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Convergence of a Fourier-Spline Representation for the Full-turn Map Generator*

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Abstract

Single-turn data from a symplectic tracking code can be used to construct a canonical generator for a full-turn symplectic map. This construction has been carried out numerically in canonical polar coordinates, the generator being obtained as a Fourier series in angle coordinates with coefficients that are spline functions of action coordinates. Here we provide a mathematical basis for the procedure, finding sufficient conditions for the existence of the generator and convergence of the Fourier-spline expansion. The analysis gives insight concerning analytic properties of the generator, showing that in general there are branch points as a function of angle and inverse square root singularities at the origin as a function of action.

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INTRODUCTION

Fast symplectic mapping is a powerful tool for study of long-term stability in accelerators, especially in large hadron storage rings such as the LHC (1),(2). Here we are concerned with a representation of the full-turn map in terms of a canonical mixed-variable generator, which can be constructed using many single-turn data from a symplectic tracking code (3). In numerical work to date, the generator has been expanded in a Fourier series in angle variables, with coefficients given as spline functions of action variables. We wish to find conditions so that this expansion converges (in the limit of infinitely many Fourier modes and spline interpolation points) to the exact generator of the full-turn evolution defined by the tracking code. We adopt canonical polar cooordinates (I, Φ), where I and Φ are n-component action and angle vectors, respectively. These are usually action-angle coordinates of an underlying linear system, but need not be such. The full-turn map $M : (I, \Phi) \mapsto (I', \Phi')$ as defined by the tracking code is denoted as follows:

$$I' = I + R(I, \Phi) \ , \tag{1}$$
$$\Phi' = \Phi + \Theta(I, \Phi) \ . \tag{2}$$

The existence of the inverse of the angular map (2) at fixed I is important in our analysis. We write it as

$$\Phi = \Phi' + F(I, \Phi') \ . \tag{3}$$

The function F is 2π-periodic in each component of Φ', as are R and Θ in each component of Φ. We assume that the tracking code is symplectic, so that the Jacobian matrix D satisfies

$$DJD^T = J \ , \tag{4}$$

where T denotes transpose and

$$D = \begin{bmatrix} \partial I'/\partial I & \partial I'/\partial \Phi \\ \partial \Phi'/\partial I & \partial \Phi'/\partial \Phi \end{bmatrix} \ , \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \ . \tag{5}$$

If it exists the generator $G(I, \Phi')$ defines the same map implicitly through the equations

$$I' = I + G_{\Phi'}(I, \Phi') \ , \quad \Phi = \Phi' + G_I(I, \Phi') \ , \tag{6}$$

where subscripts denote partial derivatives. By comparison of Eqs. (1) and (2) with Eqs. (6) we see that G must satisfy the partial differential equations

$$G_{\Phi'}(I, \Phi') = R(I, \Phi) \ , \quad G_I(I, \Phi') = -\Theta(I, \Phi) \ , \tag{7}$$

or

$$G_{\Phi'}(I,\Phi') = f(I,\Phi') \ , \quad G_I(I,\Phi') = g(I,\Phi') \ , \tag{8}$$

where

$$f(I,\Phi') = R(I,\Phi' + F(I,\Phi')) \ , \quad g(I,\Phi') = F(I,\Phi') \ . \tag{9}$$

Note that if $G^{(1)}$ and $G^{(2)}$ both satisfy Eqs. (8), then the two solutions differ by a constant at most. Since a constant does not affect the map defined through Eqs. (6), we see that a generator, if any, is essentially unique if F is unique. In Ref.(3) a formula for G was derived as a necessary condition on any solution of Eqs.(8), but there was no proof that the formula actually satisfied all of the equations, and in fact no proof that G exists. One gets more insight, as well as new ideas for computational methods, by first establishing the existence of G.

EXISTENCE OF THE GENERATING FUNCTION

In this section we make minimal assumptions about the given functions R and Θ; namely, that they are in class C^1 (have a continuous first derivative in each of the $2n$ variables), and that Eq.(2) has a unique solution $\Phi = \Phi' + F(I,\Phi')$, where F is in C^1 and is 2π-periodic in Φ'. We also suppose that the Jacobian matrix of the angular map, $1+\Theta_\Phi$, is nonsingular. These conditions are to hold for I in some open, simply connected set Ω. For the present discussion the angular part of the map given by Eq.(2) is best regarded as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, although in computations one would usually define angles modulo 2π. In the following sections we shall impose more specific conditions on R, Θ, and Ω, those that arise naturally in an accelerator tracking code. The conditions stated above will then hold automatically.

We seek a solution $G \in C^2$ of Eqs.(7) in the region $\mathcal{D} = \Omega \times \mathbb{R}^n$. Let us define the $2n$-dimensional vector $z = (I,\Phi')$ and write the equations in the form

$$G_z(z) = \gamma(z) \ . \tag{10}$$

An obvious necessary condition for Eq.(10) to have a solution $G \in C^2$ in an open region is that $\partial\gamma_i/\partial z_j = \partial\gamma_j/\partial z_i$, all i,j; i.e., the tensor equation curl $\gamma = 0$ holds. If that region is also simply connected (as is the region $\mathcal{D}$ in which we work) this condition is sufficient as well. Let ω be the 1-form associated with γ, that is $\omega = \gamma_1 dz_1 + \cdots + \gamma_{2n} dz_{2n}$, and C be a suitably well-behaved curve in $\mathcal{D}$. Then since $\gamma \in C^1$, the curl condition gives $d\omega = 0$ and the generalized Stokes theorem in $2n$ dimensions (Ref.(4),Theorem 6, p.478) gives $\int_C \omega = 0$. It follows that the integral of ω between z_0 and z is independent of path and

$$G(z) = \int_{z_0}^{z} \omega \ . \tag{11}$$

One sees that (11) satisfies (10) by differentiation, taking account of path independence. As was mentioned above, this solution is unique up to a constant addend.

To complete the proof that G exists, we show that curl $\gamma = 0$ follows from the symplectic condition Eq.(4). In the notation of Eqs. (8), (9) the equations to be verified are

$$f_I = g_{\Phi'} \ , \quad f_{\Phi'} = f_{\Phi'}^T \ , \quad g_I = g_I^T \ , \tag{12}$$

which is to say

$$R_I + R_\Phi F_I = F_{\Phi'}^T \ , \tag{13}$$
$$R_\Phi(1 + F_{\Phi'}) = (1 + F_{\Phi'}^T)R_\Phi^T \ , \tag{14}$$
$$F_I = F_I^T \ . \tag{15}$$

In terms of R and Θ the symplectic conditions are

$$(1 + \Theta_\Phi)(1 + R_I^T) - \Theta_I R_\Phi^T = 1 \ , \tag{16}$$
$$R_\Phi(1 + R_I^T) - (1 + R_I)R_\Phi^T = 0 \ , \tag{17}$$
$$\Theta_I(1 + \Theta_\Phi^T) - (1 + \Theta_\Phi)\Theta_I^T = 0 \ . \tag{18}$$

To get expressions for the derivatives of F, we invoke the definition of F,

$$F(I, \Phi') = -\Theta(I, F(I, \Phi') + \Phi') \ , \tag{19}$$

and differentiate and solve to get

$$F_I = -(1 + \Theta_\Phi)^{-1}\Theta_I \ , \tag{20}$$
$$F_{\Phi'} = -(1 + \Theta_\Phi)^{-1}\Theta_\Phi \ . \tag{21}$$

Now Eq.(15) follows from (20) and (18). To prove Eq.(13), write it with F_I^T replacing F_I, and substitute the derivatives of F from (20) and (21). The result is the same as (16). Finally, to prove Eq.(14), substitute R_I from (13) in (17), and again use the symmetry of F_I. Thus, curl $\gamma = 0$ has been established.

Even without invoking the argument based on Stokes's theorem, one can derive an explicit formula for G, one that obviously satisfies all of the equations (8). Integrate (8) with respect to one variable at a time, using the remaining differential equations and relations (12) to determine the unknown functions of the variables that remain after each integration. For instance, in the case of two degrees of freedom one formula from such a procedure is

$$\begin{aligned} G(I, \Phi') = &\int_0^{\Phi'_1} f_1(I_1, I_2, u, \Phi'_2) du + \int_0^{\Phi'_2} f_2(I_1, I_2, 0, u) du + \\ &\int_{I_{10}}^{I_1} g_1(u, I_2, 0, 0) du + \int_{I_{20}}^{I_2} g_2(I_{10}, u, 0, 0) du \ . \end{aligned} \tag{22}$$

This is easily recognized as a path integral of the form (11).

Another formula for G, the one proposed in Ref.(3) and used in all numerical work to date, is based on the Fourier expansion

$$G(I, \Phi') = \sum_m g_m(I) e^{im \cdot \Phi'} \ . \tag{23}$$

For all m with at least one non-zero component m_α, the Fourier amplitude may be expressed as

$$\begin{aligned} g_m(I) = &\frac{1}{im_\alpha (2\pi)^n} \int_{T^n} e^{im\cdot\Phi'} R_\alpha(I, \Phi' + F(I, \Phi')) d\Phi' \\ = &\frac{1}{im_\alpha (2\pi)^n} \int_{T^n} e^{im\cdot(\Phi + \Theta(I,\Phi))} R_\alpha(I, \Phi) \det\left[1 + \Theta_\Phi(I, \Phi)\right] d\Phi \ , \end{aligned} \tag{24}$$

where $T^n = [0, 2\pi]^n$ is the n-torus. To obtain (24) we differentiate (23) with respect to Φ_α, make use of the first equation in (7), and compute $im_\alpha g_m$ by orthogonality in the usual way. In the second expression for the integral, we avoid having to know the function F explicitly by making a change of integration variable $\Phi' \mapsto \Phi$. This is advantageous in numerical computations of the generator, and also convenient in the following analysis. The corresponding expression for $m = 0$ is obtained by integrating differential equations from the other equation of (7), namely

$$\frac{\partial g_0}{\partial I} = -\int_{T^n} \Theta(I, \Phi) \det\left[1 + \Theta_\Phi(I, \Phi)\right] d\Phi \ . \tag{25}$$

In confirmation of our general arguments one can show by direct computation using Eqs.(12) that Fourier amplitudes of integrals such as (22) agree with g_m as expressed through (24) and (25). The choice of α is correlated with the path, i.e., the order of integrations over single variables. In the course of the calculation one also shows that (22) is 2π-periodic in Φ'.

PROPERTIES OF THE MAP IN POLAR COORDINATES

We now describe more closely the map functions R and Θ that arise from a tracking code built on a symplectic integrator. We first treat the case of betatron motion (oscillations transverse to the beam direction) in one degree of freedom, for which the map in Cartesian coordinates takes the form

$$\begin{aligned} x' &= \cos(2\pi\nu)\ x + \beta \sin(2\pi\nu)\ p + X(x,p)\ , \\ p' &= -\frac{1}{\beta}\sin(2\pi\nu)\ x + \cos(2\pi\nu)\ p + P(x,p)\ , \end{aligned} \tag{26}$$

where ν and β are positive constants (tune, and beta function at the ring position to which the map refers, respectively). Since the full-turn evolution provided by a symplectic integrator amounts to a composition of a large number of polynomial maps, the functions X and P have the form

$$X(x,p) = \sum_{q=2}^{N} X_q(x,p)\ , \quad P(x,p) = \sum_{q=2}^{N} P_q(x,p)\ , \tag{27}$$

where X_q and P_q are homogeneous polynomials of degree q. The transformation to canonical polar coordinates (I,Φ) is given by

$$x + i\beta p = (2\beta I)^{1/2} e^{-i\Phi}\ . \tag{28}$$

Since the map can be written as

$$x' + i\beta p' = e^{-2\pi i\nu}(x + i\beta p) + X + i\beta P\ , \tag{29}$$

it is easy to derive the following expressions for R and Θ of Eqs.(1) and (2):

$$\begin{aligned} R(I,\Phi) =& \big[I/2\beta\big]^{1/2}\big[e^{i(2\pi\nu+\Phi)}(X + i\beta P) + e^{-i(2\pi\nu+\Phi)}(X - i\beta P)\big] \\ &+ \frac{1}{2\beta}\big[X^2 + \beta^2 P^2\big]\ , \end{aligned} \tag{30}$$

$$\Theta(I,\Phi) = 2\pi\nu + \frac{1}{2i}\ln\left[\frac{1 + e^{-i(2\pi\nu+\Phi)}(X - i\beta P)(2\beta I)^{-1/2}}{1 + e^{\ i(2\pi\nu+\Phi)}(X + i\beta P)(2\beta I)^{-1/2}}\right]\ . \tag{31}$$

At real I and Φ the logarithm is purely imaginary. To be definite we choose the branch to be such that $\Theta - 2\pi\nu \ \in\ [-\pi/2, \pi/2]$.

Because X and P are polynomials of second or higher order, the following features are obvious from (28),(30), and (31):

–1– $R(I,\Phi)$ is a polynomial in $I^{1/2}$ and in $\exp(\pm i\Phi)$.

–2– $\Theta(I,\Phi)$ is analytic in $I^{1/2}$ and in $\exp(\pm i\Phi)$ in any region in which $|\exp(\pm i\Phi)(X \pm i\beta P)(2\beta I)^{-1/2}| < 1$.

Now we can discuss inversion of the transformation (2) on $\mathbb{R}$ at real I, an issue in the previous section. Given $\Phi' \in \mathbb{R}$ we show that there is a unique solution $\Phi \in \mathbb{R}$ of (2), provided that I is sufficiently small. We use a method that works as well in the case of two degrees of freedom, even though a simpler argument based on requiring that $\Phi + \Theta(I,\Phi)$ be monotonic can be used in the present case. Let $x = \Phi' - 2\pi\nu - \Phi$, $\Theta(I,\Phi) = 2\pi\nu + T(I,\Phi)$ and write Eq.(2) as a fixed point problem to be solved on $\mathbb{R}$, namely

$$x = A(x) \ , \qquad A(x) = T(I, \Phi' - 2\pi\nu - x) \ . \tag{32}$$

One can see from (31) that $T(I,\Phi)$ and $T_\Phi(I,\Phi)$ are continuous, periodic functions of Φ on $\mathbb{R}$ at sufficiently small I. Since $(X \pm i\beta P)I^{-1/2} = \mathcal{O}(I^{1/2})$, the singularities of the logarithm in (31) are avoided if we choose $\bar{I}$ so that

$$|X(I,\Phi) \pm i\beta P(I,\Phi)|(2\beta I)^{-1/2} \leq \eta < 1 \ , \tag{33}$$

for $I < \bar{I}$. (Here we have written $X(I,\Phi)$ for what was previously called $X(x,p)$, and similarly for P.) Moreover, T_Φ has the form $\mathcal{O}(I^{1/2})/(1+\mathcal{O}(I^{1/2})$ as far as its I-dependence is concerned, and is small for small I, uniformly in Φ. Let us then redefine $\bar{I}$, making it smaller if necessary, so that $|T_\Phi(I,\Phi)| \leq \alpha < 1$, $\Phi \in \mathbb{R}$ for $I < \bar{I}$. It then follows from the contraction mapping theorem that the fixed point problem (32) has a unique solution if $I < \bar{I}$, since $A : \mathbb{R} \to \mathbb{R}$ and $|A(x) - A(y)| \leq \alpha |x - y|$, all $x, y \in \mathbb{R}$, the latter by Taylor's theorem. The corresponding solution Φ of (2) can be written as $\Phi = \Phi' + F(I,\Phi')$, where F is periodic. This follows from (32) and the periodicity of T.

We now have existence and uniqueness of the function F of the previous section, but we also need to know that $F \in C^1$. That may be established by an implicit function argument applied to Eq.(2) written as $H(\Phi,\Phi',I) = 0$. We have already seen that this equation has a unique solution $\Phi(\Phi',I)$ if $I < \bar{I}$. We can conclude, by an appropriate form of the implicit function theorem (Ref.(5), Sections 10.2.2, 10.2.3), that the solution has continuous derivatives in both variables if H has a continuous derivative in each of its three arguments, in a neighborhood of the solution. We have already taken care of $\partial H/\partial\Phi$ by requiring $I < \bar{I}$, and $\partial H/\partial\Phi' = 1$. For $\partial H/\partial I$ we have to add a new requirement on the region, namely that $I > \underline{I} > 0$. This is required since $\partial T/\partial I$ involves a factor $(X \pm i\beta P)I^{-3/2}$ which in general blows up at $I = 0$. In the present case, the region Ω mentioned in the previous section is $\{I | 0 < \underline{I} < I < \bar{I}\}$.

Next consider betatron motion in two degrees of freedom, but with the two motions uncoupled at the linear level. Then the map has the form

$$\begin{aligned} x_j' &= \cos(2\pi\nu_j)\ x_j + \beta_j \sin(2\pi\nu_j)\ p_j + X_j(x,p) \ , \\ p_j' &= -\frac{1}{\beta_j}\sin(2\pi\nu_j)\ x_j + \cos(2\pi\nu_j)\ p_j + P_j(x,p) \ , \\ &\qquad j = 1, 2 \ , \end{aligned} \tag{34}$$

where now X_j and P_j are sums of homogenous polynomials of degree 2 and greater in $x_1,\ p_1,\ x_2,\ p_2$. The corresponding map functions, R and Θ, are given by expressions just like (30) and (31), except that all ingredients of the formulas acquire a subscript j, for instance

$$\Theta_j(I,\Phi) = 2\pi\nu_j + \frac{1}{2i}\ln\left[\frac{1+e^{-i(2\pi\nu_j+\Phi_j)}(X_j - i\beta_j P_j)(2\beta_j I_j)^{-1/2}}{1+e^{\ i(2\pi\nu_j+\Phi_j)}(X_j + i\beta_j P_j)(2\beta_j I_j)^{-1/2}}\right] . \qquad (35)$$

Now it is clear that claims totally analogous to (–1–) and (–2–) above are valid in the present case; we have polynomial or analytic behavior in $I_j^{1/2}$ and $\exp(\pm i\Phi_j),\ j=1,2$. An important difference arises, however, when we try to verify the condition

$$|\exp(\pm i\Phi_j)(X_j \pm i\beta_j P_j)(2\beta_j I_j)^{-1/2}| \le \eta < 1\ . \qquad (36)$$

Here the coefficient of $I_j^{-1/2}$ does not necessarily vanish as $I_j \to 0$. It may contain terms like $x_k^2,\ p_k^2,\ x_k p_k$ with $k \neq j$, which are proportional to I_k. In general there is a pole in $I_j^{1/2}$ at $I_j^{1/2} = 0$.

Let us see how to deal with this situation when we turn again to the solution of Eq.(2), now in $\mathbb{R}^2$ at real I. We must first find a sufficient condition for inequality (36). For that we ask that I_1 and I_2 be not only small but also not too dissimilar, for instance by requiring

$$\begin{aligned} I\ \in\ \mathcal{K}(\lambda,\mu,\bar I) = \{\ I\ \big|\ \lambda I_1 < I_2 < \mu I_1\ ,\ \|\ I\ \| < \bar I\}, \\ 0<\lambda<1\ ,\quad 1<\mu<\infty\ , \end{aligned} \qquad (37)$$

where $\|\cdot\|$ is the Euclidian norm. Then (36) certainly holds for $\bar I$ sufficiently small. Following the plan of the one-dimensional case, we again solve the fixed point problem analogous to (32), taking some vector norm $\|x\|$ and a compatible matrix norm to bound the Jacobian matrix A_x. After a possible downward adjustment of $\bar I$ to a new value $\bar I_1$, we guarantee that $\|A_x\| \le \alpha < 1$ for all $I \in \mathcal{K}$, so that a unique solution x is implied by the contraction mapping principle. Again, we have $\Phi = \Phi' + F(I,\Phi')$ with F periodic in Φ'. We can again apply the implicit function theorem to show that $F \in C^1$, provided that we bound $\|I\|$ below to avoid the singularity of $\partial T/\partial I$, which can be more severe by one power than in the one-dimensional case. Thus we require

$$I\ \in\ \mathcal{L}(\lambda,\mu,\underline{I},\bar I_1) = \{\ I\ \big|\ \lambda I_1 < I_2 < \mu I_1\ ,\ 0<\underline{I}<\|\ I\ \|<\bar I_1\}, \qquad (38)$$

where $\underline{I}$ can have any positive value less than $\bar I_1$. Now $\mathcal{L}$ is the region Ω of the previous section.

CONVERGENCE OF THE FOURIER SERIES

Since R and Θ are analytic in Φ_j, it is natural to use a complex variable method to study the convergence of the series (23). Recall that if a function $f(\phi)$ is 2π-periodic and analytic in a strip $|\mathrm{Im}\ \phi| < \sigma$, and continuous on the closure of the strip, $|\mathrm{Im}\ \phi| \leq \sigma$, then its Fourier coefficients obey the bound $|f_m| \leq \|f\| \exp(-|m|\sigma)$. This is seen by distorting the contour in the integral that defines f_m, and applying Cauchy's theorem. For $m > 0$, say, the interval of integration $[0, 2\pi]$ can be replaced by the straight line segment between $-i\sigma$ and $-i\sigma + 2\pi$, since the contributions of vertical paths leading to and from the displaced interval cancel by periodicity. Conversely, if f_m is bounded as stated, then $f(\phi)$ is analytic in the strip, since its Fourier series converges uniformly for $|\mathrm{Im}\ \phi| \leq \delta < \sigma$. The generalization to a function of several variables, 2π-periodic in each, is obvious.

We give the proof for a map of type (34). As we have already noted, $R(I, \Phi)$ is a polynomial as a function of each $I_j^{1/2}$ and each $\exp(\pm i\Phi_j)$. To discuss analyticity of Θ, we fix σ and suppose that

$$I \in \mathcal{L}(\lambda, \mu, \underline{I}, \bar{I}_2) \ , \tag{39}$$

where $\bar{I}_2$ is to be determined, and $\underline{I}$ can have any positive value less than $\bar{I}_2$. Then since X_j and P_j are sums of homogeneous polynomials of degree 2 or greater, there is some $M(\sigma)$ so that

$$\begin{gathered} \left| e^{\pm i\Phi_j} \left[X_j(I, \Phi) \pm \beta_j P_j(I, \Phi) \right] \right| (2\beta_j I_j)^{-1/2} < M(\sigma) I_j^{1/2} \ , \\ |\mathrm{Im}\ \Phi| \leq \sigma \ . \end{gathered} \tag{40}$$

(We write $|\mathrm{Im}\ \Phi| \leq \sigma$ to mean $|\mathrm{Im}\ \Phi_j| \leq \sigma \ , \ j = 1, 2$). We then choose $\bar{I}_2 \leq \eta M(\sigma)^{-2}$, where $\eta < 1$. Then $M(\sigma) I_j^{1/2} \leq M(\sigma) \|I\|^{1/2} < \eta < 1$. By (40) and (35) we are then assured that Θ and Θ_Φ are analytic in each Φ_j for $|\mathrm{Im}\ \Phi| < \sigma$ and continuous on $|\mathrm{Im}\ \Phi| \leq \sigma$.

Now to derive exponential decrease of the Fourier coefficients, we work with Eq. (24), which holds under condition (38). Imposing also condition (39), we can now displace the contour of each integration variable Φ_j in Eq.(24), moving it into the lower (upper) half-plane a distance σ, according as m_j is positive (negative). If any m_j is zero, we need not move the corresponding contour. Notice that we need not be concerned about hitting possible zeros of $\det(1 + \Theta_\Phi)$ at complex Φ. The formula (24) is correct with nonzero determinant at real Φ, and moving the contour to complex Φ is justified by Cauchy's theorem, since the determinant and all other ingredients of the integrand are analytic. To extract exponential decrease of g_m at large m_j, it is sufficient to show existence of a τ such that

$$|\mathrm{Im}\ \Theta_j(I, \Phi)| \leq \tau < \sigma \ , \quad |\mathrm{Im}\ \Phi| \leq \sigma \ , \tag{41}$$

because then on the complex contour

$$\mathrm{Re}\,\left[-im_j\big(\Phi_j+\Theta_j(I,\Phi)\big)\right]\le -|m_j|(\sigma-\tau)\ . \tag{42}$$

To find a sufficient condition for (41) to hold, we note that

$$\mathrm{Im}\ \Theta_j=\frac{1}{2}\left[\ln\left|1+e^{i(2\pi\nu_j+\Phi_j)}(X_j+i\beta_jP_j)(2\beta_jI_j)^{-1/2}\right|-(i\to -i)\right]\ . \tag{43}$$

Since $\ln(1+x)$ is monotonic and less than x for $x>0$, we have by Eq.(40) that

$$\left|\mathrm{Im}\ \Theta_j(I,\Phi)\right|\le M(\sigma)I_j^{1/2}\le\tau<\sigma\ , \tag{44}$$

for $I\in\mathcal{L}(\lambda,\mu,\underline{I},\bar{I})$ and $|\mathrm{Im}\ \Phi|\le\sigma$, for some sufficiently small $\bar{I}$. Let us take $\tau<1$, so that our previous condition (40) also holds, and denote the resulting value of $\bar{I}$ by $\bar{I}_2$, thus possibly redefining the previous I_2. Now all conditions on I in this and the previous section are met if

$$I\ \in\ \mathcal{L}(\lambda,\mu,\underline{I},\bar{I})\ ,\quad \bar{I}=\min(\bar{I}_1,\ \bar{I}_2)\ . \tag{45}$$

When (45) holds, we can be sure that g_m decreases exponentially with $|m|=\max_j|m_j|$, and that the same is true for all its I derivatives (modulo powers of $|m|$). The latter is true because we can differentiate the second integral in (24) any number of times, after displacement of the contour, each time bringing down one power of m_j but retaining the exponentially decreasing factor. Thus, under condition (45), all derivatives $g_m^{(i,j)}=\partial g_m^{i+j}/\partial I_i\partial I_j$ are continuous and bounded, and

$$|g_m^{(i,j)}(I)|\le\kappa_{ij}|m|^{i+j}\exp\left(-|m|(\sigma-\tau)\right)\ , \tag{46}$$

for some κ_{ij} independent of I, and $i,j=0,1,\ldots$. It is not difficult to specify a region in which g_m is analytic as a function of two complex variables I_1,I_2, but we shall omit that discussion in this paper since it is not needed in our applications. We have finished the proof of

Theorem 1: For a system in two degrees of freedom, let the map be as described in Eq.(34). For I in the region $\mathcal{L}(\lambda,\mu,\underline{I},\bar{I})$, with some sufficiently small $\bar{I}$, the generator $G(I,\Phi')$ exists and is unique up to a constant addend, and its derivatives of any order are continuous and bounded. It is given by a Fourier series (23) that converges absolutely and uniformly for $|\mathrm{Im}\ \Phi'|\le\delta$ and $I\in\mathcal{L}(\lambda,\mu,\underline{I},\bar{I}(\delta))$, for any $\delta>0$ but with $\bar{I}(\delta)$ tending to zero as δ increases. The same region of convergence occurs for the Fourier series of all derivatives of G, although the convergence may be slower by powers of $|m|$. A suitable $\bar{I}$ can be computed from a knowledge of the map, following the steps of the proof.

A similar statement is of course true for one degree of freedom, in which case $\mathcal{L}$ is replaced by the interval $(\underline{I}, \bar{I})$, $\underline{I} > 0$. Let us illustrate the choice of $\bar{I}$ for the Hénon Map in one degree of freedom, which describes the effect of a thin sextupole magnet together with the rotation in phase space caused by linear forces. The term in the Hamiltonian giving the impulsive sextupole force ("kick") at location $s = 0$ in the ring is $(\lambda/3)x^3\delta(s)$. The kick changes p by $\Delta p = -\lambda x^2$ while leaving x unchanged. The kick followed by the rotation with tune ν gives

$$X - i\beta P = \lambda\beta i e^{2\pi\nu i}x^2 \ ,$$
$$\Theta(I,\Phi) = 2\pi\nu + \frac{1}{2i}\ln\left[\frac{1 + i\lambda\beta\cos^2\Phi e^{-i\Phi}(2\beta I)^{1/2}}{1 - i\lambda\beta\cos^2\Phi e^{\ i\Phi}(2\beta I)^{1/2}}\right] \ . \tag{47}$$

The equation (2) will have a unique solution in $\mathbb{R}$ if $1 + \Theta_\Phi$ is positive, and we can guarantee that by requiring

$$\left|\lambda\beta\mathrm{Re}\left[\frac{\partial_\Phi(\cos^2\Phi e^{-i\Phi})(2\beta I)^{1/2}}{1 + i\lambda\beta\cos^2\Phi e^{-i\Phi}(2\beta I)^{1/2}}\right]\right| \leq \alpha < 1 \ . \tag{48}$$

This inequality holds for $I < \bar{I}_1$. Next, to ensure analyticity of Θ and Θ_Φ for $|\mathrm{Im}\ \Phi| < \sigma$ and continuity in $|\mathrm{Im}\ \Phi| \leq \sigma$, note first that

$$\left|\lambda\beta(2\beta I)^{1/2}\cos^2\Phi e^{\pm i\Phi}\right| < M(\sigma)I^{1/2} \ ,$$
$$|\mathrm{Im}\ \Phi| \leq \sigma \ , \quad M(\sigma) = \lambda(\beta/2)^{3/2}(3e^\sigma + e^{3\sigma}) \ . \tag{49}$$

A nice choice for σ is $\bar{\sigma} = 0.5100\ldots$, the value that maximizes $\sigma/(3\exp\sigma + \exp 3\sigma)$. Choose $\bar{I}_2$ so that $M(\bar{\sigma})(\bar{I}_2)^{1/2} = \tau < \bar{\sigma}$. Then with $\bar{I} = \min(\bar{I}_1, \bar{I}_2)$, a generator G exists, unique up to a constant, and all of its derivatives are continuous and bounded. The Fourier series for G and its derivatives will converge, absolutely and uniformly for $|\mathrm{Im}\ \Phi| \leq \delta < \bar{\sigma} - \tau$ and $I \in (\underline{I}, \bar{I})$.

Although our sufficient condition for analyticity of Θ is hardly necessary, one can show in the present example that Θ certainly has branch points at sufficiently large Im Φ, regardless of the value of I. With $\Phi = u + iv$ and $\hat{\lambda} = \lambda\beta(\beta I/2)^{1/2}$, the logarithm of (47) has a branch point where

$$\hat{\lambda}[e^{-v}\sin u - e^{3v}\sin 3u - 2e^v\sin u] = 1 \ ,$$
$$e^{-v}\cos u + e^{3v}\cos 3u + 2e^v\cos u = 0 \ . \tag{50}$$

With $u = \pi/2$ and $\lambda > 0$ there is a solution where $4\hat{\lambda}e^v\sinh^2 v = 1$; for $\lambda < 0$ take $u = 3\pi/2$.

CONVERGENCE OF THE SPLINE APPROXIMATION

It remains to discuss the approximation of the Fourier coefficients $g_m(I)$ by spline functions of I, of course for I in the region $\mathcal{L}$ specified in the previous section. We need do this only for finite m, say for $|m| < M$, since the various Fourier series converge uniformly. We are mainly interested in the derivatives of G, which occur in the equations (6) that define the map induced by the generator. For fixed $\epsilon > 0$ we choose M so large that

$$\begin{aligned} &\big| G_{\Phi'_j}(I, \Phi') - \sum_{|m|<M} i m_j g_m(I) e^{im\cdot\Phi'} \big| < \epsilon/2 \ , \\ &\big| G_{I_j}(I, \Phi') - \sum_{|m|<M} \partial g_m(I)/\partial I_j e^{im\cdot\Phi'} \big| < \epsilon/2 \ , \end{aligned} \tag{51}$$

for $j = 1, 2$ and all $(I, \Phi') \in \mathcal{L} \times T^2$.

Explicit error bounds for approximation of a univariate function and its derivatives by interpolating cubic splines are known for the case of the Hermite boundary conditions, which require that the derivative of the spline match the derivative of the function at the first and last knots (6) . We assume this case, partly to avoid a longer story concerning approximation theorems with more general conditions (7). Numerical computations could be done with Hermite conditions if automatic differentiation (8) (algebra of truncated power series) were used to find first derivatives of the map defined by the tracking code. To date, a different spline definition without derivative data has been used (3, 1) , but it would be interesting to try the Hermite scheme as well.

Given an ascending sequence of spline knots, $x_0 < x_1 < \cdots < x_n$, define $h = \max|x_{i+1} - x_i|$, and $\|f\| = \sup_{x\in[x_0,x_n]} |f(x)|$. Let $s(x)$ be the unique piecewise cubic polynomial function such that $s(x_i) = f(x_i)$, $i = 0, 1, \ldots n$, $s^{(1)}(x_i) = f^{(1)}(x_i)$, $i = 0, n$, and $s \in C^2[x_0, x_n]$. For $f \in C^4[x_0, x_n]$, Hall and Meyer (9) have proved that the following is true, irrespective of the distribution of the knots:

$$\begin{aligned} &\|f^{(i)} - s^{(i)}\| \le \kappa_i h^{4-i} \|f^{(4)}\| \ , \quad i = 0, 1, 2 \ , \\ &(\kappa_0, \kappa_1, \kappa_2) = \big(\frac{5}{384}, \frac{1}{24}, \frac{3}{8}\big) \ . \end{aligned} \tag{52}$$

For bivariate spline interpolation of a function $f(x, y)$ we take two knot sequences, $x_0 < x_1 < \cdots < x_n$ and $y_0 < y_1 < \cdots < y_m$, and define $h_x = \max|x_{i+1} - x_i|$, $h_y = \max|y_{i+1} - y_i|$, $\mathcal{R} = [x_0, x_n] \times [y_0, y_m]$, $\|f\| = \sup_{\mathcal{R}} |f(x, y)|$. Define the operator $\mathcal{P}_x$ so that $\mathcal{P}_x g(x)$ is the cubic spline interpolant of $g(x)$ with Hermite boundary conditions, and similarly for $\mathcal{P}_y$. Then the bicubic spline interpolant of $f(x, y)$ with Hermite boundary conditions is defined to be $s(x, y) = \mathcal{P}_y(\mathcal{P}_x f(x, y)) = \mathcal{P}_x(\mathcal{P}_y f(x, y))$. That is, we first interpolate in one variable, and then interpolate the resulting spline coefficients in the other variable. This is equivalent to expressing $s(x,y)$ in terms of the tensor product basis formed from the cardinal spline basis functions for the two dimensions (6). For $f \in C^4[\mathcal{R}]$, Carlson and Hall (10) showed that, irrespective of knot distribution,

$$\begin{aligned}\|(f-s)^{(i,j)}\| \leq & \epsilon_{4-j,i} h_x^{4-i} \|f^{(4-j,j)}\| + \epsilon_{2i}\epsilon_{2j} h_x^{2-i} h_y^{2-j} \|f^{(2,2)}\| \\ & + \epsilon_{4-i,j} h_y^{4-j} \|f^{(i,4-i)}\| \ , \qquad 0 \leq i,j \leq 2 \ , \end{aligned} \tag{53}$$

where $g^{(i,j)} = \partial^{i+j} g/\partial x_i \partial x_j$. Values of the $\epsilon_{i,j}$ are given in Table 1 of Ref.(10). In our application we require only the following cases:

$$\|f-s\| \leq \frac{5}{384}\left[h_x^4\|f^{(4,0)}\| + h_y^4\|f^{(0,4)}\|\right] + \frac{81}{64}h_x^2 h_y^2 \|f^{(2,2)}\| \ , \tag{54}$$

$$\|(f-s)^{(1,0)}\| \leq \frac{9+\sqrt{3}}{216} h_x^3 \|f^{(4,0)}\| + \frac{9}{2} h_x h_y^2 \|f^{(2,2)}\| + \frac{71}{216} h_y^4 \|f^{(1,3)}\| \ . \tag{55}$$

We wish to approximate $g_m(I_1, I_2)$ by bicubic spline interpolation, on some rectangle $\mathcal{R} = [I_{10}, I_{1n}] \times [I_{20}, I_{2m}] \in \mathcal{L}(\lambda, \mu, \underline{I}, \bar{I})$. For notational convenience, we suppose that the two mesh step bounds are equal and write $h_{I_1} = h_{I_2} = h$. We denote the spline interpolation of a function $f(I)$ by $f^s(I)$. In numerical construction and application of the generator we approximate $g_m(I)$ by $g_m^s(I)$, and then use $\partial g_m^s(I)/\partial I$, calculated analytically, as the approximation to $\partial g_m(I)/\partial I$. If we were to approximate $g_m(I)$ and $\partial g_m(I)/\partial I$ *independently* by cubic splines, then the symplectic condition would not be maintained exactly, and the whole rationale of the generating function method would be undermined. Thus, we shall find an application for (55) as well as (54).

By applying Eqs. (46, 51, 54, 55), we bound the errors for the final approximations to $G_{\Phi'}$ and G_I as follows:

$$\begin{aligned} & \Big|G_{\Phi'_j}(I,\Phi') - \sum_{|m|<M} i m_j g_m^s(I) e^{i m\cdot\Phi'}\Big| \leq \\ & \Big|G_{\Phi'_j}(I,\Phi') - \sum_{|m|<M} i m_j g_m(I) e^{i m\cdot\Phi'}\Big| + \Big|\sum_{|m|<M} i m_j \big((g_m(I) - g_m^s(I)\big) e^{i m\cdot\Phi'}\Big| \leq \\ & \frac{\epsilon}{2} + h^4\Big(\frac{5}{384}(\kappa_{40}+\kappa_{04}) + \frac{81}{64}\kappa_{22}\Big) \sum_{|m|<M} |m|^5 e^{-|m|(\sigma-\tau)} \ , \end{aligned} \tag{56}$$

$$\begin{aligned} & \Big|G_{I_1}(I,\Phi') - \sum_{|m|<M} g_m^{s\,(1,0)}(I) e^{i m\cdot\Phi'}\Big| \leq \\ & \Big|G_{I_1}(I,\Phi') - \sum_{|m|<M} g_m^{(1,0)}(I) e^{i m\cdot\Phi'}\Big| + \Big|\sum_{|m|<M} \big((g_m^{(1,0)}(I) - g_m^{s\,(1,0)}(I)\big) e^{i m\cdot\Phi'}\Big| \leq \\ & \frac{\epsilon}{2} + h^3\Big(\frac{9+\sqrt{3}}{216}\kappa_{40} + \frac{9}{2}\kappa_{22} + \frac{71}{216}h\kappa_{13}\Big) \sum_{|m|<M} |m|^4 e^{-|m|(\sigma-\tau)} \ , \end{aligned} \tag{57}$$

and similarly for G_{I_2}. Each of the right hand sides can be made less than ϵ, by taking $h = \mathcal{O}(\epsilon^{1/3})$ sufficiently small. This completes the proof of

Theorem 2: Let G be the generating function for the map (34) with $I \in \mathcal{L}$, as described in Theorem 1. The Fourier series for G can be approximated by the series $G^{(h,M)}$, which is obtained by truncating the series for G at $|m| = M$, then approximating the coefficients by bicubic spline interpolation with Hermite boundary conditions on a rectangle $\mathcal{R} \in \mathcal{L}$, the parameter h being the maximum mesh step in either direction. For sufficiently large $M = M(\epsilon)$ and sufficiently small $h = \mathcal{O}(\epsilon^{1/3})$, the series $G_{\Phi'}^{(h,M)}$ and $G_I^{(h,M)}$ approximate $G_{\Phi'}$ and G_I within an error ϵ, uniformly for $(I, \Phi') \in \mathcal{R} \times T^2$.

COMMENTS

We have seen that elementary arguments prove convergence of the Fourier-spline representation of the generating function of a full turn map as defined by a symplectic tracking code. As is usual in analyses of this sort, the specific estimates for the region of validity of the Fourier-spline series are probably somewhat pessimistic from a practical stand point. Nevertheless, the analysis reveals the analytic structure of the generating function and gives rates of convergence, results that should be useful in a search for improvements in the practical realization of the method. One feature of the proof shows up very clearly in numerical work (1), (3), namely the restriction to a region in the I_1, I_2 plane that excludes neighborhoods of the coordinate axes. A high priority for further work is to avoid this problem, which is really a question of a coordinate singularity, by using Cartesian coordinates. One possibility is a straightforward adaptation of the present method, replacing the Fourier development by an expansion in Hermite polynomials.

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