,4)$, enter the scalar parts of the three subleading-color amplitudes with opposite gluon helicities. We have decomposed the latter two master functions further, extracting M_{2a} and M_{3a}, respectively, because these pieces appear separately as well.

The cut-constructible master function is given by

$$
\begin{aligned}
&M_1(1,2,3,4) = \\
&\quad \frac{[13]}{\langle 1|(3+4)|2\rangle\langle 3|(1+2)|4\rangle\,\Delta_3}\Big(2\,\langle 12\rangle\,\langle 5|2|6\rangle(t_{123}\,\delta_{12}+s_{56}\,\delta_{56})+2\,\langle 12\rangle\,\langle 15\rangle\,\langle 4|5|6\rangle\big([14]\;\delta_{56}-2\,[1|23|4]\big) \\
&\qquad\qquad + s_{56}\,\langle 24\rangle\,\big(\langle 15\rangle\,[46]\;\delta_{56}-2\,\langle 1|2|6\rangle\langle 5|3|4\rangle\big)\Big)\,I_3^{3\mathrm{m}}(s_{12},s_{34},s_{56}) \\
&\quad + \frac{2}{\langle 1|(3+4)|2\rangle\langle 3|(1+2)|4\rangle\,\Delta_3}\Bigg[\frac{[13]}{\langle 56\rangle}\big(\langle 2|1(3+4)|5\rangle-\langle 2|(3+4)(1+2)|5\rangle\big)\big(\langle 15\rangle\,t_{124}+\langle 1|26|5\rangle\big) \\
&\qquad\qquad - \frac{[16]\,\langle 24\rangle}{\langle 34\rangle\,[56]}\Big((s_{14}-s_{23})\big(\langle 1|2|6\rangle\,\delta_{12}-\langle 1|5|6\rangle\,\delta_{56}\big)+\langle 1|(3+4)|2\rangle\big(\langle 2|1|6\rangle\delta_{12}-\langle 2|5|6\rangle\delta_{56}\big)\Big) \\
&\qquad\qquad + 2\,\langle 2|1|3\rangle\langle 5|(1+2)|6\rangle(s_{14}-s_{23})\Bigg]\ln\!\Big(\frac{-s_{12}}{-s_{56}}\Big)\,.
\end{aligned}
\tag{7.1}
$$

The scalar master functions are given by

$$
\begin{aligned}
&M_{2a}(1,2,3,4) = \\
&\quad \frac{1}{2}\frac{[12]}{[23]\,\langle 4|(1+2)|3\rangle}\Bigg\{6\,\frac{\langle 12\rangle\,\langle 3|(1+2)|4\rangle\langle 5|(1+2)|6\rangle\big([13]\,\delta_{56}-2\,[1|24|3]\big)}{\Delta_3^2} \\
&\quad +\frac{1}{\Delta_3}\Bigg[\frac{\langle 2|1|4\rangle\,\langle 35\rangle\,\langle 5|(1+2)|3\rangle}{\langle 56\rangle}-\frac{\langle 12\rangle\,\langle 45\rangle}{\langle 34\rangle\,\langle 56\rangle}\left(\frac{[13]\,\langle 45\rangle\,\langle 3|(1+2)|4\rangle\delta_{12}}{\langle 4|(1+2)|3\rangle}-\langle 35\rangle\,\big(2\,[14]\,\delta_{12}-[1|23|4]\big)\right) \\
&\quad -\frac{\langle 24\rangle\,\langle 35\rangle\,\delta_{12}}{\langle 34\rangle\,\langle 56\rangle}\big(\langle 5|(1+2)|4\rangle-\langle 5|6|4\rangle\big)+\frac{[46]}{[56]}\Big(2\,\big(\langle 2|3|6\rangle\,\delta_{34}-\langle 2|5|6\rangle\,\delta_{56}\big)+\langle 3|4|6\rangle\langle 2|(1+4)|3\rangle\Big) \\
&\quad -\frac{\langle 2|1|3\rangle\langle 5|(3-4)|6\rangle\,\delta_{12}}{\langle 4|(1+2)|3\rangle}-4\,\langle 12\rangle\,[36]\,\left(\langle 35\rangle\,[14]+\frac{\langle 34\rangle\,\langle 56\rangle\,[13]\,[46]}{\langle 4|(1+2)|3\rangle}\right)+[46]\,\big(\langle 2|46|5\rangle-\langle 2|13|5\rangle\big)\Bigg] \\
&\quad +\frac{\langle 5|(1+2)|3\rangle\,\big(\langle 25\rangle\,\langle 34\rangle-\langle 23\rangle\,\langle 45\rangle\big)}{\langle 34\rangle\,\langle 56\rangle\,\langle 4|(1+2)|3\rangle}\Bigg\}\ln\left(\tfrac{-s_{12}}{-s_{56}}\right) \\
&\quad -\frac{1}{2}\,\frac{1}{[23]\,\langle 4|(1+2)|3\rangle\,\Delta_3}\Bigg\{-6\,\frac{[12]\,\langle 5|(1+2)|6\rangle[4|3(1+2)|4]\big(\langle 24\rangle\,\delta_{56}-2\,\langle 2|13|4\rangle\big)}{\Delta_3} \\
&\quad +\frac{[13]}{\langle 4|(1+2)|3\rangle}\big(\delta_{34}\,(t_{123}-t_{124})-\Delta_3\big)\left(\frac{\langle 5|34|5\rangle}{\langle 56\rangle}+\frac{[6|34|6]}{[56]}\right) \\
&\quad +\frac{\langle 5|3|4\rangle}{\langle 56\rangle}\Big(2\,\big(\langle 5|4|1\rangle\,\delta_{34}-\langle 5|6|1\rangle\,\delta_{56}\big)+\langle 4|(2+3)|1\rangle\langle 5|(1+2)|4\rangle\Big) \\
&\quad -\frac{[46]}{[56]}\Big(2\,\langle 3|2|1\rangle\big([36]\,\,\delta_{12}-2\,[3|45|6]\big)+[1|(2+3)43(1+2)|6]-3\,[6|43|1]\,\delta_{34}\Big)\Bigg\}\ln\left(\tfrac{-s_{34}}{-s_{56}}\right),
\end{aligned}
\tag{7.2}
$$

$$
\begin{aligned}
&M_2(1,2,3,4) = \\
&\quad M_{2a}(1,2,3,4)+\frac{\langle 4|(2+3)|1\rangle\langle 5|(1+2)|3\rangle^2}{[23]\,\langle 56\rangle\,\langle 4|(1+2)|3\rangle^3}\;\mathrm{Ls}^{2\mathrm{m}h}_{-1}\,(s_{34},t_{123};s_{12},s_{56}) \\
&\quad +\frac{[12]}{[23]\,\langle 4|(1+2)|3\rangle}\Bigg\{3\,\frac{\langle 12\rangle\,[34]\,\langle 3|(1+2)|4\rangle\langle 5|(1+2)|6\rangle}{\Delta_3^2}\big(\langle 4|2|1\rangle\,\delta_{12}-\langle 4|3|1\rangle\,\delta_{34}\big) \\
&\quad -\frac{1}{\Delta_3}\Bigg[\frac{\langle 2|13(1+2)4|5\rangle\,[36]}{\langle 4|(1+2)|3\rangle}\,(t_{124}-t_{123})+\langle 23\rangle\,[46]\,\big(\langle 5|4|3\rangle\,\delta_{34}-\langle 5|6|3\rangle\,\delta_{56}\big) \\
&\quad +\langle 3|(1+2)|4\rangle\,\big(\langle 2|4|3\rangle\langle 5|(1+2)|6\rangle+3\,\langle 2|1|3\rangle\langle 5|4|6\rangle\big) \\
&\quad -\langle 2|1|4\rangle\,\langle 35\rangle\,\,\big([6|4(1+2)|3]+2\,[6|54|3]+[36]\,\,\delta_{12}\big)\Bigg]+\frac{\langle 2|3(1+2)|5\rangle\,[46]}{t_{123}}\Bigg\}\,I_3^{3\mathrm{m}}\,(s_{12},s_{34},s_{56}) \\
&\quad +\frac{[46]\,\langle 5|4|1\rangle}{[23]\,\langle 4|(1+2)|3\rangle}\,\frac{\mathrm{L}_1\left(\frac{-s_{56}}{-t_{123}}\right)}{t_{123}}-\frac{\langle 3|2|1\rangle\langle 5|(1+2)|3\rangle^2}{[23]\,\langle 56\rangle\,\langle 4|(1+2)|3\rangle^2}\,\frac{\mathrm{L}_0\left(\frac{-t_{123}}{-s_{12}}\right)}{s_{12}}-\frac{\langle 5|4|1\rangle\langle 5|(1+2)|3\rangle t_{123}}{[23]\,\langle 56\rangle\,\langle 4|(1+2)|3\rangle^2}\,\frac{\mathrm{L}_0\left(\frac{-t_{123}}{-s_{56}}\right)}{s_{56}} \\
&\quad -\frac{1}{2}\,\frac{\langle 4|2|1\rangle\,\delta_{12}-\langle 4|3|1\rangle\,\delta_{34}}{[23]\,\langle 34\rangle\,\langle 4|(1+2)|3\rangle\,\Delta_3}\left(\frac{\langle 34\rangle\,[46]^2}{[56]}+\frac{\langle 35\rangle^2\,[34]}{\langle 56\rangle}\right)-\frac{\langle 35\rangle\,[46]\,\big([13]\,\delta_{56}-2\,[1|24|3]\big)}{[23]\,\langle 4|(1+2)|3\rangle\,\Delta_3},
\end{aligned}
\tag{7.3}
$$

$$
\begin{aligned}
&M_{3a}(1,2,3,4) = \\
&\quad \frac{1}{2}\,\frac{[24]}{[23]\,s_{34}\langle 1|(3+4)|2\rangle}\Bigg[6\,\frac{s_{12}s_{34}}{\Delta_3^2}(t_{134}-t_{234})\langle 3|(1+2)|4\rangle\langle 5|(1+2)|6\rangle \\
&\qquad + \frac{\langle 12\rangle\;\delta_{12}}{[56]\;\Delta_3}\bigg(\langle 3|(1+2)|6\rangle\,[14]\,[26] + \langle 3|(2+4)|6\rangle\,[12]\,[46] - \langle 13\rangle\,[24]\,\bigg([16]^2 + [26]^2\,\frac{\langle 2|(3+4)|1\rangle}{\langle 1|(3+4)|2\rangle}\bigg)\bigg) \\
&\qquad + \frac{[12]\;\delta_{12}}{\langle 56\rangle\;\Delta_3}\bigg(\langle 5|2|4\rangle\big(\langle 12\rangle\,\langle 35\rangle - \langle 23\rangle\,\langle 15\rangle\big) - \langle 15\rangle\,\langle 25\rangle\,\langle 3|(1+2)|4\rangle + \frac{\langle 15\rangle^2\,\langle 13\rangle\,[24]\,\langle 2|(3+4)|1\rangle}{\langle 1|(3+4)|2\rangle}\bigg) \\
&\qquad - 2\,\frac{\langle 12\rangle\,\langle 35\rangle}{\Delta_3}\,([16]\;[24] + [14]\,[26])\,\delta_{56} - \Big(\frac{\langle 34\rangle\,[46]^2}{[56]} + \frac{\langle 35\rangle^2\,[34]}{\langle 56\rangle}\Big)\frac{s_{12}\,t_{234}}{\Delta_3} \\
&\qquad + \frac{\langle 3|2|6\rangle\,[46]}{[56]} - \frac{\langle 13\rangle\,[24]}{\langle 1|(3+4)|2\rangle}\Big(\frac{\langle 5|12|5\rangle}{\langle 56\rangle} + \frac{[6|12|6]}{[56]}\Big)\Bigg]\,\ln\!\Big(\tfrac{-s_{12}}{-s_{56}}\Big) \\
&\quad + \frac{1}{2}\,\frac{[24]}{[23]\,\langle 1|(3+4)|2\rangle\Delta_3}\Bigg[3\,\frac{\delta_{56}\,(t_{134}-t_{234})\,\langle 3|(1+2)|4\rangle\langle 5|(1+2)|6\rangle}{\Delta_3} \\
&\qquad - \bigg(\frac{\langle 3|2|6\rangle\,[46]}{[56]} - \frac{\langle 5|1|4\rangle\,\langle 35\rangle}{\langle 56\rangle} + 2\,\frac{\langle 15\rangle\,[24]}{\langle 56\rangle\,\langle 1|(3+4)|2\rangle}\Big(\langle 3|(2+4)1|5\rangle - \langle 3|26|5\rangle\Big)\bigg)\,\delta_{34} \\
&\qquad + \Big(\langle 3|(1+2)|4\rangle + 4\,\frac{\langle 13\rangle\,[24]\,s_{56}}{\langle 1|(3+4)|2\rangle}\Big)\,\Big(\frac{\langle 5|12|5\rangle}{\langle 56\rangle} - \frac{[6|12|6]}{[56]}\Big) + 8\,\frac{s_{12}\;\langle 15\rangle\,[24]\,\langle 3|5|6\rangle}{\langle 1|(3+4)|2\rangle} \\
&\qquad - 2\,\langle 3|(1-2)|4\rangle\,\langle 5|(1+2)|6\rangle + 4\,\big([4|12|6]\,\langle 35\rangle + \langle 5|21|3\rangle\,[46]\big)\Bigg]\ln\!\Big(\tfrac{-s_{34}}{-s_{56}}\Big)\,,
\end{aligned}
\tag{7.4}
$$

$$
\begin{aligned}
&M_3(1,2,3,4) = \\
&\quad M_{3a}(1,2,3,4) - \frac{\langle 15\rangle^2[24]^3 t_{234}}{[23]\,[34]\,\langle 56\rangle\,\langle 1|(3+4)|2\rangle^3}\;\mathrm{Ls}^{2\mathrm{m}h}_{-1}\,(s_{12},t_{234};s_{34},s_{56}) \\
&\quad + \frac{[24]}{[23]\,\langle 1|(3+4)|2\rangle\,\Delta_3}\bigg\{\frac{3}{2}\,\frac{s_{12}\,\delta_{12}\,(t_{134}-t_{234})\langle 3|(1+2)|4\rangle\langle 5|(1+2)|6\rangle}{\Delta_3} + \frac{1}{2}\,s_{12}\,\langle 3|(1+2)|4\rangle\langle 5|(1+2)|6\rangle \\
&\qquad - \frac{\langle 13\rangle\,\langle 15\rangle\,[24]\,\langle 2|(3+4)|1\rangle}{\langle 1|(3+4)|2\rangle}\big([2|1(3+4)|6] - [2|(3+4)(1+2)|6]\big) \\
&\qquad + \langle 2|1|4\rangle\big(\langle 3|(1+4)|2\rangle\langle 5|(1+2)|6\rangle + \langle 5|1|2\rangle\langle 3|4|6\rangle\big) \\
&\qquad + \langle 15\rangle\,\langle 23\rangle\,\bigg([24]\,\Big([1|2(3+4)|6] - [1|(3+4)(1+2)|6]\Big) - [12]\,[4|(1+2)(3+4)|6]\bigg)\bigg\}I_3^{3\mathrm{m}}\,(s_{12},s_{34},s_{56}) \\
&\quad + \frac{1}{2}\,\frac{\langle 15\rangle^2\,[14]\,[24]}{[23]\,[34]\,\langle 56\rangle\,\langle 1|(3+4)|2\rangle}\bigg(-2\,\frac{[24]}{\langle 1|(3+4)|2\rangle}\mathrm{L}_0\Big(\frac{-s_{56}}{-t_{234}}\Big) + [14]\;\frac{\mathrm{L}_1\Big(\frac{-s_{56}}{-t_{234}}\Big)}{t_{234}}\bigg) \\
&\quad - \frac{1}{2}\,\frac{\langle 23\rangle\,[24]\,\langle 5|(3+4)|2\rangle}{[23]\,\langle 56\rangle\,\langle 1|(3+4)|2\rangle}\bigg(\Big(\frac{\langle 5|(2+3)|4\rangle}{t_{234}} - 2\,\frac{\langle 15\rangle\,[24]}{\langle 1|(3+4)|2\rangle}\Big)\,\frac{\mathrm{L}_0\Big(\frac{-t_{234}}{-s_{34}}\Big)}{s_{34}} + \langle 5|2|4\rangle\;\frac{\mathrm{L}_1\Big(\frac{-s_{34}}{-t_{234}}\Big)}{t_{234}^2}\bigg) \\
&\quad + \frac{1}{2}\,\frac{[24]}{[23]\,\langle 1|(3+4)|2\rangle\Delta_3}\bigg[-\frac{\langle 35\rangle}{\langle 34\rangle}\Big(\frac{\langle 25\rangle}{\langle 56\rangle}\big(\langle 3|1|2\rangle\,\delta_{12} - \langle 3|4|2\rangle\,\delta_{34}\big) - [16]\,\big(\langle 13\rangle\;\delta_{56} - 2\,\langle 1|24|3\rangle\big)\Big) \\
&\qquad + \frac{[46]}{[34]}\Big(\frac{[16]}{[56]}\,\big(\langle 1|2|4\rangle\,\delta_{12} - \langle 1|3|4\rangle\,\delta_{34}\big) + \langle 25\rangle\,\big([24]\;\delta_{56} - 2\,[2|13|4]\big)\Big)\bigg]\,.
\end{aligned}
\tag{7.5}
$$

8. Results for Primitive Amplitudes: $A_6(1_q, 2, 3, 4_{\bar{q}})$ and $A_6^{s,f,t}(1_q, 2, 3, 4_{\bar{q}})$

In this section we present the independent $qgg\bar{q}$ primitive amplitudes where in the parent diagrams neither of external gluons are attached to the external fermion lines, i.e. the helicity configurations

$$1_q^+, 2^+, 3^+, 4_{\bar{q}}^-, 5_{\bar{e}}^-, 6_e^+; \qquad 1_q^+, 2^+, 3^-, 4_{\bar{q}}^-, 5_{\bar{e}}^-, 6_e^+; \qquad 1_q^+, 2^-, 3^+, 4_{\bar{q}}^-, 5_{\bar{e}}^-, 6_e^+. \tag{8.1}$$

We suppress the lepton labels $(5_{\bar{e}}^-, 6_e^+)$ below. Representative parent diagrams are given in figs. 1a and b.

8.1 The Primitive Amplitudes: $A_6^{s,f,t}(1_q, 2, 3, 4_{\bar{q}})$

By far the simplest of the primitive amplitudes are those proportional to the number of scalars n_s or fermions n_f. This simplicity follows from the fact that only two Feynman diagrams contribute, each with a triangle integral. (See fig. 1a.) It may also be understood in terms of the rather simple cut and factorization properties that must be satisfied.

The results for these primitive amplitude are

$$\begin{aligned}
&A_6^s(1_q^+, 2^+, 3^-, 4_{\bar{q}}^-) = A_6^f(1_q^+, 2^+, 3^-, 4_{\bar{q}}^-) = 0\,,\\
&A_6^s(1_q^+, 2^-, 3^+, 4_{\bar{q}}^-) = A_6^f(1_q^+, 2^-, 3^+, 4_{\bar{q}}^-) = 0\,,\\
&A_6^f(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-) = 0\,,\\
&A_6^s(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-) = i\,\frac{c_\Gamma}{3}\frac{1}{\langle 23\rangle^2 s_{56}}\left[-\frac{\langle 45\rangle\,[6|(1+2)3|1]}{t_{123}} + \frac{[16]\,\langle 5|(4+2)3|4\rangle}{t_{234}}\right].
\end{aligned} \tag{8.2}$$

Although it is not manifest, $A_6^s(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-)$ is antisymmetric in the exchange of 2 and 3, as required by charge conjugation invariance. This fact accounts for the absence of n_f or n_s terms in $A_{6;3}$ in eq. (2.13).

The contribution of a virtual top quark, through bubble and triangle graphs, is simply related to the above function A_6^s,

$$\begin{aligned}
A_6^t(1_q^+, 2^+, 3^-, 4_{\bar{q}}^-) &= A_6^t(1_q^+, 2^-, 3^+, 4_{\bar{q}}^-) = 0\,,\\
A_6^t(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-) &= \frac{1}{20}\frac{s_{23}}{m_t^2}A_6^s(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-)\,,
\end{aligned} \tag{8.3}$$

neglecting $1/m_t^4$ corrections.

8.2 The Helicity Configuration $q^+\,g^+\,g^+\,\bar{q}^-$

The simplest of the non-$n_{s,f}$ helicity amplitudes is the one where both gluons have the same helicity, $A_6(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-)$. The tree amplitude in this case is

$$A_6^{\rm tree} = -i\,\frac{\langle 45\rangle^2}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 56\rangle}\,. \tag{8.4}$$

The contributions to the amplitude in terms of the decomposition (6.1) are

$$V^{cc} = -\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{-s_{12}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{23}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{34}}\right)^\epsilon\right) - \frac{2}{\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^\epsilon - 4\,, \tag{8.5}$$

$$\begin{aligned} F^{cc} &= \frac{A_6^{\rm tree}}{i}\left[-{\rm Ls}_{-1}\left(\tfrac{-s_{12}}{-t_{123}},\tfrac{-s_{23}}{-t_{123}}\right) - {\rm Ls}_{-1}\left(\tfrac{-s_{23}}{-t_{234}},\tfrac{-s_{34}}{-t_{234}}\right) - {\rm Ls}_{-1}^{\rm 2me}(t_{123},t_{234};s_{23},s_{56})\right] \\ &\quad + 2\,\frac{\langle 45\rangle\,\langle 5|2|3\rangle}{\langle 12\rangle\,\langle 23\rangle\,\langle 56\rangle}\frac{{\rm L}_0\left(\frac{-t_{234}}{-s_{34}}\right)}{s_{34}} + 2\frac{\langle 45\rangle\,\langle 5|1(2+3)|4\rangle}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 56\rangle}\frac{{\rm L}_0\left(\frac{-s_{56}}{-t_{234}}\right)}{t_{234}}\,, \end{aligned} \tag{8.6}$$

$$\begin{aligned} V^{sc} &= \frac{1}{2\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^\epsilon + \frac{1}{2}\,, \\ F^{sc} &= \frac{A_6^{\rm tree}}{i}\left[-\frac{1}{2}\left(\frac{\langle 4|32|5\rangle}{\langle 45\rangle}\right)^2\frac{{\rm L}_1\left(\frac{-s_{34}}{-t_{234}}\right)}{t_{234}^2} + \frac{1}{2}\left(\frac{\langle 4|(2+3)1|5\rangle}{\langle 45\rangle}\right)^2\frac{{\rm L}_1\left(\frac{-s_{56}}{-t_{234}}\right)}{t_{234}^2}\right] \\ &\quad + \frac{1}{2}\Bigg[-\frac{\langle 5|2|3\rangle\,\langle 54\rangle}{\langle 12\rangle\,\langle 23\rangle\,t_{234}\,\langle 56\rangle} - \frac{[23]\,\langle 45\rangle\,\langle 5|(2+4)|3\rangle}{\langle 12\rangle\,t_{123}t_{234}\,\langle 56\rangle} \\ &\quad + \frac{\langle 4|(2+3)|6\rangle^2}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,t_{234}\,[56]} + \frac{\langle 4|2|3\rangle[6|1(2+3)|6]}{\langle 12\rangle\,\langle 23\rangle\,t_{123}t_{234}\,[56]}\Bigg] + \frac{1}{i\,c_\Gamma}\,A_6^s(1_q^+,2^+,3^+,4_{\bar q}^-)\,. \end{aligned} \tag{8.7}$$

8.3 The Helicity Configuration $q^+\,g^+\,g^-\,\bar q^-$

The next simplest helicity configuration is $A_6(1_q^+,2^+,3^-,4_{\bar q}^-)$. Notice that this amplitude is symmetric under the 'flip$_1$' symmetry in eq. (6.7). The tree amplitude for this helicity configuration is

$$\begin{aligned} A_6^{\rm tree} = i\Bigg[&\frac{\langle 31\rangle\,[12]\,\langle 45\rangle\,\langle 3|(1+2)|6\rangle}{\langle 12\rangle\,s_{23}t_{123}s_{56}} - \frac{\langle 34\rangle\,[42]\,[16]\,\langle 5|(3+4)|2\rangle}{[34]\,s_{23}t_{234}s_{56}} \\ &- \frac{\langle 5|(3+4)|2\rangle\,\langle 3|(1+2)|6\rangle}{\langle 12\rangle\,[34]\,s_{23}s_{56}}\Bigg]\,. \end{aligned} \tag{8.8}$$

An alternate form for $A_6^{\rm tree}$, which has manifest behavior in the collinear limits $k_2 \parallel k_3$ and $k_5 \parallel k_6$, at the expense of more obscure behavior on the three-particle poles, is

$$\begin{aligned} A_6^{\rm tree} = i\Bigg[&\frac{\langle 54\rangle\,[42]\,[12]\,\langle 5|(3+4)|2\rangle}{[23]\,[34]\,t_{123}t_{234}\,\langle 56\rangle} + \frac{\langle 31\rangle\,[16]\,\langle 34\rangle\,\langle 3|(1+2)|6\rangle}{\langle 12\rangle\,\langle 23\rangle\,t_{123}t_{234}\,[56]} \\ &- \frac{\langle 3|(1+2)|6\rangle\,\langle 5|(3+4)|2\rangle}{\langle 12\rangle\,[34]\,t_{123}t_{234}}\Bigg]\,. \end{aligned} \tag{8.9}$$

The results for the cut-constructible and scalar pieces are

$$V^{cc} = -\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{-s_{12}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{23}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{34}}\right)^\epsilon\right) - \frac{2}{\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^\epsilon - 4\,, \tag{8.10}$$

$$F^{cc} = \left(\frac{\langle 13\rangle\langle 3|(1+2)|6\rangle^2}{\langle 12\rangle\langle 23\rangle[56]\,t_{123}\,\langle 1|(2+3)|4\rangle} + \frac{[12]^3\langle 45\rangle^2}{[23]\,[13]\,\langle 56\rangle\,t_{123}\,\langle 4|(2+3)|1\rangle}\right)\mathrm{Ls}_{-1}\left(\tfrac{-s_{12}}{-t_{123}},\tfrac{-s_{23}}{-t_{123}}\right)$$
$$+\left(\frac{\langle 13\rangle\langle 3|(1+2)|6\rangle^2}{\langle 12\rangle\langle 23\rangle[56]\,t_{123}\,\langle 1|(2+3)|4\rangle} + \frac{[12]^2\langle 45\rangle^2\langle 4|(1+3)|2\rangle}{[23]\,\langle 56\rangle\,t_{123}\,\langle 4|(2+3)|1\rangle\,\langle 4|(1+2)|3\rangle}\right)$$
$$\times\,\widetilde{\mathrm{Ls}}_{-1}^{2mh}(s_{34},t_{123};s_{56},s_{12})$$
$$+\frac{1}{2}\,\frac{[12]\left(\langle 4|(1+2)(3+4)|5\rangle^2 - s_{12}s_{34}\langle 45\rangle^2\right)}{\langle 12\rangle\,[34]\,\langle 56\rangle\,\langle 4|(2+3)|1\rangle\langle 4|(1+2)|3\rangle}\,I_3^{3m}(s_{12},s_{34},s_{56})$$
$$-\,2\,\frac{\langle 13\rangle\langle 3|(1+2)|6\rangle}{\langle 12\rangle\,[56]\,\langle 1|(2+3)|4\rangle}\left[\frac{[6|(2+3)1|2]}{t_{123}}\,\frac{\mathrm{L}_0\left(\frac{-s_{23}}{-t_{123}}\right)}{t_{123}} + \frac{\langle 3|4|6\rangle}{\langle 23\rangle}\,\frac{\mathrm{L}_0\left(\frac{-s_{56}}{-t_{123}}\right)}{t_{123}}\right]$$
$$+\ \mathrm{flip}_1\,, \tag{8.11}$$

$$V^{sc} = \frac{1}{2\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^{\epsilon} + \frac{1}{2}\,, \tag{8.12}$$

$$F^{sc} = -\frac{1}{2}\,\frac{\langle 13\rangle}{\langle 12\rangle\langle 23\rangle[56]\,t_{123}\,\langle 1|(2+3)|4\rangle}\left[\langle 3|21(2+3)|6\rangle^2\,\frac{\mathrm{L}_1\left(\frac{-t_{123}}{-s_{23}}\right)}{s_{23}^2} + \langle 3|4|6\rangle^2\,\mathrm{L}_1\left(\frac{-s_{56}}{-t_{123}}\right)\right]$$
$$+\frac{1}{2}\,\frac{[62]^2}{\langle 12\rangle\,[23]\,[34]\,[56]}\ +\ \mathrm{flip}_1\,, \tag{8.13}$$

where 'flip_1' is to be applied to all preceding terms in the given expression.

8.4 The Helicity Configuration $q^+\,g^-\,g^+\,\bar{q}^-$

The most complicated leading-color helicity configuration is $A_6(1_q^+,2^-,3^+,4_{\bar{q}}^-)$. In general, those amplitudes where the negative and positive helicities alternate around the loop are the most complicated ones. The symmetry flip_1 (6.7) holds for this helicity configuration as well. The tree amplitude in this case is

$$A_6^{\rm tree} = i\left[-\frac{[13]^2\langle 45\rangle\langle 2|(1+3)|6\rangle}{[12]\,s_{23}t_{123}s_{56}} + \frac{\langle 24\rangle^2[16]\langle 5|(2+4)|3\rangle}{\langle 34\rangle\,s_{23}t_{234}s_{56}} + \frac{[13]\langle 24\rangle[16]\langle 45\rangle}{[12]\langle 34\rangle\,s_{23}s_{56}}\right]. \tag{8.14}$$

An alternate form with more manifest behavior in two-particle channels, but less manifest behavior in three-particle channels, is

$$A_6^{\rm tree} = i\left[\frac{[13]^2\langle 45\rangle\langle 5|(2+4)|3\rangle}{[12]\,[23]\,t_{123}t_{234}\langle 56\rangle} - \frac{\langle 24\rangle^2[16]\langle 2|(1+3)|6\rangle}{\langle 23\rangle\langle 34\rangle\,t_{123}t_{234}[56]} + \frac{[13]\langle 24\rangle[16]\langle 45\rangle}{[12]\langle 34\rangle\,t_{123}t_{234}}\right]. \tag{8.15}$$

The results for the cut-constructible pieces are

$$V^{cc} = -\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} + \left(\frac{\mu^2}{-s_{23}}\right)^{\epsilon} + \left(\frac{\mu^2}{-s_{34}}\right)^{\epsilon}\right) - \frac{2}{\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^{\epsilon} - 4\,, \tag{8.16}$$

$$
\begin{aligned}
F^{cc} = \Bigg(& \frac{[13]^3\,\langle 45\rangle^2}{[12]\,[23]\,\langle 56\rangle\, t_{123}\langle 4|(2+3)|1\rangle} + \frac{\langle 12\rangle^3\,\langle 3|(1+2)|6\rangle^2}{\langle 23\rangle\,[56]\,\langle 13\rangle^3\, t_{123}\langle 1|(2+3)|4\rangle} \\
& - \frac{\langle 12\rangle\,\langle 23\rangle\,\langle 1|(2+3)|6\rangle^2}{[56]\,\langle 13\rangle^3\, t_{123}\langle 1|(2+3)|4\rangle}\Bigg) \mathrm{Ls}_{-1}\Big(\tfrac{-s_{12}}{-t_{123}}, \tfrac{-s_{23}}{-t_{123}}\Big) \\
& + \Bigg(\frac{[13]^3\,\langle 45\rangle^2}{[12]\,[23]\,\langle 56\rangle\, t_{123}\langle 4|(2+3)|1\rangle} + \frac{\langle 3|(1+2)|6\rangle^2\langle 2|(1+3)|4\rangle^3}{\langle 23\rangle\,[56]\, t_{123}\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle^3} \\
& - \frac{\langle 23\rangle\,[46]^2\, t_{123}\langle 2|(1+3)|4\rangle}{[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle^3}\Bigg) \mathrm{Ls}_{-1}^{2\mathrm{m}h}\big(s_{34}, t_{123}; s_{12}, s_{56}\big) \\
& + \Bigg[-2\,\frac{\langle 2|1|3\rangle\langle 5|4|6\rangle\langle 2|(1+3)|4\rangle}{t_{123}\langle 3|(1+2)|4\rangle\langle 1|(2+3)|4\rangle} \\
& - \frac{1}{2}\,\frac{t_{123}\,\delta_{34} + 2\, s_{12}\, s_{56}}{t_{123}^2}\Bigg(\frac{[13]^3\,\langle 45\rangle^2}{[12]\,[23]\,\langle 56\rangle\,\langle 4|(2+3)|1\rangle} + \frac{\langle 2|(1+3)|6\rangle^2\langle 2|(1+3)|4\rangle}{\langle 23\rangle\,[56]\,\langle 3|(1+2)|4\rangle\langle 1|(2+3)|4\rangle}\Bigg) \\
& + \frac{1}{2}\,\frac{1}{\langle 1|(3+4)|2\rangle\langle 3|(1+2)|4\rangle\langle 1|(2+3)|4\rangle}\Bigg(\frac{[36]\,\langle 2|4|6\rangle\langle 2|(1+3)|4\rangle\langle 1|(3+4)|2\rangle}{[56]} \\
& - \frac{\langle 2|1|6\rangle\langle 2|4|6\rangle\langle 4|(1+2+3)|4\rangle\langle 1|(3+4)|2\rangle}{\langle 34\rangle\,[56]} - \frac{\langle 1|4|6\rangle\langle 2|3|4\rangle\langle 5|61|4\rangle}{\langle 34\rangle} \\
& + \langle 15\rangle\,\langle 2|3|4\rangle\big([36]\, t_{234} - [3|4(1+3)|6]\big) - \langle 2|1|6\rangle\langle 1|4|3\rangle\langle 5|2|4\rangle \\
& - \frac{1}{2}\,\frac{s_{14}\,\langle 45\rangle\,[16]\,\delta_{56}\langle 1|(2+3)|4\rangle}{[12]\,\langle 34\rangle} + \frac{1}{2}\,(s_{14}-s_{23})\,\langle 12\rangle\,[34]\,\langle 5|(2+3)|6\rangle\Bigg)\Bigg] I_3^{3\mathrm{m}}(s_{12}, s_{34}, s_{56}) \\
& - 2\,\frac{\langle 2|1|3\rangle\langle 2|(1+3)|6\rangle}{[56]\,\langle 13\rangle\, t_{123}}\Bigg(\frac{\langle 3|(1+2)|6\rangle}{\langle 3|(1+2)|4\rangle}\,\frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{12}}\Big)}{s_{12}} + \frac{\langle 1|(2+3)|6\rangle}{\langle 1|(2+3)|4\rangle}\,\frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{23}}\Big)}{s_{23}}\Bigg) \\
& - 2\,\frac{\langle 2|4|6\rangle\langle 2|(1+3)|6\rangle\langle 2|(1+3)|4\rangle}{\langle 23\rangle\,[56]\,\langle 3|(1+2)|4\rangle\langle 1|(2+3)|4\rangle}\,\frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{56}}\Big)}{s_{56}} \;+\; M_1(1,2,3,4) \;+\; \mathrm{flip}_1\ .
\end{aligned}
\tag{8.17}
$$

The results where the gluon in the loop is replaced by a scalar are

$$
V^{sc} \;=\; \frac{1}{2\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^{\epsilon} + \frac{1}{2}\ , \tag{8.18}
$$

$$
\begin{aligned}
F^{sc} = {} & \frac{\langle 12\rangle\,\langle 23\rangle\,[13]^2\,\langle 1|(2+3)|6\rangle^2}{[56]\,\langle 13\rangle\,t_{123}\langle 1|(2+3)|4\rangle}\Bigg[\frac{\mathrm{Ls}_1\left(\frac{-s_{12}}{-t_{123}},\frac{-s_{23}}{-t_{123}}\right)}{t_{123}^2}-\frac{1}{2}\,\frac{\mathrm{L}_1\left(\frac{-t_{123}}{-s_{23}}\right)}{s_{23}^2}\Bigg] \\
& + \frac{\langle 23\rangle\,[46]^2\,t_{123}\langle 2|(1+3)|4\rangle}{[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle^3}\;\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{34},t_{123};s_{12},s_{56}) \\
& + \frac{\langle 12\rangle\,[46]}{\langle 1|(2+3)|4\rangle\,\Delta_3}\Bigg[3\,[34]\,\langle 56\rangle\,\langle 2|(3+4)|1\rangle\big(\langle 3|5|6\rangle\delta_{56}-\langle 3|4|6\rangle\delta_{34}\big)\frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle\,\Delta_3} \\
& \qquad + \big(3\,\langle 5|6|4\rangle\langle 2|3|1\rangle-\langle 5|3|4\rangle\langle 2|(3+4)|1\rangle\big)\frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle}-[13]\,\langle 24\rangle\,[36]\,\langle 56\rangle \\
& \qquad + [14]\,\langle 23\rangle\,\langle 5|6|4\rangle(t_{123}-t_{124})\frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle^2}\Bigg]I_3^{3\mathrm{m}}(s_{12},s_{34},s_{56}) \\
& + \frac{\langle 24\rangle\,[46]^2\,\langle 2|(1+3)|4\rangle t_{123}}{[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle}\Bigg(-\frac{1}{2}\frac{\langle 24\rangle}{\langle 23\rangle}\,\frac{\mathrm{L}_1\left(\frac{-s_{56}}{-t_{123}}\right)}{t_{123}^2}+\frac{1}{\langle 3|(1+2)|4\rangle}\,\frac{\mathrm{L}_0\left(\frac{-s_{56}}{-t_{123}}\right)}{t_{123}}\Bigg) \\
& + \frac{\langle 2|13|2\rangle\,[46]^2\,t_{123}}{[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle^2}\,\frac{\mathrm{L}_0\left(\frac{-t_{123}}{-s_{12}}\right)}{s_{12}}+\frac{1}{2}\,\frac{\langle 2|1|3\rangle^2\langle 3|(1+2)|6\rangle^2}{[56]\,\langle 13\rangle\,t_{123}\langle 3|(1+2)|4\rangle}\,\frac{\mathrm{L}_1\left(\frac{-t_{123}}{-s_{12}}\right)}{s_{12}^2} \\
& + \frac{\langle 12\rangle\,[46]}{\langle 1|(2+3)|4\rangle}\Bigg[3\,\big(\langle 5|3|4\rangle\,\delta_{34}-\langle 5|6|4\rangle\,\delta_{56}\big)\,\frac{\langle 2|(3+4)|1\rangle\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle\,\Delta_3^2} \\
& \qquad + \frac{1}{2}\,\frac{1}{\langle 34\rangle\,[56]\,\langle 3|(1+2)|4\rangle\,\Delta_3}\Bigg(-\langle 24\rangle\;\delta_{12}\,\big([6|53|1]+[6|4(2+3)|1]\big) \\
& \qquad + \langle 2|(3+4)|1\rangle\langle 4|(1+2)|3\rangle\langle 3|(4-5)|6\rangle-\langle 2|(3+4)|1\rangle\langle 3|4|6\rangle\,\delta_{34}\,\frac{(t_{123}-t_{124})}{\langle 3|(1+2)|4\rangle}\Bigg) \\
& \qquad + \frac{1}{2}\,\frac{[46]\,\langle 2|(3+4)|1\rangle}{[56]\,\langle 3|(1+2)|4\rangle^2}\Bigg]\ln\left(\frac{-s_{12}}{-s_{56}}\right) \\
& + \frac{[46]}{\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle}\Bigg[-3\,\frac{\langle 12\rangle\,[34]\,\langle 2|(3+4)|1\rangle\langle 4|(1+2)|3\rangle\big(\langle 35\rangle\,\delta_{12}-2\,\langle 3|46|5\rangle\big)}{\Delta_3^2} \\
& \qquad - \frac{1}{2}\,\frac{1}{[12]\,[56]\;\Delta_3}\Bigg(\langle 2|(3+4)|1\rangle\langle 4|(1+2)|3\rangle\big([4|3(1+2)|6]+[46]\,(\delta_{56}-2s_{12})\big) \\
& \qquad - \delta_{34}\,\langle 2|4|3\rangle\big([6|53|1]+[6|4(2+3)|1]\big)-2\;[46]\,t_{123}\,\Bigg(2\,\langle 2|(3-4)|1\rangle\langle 4|(1+2)|3\rangle \\
& \qquad + (t_{123}-t_{124})\Big([13]\,\langle 24\rangle+[14]\,\langle 23\rangle\,\frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle}\Big)\Bigg)\Bigg)-\frac{1}{2}\,\frac{[13]\,\langle 2|4|6\rangle}{[12]\,[56]}\Bigg]\ln\left(\frac{-s_{34}}{-s_{56}}\right) \\
& + \frac{1}{2}\,\frac{[46]\,\langle 2|(3+4)|1\rangle\langle 4|(1+2)|3\rangle\,\big(\langle 3|5|6\rangle\,\delta_{56}-\langle 3|4|6\rangle\,\delta_{34}\big)}{[12]\,\langle 34\rangle\,[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle\,\Delta_3} \\
& - \frac{1}{2}\,\frac{\langle 2|4|6\rangle\big([6|53|1]+[6|4(2+3)|1]\big)}{[12]\,\langle 34\rangle\,[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle}-\frac{1}{2}\frac{[13]^2\,\langle 1|(2+3)|6\rangle^2}{[12]\,[23]\,[56]\,\langle 13\rangle\,t_{123}\langle 1|(2+3)|4\rangle}\;+\;\mathrm{flip}_1\,.
\end{aligned}
\tag{8.19}
$$

9. Results for Primitive Amplitudes: $A_6(1_q,2,3_{\bar{q}},4)$

In this section we present the independent $A_6(1_q,2,3_{\bar{q}},4)$ primitive amplitudes, corresponding to the helicity configurations

$$
1_q^+,2^+,3_{\bar{q}}^-,4^+,5_{\bar{e}}^-,6_e^+;\qquad 1_q^+,2^+,3_{\bar{q}}^-,4^-,5_{\bar{e}}^-,6_e^+; \tag{9.1}
$$

note that the configuration $1_q^+, 2^-, 3_{\bar q}^-, 4^+, 5_{\bar e}^-, 6_e^+$ is obtained from $1_q^+, 2^+, 3_{\bar q}^-, 4^-, 5_{\bar e}^-, 6_e^+$ by the operation 'flip$_4$' defined in eq. (6.10). For these amplitudes one of the external gluons is cyclicly adjacent to the vector boson, as depicted in fig. 1c. These primitive amplitudes only contribute to subleading-in-color terms.

9.1 The Helicity Configuration $q^+ g^+ \bar q^- g^+$

We now give the primitive amplitudes for $A_6(1_q^+, 2^+, 3_{\bar q}^-, 4^+)$. These amplitudes are relatively simple because the three-mass triangle does not appear. The tree amplitude is

$$A_6^{\rm tree} = i \frac{\langle 13\rangle \langle 35\rangle^2}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle \langle 56\rangle} \,. \tag{9.2}$$

The cut-constructible contributions are

$$V^{cc} = -\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{-s_{12}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{23}}\right)^\epsilon\right) - \frac{2}{\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^\epsilon - 4\,, \tag{9.3}$$

$$\begin{aligned}
F^{cc} = {} & \frac{\langle 35\rangle \left(\langle 23\rangle \langle 45\rangle + \langle 34\rangle \langle 52\rangle\right)}{\langle 12\rangle \langle 34\rangle \langle 56\rangle \langle 24\rangle^2} \mathrm{Ls}_{-1}\Big(\tfrac{-s_{23}}{-t_{234}}, \tfrac{-s_{34}}{-t_{234}}\Big) - \frac{\langle 13\rangle \langle 35\rangle^2}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle \langle 56\rangle} \mathrm{Ls}_{-1}\Big(\tfrac{-s_{12}}{-t_{123}}, \tfrac{-s_{23}}{-t_{123}}\Big) \\
& + \frac{\langle 13\rangle \langle 35\rangle \left(\langle 13\rangle \langle 45\rangle + \langle 34\rangle \langle 51\rangle\right)}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle^2 \langle 56\rangle} \mathrm{Ls}_{-1}^{\rm 2me}(t_{123}, t_{234}; s_{23}, s_{56}) \\
& - \frac{\langle 13\rangle \langle 35\rangle^2}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle \langle 56\rangle} \mathrm{Ls}_{-1}^{\rm 2me}(t_{124}, t_{123}; s_{12}, s_{56}) - \frac{\langle 35\rangle^2}{\langle 23\rangle \langle 41\rangle \langle 56\rangle \langle 24\rangle} \mathrm{Ls}_{-1}\Big(\tfrac{-s_{14}}{-t_{124}}, \tfrac{-s_{12}}{-t_{124}}\Big) \\
& - 2\,\frac{\langle 35\rangle \langle 5|4|2\rangle}{\langle 41\rangle \langle 56\rangle \langle 24\rangle} \frac{\mathrm{L}_0\Big(\frac{-t_{234}}{-s_{23}}\Big)}{s_{23}} - 2\,\frac{\langle 35\rangle \langle 5|2|4\rangle}{\langle 12\rangle \langle 56\rangle \langle 24\rangle} \frac{\mathrm{L}_0\Big(\frac{-t_{234}}{-s_{34}}\Big)}{s_{34}} + 2\,\frac{\langle 13\rangle \langle 51\rangle \langle 53\rangle \langle 3|(2+4)|1\rangle}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle \langle 56\rangle} \frac{\mathrm{L}_0\Big(\frac{-t_{234}}{-s_{56}}\Big)}{s_{56}} \,.
\end{aligned} \tag{9.4}$$

The terms where a scalar replaces the gluon are

$$V^{sc} = \frac{1}{2\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^\epsilon + \frac{1}{2}\,, \tag{9.5}$$

$$
\begin{aligned}
F^{sc} = & \frac{\langle 34\rangle \langle 52\rangle^2 [24]^2}{\langle 12\rangle \langle 56\rangle \langle 24\rangle} \frac{\mathrm{Ls}_1\left(\frac{-s_{23}}{-t_{234}}, \frac{-s_{34}}{-t_{234}}\right)}{t_{234}^2} - \frac{\langle 13\rangle \langle 34\rangle \langle 51\rangle^2}{\langle 12\rangle \langle 23\rangle \langle 56\rangle \langle 41\rangle^3} \mathrm{Ls}_{-1}^{2\mathrm{me}}(t_{123}, t_{234}; s_{23}, s_{56}) \\
& - \frac{1}{2} \frac{\langle 13\rangle \langle 53\rangle^2}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle \langle 56\rangle} \ln\left(\frac{-t_{234}}{-s_{56}}\right) - \frac{\langle 13\rangle \langle 51\rangle^2 [12]}{\langle 12\rangle \langle 56\rangle \langle 41\rangle^2} \frac{\mathrm{L}_0\left(\frac{-t_{123}}{-s_{23}}\right)}{s_{23}} + \frac{\langle 34\rangle \langle 51\rangle^2 [24]}{\langle 12\rangle \langle 56\rangle \langle 41\rangle^2} \frac{\mathrm{L}_0\left(\frac{-t_{234}}{-s_{23}}\right)}{s_{23}} \\
& - \frac{\langle 13\rangle \langle 51\rangle \langle 54\rangle \langle 3|(1+2)|4\rangle}{\langle 12\rangle \langle 23\rangle \langle 56\rangle \langle 41\rangle^2} \frac{\mathrm{L}_0\left(\frac{-t_{123}}{-s_{56}}\right)}{s_{56}} + \frac{\langle 13\rangle [64] \langle 53\rangle}{\langle 12\rangle \langle 23\rangle \langle 41\rangle} \frac{\mathrm{L}_0\left(\frac{-t_{123}}{-s_{56}}\right)}{s_{56}} + \frac{\langle 13\rangle \langle 51\rangle^2 \langle 3|(2+4)|1\rangle}{\langle 12\rangle \langle 23\rangle \langle 56\rangle \langle 41\rangle^2} \frac{\mathrm{L}_0\left(\frac{-t_{234}}{-s_{56}}\right)}{s_{56}} \\
& - \frac{\langle 13\rangle^2 \langle 53\rangle [61]}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle} \frac{\mathrm{L}_0\left(\frac{-t_{234}}{-s_{56}}\right)}{s_{56}} - \frac{1}{2} \frac{\langle 23\rangle \langle 54\rangle^2 [42]^2}{\langle 14\rangle \langle 56\rangle \langle 24\rangle} \frac{\mathrm{L}_1\left(\frac{-s_{23}}{-t_{234}}\right)}{t_{234}^2} - \frac{1}{2} \frac{\langle 52\rangle^2 [24]^2 \langle 43\rangle}{\langle 12\rangle \langle 56\rangle \langle 24\rangle} \frac{\mathrm{L}_1\left(\frac{-s_{34}}{-t_{234}}\right)}{t_{234}^2} \\
& - \frac{\langle 13\rangle \langle 34\rangle \langle 56\rangle [64]^2}{\langle 12\rangle \langle 23\rangle \langle 41\rangle} \frac{\mathrm{L}_1\left(\frac{-t_{123}}{-s_{56}}\right)}{s_{56}^2} - \frac{1}{2} \frac{\langle 13\rangle^3 \langle 56\rangle [61]^2}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle} \frac{\mathrm{L}_1\left(\frac{-t_{234}}{-s_{56}}\right)}{s_{56}^2} \\
& + \frac{1}{2} \frac{[24] \langle 1|(2+4)|6\rangle \langle 3|(2+4)|6\rangle}{\langle 12\rangle \langle 41\rangle [56]\, t_{124} t_{234}} + \frac{1}{2} \frac{\langle 24\rangle [24]^2 \langle 51\rangle \langle 53\rangle}{\langle 12\rangle \langle 41\rangle \langle 56\rangle\, t_{124} t_{234}} + \frac{1}{2} \frac{\langle 13\rangle \langle 35\rangle^2}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle \langle 56\rangle} \,.
\end{aligned}
\tag{9.6}
$$

9.2 The Helicity Configuration $q^+ \, g^+ \, \bar{q}^- \, g^-$

We now present the primitive amplitude $A_6(1_q^+, 2^+, 3_{\bar{q}}^-, 4^-)$. The tree amplitude is

$$
A_6^{\mathrm{tree}} = i \left[-\frac{\langle 3|(1+2)|6\rangle \langle 5|(3+4)|1\rangle}{\langle 12\rangle \langle 23\rangle [34] [41]\, s_{56}} + \frac{[12] \langle 53\rangle \langle 4|(1+2)|6\rangle}{\langle 12\rangle [41]\, t_{124} s_{56}} + \frac{\langle 34\rangle [61] \langle 5|(3+4)|2\rangle}{\langle 23\rangle [34]\, t_{234} s_{56}} \right]. \tag{9.7}
$$

An alternate form is

$$
\begin{aligned}
A_6^{\mathrm{tree}} = i \Bigg[& -\frac{[12] [13] \langle 35\rangle \langle 5|(3+4)|2\rangle}{[34] [41]\, t_{124} t_{234} \langle 56\rangle} - \frac{\langle 13\rangle \langle 34\rangle [16] \langle 4|(1+2)|6\rangle}{\langle 12\rangle \langle 23\rangle\, t_{124} t_{234} [56]} \\
& - \frac{\langle 4|(1+2)|6\rangle \langle 5|(3+4)|2\rangle}{\langle 12\rangle [34]\, t_{124} t_{234}} - \frac{[12] \langle 34\rangle [16] \langle 35\rangle}{\langle 23\rangle [41]\, t_{124} t_{234}} \Bigg].
\end{aligned}
\tag{9.8}
$$

The first form has manifest t-pole behavior while the second form has manifest behavior for $k_5 \parallel k_6$.

The cut-constructible contributions are

$$
V^{cc} = -\frac{1}{\epsilon^2} \left(\left(\frac{\mu^2}{-s_{12}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{23}} \right)^\epsilon \right) - \frac{2}{\epsilon} \left(\frac{\mu^2}{-s_{56}} \right)^\epsilon - 4 \,, \tag{9.9}
$$

$$
\begin{aligned}
F^{cc} = {} & \frac{\langle 13\rangle\,\langle 3|(1+2)|6\rangle^2}{\langle 12\rangle\,\langle 23\rangle\,[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle}\ \mathrm{Ls}^{2\mathrm{m}h}_{-1}\big(s_{34},t_{123};s_{12},s_{56}\big) \\
& + \left(\frac{\langle 5|(3+4)|2\rangle^2}{[34]\,\langle 56\rangle\,t_{234}\langle 1|(2+3)|4\rangle} - \frac{\langle 34\rangle^2[16]^2}{\langle 23\rangle\,[56]\,t_{234}\langle 2|(3+4)|1\rangle}\right)\ \mathrm{Ls}^{2\mathrm{m}h}_{-1}\big(s_{12},t_{234};s_{34},s_{56}\big) \\
& + \left(\frac{\langle 13\rangle^3[46]^2t_{123}^2}{\langle 12\rangle\,\langle 23\rangle\,[56]\,\langle 1|(2+3)|4\rangle^3\langle 3|(1+2)|4\rangle} - \frac{\langle 13\rangle\,\langle 1|(2+3)|6\rangle^2\langle 3|(1+2)|4\rangle}{\langle 12\rangle\,\langle 23\rangle\,[56]\,\langle 1|(2+3)|4\rangle^3}\right)\ \mathrm{Ls}^{2\mathrm{m}h}_{-1}\big(s_{14},t_{123};s_{23},s_{56}\big) \\
& - \left(\frac{\langle 2|(1+4)|6\rangle^2\langle 4|(1+2)|3\rangle^2}{\langle 12\rangle\,[56]\,t_{124}\langle 2|(1+4)|3\rangle^3} + \frac{[12]^2\langle 35\rangle^2}{[14]\,\langle 56\rangle\,t_{124}\langle 3|(1+2)|4\rangle} - \frac{\langle 24\rangle^2[36]^2t_{124}}{\langle 12\rangle\,[56]\,\langle 2|(1+4)|3\rangle^3}\right) \\
& \qquad \times\ \mathrm{Ls}^{2\mathrm{m}h}_{-1}\big(s_{23},t_{124};s_{14},s_{56}\big) + \frac{\langle 13\rangle\,\langle 3|(1+2)|6\rangle^2}{\langle 12\rangle\,\langle 23\rangle\,[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle}\,\mathrm{Ls}_{-1}\left(\tfrac{-s_{12}}{-t_{123}},\tfrac{-s_{23}}{-t_{123}}\right) \\
& + \frac{\langle 5|(3+4)|2\rangle^2}{[34]\,\langle 56\rangle\,\langle 1|(2+3)|4\rangle}\,\frac{\mathrm{Ls}_{-1}\left(\tfrac{-s_{23}}{-t_{234}},\tfrac{-s_{34}}{-t_{234}}\right)}{t_{234}} - \frac{[12]^2\langle 35\rangle^2}{\langle 56\rangle\,[14]\,\langle 3|(1+2)|4\rangle}\,\frac{\mathrm{Ls}_{-1}\left(\tfrac{-s_{14}}{-t_{124}},\tfrac{-s_{12}}{-t_{124}}\right)}{t_{124}} \\
& + T\,I_3^{3\mathrm{m}}(s_{12},s_{34},s_{56}) + \widetilde{T}\,I_3^{3\mathrm{m}}(s_{14},s_{23},s_{56}) - M_1(1,4,2,3) - \Big[M_1(2,3,1,4)\big|_{\mathrm{flip}_1}\Big] \\
& - 2\,\frac{[12]\,\langle 14\rangle\,\langle 2|(1+4)|6\rangle\langle 4|(1+2)|6\rangle}{\langle 12\rangle\,[56]\,t_{124}\langle 2|(1+4)|3\rangle}\,\frac{\mathrm{L}_0\left(\tfrac{-t_{124}}{-s_{14}}\right)}{s_{14}} + 2\,\frac{[16]\,\langle 1|(3+4)|2\rangle\langle 5|(3+4)|2\rangle}{[34]\,t_{234}\langle 1|(2+3)|4\rangle}\,\frac{\mathrm{L}_0\left(\tfrac{-t_{234}}{-s_{56}}\right)}{s_{56}} \\
& + 2\,\frac{\langle 5|4|2\rangle\langle 5|(3+4)|2\rangle}{[34]\,\langle 56\rangle\,\langle 1|(2+3)|4\rangle}\,\frac{\mathrm{L}_0\left(\tfrac{-t_{234}}{-s_{23}}\right)}{s_{23}} + 2\,\frac{\langle 4|3|6\rangle\langle 4|(1+2)|6\rangle}{\langle 12\rangle\,[56]\,\langle 2|(1+4)|3\rangle}\,\frac{\mathrm{L}_0\left(\tfrac{-t_{124}}{-s_{56}}\right)}{s_{56}} \\
& - 2\,\frac{\langle 5|(3+4)|2\rangle^2}{[34]\,\langle 56\rangle\,t_{234}\langle 1|(2+3)|4\rangle}\,\ln\left(\tfrac{-s_{23}}{-s_{56}}\right),
\end{aligned}
\tag{9.10}
$$

where the three-mass-triangle coefficients are

$$
T = 2\,\frac{\langle 34\rangle^2\langle 5|(3+4)|2\rangle\,[16]}{\langle 23\rangle\ t_{234}^2} + \frac{\langle 13\rangle}{\langle 23\rangle\ t_{123}\,t_{234}}\left(\frac{\langle 4|(2+3)|1\rangle\langle 5|(3+4)|2\rangle\langle 3|(1+2)|6\rangle}{\langle 1|(2+3)|4\rangle} - [12]\,\langle 34\rangle\,\langle 45\rangle\,[16]\right),
\tag{9.11}
$$

$$
\begin{aligned}
\widetilde{T} = {} & \frac{1}{2}\,\frac{t_{124}\,\tilde{\delta}_{23} + 2\,s_{14}s_{56}}{t_{124}^2}\left(\frac{\langle 4|(1+2)|6\rangle^2}{\langle 12\rangle\,[56]\,\langle 2|(1+4)|3\rangle} + \frac{[12]^2\langle 35\rangle^2}{[14]\,\langle 56\rangle\,\langle 3|(1+2)|4\rangle}\right) \\
& - 2\,\frac{\langle 4|12|4\rangle\langle 5|3|6\rangle}{\langle 12\rangle\ t_{124}\,\langle 2|(1+4)|3\rangle} - \frac{\langle 3|1|2\rangle\,\langle 45\rangle\,\langle 3|(1+2)|6\rangle}{\langle 12\rangle\ t_{123}\,\langle 3|(1+2)|4\rangle} \\
& + \frac{1}{\langle 2|(1+4)|3\rangle\langle 3|(1+2)|4\rangle}\Bigg[\frac{\langle 4|12|3\rangle\langle 5|3|6\rangle}{\langle 12\rangle} + \frac{\langle 3|1|2\rangle\langle 5|4|6\rangle}{\langle 12\rangle\,\langle 1|(2+3)|4\rangle}\big(\langle 12\rangle\,(s_{13}-s_{24}) + \langle 1|3(1+4)|2\rangle\big) \\
& \quad - \frac{1}{2}\,\frac{\langle 4|(1+2)|6\rangle}{\langle 12\rangle\,[56]}\big(\langle 3|(1+2)|6\rangle\,\tilde{\delta}_{23} + 2\,\langle 3|1|6\rangle s_{56}\big) - \frac{1}{2}\,\frac{[12]\,\langle 35\rangle\,\langle 5|(3+4)|1\rangle\,\tilde{\delta}_{23}}{[14]\,\langle 56\rangle} \\
& \quad + [26]\,\big(\langle 4|2|1\rangle\,\langle 35\rangle + \langle 3|4|1\rangle\,\langle 45\rangle\big) - \langle 34\rangle\,[16]\,\langle 5|(1+3)|2\rangle\Bigg].
\end{aligned}
\tag{9.12}
$$

The contributions where a scalar replaces the gluon in the loop are

$$
V^{sc} = \frac{1}{2\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^{\epsilon} + \frac{1}{2}\,,
\tag{9.13}
$$

$$F^{sc} = \frac{\langle 13\rangle}{\langle 23\rangle}\Big[M_2(1,2,3,4)\big|_{\rm flip_4}\Big] + \frac{\langle 14\rangle}{\langle 24\rangle}\Big[M_3(1,2,3,4)\big|_{\rm flip_4}\Big]$$
$$+\frac{1}{2}\,\frac{\langle 5|1|2\rangle}{[34]\,\langle 1|(2+3)|4\rangle}\bigg(\frac{\langle 5|(3+4)|2\rangle}{\langle 56\rangle}\,\frac{\mathrm{L}_0\Big(\frac{-t_{234}}{-s_{56}}\Big)}{s_{56}} + [2|(3+4)1|6]\,\frac{\mathrm{L}_1\Big(\frac{-s_{56}}{-t_{234}}\Big)}{t_{234}^2}\bigg)$$
$$+\frac{1}{2}\,\frac{[23]\,\langle 34\rangle\,\langle 5|(2+3)|4\rangle}{[34]\,\langle 56\rangle\,\langle 1|(2+3)|4\rangle}\bigg(\frac{\langle 5|(3+4)|2\rangle}{t_{234}}\,\frac{\mathrm{L}_0\Big(\frac{-t_{234}}{-s_{23}}\Big)}{s_{23}} + \langle 5|4|2\rangle\,\frac{\mathrm{L}_1\Big(\frac{-t_{234}}{-s_{23}}\Big)}{s_{23}^2}\bigg)$$
$$-\frac{1}{2}\,\frac{[12]}{[14]\,\langle 1|(2+3)|4\rangle\langle 2|(1+4)|3\rangle}\bigg(\frac{\langle 5|(3+4)1|5\rangle}{\langle 56\rangle} - \frac{[6|(1+2)4|6]}{[56]}\bigg)\ln\Big(\frac{-s_{23}}{-s_{56}}\Big)$$
$$+\frac{1}{2}\bigg[-\frac{[12]\,\langle 15\rangle\,\langle 35\rangle}{\langle 23\rangle\,[34]\,\langle 56\rangle\,\langle 1|(2+3)|4\rangle} - \frac{[13]^2\,\langle 35\rangle^2}{[14]\,\langle 23\rangle\,[34]\,\langle 56\rangle\,\langle 2|(1+4)|3\rangle} + \frac{\langle 4|3|1\rangle\,\langle 15\rangle\,\langle 35\rangle}{\langle 23\rangle\,\langle 56\rangle\,\langle 1|(2+3)|4\rangle\langle 2|(1+4)|3\rangle}$$
$$-\frac{\langle 13\rangle\,\langle 35\rangle\,\langle 45\rangle}{\langle 12\rangle\,\langle 23\rangle\,\langle 56\rangle\,\langle 1|(2+3)|4\rangle} - \frac{s_{34}\,\langle 15\rangle\,\langle 45\rangle}{\langle 12\rangle\,\langle 56\rangle\,\langle 1|(2+3)|4\rangle\langle 2|(1+4)|3\rangle} - \frac{[12]\,\langle 15\rangle\,\langle 5|(3+4)|2\rangle}{[34]\,\langle 56\rangle\,t_{234}\,\langle 1|(2+3)|4\rangle}$$
$$-\frac{\langle 13\rangle\,[16]^2}{[14]\,\langle 12\rangle\,\langle 23\rangle\,[34]\,[56]} + \frac{\langle 3|4|6\rangle\langle 1|(2+3)|6\rangle}{\langle 12\rangle\,\langle 23\rangle\,[34]\,[56]\,\langle 1|(2+3)|4\rangle} - \frac{s_{13}\langle 3|4|6\rangle\,[36]}{\langle 23\rangle\,[34]\,[56]\,\langle 1|(2+3)|4\rangle\langle 2|(1+4)|3\rangle}$$
$$+\frac{\langle 14\rangle\,\langle 3|4|6\rangle\,[36]}{\langle 12\rangle\,[56]\,\langle 1|(2+3)|4\rangle\langle 2|(1+4)|3\rangle} + \frac{\langle 3|1|6\rangle\,[12]\,[46]}{[14]\,\langle 23\rangle\,[34]\,[56]\,\langle 1|(2+3)|4\rangle} - \frac{\langle 4|(1+2)|6\rangle\,[26]\,\langle 24\rangle}{\langle 12\rangle\,[56]\,t_{124}\,\langle 2|(1+4)|3\rangle}\bigg]. \tag{9.14}$$

10. Results for Primitive Amplitudes: $A_6(1_q, 2_{\bar q}, 3, 4)$

In this section we present the $A_6(1_q, 2_{\bar q}, 3, 4)$ primitive amplitudes, corresponding to the helicity configurations

$$1_q^+, 2_{\bar q}^-, 3^+, 4^+, 5_{\bar e}^-, 6_e^+; \qquad 1_q^+, 2_{\bar q}^-, 3^+, 4^-, 5_{\bar e}^-, 6_e^+; \qquad 1_q^+, 2_{\bar q}^-, 3^-, 4^+, 5_{\bar e}^-, 6_e^+;$$

again we shall suppress the lepton labels below. In the parent diagrams for these amplitudes, both external gluons are attached to the same fermion line as the vector boson, as depicted in fig. 1d. These primitive amplitudes only contribute to subleading-in-color terms.

10.1 The Helicity Configuration $q^+\,\bar q^-\,g^+\,g^+$

Here we present the primitive amplitude $A_6(1_q^+, 2_{\bar q}^-, 3^+, 4^+)$. The tree amplitude is

$$A_6^{\rm tree} = i\,\frac{\langle 25\rangle^2}{\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle\,\langle 56\rangle}\,. \tag{10.1}$$

The cut-constructible terms are,

$$V^{cc} = -\frac{1}{\epsilon^2}\Big(\frac{\mu^2}{-s_{12}}\Big)^{\epsilon} - \frac{2}{\epsilon}\Big(\frac{\mu^2}{-s_{56}}\Big)^{\epsilon} - 4\,, \tag{10.2}$$

$$F^{cc} = \frac{\langle 25\rangle\left(\langle 12\rangle\langle 45\rangle - \langle 24\rangle\langle 15\rangle\right)}{\langle 23\rangle\langle 34\rangle\langle 41\rangle^2\langle 56\rangle}\ \mathrm{Ls}^{2\mathrm{me}}_{-1}\big(t_{123}, t_{234}; s_{23}, s_{56}\big)$$
$$+ \frac{\langle 25\rangle\left(\langle 23\rangle\langle 15\rangle - \langle 12\rangle\langle 35\rangle\right)}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 13\rangle\langle 56\rangle}\ \mathrm{Ls}_{-1}\Big(\tfrac{-s_{12}}{-t_{123}}, \tfrac{-s_{23}}{-t_{123}}\Big)$$
$$- \frac{\langle 25\rangle\left(\langle 23\rangle\langle 45\rangle + \langle 24\rangle\langle 35\rangle\right)}{\langle 23\rangle\langle 34\rangle^2\langle 41\rangle\langle 56\rangle}\ \mathrm{Ls}^{2\mathrm{me}}_{-1}\big(t_{124}, t_{123}; s_{12}, s_{56}\big) \tag{10.3}$$
$$- \frac{\langle 52\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 56\rangle}\ \Big(\mathrm{Ls}_{-1}\Big(\tfrac{-s_{14}}{-t_{124}}, \tfrac{-s_{12}}{-t_{124}}\Big) + \mathrm{Ls}^{2\mathrm{me}}_{-1}\big(t_{134}, t_{124}; s_{14}, s_{56}\big)\Big)$$
$$- 2\,\frac{\langle 52\rangle\langle 5|4|3\rangle}{\langle 34\rangle\langle 41\rangle\langle 56\rangle}\ \frac{\mathrm{L}_0\Big(\frac{-t_{234}}{-s_{23}}\Big)}{s_{23}} + 2\,\frac{\langle 25\rangle\langle 15\rangle\langle 2|(3+4)|1\rangle}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 56\rangle}\ \frac{\mathrm{L}_0\Big(\frac{-t_{234}}{-s_{56}}\Big)}{s_{56}}\,.$$

The contributions where a scalar replaces the gluon are

$$V^{sc} = \frac{1}{2\epsilon}\Big(\frac{\mu^2}{-s_{56}}\Big)^{\epsilon} + \frac{1}{2}\,, \tag{10.4}$$

$$F^{sc} = -\frac{\langle 24\rangle^2\langle 51\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle^3\langle 56\rangle}\ \mathrm{Ls}^{2\mathrm{me}}_{-1}\big(t_{123}, t_{234}; s_{23}, s_{56}\big)$$
$$- \frac{\langle 23\rangle\langle 51\rangle^2}{\langle 34\rangle\langle 41\rangle\langle 13\rangle^2\langle 56\rangle}\mathrm{Ls}_{-1}\Big(\tfrac{-s_{12}}{-t_{123}}, \tfrac{-s_{23}}{-t_{123}}\Big) - \frac{\langle 23\rangle\langle 54\rangle^2}{\langle 34\rangle^3\langle 41\rangle\langle 56\rangle}\ \mathrm{Ls}^{2\mathrm{me}}_{-1}\big(t_{124}, t_{123}; s_{12}, s_{56}\big)$$
$$+ \frac{\big(\langle 41\rangle\langle 52\rangle - \langle 24\rangle\langle 51\rangle\big)\langle 5|4|3\rangle}{\langle 34\rangle\langle 41\rangle^2\langle 56\rangle}\ \frac{\mathrm{L}_0\Big(\frac{-t_{234}}{-s_{23}}\Big)}{s_{23}} - \frac{1}{2}\,\frac{\langle 5|4|3\rangle^2\langle 32\rangle}{\langle 34\rangle\langle 41\rangle\langle 56\rangle}\ \frac{\mathrm{L}_1\Big(\frac{-s_{23}}{-t_{234}}\Big)}{t_{234}^2}$$
$$- \frac{1}{2}\,\frac{\langle 52\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 56\rangle}\ln\Big(\tfrac{-t_{234}}{-s_{56}}\Big) + \left(\frac{\langle 24\rangle\langle 51\rangle^2\langle 2|(3+4)|1\rangle}{\langle 23\rangle\langle 34\rangle\langle 41\rangle^2\langle 56\rangle} - \frac{\langle 52\rangle\langle 2|1|6\rangle}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right)\frac{\mathrm{L}_0\Big(\frac{-t_{234}}{-s_{56}}\Big)}{s_{56}}$$
$$- \frac{1}{2}\,\frac{\langle 56\rangle\langle 2|1|6\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\ \frac{\mathrm{L}_1\Big(\frac{-t_{234}}{-s_{56}}\Big)}{s_{56}^2} + \frac{\langle 23\rangle\langle 5|(1+2)3|5\rangle}{\langle 34\rangle^2\langle 13\rangle\langle 56\rangle}\ \frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{12}}\Big)}{s_{12}} - \frac{\langle 12\rangle^2\langle 5|(2+3)1|5\rangle}{\langle 23\rangle\langle 41\rangle^2\langle 13\rangle\langle 56\rangle}\ \frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{23}}\Big)}{s_{23}}$$
$$+ \left(\frac{\langle 24\rangle\langle 5|4|6\rangle}{\langle 34\rangle^2\langle 41\rangle} - \frac{\big(2\,\langle 41\rangle\langle 52\rangle + \langle 51\rangle\langle 24\rangle\big)\langle 2|4|6\rangle}{\langle 23\rangle\langle 34\rangle\langle 41\rangle^2} + \frac{\langle 52\rangle\langle 54\rangle\langle 2|(1+3)|4\rangle}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 56\rangle}\right)\frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{56}}\Big)}{s_{56}}$$
$$- \frac{\langle 52\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 56\rangle}\ln\Big(\tfrac{-t_{123}}{-s_{56}}\Big) - \frac{\langle 56\rangle\langle 2|4|6\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\ \frac{\mathrm{L}_1\Big(\frac{-t_{123}}{-s_{56}}\Big)}{s_{56}^2} - \frac{\langle 24\rangle\langle 54\rangle\langle 5|(1+2)|4\rangle}{\langle 34\rangle^2\langle 41\rangle\langle 56\rangle}\ \frac{\mathrm{L}_0\Big(\frac{-t_{124}}{-s_{12}}\Big)}{s_{12}}$$
$$- \frac{\langle 24\rangle\langle 5|3|6\rangle}{\langle 34\rangle^2\langle 41\rangle}\ \frac{\mathrm{L}_0\Big(\frac{-t_{124}}{-s_{56}}\Big)}{s_{56}} - \frac{\langle 23\rangle\,[36]^2\,\langle 56\rangle}{\langle 34\rangle\langle 41\rangle}\ \frac{\mathrm{L}_1\Big(\frac{-t_{124}}{-s_{56}}\Big)}{s_{56}^2}$$
$$+ \frac{\langle 51\rangle\langle 52\rangle}{\langle 41\rangle\langle 13\rangle\langle 56\rangle}\left(\frac{1}{\langle 34\rangle}\ln\Big(\tfrac{-s_{12}}{-s_{56}}\Big) - \frac{\langle 12\rangle}{\langle 23\rangle\langle 41\rangle}\ln\Big(\tfrac{-s_{23}}{-s_{56}}\Big)\right)$$
$$+ \frac{1}{2}\left[\frac{[34]\,\langle 4|(1+3)|6\rangle\langle 2|(3+4)|6\rangle}{\langle 34\rangle\langle 41\rangle\, t_{234}t_{134}\,[56]} + \frac{[34]\,\langle 52\rangle\langle 5|1|3\rangle}{\langle 41\rangle\, t_{234}t_{134}\,\langle 56\rangle} + \frac{\langle 52\rangle\langle 5|4|3\rangle}{\langle 34\rangle\langle 41\rangle\, t_{234}\,\langle 56\rangle} + \frac{\langle 25\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 56\rangle}\right]. \tag{10.5}$$

10.2 The Helicity Configuration $q^+\,\bar{q}^-\,g^-\,g^+$

Here we present the primitive amplitude $A_6(1_q^+, 2_{\bar{q}}^-, 3^-, 4^+)$. This amplitude satisfies the symmetry ‘flip$_2$’ defined in eq. (6.8). The tree amplitude is

$$A_6^{\rm tree} = i \left[\frac{[41]\, \langle 13\rangle\, \langle 52\rangle\, \langle 3|(1+4)|6\rangle}{\langle 41\rangle\, s_{34} t_{134} s_{56}} + \frac{\langle 23\rangle\, [24]\, [61]\, \langle 5|(2+3)|4\rangle}{[23]\, s_{34} t_{234} s_{56}} - \frac{\langle 5|(2+3)|4\rangle \langle 3|(1+4)|6\rangle}{[23]\, \langle 41\rangle\, s_{34} s_{56}} \right]. \tag{10.6}$$

An alternate form with manifest $k_5 \parallel k_6$ behavior is

$$A_6^{\rm tree} = -i \left[\frac{[41]\, [24]\, \langle 52\rangle\, \langle 5|(2+3)|4\rangle}{[23]\, [34]\, t_{134} t_{234}\, \langle 56\rangle} + \frac{\langle 23\rangle\, \langle 13\rangle\, [61]\, \langle 3|(1+4)|6\rangle}{\langle 34\rangle\, \langle 41\rangle\, t_{134} t_{234}\, [56]} + \frac{\langle 3|(1+4)|6\rangle \langle 5|(2+3)|4\rangle}{[23]\, \langle 41\rangle\, t_{134} t_{234}} \right]. \tag{10.7}$$

The cut-constructible pieces are,

$$V^{cc} = -\frac{1}{\epsilon^2} \left(\frac{\mu^2}{-s_{12}} \right)^{\epsilon} - \frac{2}{\epsilon} \left(\frac{\mu^2}{-s_{56}} \right)^{\epsilon} - 4\,, \tag{10.8}$$

$$
\begin{aligned}
F^{cc} = & \frac{\langle 5|(2+3)|1\rangle^2}{[23]\,\langle 56\rangle\,\langle 4|(1+2)|3\rangle\langle 4|(2+3)|1\rangle}\left(\widetilde{\mathrm{Ls}}_{-1}^{2\mathrm{m}h}(s_{14},t_{123};s_{23},s_{56}) + \mathrm{Ls}_{-1}\left(\tfrac{-s_{12}}{-t_{123}},\tfrac{-s_{23}}{-t_{123}}\right)\right) \\
& + \left(\frac{[24]^3\langle 15\rangle^2\,t_{234}}{[23]\,[34]\,\langle 56\rangle\,\langle 1|(3+4)|2\rangle^3} - \frac{[24]\,\langle 1|(2+3)|4\rangle^2\langle 5|(3+4)|2\rangle^2}{[23]\,[34]\,\langle 56\rangle\;t_{234}\,\langle 1|(3+4)|2\rangle^3}\right. \\
& \qquad \left. - \frac{\langle 23\rangle^2[16]^2\langle 3|(2+4)|1\rangle}{\langle 34\rangle\,[56]\;t_{234}\,\langle 2|(3+4)|1\rangle\langle 4|(2+3)|1\rangle}\right)\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{12},t_{234};s_{34},s_{56}) \\
& + \left(\frac{[13]^2\langle 45\rangle^2 t_{123}^2}{[23]\,\langle 56\rangle\,\langle 4|(1+2)|3\rangle^3\langle 4|(2+3)|1\rangle} - \frac{\langle 4|(2+3)|1\rangle\langle 5|(1+2)|3\rangle^2}{[23]\,\langle 56\rangle\,\langle 4|(1+2)|3\rangle^3}\right)\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{34},t_{123};s_{12},s_{56}) \\
& - \frac{\langle 23\rangle^2[16]^2}{[56]\;t_{234}\;\langle 24\rangle\,\langle 4|(2+3)|1\rangle}\,\mathrm{Ls}_{-1}\left(\tfrac{-s_{23}}{-t_{234}},\tfrac{-s_{34}}{-t_{234}}\right) \\
& - \frac{1}{2}\,\frac{[14]\left(\langle 2|(1+4)(2+3)|5\rangle^2 - \langle 25\rangle^2\,s_{14}\,s_{23}\right)}{\langle 14\rangle\,[23]\,\langle 56\rangle\,\langle 2|(1+4)|3\rangle\langle 2|(3+4)|1\rangle}\,I_3^{3\mathrm{m}}(s_{14},s_{23},s_{56}) \\
& + \left\{\frac{1}{2}\,\frac{[24]\,\langle 5|(2+3)|4\rangle^2\left(2\,s_{34}\,s_{56}+\delta_{12}\,t_{234}\right)}{[23]\,[34]\,\langle 56\rangle\;t_{234}^2\,\langle 1|(3+4)|2\rangle} + \frac{\langle 23\rangle\,[16]\,\langle 3|(2+4)|1\rangle\langle 5|(2+3)|4\rangle}{t_{234}^2\,\langle 4|(2+3)|1\rangle}\right. \\
& \quad + \frac{1}{2}\,\frac{\langle 23\rangle\,[16]\,\langle 3|(1+4)|6\rangle\langle 3|(2+4)|1\rangle}{\langle 34\rangle\,[56]\;t_{234}\,\langle 4|(2+3)|1\rangle} + \frac{1}{2}\,\frac{[14]\,\langle 25\rangle\,[16]\,\langle 3|2|4\rangle}{[34]\;t_{234}\,\langle 4|(2+3)|1\rangle} + 2\,\frac{\langle 23\rangle\,[24]^2\langle 5|1|6\rangle}{[23]\;t_{234}\,\langle 1|(3+4)|2\rangle} \\
& \quad - \frac{\langle 3|2|1\rangle\langle 5|(2+3)|1\rangle\,[46]}{[23]\;t_{123}\,\langle 4|(2+3)|1\rangle} \\
& \quad + \frac{1}{2}\,\frac{1}{\langle 1|(3+4)|2\rangle\langle 4|(2+3)|1\rangle}\left[\frac{[14]\,\langle 3|1|6\rangle\langle 5|2|4\rangle}{[34]} + \frac{[14]\,\langle 5|4|2\rangle\langle 5|(2+3)|4\rangle\,\delta_{12}}{[23]\,[34]\,\langle 56\rangle}\right. \\
& \qquad - \frac{\langle 23\rangle\,[16]}{\langle 34\rangle\,[56]}\left(\langle 3|2|6\rangle\langle 3|(1+4)|2\rangle + \langle 3|1|2\rangle\langle 3|5|6\rangle\right) + 2\,\frac{[12]\,\langle 3|5|6\rangle\langle 5|(2+3)|4\rangle}{[23]} \\
& \qquad \left. + \frac{[24]\,[16]}{[23]\,[34]}\left(\langle 5|2|4\rangle(s_{56}-t_{123}) + \langle 5|3|4\rangle(s_{56}+t_{234})\right) + [14]\,\langle 35\rangle\,\left(2\,\langle 3|4|6\rangle + \langle 3|2|6\rangle\right)\right] \\
& \quad + \frac{1}{2}\,\frac{1}{\langle 1|(3+4)|2\rangle\langle 4|(1+2)|3\rangle\langle 4|(2+3)|1\rangle}\left[\frac{\langle 4|123|4\rangle}{[23]}\left(\langle 5|(1-4)|2\rangle\,[16] - 2\,[12]\,\langle 5|4|6\rangle\right)\right. \\
& \qquad + [1|(3+4)2|4]\left(\langle 35\rangle\,\langle 4|1|6\rangle + \langle 45\rangle\,\langle 3|2|6\rangle\right) + \langle 5|3|6\rangle\,\left(\langle 3|2|1\rangle\,s_{13} + \langle 3|4|1\rangle\,s_{12}\right) \\
& \qquad + [1|3(1-2)|6]\langle 3|(1+2)4|5\rangle + \langle 5|(3+4)|1\rangle\langle 3|4|6\rangle s_{23} + \langle 5|4|1\rangle\langle 3|(1+2)|6\rangle\,s_{13} \\
& \qquad + \left(3\,\langle 3|2|1\rangle + \langle 3|4|1\rangle\right)\langle 5|2|6\rangle\,s_{14} - \langle 5|3|1\rangle\langle 1|(3+4)|6\rangle\langle 3|(2-4)|1\rangle \\
& \qquad + 2\,\langle 3|4|1\rangle\,[16]\,\left(\langle 51\rangle\,(s_{12}+t_{123}) - \langle 5|23|1\rangle\right) - \langle 3|2|1\rangle\langle 5|4|6\rangle\,s_{34} + 4\,\langle 5|123|1\rangle\langle 3|4|6\rangle \\
& \qquad \left.\left. - \langle 23\rangle\,\langle 4|(2+3)|1\rangle\left(2\,\langle 5|(2-3)|4\rangle\,[26] - \langle 5|1|6\rangle\,[24]\right)\right]\right\}\,I_3^{3\mathrm{m}}(s_{12},s_{34},s_{56}) \\
& + 2\,\frac{\langle 23\rangle\,[24]\,\langle 5|(2+3)|4\rangle\langle 5|(3+4)|2\rangle}{[23]\,\langle 56\rangle\;t_{234}\,\langle 1|(3+4)|2\rangle}\,\frac{\mathrm{L}_0\left(\frac{-t_{234}}{-s_{34}}\right)}{s_{34}} - 2\,\frac{[14]\,[24]\,\langle 15\rangle\,\langle 5|(2+3)|4\rangle}{[23]\,[34]\,\langle 56\rangle\,\langle 1|(3+4)|2\rangle}\,\frac{\mathrm{L}_0\left(\frac{-t_{234}}{-s_{56}}\right)}{s_{56}} \\
& \qquad\qquad + \;\mathrm{flip}_2\,.
\end{aligned} \tag{10.9}
$$

The contributions where a scalar replaces the gluon in the loop are

$$
V^{sc} = \frac{1}{2\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^{\epsilon} + \frac{1}{2}\,, \tag{10.10}
$$

$$F^{sc} = M_2(1,2,3,4) + M_3(1,2,3,4) + \frac{1}{2}\,\frac{\langle 45\rangle\,\langle 3|(1+2)|4\rangle\langle 5|(2+3)|4\rangle}{s_{34}\,\langle 56\rangle\,\langle 1|(3+4)|2\rangle\langle 4|(1+2)|3\rangle}\ln\left(\tfrac{-s_{12}}{-s_{56}}\right)$$
$$- \frac{1}{2}\,\frac{\langle 35\rangle\,\big(\langle 23\rangle\,\langle 45\rangle - \langle 25\rangle\,\langle 34\rangle\big)}{\langle 14\rangle\,\langle 34\rangle\,\langle 56\rangle\,\langle 4|(1+2)|3\rangle} + \frac{1}{2}\,\frac{[24]\,\langle 35\rangle}{[23]\,\langle 56\rangle\,\langle 1|(3+4)|2\rangle}\left(\frac{\langle 35\rangle}{\langle 34\rangle} + \frac{\langle 5|(2+3)|4\rangle}{t_{234}}\right) \;+\; \mathrm{flip}_2\,, \tag{10.11}$$

where M_2 and M_3 are given in eqs. (7.3) and (7.5).

10.3 The Helicity Configuration $q^+\,\bar{q}^-\,g^+\,g^-$

Here we present the primitive amplitude $A_6(1_q^+, 2_{\bar{q}}^-, 3^+, 4^-)$. This amplitude has the same flip symmetry as the last one, as given in eq. (6.8). The tree amplitude is

$$A_6^{\mathrm{tree}} = -i\left[\frac{[13]^2\,\langle 52\rangle\,\langle 4|(1+3)|6\rangle}{[41]\,s_{34}t_{134}s_{56}} - \frac{\langle 24\rangle^2\,[61]\,\langle 5|(2+4)|3\rangle}{\langle 23\rangle\,s_{34}t_{234}s_{56}} + \frac{[13]\,\langle 24\rangle\,[61]\,\langle 52\rangle}{\langle 23\rangle\,[41]\,s_{34}s_{56}}\right]. \tag{10.12}$$

An alternate form with manifest $k_5 \parallel k_6$ behavior is

$$A_6^{\mathrm{tree}} = -i\left[\frac{[13]^2\,\langle 52\rangle\,\langle 5|(2+4)|3\rangle}{[34]\,[41]\,\langle 56\rangle\,t_{134}t_{234}} - \frac{\langle 24\rangle^2\,[61]\,\langle 4|(1+3)|6\rangle}{\langle 23\rangle\,\langle 34\rangle\,[56]\,t_{134}t_{234}} + \frac{[13]\,\langle 24\rangle\,[61]\,\langle 52\rangle}{[41]\,\langle 23\rangle\,t_{134}t_{234}}\right]. \tag{10.13}$$

The results for the cut-constructible pieces are

$$V^{cc} = -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} - \frac{2}{\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^{\epsilon} - 4\,, \tag{10.14}$$

$$F^{cc} = \left(\frac{\langle 12\rangle^2[46]^2\,t_{123}^2}{\langle 23\rangle\,[56]\,\langle 1|(2+3)|4\rangle^3\langle 3|(1+2)|4\rangle} - \frac{\langle 1|(2+3)|6\rangle^2\langle 2|(1+3)|4\rangle^2}{\langle 23\rangle\,[56]\,\langle 1|(2+3)|4\rangle^3\langle 3|(1+2)|4\rangle}\right)\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{14},t_{123};s_{23},s_{56})$$
$$+\left(\frac{\langle 2|(1+3)|4\rangle^2\langle 3|(1+2)|6\rangle^2}{\langle 23\rangle\,[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle^3} - \frac{\langle 23\rangle\,[46]^2\,t_{123}^2}{[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle^3}\right)\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{34},t_{123};s_{12},s_{56})$$
$$+\left(\frac{[23]^2\langle 15\rangle^2\,t_{234}\,\langle 1|(2+4)|3\rangle}{[34]\,\langle 56\rangle\,\langle 1|(2+3)|4\rangle\langle 1|(3+4)|2\rangle^3} - \frac{\langle 1|(2+4)|3\rangle^3\langle 5|(3+4)|2\rangle^2}{[34]\,\langle 56\rangle\;t_{234}\,\langle 1|(2+3)|4\rangle\langle 1|(3+4)|2\rangle^3}\right.$$
$$\left. - \frac{\langle 24\rangle^3[16]^2}{\langle 23\rangle\,\langle 34\rangle\,[56]\;t_{234}\,\langle 2|(3+4)|1\rangle}\right)\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{12},t_{234};s_{34},s_{56})$$
$$+\left(\frac{\langle 12\rangle^2\langle 3|(1+2)|6\rangle^2}{\langle 23\rangle\,[56]\,\langle 13\rangle^2\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle} - \frac{\langle 23\rangle\,\langle 1|(2+3)|6\rangle^2}{[56]\,\langle 13\rangle^2\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle}\right)\mathrm{Ls}_{-1}\left(\tfrac{-s_{12}}{-t_{123}},\tfrac{-s_{23}}{-t_{123}}\right)$$
$$+\left(\frac{\langle 5|(2+3)|4\rangle^2[23]^2}{\langle 56\rangle\;t_{234}\,[24]^3\langle 1|(2+3)|4\rangle} - \frac{\langle 5|(3+4)|2\rangle^2[34]^2}{\langle 56\rangle\;t_{234}\,[24]^3\langle 1|(2+3)|4\rangle}\right)\mathrm{Ls}_{-1}\left(\tfrac{-s_{23}}{-t_{234}},\tfrac{-s_{34}}{-t_{234}}\right)$$
$$+\,T\,I_3^{3\mathrm{m}}(s_{12},s_{34},s_{56}) + \widetilde{T}\,I_3^{3\mathrm{m}}(s_{14},s_{23},s_{56}) + 2\,\frac{\langle 5|1|3\rangle\langle 1|(2+4)|3\rangle\langle 5|(2+4)|3\rangle}{[34]\,\langle 56\rangle\,\langle 1|(2+3)|4\rangle\langle 1|(3+4)|2\rangle}\,\frac{\mathrm{L}_0\left(\frac{-t_{234}}{-s_{56}}\right)}{s_{56}}$$
$$+\,2\,\frac{\langle 4|2|3\rangle\langle 5|(2+3)|4\rangle\langle 5|(2+4)|3\rangle}{\langle 56\rangle\;t_{234}\;[24]\,\langle 1|(2+3)|4\rangle}\,\frac{\mathrm{L}_0\left(\frac{-t_{234}}{-s_{23}}\right)}{s_{23}} + 2\,\frac{\langle 4|2|3\rangle\langle 5|(2+4)|3\rangle\langle 5|(3+4)|2\rangle}{\langle 56\rangle\;t_{234}\;[24]\,\langle 1|(3+4)|2\rangle}\,\frac{\mathrm{L}_0\left(\frac{-t_{234}}{-s_{34}}\right)}{s_{34}}$$
$$+\;M_1(1,4,3,2)\;+\;\mathrm{flip}_2\,. \tag{10.15}$$

The coefficients of the triangle integrals are

$$T = -\frac{\langle 24\rangle^2\,[16]\,\langle 5|(2+4)|3\rangle}{\langle 23\rangle\; t_{234}^2} + \frac{1}{2}\frac{[13]\,\langle 24\rangle\,\langle 25\rangle\,\langle 5|(2+4)|3\rangle}{\langle 23\rangle\,[34]\,\langle 56\rangle\; t_{234}} + \frac{1}{2}\frac{\langle 24\rangle^2\,[16]\,\langle 4|(1+3)|6\rangle}{\langle 23\rangle\,\langle 34\rangle\,[56]\; t_{234}}$$
$$- \frac{\langle 45\rangle\,\langle 2|1|3\rangle\langle 2|(1+3)|6\rangle}{\langle 23\rangle\; t_{123}\,\langle 1|(2+3)|4\rangle} + \frac{1}{2}\frac{1}{\langle 1|(2+3)|4\rangle\langle 1|(3+4)|2\rangle}\Bigg[4\,\frac{\langle 5|1|6\rangle\langle 4|2|3\rangle\langle 1|(2+4)|3\rangle}{t_{234}}$$
$$+ \frac{\langle 1|(2+4)|3\rangle\langle 5|(2+4)|3\rangle^2(2\,s_{34}\,s_{56}+\delta_{12}\,t_{234})}{[34]\,\langle 56\rangle\; t_{234}^2} + \frac{\langle 5|1|3\rangle^2\big(\langle 12\rangle\,\delta_{12}+\langle 1|(2+3)4|2\rangle\big)}{\langle 23\rangle\,[34]\,\langle 56\rangle}$$
$$+ \frac{\langle 24\rangle^3[26]^2\langle 1|(2+3)|4\rangle}{\langle 23\rangle\,\langle 34\rangle\,[56]} - \frac{\langle 24\rangle\,\langle 45\rangle\,\langle 1|(2+3)4(1-2)|6\rangle}{\langle 23\rangle\,\langle 34\rangle} + \frac{\langle 12\rangle\,[36]\,\langle 5|(1-2)|3\rangle(s_{23}+t_{134})}{\langle 23\rangle\,[34]}$$
$$- \frac{\langle 12\rangle\,\langle 45\rangle}{\langle 23\rangle}\big([36]\,(2\,s_{24}+\delta_{34}) - 2\,[3|15|6]\big)\Bigg]$$
$$+ \frac{1}{2}\frac{1}{\langle 1|(2+3)|4\rangle\langle 1|(3+4)|2\rangle\langle 3|(1+2)|4\rangle}\Bigg[\langle 1|4|6\rangle\,(\langle 5|1|3\rangle\,s_{13} - \langle 5|2|3\rangle\,s_{23}) + 5\,\langle 1|241|3\rangle\langle 5|4|6\rangle$$
$$+ \langle 5|4|3\rangle\langle 1|(2+3)|6\rangle(s_{13}-s_{24}) + \langle 1|2|3\rangle\Bigg(\langle 5|2|6\rangle\,s_{34} + \langle 5|4|6\rangle\,s_{13} + 2\,\frac{\langle 2|14|3\rangle\langle 5|(1+3)|6\rangle}{\langle 23\rangle}$$
$$+ 3\,\big(\langle 5|431|6\rangle - \langle 5|3|6\rangle\,s_{24}\big) - 2\,\big(\langle 5|1|6\rangle(s_{14}+s_{24}) + \langle 5|143|6\rangle\big)\Bigg)\Bigg]\,, \tag{10.16}$$

$$\widetilde{T} = \frac{1}{\langle 1|(2+3)|4\rangle\langle 3|(1+4)|2\rangle}\Bigg[\frac{1}{2}\langle 4|(1-2)|3\rangle\langle 5|(1+4)|6\rangle + \langle 5|1|3\rangle\langle 4|2|6\rangle$$
$$+ \frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle}\big(\langle 5|2|4\rangle\langle 3|1|6\rangle - \langle 3|1|4\rangle\langle 5|(2-3)|6\rangle\big)\Bigg] - \frac{[13]\,\langle 45\rangle\,\langle 2|(1+3)|6\rangle}{\langle 3|(1+2)|4\rangle\,t_{123}}\,. \tag{10.17}$$

The contributions where a scalar replaces the gluon in the loop are

$$V^{sc} = \frac{1}{2\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^{\epsilon} + \frac{1}{2}\,, \tag{10.18}$$

$$
\begin{aligned}
F^{sc} = {} & \frac{[24]\,\langle 3|(2+4)|1\rangle}{[14]\,\langle 3|(1+2)|4\rangle}\,M_2(1,4,2,3) - \frac{[23]^2\,\langle 24\rangle^2\,\langle 5|(2+3)|4\rangle^2}{\langle 56\rangle\,[24]\,t_{234}\langle 1|(2+3)|4\rangle}\,\frac{\mathrm{Ls}_1\Big(\frac{-s_{23}}{-t_{234}},\frac{-s_{34}}{-t_{234}}\Big)}{t_{234}^2} \\
& + \frac{\langle 23\rangle^2\,[46]^2\,t_{123}^2}{\langle 23\rangle\,[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle^3}\;\mathrm{Ls}_{-1}^{\mathrm{2m}h}\big(s_{34},t_{123};s_{12},s_{56}\big) \\
& - \frac{[23]^2\,\langle 15\rangle^2\,t_{234}\,\langle 1|(2+4)|3\rangle}{[34]\,\langle 56\rangle\,\langle 1|(2+3)|4\rangle\langle 1|(3+4)|2\rangle^3}\;\mathrm{Ls}_{-1}^{\mathrm{2m}h}\big(s_{12},t_{234};s_{34},s_{56}\big) \\
& + \frac{\langle 23\rangle^2\,\langle 1|(2+3)|6\rangle^2}{\langle 23\rangle\,[56]\,\langle 13\rangle^2\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle}\mathrm{Ls}_{-1}\Big(\frac{-s_{12}}{-t_{123}},\frac{-s_{23}}{-t_{123}}\Big) + T\,I_3^{\mathrm{3m}}(s_{12},s_{34},s_{56}) + \widetilde{T}\,I_3^{\mathrm{3m}}(s_{14},s_{23},s_{56}) \\
& + \frac{[13]}{[56]\,\langle 13\rangle\,\langle 3|(1+2)|4\rangle}\Bigg(\frac{\langle 12\rangle\,\langle 3|(1+2)|6\rangle^2}{\langle 3|(1+2)|4\rangle}\,\frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{12}}\Big)}{s_{12}} - \frac{\langle 23\rangle\,\langle 1|(2+3)|6\rangle^2}{\langle 1|(2+3)|4\rangle}\,\frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{23}}\Big)}{s_{23}}\Bigg) \\
& + \frac{1}{2}\,\frac{[23]\,\langle 24\rangle^2}{\langle 56\rangle\,[24]\,t_{234}}\Bigg(\frac{[23]\,\langle 5|(2+3)|4\rangle^2}{\langle 1|(2+3)|4\rangle}\,\frac{\mathrm{L}_1\Big(\frac{-t_{234}}{-s_{23}}\Big)}{s_{23}^2} - \frac{[34]\,\langle 5|(3+4)|2\rangle^2}{\langle 1|(3+4)|2\rangle}\,\frac{\mathrm{L}_1\Big(\frac{-t_{234}}{-s_{34}}\Big)}{s_{34}^2}\Bigg) \\
& + \frac{\langle 24\rangle\,[46]^2\,t_{123}^2}{[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle^2}\,\frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{56}}\Big)}{s_{56}} - \frac{[23]^2\,\langle 24\rangle\,\langle 15\rangle^2\,t_{234}}{\langle 56\rangle\,\langle 1|(2+3)|4\rangle\langle 1|(3+4)|2\rangle^2}\,\frac{\mathrm{L}_0\Big(\frac{-t_{234}}{-s_{34}}\Big)}{s_{34}} \\
& + \frac{1}{2}\,\frac{[13]\,[23]\,\langle 15\rangle^2\,t_{234}\,\langle 1|(2+4)|3\rangle}{[34]\,\langle 56\rangle\,\langle 1|(2+3)|4\rangle\langle 1|(3+4)|2\rangle}\Bigg(\frac{[13]}{[23]}\,\frac{\mathrm{L}_1\Big(\frac{-s_{56}}{-t_{234}}\Big)}{t_{234}^2} - \frac{2}{\langle 1|(3+4)|2\rangle}\,\frac{\mathrm{L}_0\Big(\frac{-s_{56}}{-t_{234}}\Big)}{t_{234}}\Bigg) \\
& + \frac{1}{2}\,\frac{[13]\,[46]}{\langle 1|(2+3)|4\rangle\langle 3|(1+4)|2\rangle\langle 3|(1+2)|4\rangle}\Big(\langle 13\rangle\,\langle 45\rangle + \frac{\langle 3|(1+4)|6\rangle t_{123} - \langle 3|(1+2)|6\rangle t_{134}}{[14]\,[56]}\Big)\ln\Big(\frac{-s_{23}}{-s_{56}}\Big) \\
& + \Bigg\{3\,\frac{[23]\,\langle 2|(3+4)|1\rangle}{\langle 3|(1+2)|4\rangle\,\tilde{\Delta}_3^2}\Big(\langle 45\rangle\,\big(\langle 2|3|6\rangle\,\tilde{\delta}_{23} - \langle 2|5|6\rangle\,\tilde{\delta}_{56}\big) - \langle 14\rangle\,[16]\,\big(\langle 25\rangle\,\tilde{\delta}_{14} - 2\,\langle 2|36|5\rangle\big)\Big) \\
& \quad + \frac{1}{2}\,\frac{1}{\langle 3|(1+2)|4\rangle\,\tilde{\Delta}_3}\Bigg[\Big(-\frac{\langle 25\rangle\,\langle 45\rangle}{\langle 56\rangle} + \frac{[16]\,\langle 2|(1+4)|6\rangle}{[14]\,[56]}\Big)\big([13]\,\tilde{\delta}_{23} - [1|(5+6)2|3]\big) \\
& \quad - 2\,\frac{[36]\,\langle 2|(3+4)|1\rangle}{[14]\,[56]}\big([16]\,\tilde{\delta}_{23} - 2\,[1|45|6]\big) - 4\,\langle 4|1|3\rangle\,[16]\,\langle 25\rangle\Bigg]\Bigg\}\ln\Big(\frac{-s_{23}}{-s_{56}}\Big) \\
& - M_{2a}(1,2,4,3) - M_{3a}(1,2,4,3) \\
& - \frac{1}{2}\,\frac{\langle 35\rangle\,\langle 4|(1+2)|3\rangle\langle 5|(2+4)|3\rangle}{s_{34}\,\langle 56\rangle\,\langle 1|(3+4)|2\rangle\langle 3|(1+2)|4\rangle}\,\ln\Big(\frac{-s_{12}}{-s_{56}}\Big) + \frac{1}{2}\,\frac{[46]\,\langle 4|(1+2)|3\rangle}{[56]\,\langle 3|(1+2)|4\rangle\langle 1|(2+3)|4\rangle\,\Delta_3} \\
& \quad \times\Big(\frac{\langle 3|(1+2)|6\rangle\big(\langle 24\rangle\;\delta_{56} - 2\,\langle 2|13|4\rangle\big)}{\langle 34\rangle} + \frac{\langle 12\rangle\,[46]\,\big([13]\;\delta_{56} - 2\,[1|24|3]\big)}{[34]}\Big) \\
& + \frac{1}{2}\,\frac{\langle 15\rangle\,\langle 4|(1+2)|3\rangle}{\langle 56\rangle\,\langle 1|(3+4)|2\rangle\langle 1|(2+3)|4\rangle\,\Delta_3}\,\Big(\big(\langle 5|2|1\rangle\delta_{12} - \langle 5|6|1\rangle\delta_{56}\big)\frac{\langle 14\rangle}{\langle 34\rangle} - [23]\,\big(\langle 25\rangle\,\delta_{56} - 2\,\langle 2|1(3+4)|5\rangle\big)\Big) \\
& + \frac{1}{2}\,\frac{\langle 2|(3+4)|1\rangle}{\langle 3|(1+2)|4\rangle\,\tilde{\Delta}_3}\Big(\frac{1}{2}\,\frac{\tilde{\delta}_{56}\,\langle 25\rangle\,[16]}{[14]\,\langle 23\rangle} + \frac{\tilde{\delta}_{14}\,\langle 45\rangle\,\langle 25\rangle}{\langle 23\rangle\,\langle 56\rangle} + \langle 45\rangle\,[36]\Big) \\
& + \frac{1}{2}\,\frac{1}{\langle 3|(1+2)|4\rangle\langle 1|(2+3)|4\rangle}\Big(-\frac{\langle 12\rangle\,[13]^2\,[46]^2}{[14]\,[34]\,[56]} - \frac{\langle 25\rangle\,\langle 45\rangle\,\langle 2|(1+3)|4\rangle}{\langle 23\rangle\,\langle 56\rangle} + \frac{\langle 24\rangle\,\langle 45\rangle\,[46]}{\langle 34\rangle}\Big) \\
& - \frac{1}{2}\,\frac{\langle 24\rangle^2\,\langle 5|(3+4)|2\rangle\langle 5|(2+3)|4\rangle}{\langle 23\rangle\,\langle 34\rangle\,\langle 56\rangle\;t_{234}\;[24]\,\langle 1|(2+3)|4\rangle} + \frac{1}{2}\,\frac{\langle 15\rangle\,\langle 45\rangle\,\langle 4|(1+2)|3\rangle}{\langle 34\rangle\,\langle 56\rangle\,\langle 1|(3+4)|2\rangle\langle 1|(2+3)|4\rangle} \;+\; \mathrm{flip}_2\,.
\end{aligned}
\tag{10.19}
$$

The three-mass triangle coefficients are

$$
\begin{aligned}
T = 3\, & \frac{s_{12}\,[46]\,\langle 56\rangle\,\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle\langle 1|(2+3)|4\rangle\,\Delta_3^2}\Big[\langle 12\rangle\,[56]\,\big(\langle 5|6|1\rangle\,\delta_{56} - \langle 5|2|1\rangle\,\delta_{12}\big) + t_{124}\,\big(\langle 2|5|6\rangle\,\delta_{56} - \langle 2|1|6\rangle\,\delta_{12}\big)\Big] \\
& + \frac{\langle 12\rangle\,[46]}{\langle 3|(1+2)|4\rangle\langle 1|(2+3)|4\rangle\,\Delta_3}\bigg[-[12]\,\langle 56\rangle\,\Big(3\,\langle 4|(1+2)|3\rangle\langle 2|(1+3)|6\rangle - \langle 24\rangle\,[36]\,(t_{123}-t_{124})\Big) \\
& \quad + \langle 4|(1+2)|3\rangle\big(\langle 5|2|1\rangle\,\delta_{12} - \langle 5|6|1\rangle\,\delta_{56} + t_{123}\,\langle 5|(2+4)|1\rangle - t_{124}\,\langle 5|(2+3)|1\rangle\big) \\
& \quad + \langle 45\rangle\,t_{123}\big([1|(3+4)(1+2)|3] - [1|24|3]\big) - \frac{\langle 4|(1+2)3|5\rangle\,t_{123}\big([1|(3+4)(1+2)|4] - [1|23|4]\big)}{\langle 3|(1+2)|4\rangle}\bigg] \\
& - \frac{\langle 2|1|3\rangle\langle 5|4|6\rangle}{\langle 3|(1+2)|4\rangle\langle 1|(2+3)|4\rangle} \\
& + 3\,\frac{\langle 1|5|6\rangle\langle 4|(1+2)|3\rangle}{\langle 1|(2+3)|4\rangle\Delta_3^2}\bigg[\langle 2|1|3\rangle\big(\langle 5|2|1\rangle\,\delta_{12} - \langle 5|6|1\rangle\,\delta_{56}\big) + \langle 1|2|3\rangle\big(\langle 5|6|2\rangle\,\delta_{56} - \langle 5|1|2\rangle\,\delta_{12}\big)\frac{\langle 2|(3+4)|1\rangle}{\langle 1|(3+4)|2\rangle}\bigg] \\
& + \frac{\langle 15\rangle}{\langle 1|(2+3)|4\rangle\,\Delta_3}\bigg[-[13]\,\Big(\langle 2|1|3\rangle\langle 4|5|6\rangle + \langle 4|2|3\rangle\langle 2|5|6\rangle - \langle 4|(1+2)|3\rangle\langle 2|(1-5)|6\rangle\Big) \\
& \quad + [23]\,\frac{\langle 2|(3+4)|1\rangle}{\langle 1|(3+4)|2\rangle}\Big(3\,\langle 1|5|6\rangle\langle 4|2|3\rangle - \langle 1|2|6\rangle\langle 4|(1+2)|3\rangle + \frac{[23]\,\langle 14\rangle\,\langle 1|5|6\rangle(t_{234}-t_{134})}{\langle 1|(3+4)|2\rangle}\Big)\bigg]\,,
\end{aligned}
\tag{10.20}
$$

$$
\begin{aligned}
\widetilde{T} = 3\, & \frac{\langle 14\rangle\,[23]\,\langle 2|(3+4)|1\rangle}{\langle 3|(1+2)|4\rangle\,\tilde{\Delta}_3^2}\bigg[\big(\tilde{\delta}_{14}\,\langle 2|5|6\rangle - s_{56}\,\langle 2|3|6\rangle\big)\langle 5|4|1\rangle - \frac{1}{2}\,\tilde{\delta}_{56}\langle 2|5|6\rangle\langle 5|6|1\rangle\bigg] \\
& - \frac{\langle 2|(3+4)|1\rangle}{\langle 3|(1+2)|4\rangle\,\tilde{\Delta}_3}\bigg[\frac{1}{2}\,\langle 4|5|6\rangle\big(\langle 5|6|3\rangle + 2\,\langle 5|2|3\rangle\big) - \langle 5|2|3\rangle\langle 4|1|6\rangle\bigg]\,.
\end{aligned}
\tag{10.21}
$$

11. Contributions with the Vector Boson coupled to a Fermion Loop

We now consider the remaining contributions where the vector boson (γ^*, Z) couples directly to a quark loop, as depicted in figs. 1e and f. The contributions proportional to the vector and axial-vector couplings of the quark to the vector boson are separately gauge invariant and have different symmetry properties, so we separate the two contributions. Both contributions are infrared and ultraviolet finite, because there is no tree-level coupling between the vector boson and any number of gluons. Therefore for each amplitude in this section $V = 0$, and we just give the finite (F) terms in eq. (6.1).

For the vector coupling case, Furry's theorem (charge conjugation) implies that only box diagrams contribute. The three box diagrams are shown in fig. 1e. A simple way to understand the cancellation of triangle diagrams, depicted in fig. 9, is that under reversal of the fermion loop arrow the sign of the diagrams flip so that there is a pairwise cancellation. (This argument also relies on the existence of only one $SU(N_c)$-invariant combination of two gluons, that proportional to δ^{ab}.) For the axial-vector case, Furry's theorem does not apply, and triangle diagrams such as those in fig. 9 do contribute.

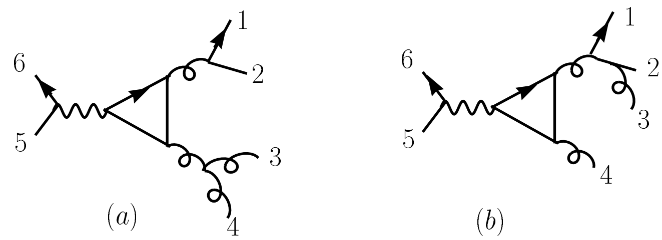

Figure 9. Two examples of triangle diagrams that contribute to the axial-vector coupling case, but cancel in the vector coupling case.

For a massless isodoublet the axial-vector contribution exactly cancels. Since we take the u, d, s, c, b quarks to be massless, but not the t quark, only the t, b pair survives the isodoublet cancellation in the loop. As a uniform approximation we keep all terms of order $1/m_t^2$ but drop terms of order $1/m_t^4$ and beyond. For the vector coupling, the contribution of the top quark loop decouples rapidly, leaving only terms of order $1/m_t^4$ which we drop. For the axial-vector part, terms of order $1/m_t^2$ appear which we keep. The $1/m_t^2$ terms are rather easily obtained (especially when compared to a diagrammatic calculation) from an effective Lagrangian (or operator product) analysis, where the coefficients are fixed via the collinear limits.

11.1 The Helicity Configuration $q^+\,\bar{q}^-\,g^+\,g^+$

We now present the results for the primitive amplitudes $A_6^{\mathrm{v}s}(1_q^+, 2_{\bar{q}}^-, 3^+, 4^+)$, $A_6^{\mathrm{v}f}(1_q^+, 2_{\bar{q}}^-, 3^+, 4^+)$, $A_6^{\mathrm{ax}}(1_q^+, 2_{\bar{q}}^-, 3^+, 4^+)$, and $A_6^{\mathrm{ax,sl}}(1_q^+, 2_{\bar{q}}^-, 3^+, 4^+)$.

The various contributions are

$$
\begin{aligned}
F^{\mathrm{v}s} = & -\frac{\langle 23\rangle\langle 24\rangle\langle 35\rangle\langle 45\rangle}{\langle 12\rangle\langle 56\rangle\langle 34\rangle^4}\,\mathrm{Ls}_{-1}^{2\mathrm{m}e}\big(t_{123}, t_{124}; s_{12}, s_{56}\big) \\
& -\frac{\langle 24\rangle\langle 35\rangle + \langle 23\rangle\langle 45\rangle}{\langle 12\rangle\langle 56\rangle\langle 34\rangle^3}\left[\langle 5|31|2\rangle\,\frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{12}}\Big)}{s_{12}} + \langle 5|64|2\rangle\,\frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{56}}\Big)}{s_{56}} + \frac{1}{2}\langle 52\rangle\ln\Big(\frac{-t_{123}}{-t_{124}}\Big)\right] \\
& -\left[\frac{\langle 12\rangle\langle 5|3|1\rangle^2}{\langle 56\rangle\langle 34\rangle^2}\,\frac{\mathrm{L}_1\Big(\frac{-t_{123}}{-s_{12}}\Big)}{s_{12}^2} + \frac{\langle 52\rangle\langle 5|3|1\rangle}{\langle 56\rangle\langle 34\rangle^2}\,\frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{12}}\Big)}{s_{12}} + \mathrm{exch}_{16,25}\right] \\
& +\frac{1}{2}\,\frac{1}{\langle 34\rangle^2}\left(\frac{\langle 25\rangle^2}{\langle 12\rangle\langle 56\rangle} - \frac{[16]^2}{[12]\,[56]}\right) + \mathrm{exch}_{34}\,,
\end{aligned}
\tag{11.1}
$$

$$
F^{\mathrm{v}f} = -\frac{\langle 25\rangle^2}{\langle 12\rangle\langle 56\rangle\langle 34\rangle^2}\,\mathrm{Ls}_{-1}^{2\mathrm{me}}\big(t_{123}, t_{124}; s_{12}, s_{56}\big)\,,
\tag{11.2}
$$

$$
\begin{aligned}
F^{\rm ax} = & -\frac{1}{2}\frac{(\langle 23\rangle\,\langle 45\rangle + \langle 24\rangle\,\langle 35\rangle)\,\langle 25\rangle}{\langle 12\rangle\,\langle 56\rangle\,\langle 34\rangle^3}\,\mathrm{Ls}_{-1}^{\rm 2me}(t_{123},t_{124};s_{12},s_{56}) + \Bigg[\frac{\langle 25\rangle^2}{\langle 12\rangle\,\langle 56\rangle\,\langle 34\rangle^2}\ln\Big(\frac{-t_{123}}{-s_{56}}\Big) \\
& - \frac{\langle 2|4|6\rangle\,\langle 25\rangle}{\langle 12\rangle\,\langle 34\rangle^2}\left(s_{34}\,\frac{\mathrm{L}_1\Big(\frac{-t_{123}}{-s_{56}}\Big)}{s_{56}^2} + \frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{56}}\Big)}{s_{56}}\right) - \frac{\langle 5|3|1\rangle\,\langle 25\rangle}{\langle 56\rangle\,\langle 34\rangle^2}\frac{\mathrm{L}_0\Big(\frac{-t_{123}}{-s_{12}}\Big)}{s_{12}} - \mathrm{exch}_{34}\Bigg] \\
& + (s_{14}+s_{34})\,\frac{\langle 25\rangle\,[46]}{\langle 13\rangle\,\langle 34\rangle}\,\frac{\mathrm{L}_1\Big(\frac{-t_{123}}{-s_{56}}\Big)}{s_{56}^2} + \frac{\langle 2|3|1\rangle\,\langle 25\rangle\,[36]}{\langle 24\rangle\,\langle 34\rangle}\,\frac{\mathrm{L}_1\Big(\frac{-t_{124}}{-s_{56}}\Big)}{s_{56}^2} \\
& - \frac{1}{12\,m_t^2}\,\frac{\langle 25\rangle}{\langle 34\rangle\,s_{56}}\left(\frac{[46]\,t_{134}}{\langle 13\rangle} + \frac{[36]\,\langle 2|(3+4)|1\rangle}{\langle 24\rangle}\right),
\end{aligned}
\tag{11.3}
$$

and

$$
F^{\rm ax,sl} = 2\,\frac{\langle 25\rangle\,[46]\,\langle 2|(1+3)|4\rangle}{\langle 13\rangle\,\langle 23\rangle\,s_{56}}\left[-\frac{1}{2}\frac{\mathrm{L}_1\Big(\frac{-t_{123}}{-s_{56}}\Big)}{s_{56}} + \frac{1}{24\,m_t^2}\right] + \mathrm{exch}_{34}\,. \tag{11.4}
$$

11.2 The Helicity Configurations $q^+\,\bar{q}^-\,g^+\,g^-$ and $q^+\,\bar{q}^-\,g^-\,g^+$

It is convenient to quote the helicity amplitudes $A_6^{\rm v,ax}(1_q^+,2_{\bar{q}}^-,3^+,4^-)$ and $A_6^{\rm v,ax}(1_q^+,2_{\bar{q}}^-,3^-,4^+)$ together because they are rather similar. In fact for the vector case Furry's theorem (charge conjugation) implies that the two $A_6^{\rm v}$ amplitudes are identical up to the relabelling $3 \leftrightarrow 4$,

$$
A_6^{\rm v}(1_q^+,2_{\bar{q}}^-,3^-,4^+) = A_6^{\rm v}(1_q^+,2_{\bar{q}}^-,3^+,4^-)\big|_{3\leftrightarrow 4}\,. \tag{11.5}
$$

For $A_6^{\rm v}(1_q^+, 2_{\bar q}^-, 3^+, 4^-)$ we have

$$
\begin{aligned}
F^{{\rm v}s} &= -2\,\frac{\langle 23\rangle\,[46]\,\langle 3|(1+2)|6\rangle\langle 2|(5+6)|4\rangle\,t_{123}}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle^4}\,{\rm Ls}_{-1}^{2{\rm m}h}\big(s_{34},t_{123};s_{12},s_{56}\big)\\
&\quad + \bigg[2\,\langle 23\rangle\,[46]\,\big([12]\,\langle 56\rangle\,\langle 23\rangle\,[46] - \langle 5|(2+4)|1\rangle\langle 3|(1+2)|4\rangle\big)\,\frac{\langle 4|(1+2)|3\rangle(t_{123}-t_{124})}{\langle 3|(1+2)|4\rangle^3\,\Delta_3}\\
&\qquad - 3\,\Big(\frac{s_{34}\,\delta_{34}\langle 2|(3+4)|1\rangle\langle 5|(3+4)|6\rangle}{\Delta_3} - \langle 2|3|1\rangle\langle 5|4|6\rangle - \langle 2|4|1\rangle\langle 5|3|6\rangle\Big)\,\frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle\,\Delta_3}\\
&\qquad - \frac{[14]\,\langle 35\rangle\,\langle 23\rangle\,[46]\,\langle 4|(1+2)|3\rangle^2}{\langle 3|(1+2)|4\rangle^2\,\Delta_3} - \frac{[13]\,\langle 45\rangle\,\langle 24\rangle\,[36]}{\Delta_3}\bigg]\,I_3^{3{\rm m}}(s_{12},s_{34},s_{56})\\
&\quad + \bigg(2\,\frac{\langle 23\rangle\,[46]\;t_{123}\,\langle 2|(1+4)|6\rangle}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle^3} - \frac{\langle 2|(1+4)|6\rangle^2 + 2\,\langle 2|3|6\rangle\langle 2|4|6\rangle}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle^2}\bigg)\,\ln\Big(\tfrac{-t_{123}}{-s_{34}}\Big)\\
&\quad + \bigg\{2\,\frac{[12]\,\langle 2|3|6\rangle}{[56]\,\langle 3|(1+2)|4\rangle^2}\bigg[\frac{\langle 3|(1+2)|6\rangle\langle 2|(1+3)|4\rangle}{\langle 3|(1+2)|4\rangle}\,\frac{{\rm L}_0\big(\frac{-t_{123}}{-s_{12}}\big)}{s_{12}} - \langle 2|3|6\rangle\Big(\frac{{\rm L}_0\big(\frac{-t_{123}}{-s_{12}}\big)}{s_{12}} - \frac{1}{2}\,\frac{{\rm L}_1\big(\frac{-t_{123}}{-s_{12}}\big)}{s_{12}}\Big)\bigg]\\
&\qquad - \bigg[3\,\frac{\delta_{56}\,\langle 2|(3+4)|1\rangle\langle 5|(3+4)|6\rangle\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle\,\Delta_3^2} + 2\,\frac{\langle 3|2|1\rangle\,[46]\,(t_{123}-t_{124})\big(\langle 25\rangle\,t_{123} + \langle 2|16|5\rangle\big)}{\langle 3|(1+2)|4\rangle^3\,\Delta_3}\\
&\qquad\quad - \Big(2\,\langle 2|3|6\rangle(t_{123}-t_{124}) + \frac{\delta_{12}\big(\langle 25\rangle\,t_{123} + \langle 2|16|5\rangle\big)}{\langle 56\rangle}\Big)\,\frac{\langle 5|2|1\rangle}{\langle 3|(1+2)|4\rangle^2\,\Delta_3}\\
&\qquad\quad + \frac{1}{2}\Big(\frac{\langle 12\rangle\,[16]^2}{[56]} + \frac{[12]\,\langle 25\rangle^2}{\langle 56\rangle} - 2\,\langle 25\rangle\,[16]\Big)\,\frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle\,\Delta_3}\bigg]\ln\Big(\tfrac{-s_{12}}{-s_{34}}\Big) \;+\; {\rm flip}_3\bigg\}\\
&\quad + \frac{[16]\,\langle 4|(1+2)|3\rangle\big([16]\,\delta_{34} - 2\,[1|25|6]\big)}{[12]\,[56]\,\langle 3|(1+2)|4\rangle\,\Delta_3} + \frac{\langle 2|(1+4)|6\rangle^2 + \langle 2|1|6\rangle\langle 2|5|6\rangle}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle^2} \;+\; {\rm flip}_2\,,
\end{aligned}
\tag{11.6}
$$

$$
\begin{aligned}
F^{{\rm v}f} &= -\frac{\langle 2|(1+3)|6\rangle^2}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle^2}\,{\rm Ls}_{-1}^{2{\rm m}h}\big(s_{34},t_{123};s_{12},s_{56}\big)\\
&\qquad\qquad - \Big[\frac{1}{2}\,\frac{[13]\,\langle 45\rangle\,\langle 2|(1+3)|6\rangle}{t_{123}\,\langle 3|(1+2)|4\rangle}\,I_3^{3{\rm m}}(s_{12},s_{34},s_{56}) \;+\; {\rm flip}_3\Big] \;+\; {\rm flip}_2\,.
\end{aligned}
\tag{11.7}
$$

The full axial-vector amplitude $A_6^{\rm ax}$ does not obey a relation as simple as eq. (11.5) under exchange of the two gluons. However, the more complicated parts of $A_6^{\rm ax}$ (the box diagram contributions) do obey such a relation, with an additional minus sign due to the γ_5 insertion. Thus it is convenient to separate the $A_6^{\rm ax}$ contributions into a common part $C^{\rm ax}$ obeying the relation, plus additional terms with no special symmetry. The common part is

$$C^{\rm ax} = -\frac{1}{2}\,\frac{\langle 2|(1+3)|4\rangle^2\langle 3|(1+2)|6\rangle^2 - \langle 23\rangle^2\,[46]^2\;t_{123}^2}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle^4}\,{\rm Ls}_{-1}^{2{\rm m}h}\big(s_{34},t_{123};s_{12},s_{56}\big)$$
$$+\Bigg[-\frac{3}{2}\big(\langle 5|2|1\rangle\langle 2|1|6\rangle+\langle 5|6|1\rangle\langle 2|5|6\rangle-\langle 5|3|1\rangle\langle 2|4|6\rangle-\langle 5|4|1\rangle\langle 2|3|6\rangle\big)\frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle\,\Delta_3}$$
$$-3\,\frac{\delta_{34}\big(\langle 5|2|1\rangle\delta_{12}-\langle 5|6|1\rangle\delta_{56}\big)\langle 4|(1+2)|3\rangle\langle 2|(1+3)|6\rangle}{\langle 3|(1+2)|4\rangle\,\Delta_3^2} - \frac{[13]\,\langle 45\rangle\,\langle 24\rangle\,[36]}{\Delta_3}$$
$$+\frac{[14]\,\langle 35\rangle\,(t_{123}-t_{124})\langle 4|(1+2)|3\rangle\langle 2|(1+3)|6\rangle}{\langle 3|(1+2)|4\rangle^2\,\Delta_3} - \frac{1}{2}\,\frac{[13]\,\langle 45\rangle\,\langle 2|(1+3)|6\rangle}{t_{123}\,\langle 3|(1+2)|4\rangle}\Bigg]\,I_3^{\rm 3m}(s_{12},s_{34},s_{56})$$
$$+\Bigg[\Bigg(-6\,\frac{[12]\,\langle 2|(1+3)|6\rangle\big(\langle 25\rangle\,\delta_{34}-2\langle 2|16|5\rangle\big)\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle\,\Delta_3^2} - \frac{[13]\,[46]\,\langle 2|(1+3)|6\rangle}{[34]\,[56]\,\langle 3|(1+2)|4\rangle^2}$$
$$+[14]\,\frac{\langle 2|(1+3)|6\rangle\big(3\,[4|(1+2)3|6]-[46]\,(t_{123}-t_{124})\big)\langle 4|(1+2)|3\rangle}{[34]\,[56]\,\langle 3|(1+2)|4\rangle^2\,\Delta_3} - \frac{[13]\,\langle 24\rangle\,[36]^2}{[34]\,[56]\,\Delta_3}\Bigg)\ln\Big(\frac{-s_{12}}{-s_{34}}\Big)$$
$$+\,{\rm exch}_{16,25}\Bigg] + \frac{\langle 2|(1+3)|6\rangle^2}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle^2}\ln\Big(\frac{-s_{56}}{-s_{34}}\Big)$$
$$+\frac{\langle 24\rangle\,[36]}{\langle 3|(1+2)|4\rangle}\Bigg(\frac{\langle 2|4|6\rangle\,\delta_{34}}{\langle 12\rangle\,[56]\,\Delta_3} - \frac{\langle 24\rangle\,\langle 35\rangle\,\delta_{56}}{\langle 12\rangle\,\langle 34\rangle\,\Delta_3} - \frac{[13]\,[46]\,\delta_{12}}{[34]\,[56]\,\Delta_3} - 2\,\frac{\langle 5|3|1\rangle}{\Delta_3} + \frac{\langle 24\rangle\,\langle 35\rangle}{\langle 12\rangle\,\langle 34\rangle\,s_{56}}\Bigg)\,, \tag{11.8}$$

where the operation ${\rm exch}_{16,25}$ is to be applied to all preceding terms within the brackets.

For $A_6^{\rm ax}(1_q^+,2_{\bar q}^-,3^+,4^-)$ we have

$$F^{\rm ax} = C^{\rm ax} - \frac{\langle 24\rangle\,\langle 1|4|6\rangle\langle 2|(1+3)|6\rangle t_{123}}{\langle 12\rangle\,\langle 13\rangle\,[56]\,\langle 3|(1+2)|4\rangle}\,\frac{{\rm L}_1\Big(\frac{-s_{56}}{-t_{123}}\Big)}{t_{123}^2} - \frac{\langle 2|(1+3)|6\rangle\langle 3|(1+2)|6\rangle\,[13]}{[56]\,\langle 3|(1+2)|4\rangle^2}\,\frac{{\rm L}_0\Big(\frac{-t_{123}}{-s_{12}}\Big)}{s_{12}}$$
$$-\frac{\langle 24\rangle\,\langle 1|(2+3)|4\rangle\langle 2|(1+3)|6\rangle\langle 3|(1+2)|6\rangle}{\langle 12\rangle\,\langle 13\rangle\,[56]\,\langle 3|(1+2)|4\rangle^2}\,\frac{{\rm L}_0\Big(\frac{-t_{123}}{-s_{56}}\Big)}{s_{56}}$$
$$+\frac{\langle 24\rangle\,\langle 35\rangle\,\langle 4|(1+3)|6\rangle}{\langle 13\rangle\,\langle 34\rangle\,s_{56}\langle 3|(1+2)|4\rangle} - \frac{1}{12m_t^2}\,\frac{\langle 4|(1+3)|6\rangle\,\langle 24\rangle\,\langle 45\rangle}{\langle 13\rangle\,\langle 34\rangle\,s_{56}} + {\rm flip}_2\,, \tag{11.9}$$

where the ${\rm flip}_2$ operation is to be applied to all terms including the $C^{\rm ax}$ terms.

Similarly, for $A_6^{\rm ax}(1_q^+,2_{\bar q}^-,3^-,4^+)$ we have

$$F^{\rm ax} = -C^{\rm ax}\big|_{3\leftrightarrow 4} + \frac{[14]^2\,\langle 45\rangle\,\langle 5|(2+3)|1\rangle t_{123}}{[12]\,[13]\,\langle 56\rangle\,\langle 4|(1+2)|3\rangle}\,\frac{{\rm L}_1\Big(\frac{-s_{56}}{-t_{123}}\Big)}{t_{123}^2} - \frac{\langle 5|(2+3)|1\rangle\langle 5|(1+2)|3\rangle\,\langle 23\rangle}{\langle 56\rangle\,\langle 4|(1+2)|3\rangle^2}\,\frac{{\rm L}_0\Big(\frac{-t_{123}}{-s_{12}}\Big)}{s_{12}}$$
$$+\frac{[14]\,\langle 4|(2+3)|1\rangle\langle 5|(2+3)|1\rangle\langle 5|(1+2)|3\rangle}{[12]\,[13]\,\langle 56\rangle\,\langle 4|(1+2)|3\rangle^2}\,\frac{{\rm L}_0\Big(\frac{-t_{123}}{-s_{56}}\Big)}{s_{56}}$$
$$+\frac{[14]^2\,\langle 25\rangle\,[36]}{[13]\,[34]\,s_{56}\langle 4|(1+2)|3\rangle} - \frac{1}{12m_t^2}\frac{[14]^2\,\langle 25\rangle\,[46]}{[13]\,[34]\,s_{56}} + {\rm flip}_2\,. \tag{11.10}$$

The subleading-color axial-vector contributions do obey a $3\leftrightarrow 4$ symmetry relation,

$$A_6^{\rm ax,sl}(1_q^+,2_{\bar q}^-,3^-,4^+) = A_6^{\rm ax,sl}(1_q^+,2_{\bar q}^-,3^+,4^-)\big|_{3\leftrightarrow 4}\,. \tag{11.11}$$

We give here $A_6^{\rm ax,sl}(1_q^+, 2_{\bar q}^-, 3^+, 4^-)$:

$$F^{\rm ax,sl} = 2\,\frac{\langle 24\rangle\,\langle 45\rangle\,\langle 2|(1+3)|6\rangle}{\langle 13\rangle\,\langle 23\rangle\,s_{56}}\left[-\frac{1}{2}\frac{{\rm L}_1\!\left(\frac{-t_{123}}{-s_{56}}\right)}{s_{56}} + \frac{1}{24\,m_t^2}\right] \;+\; {\rm flip}_2\,. \tag{11.12}$$

12. Simplified Versions of Four Quark Amplitudes

In a previous paper we have presented the one-loop helicity amplitudes for $e^+\,e^- \to \bar q q \bar Q Q$ [10]. Here we present simplified versions of these amplitudes. These versions have fewer spurious singularities and are therefore (slightly) better for implementing in a jet program. We have verified that the two forms are numerically identical. (In contrast to the $e^+\,e^- \to \bar q q g g$ case, here there is not much to gain from decomposing the amplitude into a scalar contribution and a cut-constructible part.)

The primitive amplitudes A_6^f and A_6^s are proportional to tree amplitudes and are given by

$$\begin{aligned}
A_6^{s,+\pm}(1,2,3,4) &= c_\Gamma\, A_6^{\rm tree}(1_q^+, 2_{\bar Q}^\pm, 3_Q^\mp, 4_{\bar q}; 5_{\bar e}^-, 6_e^+)\,\left[-\frac{1}{3\epsilon}\left(\frac{\mu^2}{-s_{23}}\right)^\epsilon - \frac{8}{9}\right],\\
A_6^{f,+\pm}(1,2,3,4) &= c_\Gamma\, A_6^{\rm tree}(1_q^+, 2_{\bar Q}^\pm, 3_Q^\mp, 4_{\bar q}; 5_{\bar e}^-, 6_e^+)\,\left[\frac{1}{\epsilon}\left(\frac{\mu^2}{-s_{23}}\right)^\epsilon + 2\right],
\end{aligned} \tag{12.1}$$

where $A_6^{\rm tree}$ is given below.

The top quark vacuum polarization contribution is

$$A_6^{t,+\pm}(1,2,3,4) = -\frac{2}{15}\,c_\Gamma\,\frac{s_{23}}{m_t^2}\,A_6^{\rm tree}(1_q^+, 2_{\bar Q}^\pm, 3_Q^\mp, 4_{\bar q}; 5_{\bar e}^-, 6_e^+)\,, \tag{12.2}$$

neglecting corrections of order $1/m_t^4$ and higher. The remaining contributions are decomposed further into divergent (V) and finite (F) pieces according to eq. (6.1).

12.1 The Helicity Configuration $q^+\bar Q^+ Q^- \bar q^-$

We first give the primitive amplitude $A_6^{++}(1,2,3,4)$. This amplitude is odd under the operation ${\rm flip}_1$ defined in eq. (6.7). The tree amplitude for this helicity configuration is

$$A_6^{\rm tree,\,++}(1,2,3,4) = i\,\left[\frac{[12]\,\langle 54\rangle\,\langle 3|(1+2)|6\rangle}{s_{23}s_{56}t_{123}} + \frac{\langle 34\rangle\,[61]\,\langle 5|(3+4)|2\rangle}{s_{23}s_{56}t_{234}}\right]. \tag{12.3}$$

The loop amplitude is

$$V^{++}(1,2,3,4) = -\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{-s_{12}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{34}}\right)^\epsilon\right) + \frac{2}{3\epsilon}\left(\frac{\mu^2}{-s_{23}}\right)^\epsilon - \frac{3}{2}\ln\!\left(\frac{-s_{23}}{-s_{56}}\right) + \frac{10}{9}\,, \tag{12.4}$$

$$
\begin{aligned}
F^{++}(1,2,3,4) = {} & \left(\frac{\langle 3|(1+2)|6\rangle^2}{\langle 23\rangle\,[56]\,t_{123}\langle 1|(2+3)|4\rangle} - \frac{[12]^2\,\langle 45\rangle^2}{[23]\,\langle 56\rangle\,t_{123}\langle 4|(2+3)|1\rangle} \right) \\
& \times \left[\mathrm{Ls}_{-1}\!\left(\tfrac{-s_{12}}{-t_{123}}, \tfrac{-s_{23}}{-t_{123}}\right) + \widetilde{\mathrm{Ls}}_{-1}^{2\mathrm{m}h}(s_{34}, t_{123}; s_{12}, s_{56}) \right] \\
& - 2\,\frac{\langle 3|(1+2)|6\rangle}{[56]\,\langle 1|(2+3)|4\rangle} \left[\frac{\langle 1|(2+3)|6\rangle\,[12]}{t_{123}} \frac{\mathrm{L}_0\!\left(\tfrac{-s_{23}}{-t_{123}}\right)}{t_{123}} + \frac{\langle 3|4|6\rangle}{\langle 23\rangle} \frac{\mathrm{L}_0\!\left(\tfrac{-s_{56}}{-t_{123}}\right)}{t_{123}} \right] \\
& - \frac{1}{2} \frac{1}{\langle 23\rangle\,[56]\,t_{123}\langle 1|(2+3)|4\rangle} \left[\left(\langle 3|2|1\rangle\langle 1|(2+3)|6\rangle\right)^2 \frac{\mathrm{L}_1\!\left(\tfrac{-t_{123}}{-s_{23}}\right)}{s_{23}^2} + \langle 3|4|6\rangle^2 t_{123}^2 \frac{\mathrm{L}_1\!\left(\tfrac{-s_{56}}{-t_{123}}\right)}{t_{123}^2} \right] \\
& - \ \mathrm{flip}_1\,,
\end{aligned}
\tag{12.5}
$$

where flip_1 is to be applied to all the preceding terms in F^{++}. The structure of this amplitude is already rather simple, and it is unchanged from ref. [10].

12.2 The Helicity Configuration $q^+\bar{Q}^-Q^+\bar{q}^-$

We now give the result for $A_6^{+-}(1,2,3,4)$. This amplitude is odd under the same flip symmetry (6.7) as A_6^{++}. The tree amplitude is

$$
A_6^{\mathrm{tree},+-}(1,2,3,4) = -i \left[\frac{[13]\,\langle 54\rangle\,\langle 2|(1+3)|6\rangle}{s_{23}s_{56}t_{123}} + \frac{\langle 24\rangle\,[61]\,\langle 5|(2+4)|3\rangle}{s_{23}s_{56}t_{234}} \right] . \tag{12.6}
$$

Note that $A_6^{\mathrm{tree},+-}(1,2,3,4) = -A_6^{\mathrm{tree},++}(1,3,2,4)$. The simpler form for the loop amplitude is

$$
V^{+-}(1,2,3,4) = V^{++}(1,2,3,4)\,, \tag{12.7}
$$

$$
\begin{aligned}
F^{+-}(1,2,3,4) &= \left(-\frac{[13]^2\,\langle 45\rangle^2}{[23]\,\langle 56\rangle\,t_{123}\langle 4|(2+3)|1\rangle} + \frac{\langle 12\rangle^2\,\langle 3|(1+2)|6\rangle^2}{\langle 23\rangle\,[56]\,t_{123}\,\langle 13\rangle^2\,\langle 1|(2+3)|4\rangle}\right)\mathrm{Ls}_{-1}\!\left(\tfrac{-s_{12}}{-t_{123}},\tfrac{-s_{23}}{-t_{123}}\right)\\
&+\left(-\frac{[13]^2\,\langle 45\rangle^2}{[23]\,\langle 56\rangle\,t_{123}\langle 4|(2+3)|1\rangle} + \frac{\langle 3|(1+2)|6\rangle^2\langle 2|(1+3)|4\rangle^2}{\langle 23\rangle\,[56]\,t_{123}\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle^2}\right)\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{34},t_{123};s_{12},s_{56})\\
&+\Bigg[\frac{1}{2}\frac{\delta_{56}\langle 5|(1+2)|6\rangle(s_{56}\langle 2|1|4\rangle - t_{123}\langle 2|(1+3)|4\rangle)}{\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle\Delta_3} - \frac{1}{2}\frac{(\langle 2|1|4\rangle\langle 5|(3-4)|6\rangle + \langle 2|3|4\rangle\langle 5|(3+4)|6\rangle)}{\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle}\\
&\qquad+\frac{1}{2}\frac{t_{123}\delta_{34}+2s_{12}s_{56}}{t_{123}^2}\left(\frac{[13]^2\langle 45\rangle^2}{[23]\,\langle 56\rangle\,\langle 4|(2+3)|1\rangle} - \frac{\langle 2|(1+3)|6\rangle^2}{\langle 23\rangle\,[56]\,\langle 1|(2+3)|4\rangle}\right) - 2\,\frac{\langle 2|1|3\rangle\langle 5|4|6\rangle}{t_{123}\langle 1|(2+3)|4\rangle}\Bigg]\\
&\qquad\quad\times I_3^{3\mathrm{m}}(s_{12},s_{34},s_{56})\\
&+\frac{[13]\,\langle 12\rangle\,\langle 3|(1+2)|6\rangle^2}{[56]\,t_{123}\,\langle 13\rangle\,\langle 3|(1+2)|4\rangle}\frac{\mathrm{L}_0\!\left(\frac{-t_{123}}{-s_{12}}\right)}{s_{12}} - \frac{1}{2}\frac{[13]^2\,\langle 23\rangle\,\langle 1|(2+3)|6\rangle^2}{[56]\,t_{123}\langle 1|(2+3)|4\rangle}\frac{\mathrm{L}_1\!\left(\frac{-t_{123}}{-s_{23}}\right)}{s_{23}^2}\\
&+\frac{[13]\,\langle 1|(2+3)|6\rangle\langle 2|(1+3)|6\rangle}{[56]\,t_{123}\langle 1|(2+3)|4\rangle}\frac{\mathrm{L}_0\!\left(\frac{-t_{123}}{-s_{23}}\right)}{s_{23}} + \frac{[13]\,\langle 12\rangle\,\langle 1|(2+3)|6\rangle\langle 3|(1+2)|6\rangle}{[56]\,t_{123}\,\langle 13\rangle\,\langle 1|(2+3)|4\rangle}\frac{\mathrm{L}_0\!\left(\frac{-t_{123}}{-s_{23}}\right)}{s_{23}}\\
&-\frac{1}{2}\frac{[64]^2\,\langle 42\rangle^2\,t_{123}}{\langle 23\rangle\,[56]\,\langle 1|(2+3)|4\rangle}\frac{\mathrm{L}_1\!\left(\frac{-s_{56}}{-t_{123}}\right)}{t_{123}^2} - \frac{[64]^2\,\langle 42\rangle\,t_{123}}{[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle}\frac{\mathrm{L}_0\!\left(\frac{-t_{123}}{-s_{56}}\right)}{s_{56}}\\
&-2\,\frac{[64]\,\langle 42\rangle\,\langle 2|(1+3)|6\rangle}{\langle 23\rangle\,[56]\,\langle 1|(2+3)|4\rangle}\frac{\mathrm{L}_0\!\left(\frac{-t_{123}}{-s_{56}}\right)}{s_{56}}\\
&+\frac{1}{\langle 3|(1+2)|4\rangle\Delta_3}\Bigg[\frac{\langle 12\rangle\,t_{123}}{[56]}\left([16]^2+[26]^2\frac{\langle 2|(3+4)|1\rangle}{\langle 1|(3+4)|2\rangle}\right) + \frac{[12]\,t_{124}}{\langle 56\rangle}\left(\langle 25\rangle^2+\langle 15\rangle^2\frac{\langle 2|(3+4)|1\rangle}{\langle 1|(3+4)|2\rangle}\right)\\
&\qquad+2s_{12}\left(-\langle 25\rangle\,[16]+\langle 15\rangle\,[26]\,\frac{\langle 2|(3+4)|1\rangle}{\langle 1|(3+4)|2\rangle}\right)\Bigg]\ln\!\left(\tfrac{-s_{12}}{-s_{56}}\right)\\
&\quad-\ \mathrm{flip}_1\,,
\end{aligned}
\tag{12.8}
$$

where flip_1 is to be applied to all the preceding terms in F^{+-}.

12.3 Subleading-Color Primitive Amplitude

The primitive amplitude $A_6^{\rm sl}(1,2,3,4)$, contributes only at subleading order in N_c. The "tree amplitude" appearing in eq. (6.1) is

$$
A_6^{\rm tree,sl}(1,2,3,4) = i\,\left[\frac{[13]\,\langle 54\rangle\,\langle 2|(1+3)|6\rangle}{s_{12}s_{56}t_{123}} - \frac{\langle 24\rangle\,[63]\,\langle 5|(2+4)|1\rangle}{s_{12}s_{56}t_{412}}\right], \tag{12.9}
$$

and satisfies $A_6^{\rm tree,sl}(2,3,1,4) = -A_6^{\rm tree,++}(1,2,3,4)$.

To describe the loop amplitude, we use the operations flip_3 and $\mathrm{exch}_{16,25}$ defined in eqs. (6.9) and (6.11). The loop amplitude is given by

$$
V^{\rm sl}(1,2,3,4) = \left[-\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{34}}\right)^{\epsilon} - \frac{3}{2\epsilon}\left(\frac{\mu^2}{-s_{34}}\right)^{\epsilon} - 4\right] + \left[-\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} - \frac{3}{2\epsilon}\left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} - \frac{7}{2}\right], \tag{12.10}
$$

$$
\begin{aligned}
F^{\rm sl}(1,2,3,4) = {}& \left(\frac{[13]^2\,\langle 45\rangle^2}{[12]\,\langle 56\rangle\, t_{123}\langle 4|(1+2)|3\rangle} - \frac{\langle 3|(1+2)|6\rangle^2 \langle 2|(1+3)|4\rangle^2}{\langle 12\rangle\,[56]\,t_{123}\langle 3|(1+2)|4\rangle^3} \right) \mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{34},t_{123};s_{12},s_{56}) \\
& + T\, I_3^{3\mathrm{m}}(s_{12},s_{34},s_{56}) + \Bigg[\frac{1}{2} \frac{[64]^2\,\langle 42\rangle^2\, t_{123}}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle} \frac{\mathrm{L}_1\!\left(\frac{-s_{56}}{-t_{123}}\right)}{t_{123}^2} + 2\, \frac{[64]\,\langle 42\rangle\,\langle 2|(1+3)|6\rangle}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle} \frac{\mathrm{L}_0\!\left(\frac{-t_{123}}{-s_{56}}\right)}{s_{56}} \\
& - \frac{\langle 23\rangle\,\langle 24\rangle\,[64]^2\,t_{123}}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle^2} \frac{\mathrm{L}_0\!\left(\frac{-t_{123}}{-s_{56}}\right)}{s_{56}} - \frac{1}{2} \frac{\langle 23\rangle\,[64]\,\langle 2|(1+3)|6\rangle}{\langle 12\rangle\,[56]\,\langle 3|(1+2)|4\rangle^2} \ln\!\left(\frac{(-t_{123})(-s_{34})}{(-s_{56})^2}\right) \\
& - \frac{3}{4} \frac{\langle 2|(1+3)|6\rangle^2}{\langle 12\rangle\,[56]\,t_{123}\langle 3|(1+2)|4\rangle} \ln\!\left(\frac{(-t_{123})(-s_{34})}{(-s_{56})^2}\right) \\
& + \Bigg(\frac{3}{2} \frac{\delta_{56}\,(t_{123}-t_{124})\,\langle 2|(3+4)|1\rangle\langle 5|(3+4)|6\rangle}{\langle 3|(1+2)|4\rangle \Delta_3^2} \\
& \quad - \frac{[12]\,\langle 23\rangle\,[46]}{\langle 3|(1+2)|4\rangle^2 \Delta_3} \Big(\langle 25\rangle\,(t_{123}-t_{124}) - 2\,\langle 2|1\,6|5\rangle \Big) + \frac{[12]\,\langle 25\rangle\,\langle 2|(3-4)|6\rangle}{\langle 3|(1+2)|4\rangle \Delta_3} \\
& \quad + \frac{[16]}{[56]\,\langle 3|(1+2)|4\rangle \Delta_3} \Big(\langle 2|3|6\rangle\, t_{123} - \langle 2|4|6\rangle\, t_{124} + \frac{\langle 23\rangle\,[46]\;\delta_{34}\,t_{123}}{\langle 3|(1+2)|4\rangle} \Big) \Bigg) \ln\!\left(\frac{-s_{12}}{-s_{34}}\right) \\
& - \frac{1}{4} \frac{[16]\;(t_{123}-t_{124}) \big([16]\;\delta_{34} - 2\,[1|25|6] \big)}{[12]\,[56]\,\langle 3|(1+2)|4\rangle \Delta_3} \; - \; \mathrm{flip}_3 \Bigg] \; + \; \mathrm{exch}_{16,25}\,,
\end{aligned}
\tag{12.11}
$$

where $\mathrm{exch}_{16,25}$ is to be applied to all the preceding terms in $F^{\rm sl}$, but flip_3 is to be applied only to the terms inside the brackets ([ ]) in which it appears. The three-mass triangle coefficient T is given by

$$
\begin{aligned}
T = {}& \frac{1}{2} \frac{(3 s_{34}\delta_{34} - \Delta_3)(t_{123}-t_{124})\langle 2|(3+4)|1\rangle\langle 5|(3+4)|6\rangle}{\langle 3|(1+2)|4\rangle \Delta_3^2} + \frac{1}{2} \frac{s_{34}\,(t_{123}-t_{124})\,\langle 25\rangle\,[16]}{\langle 3|(1+2)|4\rangle \Delta_3} \\
& - \frac{[12]\,\langle 56\rangle\,\langle 2|(3+4)|6\rangle^2}{\langle 3|(1+2)|4\rangle \Delta_3} + \frac{\langle 23\rangle\,[46]\,\delta_{34} \big(\langle 5|2|1\rangle\,\delta_{12} - \langle 5|6|1\rangle\,\delta_{56} \big)}{\langle 3|(1+2)|4\rangle^2 \Delta_3} \\
& + \frac{\langle 23\rangle\,[46]\,\langle 5|(2+3)|1\rangle}{\langle 3|(1+2)|4\rangle^2} - 2\, \frac{[13]\,\langle 45\rangle\,\langle 2|(1+3)|4\rangle\langle 3|(1+2)|6\rangle}{t_{123}^2\,\langle 3|(1+2)|4\rangle}\,.
\end{aligned}
\tag{12.12}
$$

12.4 Axial-Vector Quark Loop

The axial-vector quark-triangle contribution $A_6^{\rm ax}(1,2,3,4)$ is easily obtained from the fully off-shell Zgg vertex presented in ref. [49]. The infrared- and ultraviolet-finite result is

$$
\begin{aligned}
A_6^{\rm ax}(1,2,3,4) \;=\; & -\frac{2i}{(4\pi)^2} \frac{f(m_t;s_{12},s_{34},s_{56}) - f(m_b;s_{12},s_{34},s_{56})}{s_{56}} \left(\frac{[63]\,\langle 42\rangle\,\langle 25\rangle}{\langle 12\rangle} - \frac{[61]\,[13]\,\langle 45\rangle}{[12]} \right) \\
& + (1 \leftrightarrow 3,\;\; 2 \leftrightarrow 4)\;,
\end{aligned}
\tag{12.13}
$$

where the integral $f(m)$ is defined in eq. (II.14). We need only the large mass expansion (for $m = m_t$) and the $m = 0$ limit (for $m = m_b$) of this integral; these are presented in eq. (II.15).

13. Summary and Conclusions

In this paper we presented explicit formulæ for all one-loop helicity amplitudes which enter into numerical programs for the next-to-leading order QCD corrections to $e^+ e^- \to (\gamma^*, Z) \to 4$ jets. These amplitudes have already been incorporated into the first such program to be constructed [12]; the result has been a significant reduction in theoretical uncertainties in the four-jet cross-section and associated quantities. With appropriate modifications to the coupling constants, these same amplitudes enter into the computation of next-to-leading order contributions to the production of a vector boson (W, Z, or Drell-Yan pair) in association with two jets at hadron colliders, and three-jet production at ep colliders.

Following the methods reviewed in ref. [25], we have obtained the loop amplitudes by demanding that their functional forms satisfy unitarity and factorization. This approach makes use of a color decomposition [32] into gauge invariant primitive amplitudes [19,10] which are expressed in terms of a helicity basis [31]. The color decomposition limits the analytic functions that may appear, greatly simplifying the reconstruction of the amplitudes from their analytic properties. The helicity basis results in relatively compact expressions, from which spurious poles can be systematically removed to further simplify the results. A further advantage of the helicity basis is that one retains all spin information.

As a check on the methods we have verified that the amplitudes are numerically identical to ones obtained by a (numerical) Feynman diagram calculation that we have performed. This diagrammatic calculation made use of a number of string-motivated ideas reviewed in ref. [25]. We have also numerically verified that the typed form of the amplitudes appearing in this paper agree with our Maple and Mathematica expressions for the same quantities.

We expect that the amplitudes presented in this paper should lead to an improved knowledge of QCD predictions for a wider class of observables, and thus of the QCD background to searches for new physics in various processes.

Acknowledgements

Z.B. wishes to thank SLAC, Saclay and the Aspen Center for Physics for their hospitality during the period of this work; L.D. similarly wishes to thank Rutgers University, the Aspen Center for Physics, Saclay and UCLA; and D.A.K. wishes to thank SLAC and UCLA. We thank Dave Dunbar, Adrian Signer and Stefan Weinzierl for useful conversations. This work was supported in part by the US Department of Energy under grants DE-FG03-91ER40662 and DE-AC03-76SF00515, by the Alfred P. Sloan Foundation under grant BR-3222, and by the *Direction des Sciences de la Matière* of the *Commissariat à l'Energie Atomique* of France. L.D. and D.A.K. gratefully acknowledge the support of NATO Collaborative Research Grant CRG–921322.

Appendix I. One-Loop Integrals

I.1 General Properties and Reduction Formulæ

The loop momentum integrals that appear in either a Feynman diagram or cut-based analysis are of the form

$$\mathcal{I}_n^D[\ell^{\alpha_1}\cdots\ell^{\alpha_m}] = \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^{\alpha_1}\cdots\ell^{\alpha_m}}{\ell^2(\ell-K_1)^2\cdots(\ell-\sum_{i=1}^{n-1}K_i)^2}\,, \tag{I.1}$$

where the K_i are external momenta, or sums of external momenta. The momentum ℓ flows through the propagator between external legs n and 1. For convenience we define

$$I_n^D[\ell^{\alpha_1}\cdots\ell^{\alpha_m}] = i(-1)^{n+1}(4\pi)^{D/2}\mathcal{I}_n^D[\ell^{\alpha_1}\cdots\ell^{\alpha_m}]\,. \tag{I.2}$$

Integrals with powers of loop momenta ℓ^α in the numerator are known as tensor integrals; integrals with no powers of loop momenta in the numerators are known as scalar integrals and are denoted by $I_n^D \equiv I_n^D[1]$. When the superscript D is omitted below, D is implicitly equal to $4-2\epsilon$.

In general, any one-loop amplitude may be expressed as a linear combination of bubble, triangle and box (i.e. two-, three- and four-point) scalar integrals. This follows from the Passarino-Veltman reduction [39] of any tensor n-point amplitude with m powers of loop momenta to a linear combination of $n-1$ and n-point integrals with no more than $m-1$ powers of loop momenta. The resulting scalar integrals with $n>4$ legs can be further reduced to scalar box integrals using an additional set of reduction formulæ [40,41,42].

It is useful to review briefly the conventional reduction of integrals to explain the origin and types of determinantal poles that can appear. Consider the five-point tensor integral with the kinematic configuration depicted in fig. 6. Following the Passarino-Veltman reduction technique, we expand the loop momentum in terms of four independent external momenta,

$$I_5[\ell^{\alpha_1}\ell^{\alpha_2}\ldots\ell^{\alpha_j}] = \sum_{k=1}^{4} p_k^{\alpha_1} A_k \tag{I.3}$$

where

$$p_1 = K_1\,, \qquad p_2 = K_1+K_2\,, \qquad p_3 = K_1+K_2+K_3\,, \qquad p_4 = -K_5\,, \tag{I.4}$$

and we have suppressed the Lorentz indices $\alpha_2\ldots\alpha_j$ on the right-hand side of eq. (I.3) in A_k. The functions A_i are found by first contracting eq. (I.3) with the momentum sums p_i, generating the four linearly independent equations,

$$2I_5[\ell\cdot p_i\;\ell^{\alpha_2}\ldots\ell^{\alpha_j}] = \sum_{k=1}^{4} t_{ik}A_k\,, \qquad (i=1,2,3,4)\,, \tag{I.5}$$

where $t_{ik} \equiv 2p_i \cdot p_k$. Using $2\ell \cdot p_i = -(\ell - p_i)^2 + \ell^2 + p_i^2$, and recognizing that $(\ell - p_i)^2$ and ℓ^2 are inverse propagators of the pentagon integral, one can reduce the left-hand-side to a sum of two four-point integrals plus a five-point integral with one less power of loop momentum. Inverting t_{ik} we have

$$A_i = \sum_{k=1}^{4} t_{ik}^{-1} \left(I_4^{(k+1)}[\ell^{\alpha_2}\ell^{\alpha_3}\cdots\ell^{\alpha_j}] - I_4^{(1)}[\ell^{\alpha_2}\ell^{\alpha_3}\cdots\ell^{\alpha_j}] + p_k^2 I_5[\ell^{\alpha_2}\ell^{\alpha_3}\cdots\ell^{\alpha_j}] \right), \qquad \text{(I.6)}$$

where $I_4^{(i)}$ denotes the box integral that is obtained from the pentagon integral by canceling the propagator factor $1/\ell_i^2$. The coefficients t_{ik}^{-1} contain in their denominators the pentagon Gram determinant Δ_5 given in eq. (I.10).

The reduction of tensor boxes, triangles, and bubbles is similar except that one must also include the Kronecker-delta in the expansion of the integral function since there are less than four independent momenta. More powerful reduction techniques which lead to square-roots of Gram determinants in denominators, instead of single powers, have also been developed [40].

The scalar pentagon integrals may be reduced to scalar box integrals using the scalar integral recursion formula [40,41,42] (valid for $n \leq 6$)

$$I_n = \frac{1}{2}\left[\sum_{i=1}^{n} c_i \; I_{n-1}^{(i)} + (n - 5 + 2\epsilon)\, c_0 \; I_n^{D=6-2\epsilon}\right], \qquad \text{(I.7)}$$

where

$$c_i = \sum_{j=1}^{n} S_{ij}^{-1}, \qquad\qquad c_0 = \sum_{i=1}^{n} c_i \;=\; \sum_{i,j=1}^{n} S_{ij}^{-1}. \qquad \text{(I.8)}$$

The matrix S_{ij} is defined in eq. (4.4) and the integral $I_n^{D=6-2\epsilon}$ is the scalar n-point integral evaluated in $(6 - 2\epsilon)$ dimensions. Observe that for $n = 5$ the prefactor of $I_5^{D=6-2\epsilon}$ is of $\mathcal{O}(\epsilon)$; since $I_5^{D=6-2\epsilon}$ is finite as $\epsilon \to 0$ we may drop this term. This equation is also useful for rewriting $D = 4 - 2\epsilon$ box integrals as $D = 6 - 2\epsilon$ box integrals, which turns out to be a convenient way to represent the amplitudes. Indeed, the Ls_{-1} functions defined in appendix II are $D = 6 - 2\epsilon$ box integrals from which a simple overall kinematic factor has been removed. The explicit forms of the higher-dimension box integrals may be obtained by solving eq. (I.7) for $I_4^{D=6-2\epsilon}$

For the case of a single power of loop momentum we may avoid a Gram determinant in the denominator by making use of the reduction formula,

$$I_n[\ell^\mu] \;=\; \frac{1}{2}\sum_{i=2}^{n} p_{i-1}^{\mu}\left[\sum_{j=1}^{n} S_{ij}^{-1}\; I_{n-1}^{(j)} \;+\; (n - 5 + 2\epsilon)\, c_i \; I_n^{D=6-2\epsilon}\right], \qquad \text{(I.9)}$$

where $p_i = K_1 + K_2 + \cdots + K_i$ and the K_j are the external momenta of the integrals. For pentagon integrals $(n = 5)$ the $I_5^{D=6-2\epsilon}$ term may be dropped since it is of $\mathcal{O}(\epsilon)$.

Since the integral recursion formula (I.7) contains inverses of the matrix S defined in eq. (4.4), the amplitudes will contain poles in $\det S$. As discussed in the text, these poles do not correspond to the propagation of physical states, and so their residues in a full amplitude must vanish.

In summary, when reducing tensor and scalar integrals to linear combinations of scalar box, triangle and bubble integrals, one encounters a set of spurious determinantal poles. Some of these spurious poles, such as the pentagon Gram determinant (see section 4.2), are artifacts of the reduction procedure, but others are inherently part of the amplitude when it is expressed in terms of logarithms and dilogarithms.

I.2 Determinants Appearing in Amplitudes

Here we explicitly list the determinants that appear in the denominators of coefficients in the general integral reduction formulæ of the preceding subsection. As discussed in section 4, knowledge of these determinants — and how they can be factored — is useful in deciding which spinor factors to multiply and divide by in order to simplify the integral reduction of specific cuts.

The Gram determinants are defined in eq. (4.2). The explicit forms of the Gram determinants associated with pentagon and box integrals whose external legs follow the ordering 123456 is

$$\begin{aligned}\Delta_5 = & -s_{23}^2\, s_{34}^2 - s_{34}^2\, t_{123}^2 - t_{123}^2\, t_{234}^2 - s_{12}^2\, t_{234}^2 - s_{23}^2\, s_{56}^2 - s_{12}^2\, s_{23}^2 \\ & + 2\, s_{34}^2\, t_{123}\, s_{23} + 2\, t_{123}^2\, t_{234}\, s_{34} + 2\, s_{12}^2\, t_{234}\, s_{23} + 2\, s_{56}\, s_{23}^2\, s_{34} + 2\, s_{56}\, s_{23}^2\, s_{12} + 2\, s_{23}^2\, s_{12}\, s_{34} \\ & - 2\, t_{123}\, t_{234}\, s_{23}\, s_{34} - 2\, t_{234}\, s_{12}\, s_{23}\, s_{34} - 2\, t_{234}\, s_{34}\, t_{123}\, s_{12} - 2\, t_{123}\, s_{12}\, s_{23}\, s_{34} + 2\, t_{234}^2\, t_{123}\, s_{12} \\ & - 2\, t_{123}\, t_{234}\, s_{12}\, s_{23} - 2\, s_{56}\, s_{23}\, s_{34}\, t_{123} + 2\, s_{56}\, s_{23}\, t_{123}\, t_{234} - 2\, s_{56}\, s_{23}\, s_{12}\, t_{234} + 4\, s_{56}\, s_{23}\, s_{12}\, s_{34}\,,\end{aligned} \tag{I.10}$$

$$\begin{aligned}\Delta_4^{(1)} &= -2\, s_{23} s_{34} s_{24}\,, \\ \Delta_4^{(2)} &= 2\, s_{34}(s_{12}\, s_{56} - t_{123}\, t_{124})\,, \\ \Delta_4^{(3)} &= 2\, s_{14}(s_{23}\, s_{56} - t_{123}\, t_{234})\,, \\ \Delta_4^{(4)} &= 2\, s_{12}(s_{34}\, s_{56} - t_{134}\, t_{234})\,, \\ \Delta_4^{(5)} &= -2\, s_{12} s_{23} s_{13}\,,\end{aligned} \tag{I.11}$$

where the superscript on the box Gram determinants labels the propagator that has been canceled in the pentagon integral to obtain the box. Similarly, the triangle Gram determinants are labeled by a pair of indices, for the two canceled propagators,

$$\begin{aligned}& \Delta_3^{(1,2)} = s_{34}^2\,, \qquad \Delta_3^{(1,3)} = (t_{234} - s_{23})^2\,, \qquad \Delta_3^{(1,4)} = (t_{234} - s_{34})^2\,, \qquad \Delta_3^{(1,5)} = s_{23}^2\,, \\ & \Delta_3^{(2,3)} = (t_{123} - s_{56})^2\,, \qquad \Delta_3^{(2,4)} = s_{12}^2 + s_{34}^2 + s_{56}^2 - 2\, s_{12}\, s_{34} - 2\, s_{34}\, s_{56} - 2\, s_{56}\, s_{12}\,, \\ & \Delta_3^{(2,5)} = (t_{123} - s_{12})^2\,, \qquad \Delta_3^{(3,4)} = (t_{234} - s_{56})^2\,, \qquad \Delta_3^{(3,5)} = (t_{123} - s_{23})^2\,, \\ & \Delta_3^{(4,5)} = s_{12}^2\,.\end{aligned} \tag{I.12}$$

The bubble Gram determinants are trivial. As discussed in section 4, the pentagon Gram determinant does not appear at all in the cut calculations (assuming the integral reductions are performed as discussed). However, the triangle and box Gram determinants do appear in the final results, when expressed in terms of logarithms and dilogarithms.

Equations (I.7) and (I.8) show that another source of spurious singularities is the determinant of the matrix S_{ij}. Again labeling the pentagon and box matrices by the canceled propagators we have

$$\det S = \frac{1}{16} s_{12}\, s_{23}\, s_{34} \big(s_{23}\, s_{56} - t_{123}\, t_{234} \big) \tag{I.13}$$

$$\begin{aligned} &\det S^{(1)} = \frac{1}{16}\, s_{23}^2\, s_{34}^2\,, \qquad \det S^{(2)} = \frac{1}{16}\, s_{34}^2\, t_{123}^2\,, \qquad \det S^{(3)} = \frac{1}{16} \big(s_{23} s_{56} - t_{123} t_{234} \big)^2\,, \\ &\det S^{(4)} = \frac{1}{16}\, s_{12}^2\, t_{234}^2\,, \qquad \det S^{(5)} = \frac{1}{16}\, s_{12}^2\, s_{23}^2\,. \end{aligned} \tag{I.14}$$

Since the primitive amplitudes contain loop integrals where the external legs follow the orderings 123564 and 125634 instead of 123456, an additional set of possible spurious poles are obtained from the above set via the relabelings of external legs:

$$1234 \to 4123\,, \qquad\qquad 1234 \to 3412\,. \tag{I.15}$$

Observe the appearance of the combinations

$$\begin{aligned} s_{12}\, s_{56} - t_{123}\, t_{124} &= -\langle 4|(1+2)|3\rangle \langle 3|(1+2)|4\rangle\,, \\ s_{23}\, s_{56} - t_{123}\, t_{234} &= -\langle 1|(2+3)|4\rangle \langle 4|(2+3)|1\rangle\,, \\ s_{34}\, s_{56} - t_{134}\, t_{234} &= -\langle 1|(3+4)|2\rangle \langle 2|(3+4)|1\rangle\,, \end{aligned} \tag{I.16}$$

which we have factored into products of spinor strings. The above factored form is quite useful in simplifying the cuts. As discussed in section 4.2, we insert such 'back-to-back' factors into tensor integrals by hand in order to help simplify the expression by forming appropriate spinor strings from which inverse propagators can be extracted.

Although the three-mass triangle Gram determinant $\Delta_3 \equiv \Delta_3^{(2,4)}$ cannot be factored simply like eq. (I.16), there are many kinematic combinations that vanish whenever Δ_3 does. Using the fact that the three four-vectors $k_1 + k_2$, $k_3 + k_4$ and $k_5 + k_6$ all become proportional in the limit $\Delta_3 \to 0$, and relations like $\delta_{56} = 2\,(k_1 + k_2) \cdot (k_3 + k_4)$, it is easy to verify that the expressions

$$\begin{aligned} &\langle 1|(3+4)|2\rangle\,, \qquad \langle 14\rangle\, \delta_{56} - 2\,\langle 1|23|4\rangle\,, \qquad \langle 1|2|4\rangle\, \delta_{12} - \langle 1|3|4\rangle\, \delta_{34}\,, \\ &\langle 1|(1+2)(3+4)|6\rangle - \langle 1|(3+4)(1+2)|6\rangle\,, \end{aligned} \tag{I.17}$$

plus their complex conjugates and a variety of permutations of them, all vanish in this limit. In the amplitudes we present, the appearance of such combinations in the numerator of coefficients, when Δ_3 appears in the denominator, alleviates the amplitudes' spurious singularities as $\Delta_3 \to 0$.

Appendix II. Integral Functions Appearing in Amplitudes

We collect here the integral functions appearing in the text, which contain all logarithms and dilogarithms present in the amplitudes. Most of the functions have already appeared in previous papers [17,10], but for completeness we list them all in this appendix. Except for the contribution of the top quark to vacuum polarization contributions and to the axial-vector contribution $A_6^{\rm ax}$, all internal lines are taken to be massless. The following functions arise from box integrals with one external mass:

$$\begin{aligned}
\mathrm{L}_0(r) &= \frac{\ln(r)}{1-r}\,, \qquad\qquad \mathrm{L}_1(r) = \frac{\mathrm{L}_0(r)+1}{1-r}\,,\\
\mathrm{Ls}_{-1}(r_1,r_2) &= \mathrm{Li}_2(1-r_1)+\mathrm{Li}_2(1-r_2)+\ln r_1\,\ln r_2-\frac{\pi^2}{6}\,,\\
\mathrm{Ls}_0(r_1,r_2) &= \frac{1}{(1-r_1-r_2)}\ \mathrm{Ls}_{-1}(r_1,r_2)\,,\\
\mathrm{Ls}_1(r_1,r_2) &= \frac{1}{(1-r_1-r_2)}\ \left[\mathrm{Ls}_0(r_1,r_2)+\mathrm{L}_0(r_1)+\mathrm{L}_0(r_2)\right]\,,
\end{aligned} \tag{II.1}$$

where the dilogarithm is

$$\mathrm{Li}_2(x) = -\int_0^x dy\,\frac{\ln(1-y)}{y}\,. \tag{II.2}$$

The function Ls_{-1} is simply related to the scalar box integral with one external mass, evaluated in six space-time dimensions where it is infrared- and ultraviolet-finite. The above functions have the property that they are finite as as their denominators vanish. Generalizations of the Ls_0 and Ls_1 functions to the case of box integrals with two or more external masses have been presented in ref. [38].

The box function analogous to Ls_{-1}, but for two adjacent external masses, is

$$\begin{aligned}
\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s,t;m_1^2,m_2^2) = &-\mathrm{Li}_2\left(1-\frac{m_1^2}{t}\right)-\mathrm{Li}_2\left(1-\frac{m_2^2}{t}\right)-\frac{1}{2}\ln^2\left(\frac{-s}{-t}\right)+\frac{1}{2}\ln\left(\frac{-s}{-m_1^2}\right)\ln\left(\frac{-s}{-m_2^2}\right)\\
&+\left[\frac{1}{2}(s-m_1^2-m_2^2)+\frac{m_1^2m_2^2}{t}\right]I_3^{3\mathrm{m}}(s,m_1^2,m_2^2)\,,
\end{aligned} \tag{II.3}$$

where $I_3^{3\mathrm{m}}$ is the three-mass scalar triangle integral. This integral vanishes in the appropriate 'back-to-back' kinematic limit. We also employ a version of this box function with $I_3^{3\mathrm{m}}$ removed,

$$\widetilde{\mathrm{Ls}}_{-1}^{2\mathrm{m}h}(s,t;m_1^2,m_2^2) = -\mathrm{Li}_2\left(1-\frac{m_1^2}{t}\right)-\mathrm{Li}_2\left(1-\frac{m_2^2}{t}\right)-\frac{1}{2}\ln^2\left(\frac{-s}{-t}\right)+\frac{1}{2}\ln\left(\frac{-s}{-m_1^2}\right)\ln\left(\frac{-s}{-m_2^2}\right)\,. \tag{II.4}$$

The label 'h' refers to the fact that this integral is relatively hard to obtain [42].

The box function with two non-adjacent external masses is

$$\begin{aligned}
\mathrm{Ls}_{-1}^{2\mathrm{me}}(s,t;m_1^2,m_3^2) = &-\mathrm{Li}_2\left(1-\frac{m_1^2}{s}\right)-\mathrm{Li}_2\left(1-\frac{m_1^2}{t}\right)-\mathrm{Li}_2\left(1-\frac{m_3^2}{s}\right)-\mathrm{Li}_2\left(1-\frac{m_3^2}{t}\right)\\
&+\mathrm{Li}_2\left(1-\frac{m_1^2m_3^2}{st}\right)-\frac{1}{2}\ln^2\left(\frac{-s}{-t}\right)\,.
\end{aligned} \tag{II.5}$$

In this case the label 'e' refers to the fact that this integral is relatively easy to obtain. This integral vanishes as $s+t-m_1^2-m_3^2 \to 0$.

The kinematic region in which an Ls_{-1} function vanishes always turns out to be related to the spinor product (or string) required to reduce the corresponding box integral: $\langle 24\rangle$ for $I_4^{(1)}$, $\langle 3|(1{+}2)|4\rangle$ for $I_4^{(2)}$, $\langle 14\rangle$ for $I_4^{(3)}$, $\langle 1|(3{+}4)|2\rangle$ for $I_4^{(4)}$, and $\langle 13\rangle$ for $I_4^{(5)}$ (or their complex conjugates). Curiously, no single helicity amplitude contains both $\mathrm{Ls}_{-1}^{2\mathrm{m}h}$ and $\mathrm{Ls}_{-1}^{2\mathrm{m}e}$ functions simultaneously.

The analytic properties of these integrals are straightforward to obtain from the prescription of adding a small positive imaginary part to each invariant, $s_{ij} \to s_{ij} + i\varepsilon$. One expands the logarithmic ratios, $\ln(r) \equiv \ln(\frac{-s}{-s'}) = \ln(-s) - \ln(-s')$, and then uses

$$\ln(-s-i\varepsilon) \;=\; \ln|s| - i\pi\Theta(s)\,. \tag{II.6}$$

where $\Theta(s)$ is the step function: $\Theta(s>0)=1$ and $\Theta(s<0)=0$. The imaginary part of the dilogarithm $\mathrm{Li}_2(1-r)$ is given in terms of the logarithmic ratio,

$$\mathrm{Im}\,\mathrm{Li}_2(1-r) \;=\; -\ln(1-r)\;\mathrm{Im}\ln(r)\,. \tag{II.7}$$

For $r>0$ the real part of $\mathrm{Li}_2(1-r)$ is given directly by eq. (II.2). For $r<0$ one may use [50]

$$\mathrm{Re}\,\mathrm{Li}_2(1-r) \;=\; \frac{\pi^2}{6} - \ln|r|\ln|1-r| - \mathrm{Re}\,\mathrm{Li}_2(r)\,, \tag{II.8}$$

with $\mathrm{Re}\,\mathrm{Li}_2(r)$ given by eq. (II.2).

The analytic structure of $I_3^{3\mathrm{m}}$ is more complicated [51,52,42], and the numerical representation we use depends on the kinematics. The integral is defined by

$$I_3^{3\mathrm{m}}(s_{12},s_{34},s_{56}) \;=\; \int_0^1 d^3a_i\,\delta(1-a_1-a_2-a_3)\,\frac{1}{-s_{12}a_1a_2 - s_{34}a_2a_3 - s_{56}a_3a_1}\,. \tag{II.9}$$

This integral is symmetric under any permutation of its three arguments, and acquires a minus sign when the signs of all three arguments are simultaneously reversed. Therefore we only have to consider two cases,

1. The Euclidean region $s_{12}, s_{34}, s_{56} < 0$, which is related by the sign flip to the pure Minkowski region ($s_{12}, s_{34}, s_{56} > 0$) relevant for e^+e^- annihilation. Here the imaginary part vanishes. This region has two sub-cases, depending on the sign of the Gram determinant $\Delta_3(s_{12},s_{34},s_{56})$ defined in eq. (6.6):

   1a. $\Delta_3 < 0$,

   1b. $\Delta_3 > 0$.

2. The mixed region $s_{12}, s_{56} < 0$, $s_{34} > 0$, for which Δ_3 is always positive.

In region 1a one may use a symmetric representation found by Lu and Perez [51], which is closely related to that given in ref. [42]:

$$I_3^{3\mathrm{m}} = \frac{2}{\sqrt{-\Delta_3}} \left[\mathrm{Cl}_2\left(2\tan^{-1}\left(\frac{\sqrt{-\Delta_3}}{\delta_{12}}\right)\right) + \mathrm{Cl}_2\left(2\tan^{-1}\left(\frac{\sqrt{-\Delta_3}}{\delta_{34}}\right)\right) + \mathrm{Cl}_2\left(2\tan^{-1}\left(\frac{\sqrt{-\Delta_3}}{\delta_{56}}\right)\right)\right], \tag{II.10}$$

where the δ_{ij} are defined in eq. (6.6) and the Clausen function $\mathrm{Cl}_2(x)$ is defined by

$$\mathrm{Cl}_2(x) \;\equiv\; \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} \;=\; -\int_0^x dt\, \ln\big(|2\sin(t/2)|\big). \tag{II.11}$$

In regions 1b and 2 a convenient representation is given by Ussyukina and Davydychev [52],

$$\begin{aligned} I_3^{3\mathrm{m}} \;&=\; -\frac{1}{\sqrt{\Delta_3}} \mathrm{Re}\left[2\left(\mathrm{Li}_2(-\rho x) + \mathrm{Li}_2(-\rho y)\right) + \ln(\rho x)\ln(\rho y) + \ln\left(\frac{y}{x}\right)\ln\left(\frac{1+\rho y}{1+\rho x}\right) + \frac{\pi^2}{3}\right] \\ &\quad - \frac{i\pi\Theta(s_{34})}{\sqrt{\Delta_3}} \ln\left(\frac{(\delta_{12}+\sqrt{\Delta_3})(\delta_{56}+\sqrt{\Delta_3})}{(\delta_{12}-\sqrt{\Delta_3})(\delta_{56}-\sqrt{\Delta_3})}\right), \end{aligned} \tag{II.12}$$

where

$$x = \frac{s_{12}}{s_{56}}, \quad y = \frac{s_{34}}{s_{56}}, \quad \rho = \frac{2 s_{56}}{\delta_{56}+\sqrt{\Delta_3}}. \tag{II.13}$$

Finally, in the top quark contribution to A_6^{ax} the combination $f(m_t) - f(m_b)$ appears, where $f(m)$ is the integral

$$f(m; s_{12}, s_{34}, s_{56}) \;=\; \int_0^1 d^3a_i\, \delta(1-a_1-a_2-a_3)\, \frac{a_2 a_3}{m^2 - s_{12}a_1a_2 - s_{34}a_2a_3 - s_{56}a_3a_1}\,. \tag{II.14}$$

This integral is complicated for arbitrary mass m; however, the large and small mass limits of it suffice for m_t and m_b respectively. For $m = m_t$ we simply Taylor expand the integrand in $1/m$; for $m = m_b$ we set m_b to zero, and reduce $f(0)$ to a linear combination of the massless scalar triangle integral $I_3^{3\mathrm{m}}$ given above, logarithms and rational functions. We get

$$\begin{aligned} f(m_t; s_{12}, s_{34}, s_{56}) &= \frac{1}{24 m_t^2} + \frac{(2s_{34}+s_{12}+s_{56})}{360 m_t^4} + \cdots, \\ f(0; s_{12}, s_{34}, s_{56}) &= \left(\frac{3 s_{34}\delta_{34}}{\Delta_3^2} - \frac{1}{\Delta_3}\right) s_{12}s_{56} I_3^{3\mathrm{m}}(s_{12}, s_{34}, s_{56}) + \left(\frac{3 s_{56}\delta_{56}}{\Delta_3^2} - \frac{1}{2\Delta_3}\right) s_{12} \ln\left(\frac{-s_{12}}{-s_{34}}\right) \\ &\quad + \left(\frac{3 s_{12}\delta_{12}}{\Delta_3^2} - \frac{1}{2\Delta_3}\right) s_{56} \ln\left(\frac{-s_{56}}{-s_{34}}\right) - \frac{\delta_{34}}{2\Delta_3}\,. \end{aligned} \tag{II.15}$$

Note that the limit where one of the invariants vanishes,

$$f(m_t; 0, t_{123}, s_{56}) - f(0; 0, t_{123}, s_{56}) = -\frac{1}{2}\frac{\mathrm{L}_1\left(\frac{-t_{123}}{-s_{56}}\right)}{s_{56}} + \frac{1}{24\, m_t^2}, \tag{II.16}$$

appears in two amplitudes, eqs. (11.4) and (11.12).

Appendix III. Spinor Identities for Simplifying Spurious Poles

In order to simplify or remove spurious poles a number of spinor identities are of great utility. In this appendix we collect these identities. As discussed in section 4 some of the spurious poles can be completely removed from the amplitudes, but others are an inherent part of the amplitude when it is expressed in terms of logarithms and dilogarithms.

The most difficult spurious poles to simplify are those appearing as coefficients of three external mass triangle integrals and associated logarithms. In simplifying these spurious poles, it is useful to have identities for rewriting commonly occurring expressions to make one behavior or another more manifest. The following identities are used in different 'directions', depending on the situation.

Particularly important spurious singularities are where Δ_3, $\langle 3|(1+2)|4\rangle$ and $t_{123} - s_{12}$ (and the label-permuted objects) vanish. In order to obtain relatively compact expressions it is essential to simplify these spurious poles.

One important quantity is $(t_{123}\delta_{34} + 2s_{12}s_{56})$, which appears as a factor in the coefficient of $I_3^{3\mathrm{m}}(s_{12}, s_{34}, s_{56})$ within the function $\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{34}, t_{123}; s_{12}, s_{56})$ defined in eq. (II.3). If we shift from using the function $\mathrm{Ls}_{-1}^{2\mathrm{m}h}$ in an amplitude, to $\widetilde{\mathrm{Ls}}_{-1}^{2\mathrm{m}h}$ instead (or vice-versa), then we also have to shift the coefficient of the three-mass triangle by a quantity which contains this factor, as well as the spurious singularity factor $\langle 3|(1+2)|4\rangle$ typically associated with the coefficient of $\mathrm{Ls}_{-1}^{2\mathrm{m}h}(s_{34}, t_{123}; s_{12}, s_{56})$. The identities

$$
\begin{aligned}
& t_{123}\,\delta_{34} + 2\,s_{12}\,s_{56} = -t_{123}(t_{123} - t_{124}) - 2\,\langle 3|(1+2)|4\rangle\langle 4|(1+2)|3\rangle\,, \\
& t_{123}\,\delta_{34} + 2\,s_{12}\,s_{56} = \tfrac{1}{2}\,\delta_{34}(t_{123} - t_{124}) - \tfrac{1}{2}\,\Delta_3\,, \\
& \Delta_3 = -(t_{123}\,\delta_{34} + 2\,s_{12}\,s_{56}) - (t_{124}\,\delta_{34} + 2\,s_{12}\,s_{56}) \\
& \Delta_3 = (\delta_{34})^2 - 4\,s_{12}\,s_{56}\,, \\
& \Delta_3 = (t_{123} - t_{124})^2 + 4\,\langle 3|(1+2)|4\rangle\langle 4|(1+2)|3\rangle\,,
\end{aligned}
\qquad \text{(III.1)}
$$

are useful in manipulating leading $1/\langle 3|(1+2)|4\rangle$ singularities in order to remove, for example, 'extra' $1/t_{123}$ poles. Also note that

$$
\delta_{34} + t_{123} + t_{124} = 0\,, \qquad \delta_{12} + t_{234} + t_{134} = 0\,, \qquad \delta_{12} + \delta_{34} + 2s_{56} = 0\,. \qquad \text{(III.2)}
$$

The identities

$$
\begin{aligned}
& s_{12}\,\delta_{12} + s_{34}\,\delta_{34} + s_{56}\,\delta_{56} = \Delta_3\,, \\
& (\delta_{12})^2 = 4\,s_{34}\,s_{56} + \Delta_3\,, \\
& s_{56}\,\delta_{56} = \tfrac{1}{2}\,\delta_{12}\,\delta_{34} + \tfrac{1}{2}\,\Delta_3\,,
\end{aligned}
\qquad \text{(III.3)}
$$

are useful in simplifying the $1/\Delta_3$ poles. Also useful in this regard are the identities,

$$\begin{aligned}
\delta_{12}\,\langle 35\rangle\,[46] &= \langle 34\rangle\,[46]^2\,\langle 65\rangle + [43]\,\langle 35\rangle^2\,[56] + \langle 3|(1+2)|4\rangle\langle 5|(3+4)|6\rangle\,,\\
\delta_{34}\,\langle 25\rangle\,[16] &= \langle 21\rangle\,[16]^2\,\langle 65\rangle + [12]\,\langle 25\rangle^2\,[56] - \langle 2|(3+4)|1\rangle\langle 5|(3+4)|6\rangle\,,\\
\delta_{56}\,\langle 32\rangle\,[41] &= \langle 34\rangle\,[41]^2\,\langle 12\rangle + [43]\,\langle 32\rangle^2\,[21] - \langle 3|(1+2)|4\rangle\langle 2|(3+4)|1\rangle\,.
\end{aligned} \tag{III.4}$$

where we show a few permutations. A consequence of the last identity is

$$(\Delta_3 + 4\,s_{12}\,s_{56})\,\langle 25\rangle\,[16] = \delta_{34}\,(\langle 21\rangle\,[16]^2\,\langle 65\rangle + [12]\,\langle 25\rangle^2\,[56]) - \delta_{34}\,\langle 2|(3+4)|1\rangle\langle 5|(3+4)|6\rangle\,. \tag{III.5}$$

Another related identity is (see also eq. (5.21))

$$[64]\,\langle 32\rangle\,t_{123} = \langle 2|(1+3)|6\rangle\langle 3|(1+2)|4\rangle + \langle 3|(1+2)|6\rangle\langle 2|(5+6)|4\rangle\,. \tag{III.6}$$

Appendix IV. $e^+\,e^- \to \bar{q}qg$ Helicity Amplitudes

In this appendix we present the $e^+\,e^- \to \bar{q}qg$ primitive amplitudes, which appear in the collinear limits discussed in section 3.2. These amplitudes were first calculated in the spinor helicity formalism by Giele and Glover [7]. We present these amplitudes in the same primitive amplitude format as for the $e^+\,e^- \to \bar{q}qgg$ amplitudes discussed in section 2.3, including a separation into cut-constructible and scalar pieces. The parent diagrams associated with these amplitudes are depicted in fig. 10. Using parity and charge conjugation invariance, there is only one independent helicity configuration for each of two independent color configurations.

These amplitudes appear in the singular collinear limits of the $e^+\,e^- \to \bar{q}qgg$ and $e^+\,e^- \to \bar{q}q\bar{Q}Q$ amplitudes, except for the $k_5 \parallel k_6$ channel where the lepton pair becomes collinear. For the primitive amplitudes with closed fermion loops, the amplitudes $e^+\,e^- \to ggg$ also appear in the collinear limits. Although we do not present these amplitudes here, they may be obtained from ref. [53] after using the spinor helicity representation for the polarization vectors. One may also obtain the vector coupling results from the $\bar{q}qggg$ amplitudes of ref. [19] after summing over permutations of the quark lines, which effectively removes their color charge.

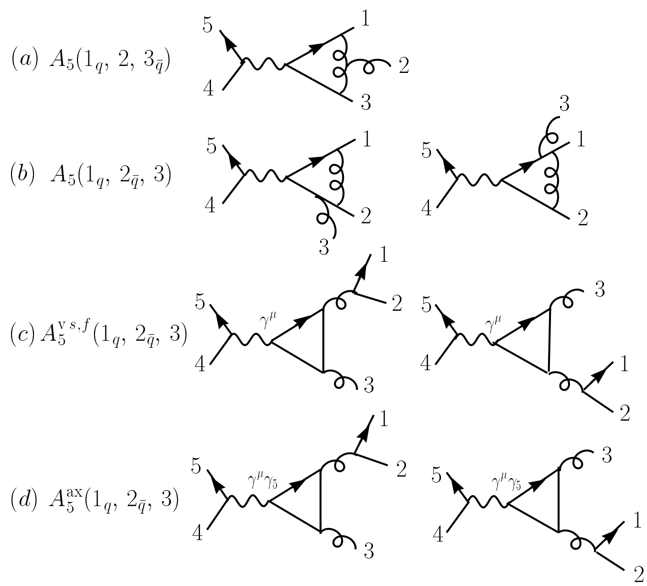

Figure 10. Parent diagrams for the various two quark and one gluon primitive amplitudes. Straight lines represent fermions, curly lines gluons, and wavy lines a vector boson (γ^* or Z).

First consider the primitive amplitude $A_5(1_q^+, 2^+, 3_{\bar{q}}^-, 4_{\bar{e}}^-, 5_e^+)$, which contributes at leading order in $1/N_c$. The tree amplitude in this case is

$$A_5^{\rm tree} = -i\, \frac{\langle 34\rangle^2}{\langle 12\rangle\, \langle 23\rangle\, \langle 45\rangle}\,. \tag{IV.1}$$

The results for the cut-constructible and scalar pieces are

$$V^{cc} = -\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{-s_{12}}\right)^\epsilon + \left(\frac{\mu^2}{-s_{23}}\right)^\epsilon\right) - \frac{2}{\epsilon}\left(\frac{\mu^2}{-s_{23}}\right)^\epsilon - 4\,, \tag{IV.2}$$

$$F^{cc} = \frac{\langle 34\rangle^2}{\langle 12\rangle\, \langle 23\rangle\, \langle 45\rangle}\left[\mathrm{Ls}_{-1}\Big(\tfrac{-s_{12}}{-s_{45}}, \tfrac{-s_{23}}{-s_{45}}\Big) - 2\,\frac{\langle 3|1 5|4\rangle}{\langle 34\rangle}\,\frac{\mathrm{L}_0\Big(\tfrac{-s_{23}}{-s_{45}}\Big)}{s_{45}}\right], \tag{IV.3}$$

$$V^{sc} = \frac{1}{2\epsilon}\left(\frac{\mu^2}{-s_{23}}\right)^\epsilon + 1\,, \tag{IV.4}$$

$$F^{sc} = \frac{\langle 34\rangle\, \langle 3|1 5|4\rangle}{\langle 12\rangle\, \langle 23\rangle\, \langle 45\rangle}\, \frac{\mathrm{L}_0\Big(\tfrac{-s_{23}}{-s_{45}}\Big)}{s_{45}} + \frac{1}{2}\, \frac{\langle 3|1 5|4\rangle^2}{\langle 12\rangle\, \langle 23\rangle\, \langle 45\rangle}\, \frac{\mathrm{L}_1\Big(\tfrac{-s_{23}}{-s_{45}}\Big)}{s_{45}^2}\,. \tag{IV.5}$$

Similarly, for the primitive amplitude $A_6(1_q^+, 2_{\bar{q}}^-, 3^+, 4_{\bar{e}}^-, 5_e^+)$, which contributes only at sub-leading order in N_c, the tree amplitude is

$$A_5^{\rm tree} = i\,\frac{\langle 24\rangle^2}{\langle 23\rangle\, \langle 31\rangle\, \langle 45\rangle}\,. \tag{IV.6}$$

The cut-constructible and scalar contributions are

$$V^{cc} = -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} - \frac{2}{\epsilon}\left(\frac{\mu^2}{-s_{45}}\right)^{\epsilon} - 4\,, \tag{IV.7}$$

$$\begin{aligned} F^{cc} = &-\frac{\langle 24\rangle^2}{\langle 23\rangle\langle 31\rangle\langle 45\rangle}\,\mathrm{Ls}_{-1}\Big(\tfrac{-s_{12}}{-s_{45}},\tfrac{-s_{13}}{-s_{45}}\Big) + \frac{\langle 24\rangle\left(\langle 12\rangle\langle 34\rangle - \langle 14\rangle\langle 23\rangle\right)}{\langle 23\rangle\langle 13\rangle^2\langle 45\rangle}\,\mathrm{Ls}_{-1}\Big(\tfrac{-s_{12}}{-s_{45}},\tfrac{-s_{23}}{-s_{45}}\Big) \\ &+ 2\,\frac{[13]\langle 14\rangle\langle 24\rangle}{\langle 13\rangle\langle 45\rangle}\,\frac{\mathrm{L}_0\Big(\tfrac{-s_{23}}{-s_{45}}\Big)}{s_{45}}\,, \end{aligned} \tag{IV.8}$$

$$V^{sc} = \frac{1}{2\epsilon}\left(\frac{\mu^2}{-s_{45}}\right)^{\epsilon} + \frac{1}{2}\,, \tag{IV.9}$$

$$\begin{aligned} F^{sc} = &\frac{\langle 14\rangle^2\langle 23\rangle}{\langle 13\rangle^3\langle 45\rangle}\,\mathrm{Ls}_{-1}\Big(\tfrac{-s_{12}}{-s_{45}},\tfrac{-s_{23}}{-s_{45}}\Big) - \frac{1}{2}\,\frac{\langle 4|1|3\rangle^2\langle 23\rangle}{\langle 13\rangle\langle 45\rangle}\,\frac{\mathrm{L}_1\Big(\tfrac{-s_{45}}{-s_{23}}\Big)}{s_{23}^2} + \frac{\langle 14\rangle^2\langle 2|3|1\rangle}{\langle 13\rangle^2\langle 45\rangle}\,\frac{\mathrm{L}_0\Big(\tfrac{-s_{45}}{-s_{23}}\Big)}{s_{23}} \\ &- \frac{\langle 2|1|3\rangle\langle 4|3|5\rangle}{\langle 13\rangle}\,\frac{\mathrm{L}_1\Big(\tfrac{-s_{45}}{-s_{12}}\Big)}{s_{12}^2} - \frac{\langle 2|13|4\rangle\langle 14\rangle}{\langle 13\rangle^2\langle 45\rangle}\,\frac{\mathrm{L}_0\Big(\tfrac{-s_{45}}{-s_{12}}\Big)}{s_{12}} - \frac{1}{2}\,\frac{[35]\left([13]\,[25] + [23]\,[15]\right)}{[12]\,[23]\langle 13\rangle[45]}\,. \end{aligned} \tag{IV.10}$$

The contributions with a closed fermion or scalar loop are rather simple. By Furry's theorem, the cases with vector-like couplings vanish, so $A_5^{\mathrm{v}\,s,f}(1_q,2_{\bar{q}},3) = 0$. The axial contribution $A_5^{\mathrm{ax}}(1_q^+,2_{\bar{q}}^-,3^+)$ are also simple [49] and are given by

$$F^{\mathrm{ax}} = -[53]\,[31]\langle 24\rangle\left(\frac{\mathrm{L}_1\Big(\tfrac{-s_{12}}{-s_{45}}\Big)}{s_{45}^2} - \frac{1}{12 s_{45}\,m_t^2}\right). \tag{IV.11}$$

References

[1] J. Ellis, M.K. Gaillard and G.G. Ross, Nucl. Phys. B111:253 (1976).

[2] A. Ali, et al., Phys. Lett. 82B:285 (1979); Nucl. Phys. B167:454 (1980).

[3] R.K. Ellis, D.A. Ross and A.E. Terrano, Phys. Rev. Lett. 45:1226 (1980); Nucl. Phys. B178:421 (1981);
K. Fabricius, I. Schmitt, G. Kramer and G. Schierholz, Phys. Lett. B97:431 (1980); Z. Phys. C11:315 (1981).

[4] K. Hagiwara and D. Zeppenfeld, Nucl. Phys. B313:560 (1989);
N.K. Falck, D. Graudenz and G. Kramer, Nucl. Phys. B328:317 (1989).

[5] F.A. Berends, W.T. Giele and H. Kuijf, Nucl. Phys. B321:39 (1989).

[6] Z. Kunszt and P. Nason, in Z Physics at LEP1, CERN Yellow Report 89-08;
G. Kramer and B. Lampe, Z. Phys. C34:497 (1987); C42:504(E) (1989); Fortschr. Phys. 37:161 (1989);
S. Catani and M.H. Seymour, Phys. Lett. B378:287 (1996) [hep-ph/9602277].

[7] W.T. Giele and E.W.N. Glover, Phys. Rev. D46:1980 (1992).

[8] OPAL Collab., P.D. Acton et al., Z. Phys. C55:1 (1992);
ALEPH Collab., D. Decamp et al., Phys. Lett. B284:163 (1992);
L3 Collab., O. Adriani et al., Phys. Lett. B284:471 (1992);
DELPHI Collab., P. Abreu et al., Z. Phys. C59:21 (1993);
SLD Collab., K. Abe et al., Phys. Rev. D51:962 (1995).

[9] S. Dawson, E. Eichten and C. Quigg, Phys. Rev. D31:1581 (1985);
R.M. Barnett, H.E. Haber and G.L. Kane, Nucl. Phys. B267:625 (1986);
L. Clavelli, Phys. Rev. D46:2112 (1992);
J. Ellis, D. Nanopoulos, and D. Ross, Phys. Lett. B305:375 (1993) [hep-ph/9303273];
L. Clavelli, P. Coulter and K. Yuan, Phys. Rev. D47:1973 (1993) [hep-ph/9205237];
R. Muñoz-Tapia and W.J. Stirling, Phys. Rev. D49:3763 (1994) [hep-ph/9309246];
G.R. Farrar, Phys. Rev. D51:3904 (1995) [hep-ph/9407401]; preprint hep-ph/9504295; preprint hep-ph/9508291; preprint hep-ph/9508292; preprint hep-ph/9707467;
L. Clavelli, I. Terekhov, Phys. Rev. Lett. 77:1941 (1996) [hep-ph/9605463]; Phys. Lett. B385:139 (1996) [hep-ph/9603390];
A. de Gouvea and H. Murayama, Phys. Lett. B400:117 (1997) [hep-ph/9606449];
Z. Bern, A.K. Grant and A.G. Morgan, Phys. Lett. B387:804 (1996) [hep-ph/9606466];
J.L. Hewett, T.G. Rizzo and M.A. Doncheski, preprint hep-ph/9612377.

[10] Z. Bern, L. Dixon, D.A. Kosower and S. Weinzierl, Nucl. Phys. B489:3 (1997) [hep-ph/9610370].

[11] Z. Bern, L. Dixon and D.A. Kosower, Nucl. Phys. Proc. Suppl. 51C:243 (1996) [hep-ph/9606378].

[12] A. Signer and L. Dixon, Phys. Rev. Lett. 78:811 (1997) [hep-ph/9609460]; preprint hep-ph/9706285, to appear in Phys. Rev. D.

[13] E.W.N. Glover and D.J. Miller, Phys. Lett. B396:257 (1997) [hep-ph/9609474].

[14] J.M. Campbell, E.W.N. Glover and D.J. Miller, preprint hep-ph/9706297.

[15] Z. Kunszt, A. Signer and Z. Trócsányi, Nucl. Phys. B411:397 (1994) [hep-ph/9305239];
A. Signer, Ph.D. thesis, ETH Zürich (1995);
S. Catani, M.H. Seymour and Z. Trócsányi, Phys. Rev. D55:6819 (1997) [hep-ph/9610553].

[16] Z. Nagy and Z. Trócsányi, preprint hep-ph/9707309.

[17] Z. Bern, L. Dixon and D.A. Kosower, Phys. Rev. Lett. 70:2677 (1993) [hep-ph/9302280].

[18] Z. Kunszt, A. Signer and Z. Trócsányi, Phys. Lett. B336:529 (1994) [hep-ph/9405386].

[19] Z. Bern, L. Dixon and D.A. Kosower, Nucl. Phys. B437:259 (1995) [hep-ph/9409393].

[20] Z. Bern, G. Chalmers, L. Dixon and D.A. Kosower, Phys. Rev. Lett. 72:2134 (1994) [hep-ph/9312333].

[21] G.D. Mahlon, Phys. Rev. D49:2197 (1994) [hep-ph/9311213]; Phys. Rev. D49:4438 (1994) [hep-ph/9312276].

[22] Z. Bern, L. Dixon, D.C. Dunbar and D.A. Kosower, Nucl. Phys. B425:217 (1994) [hep-ph/9405248].

[23] Z. Bern, L. Dixon, D.C. Dunbar and D.A. Kosower, Nucl. Phys. B435:39 (1995) [hep-ph/9409265].

[24] Z. Bern, J. Rozowsky and B. Yan, Phys. Lett. B401:273 (1997) [hep-ph/9702424].

[25] Z. Bern, L. Dixon and D.A. Kosower, Ann. Rev. Nucl. Part. Sci. 46:109 (1996) [hep-ph/9602280].

[26] L.D. Landau, Nucl. Phys. 13:181 (1959);
S. Mandelstam, Phys. Rev. 112:1344 (1958), 115:1741 (1959);
R.E. Cutkosky, J. Math. Phys. 1:429 (1960).

[27] Z. Bern and A.G. Morgan, Nucl. Phys. B467:479 (1996) [hep-ph/9511336].

[28] S.J. Parke and T.R. Taylor, Phys. Rev. Lett. 56:2459 (1986).

[29] M. Mangano and S.J. Parke, Phys. Rep. 200:301 (1991);
L. Dixon, in *QCD & Beyond: Proceedings of TASI '95*, ed. D.E. Soper (World Scientific, 1996) [hep-ph/9601359].

[30] Z. Bern and G. Chalmers, Nucl. Phys. B447:465 (1995) [hep-ph/9503236].

[31] F.A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans and T. T. Wu, Phys. Lett. 103B:124 (1981);
P. De Causmaeker, R. Gastmans, W. Troost and T.T. Wu, Nucl. Phys. B206:53 (1982);
R. Kleiss and W.J. Stirling, Nucl. Phys. B262:235 (1985);
R. Gastmans and T.T. Wu, *The Ubiquitous Photon: Helicity Method for QED and QCD*

(Clarendon Press, 1990);
Z. Xu, D.-H. Zhang and L. Chang, Nucl. Phys. B291:392 (1987).

[32] F.A. Berends and W.T. Giele, Nucl. Phys. B294:700 (1987);
D.A. Kosower, B.-H. Lee and V.P. Nair, Phys. Lett. 201B:85 (1988);
M. Mangano, S. Parke and Z. Xu, Nucl. Phys. B298:653 (1988);
Z. Bern and D.A. Kosower, Nucl. Phys. B362:389 (1991).

[33] W.H. Furry, Phys. Rev. 51:125 (1937).

[34] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, 1995).

[35] G. 't Hooft, in Acta Universitatis Wratislavensis no. 38, 12th Winter School of Theoretical Physics in Karpacz, *Functional and Probabilistic Methods in Quantum Field Theory*, Vol. 1 (1975);
B.S. DeWitt, in *Quantum gravity II*, eds. C. Isham, R. Penrose and D. Sciama (Oxford, 1981);
L.F. Abbott, Nucl. Phys. B185:189 (1981);
L.F. Abbott, M.T. Grisaru and R.K. Schaefer, Nucl. Phys. B229:372 (1983).

[36] W.L. van Neerven, Nucl. Phys. B268:453 (1986).

[37] Z. Bern, L. Dixon, D.C. Dunbar and D.A. Kosower, in *Continuous advances in QCD*, ed. A.V. Smilga (World Scientific, 1994) [hep-ph/9405248] .

[38] J.M. Campbell, E.W.N. Glover and D.J. Miller, preprint hep-ph/9612413.

[39] L.M. Brown and R.P. Feynman, Phys. Rev. 85:231 (1952);
L.M. Brown, Nuovo Cimento 21:3878 (1961);
G. Passarino and M. Veltman, Nucl. Phys. B160:151 (1979);
R. Stuart, Comput. Phys. Commun. 48:367 (1988);
G. Devaraj and R.G. Stuart, preprint hep-ph/9704308.

[40] W. van Neerven and J.A.M. Vermaseren, Phys. Lett. 137B:241 (1984).

[41] Z. Bern, L. Dixon and D.A. Kosower, Phys. Lett. 302B:299 (1993), erratum *ibid.* 318:649 (1993) [hep-ph/9212308].

[42] Z. Bern, L. Dixon and D.A. Kosower, Nucl. Phys. B412:751 (1994) [hep-ph/9306240].

[43] Z. Bern and D.A. Kosower, Nucl. Phys. B379:451 (1992).

[44] W. Siegel, Phys. Lett. 84B:193 (1979);
D.M. Capper, D.R.T. Jones and P. van Nieuwenhuizen, Nucl. Phys. B167:479 (1980);
L.V. Avdeev and A.A. Vladimirov, Nucl. Phys. B219:262 (1983).

[45] R. Pittau, preprint hep-ph/9607309;
Z. Bern, P. Gondolo and M. Perelstein, preprint hep-ph/9706538.

[46] M.T. Grisaru, H.N. Pendleton and P. van Nieuwenhuizen, Phys. Rev. D15:996 (1977);
M.T. Grisaru and H.N. Pendleton, Nucl. Phys. B124:81 (1977);
S.J. Parke and T. Taylor, Phys. Lett. B157:81 (1985);
Z. Kunszt, Nucl. Phys. B271:333 (1986).

[47] Z. Kunszt and D.E. Soper, Phys. Rev. D46:192 (1992).

[48] J.M. Campbell and E.W.N. Glover, private communication.

[49] K. Hikasa, Mod. Phys. Lett. A5:1801 (1990);
K. Hagiwara, T. Kuruma and Y. Yamada, Nucl. Phys. B358:80 (1991).

[50] L. Lewin, *Dilogarithms and Associated Functions* (Macdonald, 1958).

[51] H.-J. Lu and C. Perez, preprint SLAC–PUB–5809 (1992);
A.I. Davydychev and J.B. Tausk, Nucl. Phys. B397:123 (1993); Phys. Rev. D53:7381 (1996).

[52] N.I. Ussyukina and A.I. Davydychev, Phys. Lett. 298B:363 (1993).

[53] J.J. van der Bij and E.W.N. Glover, Nucl. Phys. B313:237 (1989).