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A Model Independent Construction of the Unitarity Triangle

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In a large class of models, the only significant new physics effect on the CP asymmetries in $B \rightarrow \psi K_S$ and $B \rightarrow \pi\pi$ decays is a new contribution to the $B - \bar{B}$ mixing amplitude. This allows a model independent construction of the CKM Unitarity Triangle (up to hadronic uncertainties). Furthermore, the contributions to the mixing from the Standard Model and from the new physics can be disentangled. A serious obstacle to this analysis is an eightfold discrete ambiguity in solving for the angles of the triangle. Several ways to reduce the ambiguity either by making further measurements, or by making further assumptions about the nature of the new physics are described.

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1. The Basic Assumptions and Results

The first two CP asymmetries to be measured in a B factory are likely to be [1]

$$\frac{\Gamma(B_{\text{phys}}^0(t) \rightarrow \psi K_S) - \Gamma(\bar{B}_{\text{phys}}^0(t) \rightarrow \psi K_S)}{\Gamma(B_{\text{phys}}^0(t) \rightarrow \psi K_S) + \Gamma(\bar{B}_{\text{phys}}^0(t) \rightarrow \psi K_S)} = -a_{\psi K_S} \sin(\Delta m_B t), \quad (1.1)$$

$$\frac{\Gamma(B_{\text{phys}}^0(t) \rightarrow \pi\pi) - \Gamma(\bar{B}_{\text{phys}}^0(t) \rightarrow \pi\pi)}{\Gamma(B_{\text{phys}}^0(t) \rightarrow \pi\pi) + \Gamma(\bar{B}_{\text{phys}}^0(t) \rightarrow \pi\pi)} = -a_{\pi\pi} \sin(\Delta m_B t). \quad (1.2)$$

In addition, the B factory will improve our knowledge of the $B - \bar{B}$ mixing parameter, $x_d \equiv \frac{\Delta m_B}{\Gamma_B}$, and of the charmless semileptonic branching ratio of the B mesons.

Within the Standard Model, these four measurements are useful in constraining the unitarity triangle. The asymmetries (1.1) and (1.2) measure angles of the unitarity triangle:

$$a_{\psi K_S} = \sin 2\beta, \quad (1.3)$$

$$a_{\pi\pi} = \sin 2\alpha, \quad (1.4)$$

where

$$\alpha \equiv \arg \left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*} \right], \quad \beta \equiv \arg \left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*} \right]. \quad (1.5)$$

In (1.3) we have taken into account the fact that the final state is CP-odd. In (1.4) we have ignored possible penguin contamination which can, in principle, be eliminated by isospin analysis [2]. The measurement of x_d determines one side of the unitarity triangle (R_t):

$$x_d = C_t R_t^2, \quad (1.6)$$

where

$$R_t \equiv \left| \frac{V_{tb}^* V_{td}}{V_{cb}^* V_{cd}} \right|, \quad (1.7)$$

and $C_t = \tau_b \frac{G_F^2}{6\pi^2} \eta_B m_B (B_B f_B^2) m_t^2 f_2 (m_t^2/m_W^2) |V_{cb}^* V_{cd}|^2$ (for definitions and notations see [1]). The present values are $x_d = 0.73 \pm 0.05$ and $C_t \sim 0.4 - 0.8$ for $\sqrt{B_B} f_B = 140 - 200$ MeV [3]. Measurements of various inclusive and exclusive $b \rightarrow u\ell\nu$ processes will determine (up to uncertainties arising from various hadronic models) the length of the other side of the unitarity triangle (R_u):

$$\frac{\Gamma(b \rightarrow u\ell\nu)}{\Gamma(b \rightarrow c\ell\nu)} = \frac{1}{F_{\text{ps}}} \left| \frac{V_{cd}}{V_{ud}} \right|^2 R_u^2, \quad (1.8)$$

where

$$R_u \equiv \left| \frac{V_{ub}^* V_{ud}}{V_{cb}^* V_{cd}} \right| \quad (1.9)$$

and $F_{\text{ps}} \approx 0.5$ is a phase space factor. The present value for R_u ranges from 0.27 to 0.45 depending on the hadronic model used to relate the measurement at the endpoint region, or of some exclusive mode, to the total $b \rightarrow u$ inclusive rate [3].

In the presence of new physics it is quite possible that the Standard Model predictions (1.3), (1.4) and (1.6) are violated. The most likely reason is a new, significant contribution to $B - \bar{B}$ mixing that carries a CP violating phase different from the Standard Model one. Other factors that could affect the construction of the unitarity triangle from these four measurements are unlikely to be significant [4-5]:

- a. The $\bar{b} \rightarrow \bar{c}c\bar{s}$ and $\bar{b} \rightarrow \bar{u}u\bar{d}$ decays for $a_{\psi K_S}$ and $a_{\pi\pi}$ respectively, as well as the semileptonic B decays for R_u , are mediated by Standard Model tree level diagrams. In most extensions of the Standard Model there is no decay mechanism that could significantly compete with these contributions. (For exceptions, which could affect the $\bar{b} \rightarrow \bar{u}u\bar{d}$ decay, see [6].)
- b. New physics could contribute significantly to $K - \bar{K}$ mixing. However, the small value of ε_K forbids large deviations from the Standard Model phase of the mixing amplitude.
- c. Unitarity of the three generation CKM matrix is maintained if there are no quarks beyond the three generations of the Standard Model. Even in models with an extended quark sector the effect on $B - \bar{B}$ mixing is always larger than the violation of CKM unitarity.

Our analysis below applies to models where the above three conditions are not significantly violated. Under these circumstances the relevant new physics effects can be described by two new parameters, r_d and θ_d [7-10], defined by

$$(r_d e^{i\theta_d})^2 \equiv \frac{\langle B^0 | \mathcal{H}_{\text{eff}}^{\text{full}} | \bar{B}^0 \rangle}{\langle B^0 | \mathcal{H}_{\text{eff}}^{\text{SM}} | \bar{B}^0 \rangle}, \quad (1.10)$$

where $\mathcal{H}_{\text{eff}}^{\text{full}}$ is the effective Hamiltonian including both Standard Model and new physics contributions, and $\mathcal{H}_{\text{eff}}^{\text{SM}}$ only includes the Standard Model box diagrams. In particular,

with this definition, the modification of the two CP asymmetries in (1.3) and (1.4) depends on a *single* new parameter, the phase θ_d :

$$a_{\psi K_S} = \sin(2\beta + 2\theta_d), \quad (1.11)$$

$$a_{\pi\pi} = \sin(2\alpha - 2\theta_d), \quad (1.12)$$

while the modification of the $B - \bar{B}$ mixing parameter x_d in (1.6) is given by the magnitude rescaling parameter, r_d :

$$x_d = C_t R_t^2 r_d^2. \quad (1.13)$$

Furthermore, since the determination of R_u from the semileptonic B decays is not affected by the new physics, and since the unitarity triangle remains valid, we have the following relations between the length of its sides and its angles:

$$R_u = \frac{\sin \beta}{\sin \alpha}, \quad (1.14)$$

$$R_t = \frac{\sin \gamma}{\sin \alpha}, \quad (1.15)$$

where

$$\gamma \equiv \arg \left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} \right]. \quad (1.16)$$

When α , β and γ are defined to lie in the $\{0, 2\pi\}$ range, they satisfy

$$\alpha + \beta + \gamma = \pi \text{ or } 5\pi. \quad (1.17)$$

The four measured quantities $a_{\psi K_S}$, $a_{\pi\pi}$, x_d and R_u can be used to achieve the following [7]:

- (i) Fully reconstruct the unitarity triangle and, in particular, find α , β and R_t ;
- (ii) Find the magnitude and phase of the new physics contribution to $B - \bar{B}$ mixing, namely determine r_d and θ_d .

It is straightforward to show that the above tasks are possible. Eqs. (1.11), (1.12) and (1.14) give three equations for three unknowns, α , β and θ_d . Once α and β are known, γ can be extracted from (1.17), R_t can then be deduced from (1.15), and finally r_d is found from (1.13).

In practice, however, it is quite likely that the combination of experimental and theoretical uncertainties (particularly in the x_d and R_u constraints) and discrete ambiguities will limit the usefulness of the above method rather significantly. In the next section we discuss the discrete ambiguities that arise in this calculation. We then describe how to determine the parameters, both in the $\rho - \eta$ plane (section 3), and in the $\sin 2\alpha - \sin 2\beta$ plane (section 4). We mention ways to resolve some of the ambiguities in the concluding section.

2. Discrete Ambiguities

A major obstacle in carrying out the above program will be the discrete ambiguities in determining γ . We now describe these ambiguities.

A physically meaningful range for an angle is 2π . We choose this range to be $\{0, 2\pi\}$. Measurement of any single asymmetry, $\sin 2\phi$, determines the corresponding angle only up to a fourfold ambiguity: ϕ , $\pi/2 - \phi$, $\pi + \phi$ and $3\pi/2 - \phi \pmod{2\pi}$. Specifically, let us denote by $\bar{\alpha}$ and $\bar{\beta}$ some solution of the equations

$$a_{\psi K_S} = \sin 2\bar{\beta}, \quad a_{\pi\pi} = \sin 2\bar{\alpha}. \quad (2.1)$$

Thus, measurements of the two asymmetries leads to a sixteenfold ambiguity in the values of the $\{\bar{\alpha}, \bar{\beta}\}$ pair. However, since $\bar{\alpha} = \alpha - \theta_d$ and $\bar{\beta} = \beta + \theta_d$, and unitarity is not violated, γ still satisfies the condition

$$\bar{\alpha} + \bar{\beta} + \gamma = \pi \pmod{2\pi}. \quad (2.2)$$

Then, the sixteen possibilities for γ are divided into two groups of eight that are related by the combined operation $\bar{\alpha} \rightarrow \bar{\alpha} + \pi$ and $\bar{\beta} \rightarrow \bar{\beta} + \pi$. This, in turn shifts the value of γ by 2π . However, since γ is only defined modulo 2π , the ambiguity in γ is reduced to eightfold. We emphasize that this reduction of the ambiguity depends only on the definition of γ . Defining

$$\phi_{\pm} = \bar{\alpha} \pm \bar{\beta}, \quad (2.3)$$

the eight possible solutions for γ are

$$\gamma = \pm\phi_+, \pi \pm \phi_+, \pi/2 \pm \phi_-, 3\pi/2 \pm \phi_- \pmod{2\pi}. \quad (2.4)$$

Note that the eight solutions come in pairs of $\pm\gamma$. This in turn implies that the ambiguity on R_t is only fourfold.

In any model where the three angles $\bar{\alpha}$, $\bar{\beta}$, and γ form a triangle, the ambiguity is further reduced [11]: the requirement that the angles are either all in the range $\{0, \pi\}$ or all in the range $\{\pi, 2\pi\}$ reduces the ambiguity in γ to fourfold. It is enough to know the signs of $a_{\psi K_S}$ and $a_{\pi\pi}$ to carry out this step. Finally, within the Standard Model, the bound $0 < \beta < \pi/4$ (obtained from the sign of ε_K and from $R_u < 1/\sqrt{2}$) reduces the ambiguity in γ to twofold.

When we allow for the possibility of new physics effects in the mixing, knowing the signs of $a_{\psi K_S}$ and $a_{\pi\pi}$ does not lead to further reduction in the ambiguity, which remains eightfold. The three angles $\bar{\alpha}$, $\bar{\beta}$ and γ are not angles that define a triangle and therefore further constraints cannot be imposed. It is possible, for example, that both γ and $\bar{\beta}$ lie in the range $\{\pi/2, \pi\}$. Further the sign of ε_K may not be related to the sign of η .

The following example will make the situation clear. Take

$$a_{\pi\pi} = 1/2, \quad a_{\psi K_S} = \sqrt{3}/2. \quad (2.5)$$

Then, we could have

$$\bar{\alpha} = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \quad \bar{\beta} = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}. \quad (2.6)$$

The eight solutions for γ are

$$\gamma = \frac{\pi}{4}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{7\pi}{4}. \quad (2.7)$$

If $\bar{\alpha}, \bar{\beta}, \gamma$ define a triangle, then only four solutions are allowed:

$$(\bar{\alpha}, \bar{\beta}, \gamma) = \left(\frac{\pi}{12}, \frac{\pi}{6}, \frac{3\pi}{4} \right), \left(\frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12} \right), \left(\frac{5\pi}{12}, \frac{\pi}{6}, \frac{5\pi}{12} \right), \left(\frac{5\pi}{12}, \frac{\pi}{3}, \frac{\pi}{4} \right). \quad (2.8)$$

Assuming $0 < \bar{\beta} < \pi/4$ as in the Standard Model leaves only the first two choices.

In various specific cases, the discrete ambiguity is smaller. If the two asymmetries are equal in magnitude, there is only a sixfold ambiguity:

$$\begin{aligned} a_{\pi\pi} = a_{\psi K_S} &\implies \gamma = \pm 2\bar{\beta}, \pi \pm 2\bar{\beta}, \pi/2, 3\pi/2 \pmod{2\pi}, \\ a_{\pi\pi} = -a_{\psi K_S} &\implies \gamma = 0, \pi, \pi/2 \pm 2\bar{\beta}, 3\pi/2 \pm 2\bar{\beta} \pmod{2\pi}. \end{aligned} \quad (2.9)$$

If one of the asymmetries is maximal, there is a fourfold ambiguity, e.g.

$$\begin{aligned} a_{\pi\pi} = +1 &\implies \gamma = \pm(\pi/4 + \bar{\beta}), \pm(3\pi/4 - \bar{\beta}) \pmod{2\pi}, \\ a_{\pi\pi} = -1 &\implies \gamma = \pm(\pi/4 - \bar{\beta}), \pm(3\pi/4 + \bar{\beta}) \pmod{2\pi}. \end{aligned} \quad (2.10)$$

If both asymmetries are maximal, the ambiguity is twofold. If the two asymmetries vanish, there is only a fourfold ambiguity:

$$a_{\pi\pi} = a_{\psi K_S} = 0 \implies \gamma = 0, \pi/2, \pi, 3\pi/2. \quad (2.11)$$

This is an interesting case, because it is predicted by models with approximate CP symmetry (e.g. in some supersymmetric models [12]). Only two of the solutions $(0, \pi)$ correspond to the CP symmetric case while in the other two $(\pi/2, 3\pi/2)$, the zero asymmetries are accidental.

So far we have ignored the penguin contamination in $a_{\pi\pi}$. The isospin analysis eliminates the penguin contamination only up to a four fold ambiguity [2]. Therefore, if the isospin analysis is needed, the ambiguities are increased.

In addition, for each value of γ there are two possibilities for θ_d related by $\theta_d \rightarrow \theta_d + \pi$. As long as the new physics is such that the $\Delta b = 2$ operator that contributes to $B - \bar{B}$ mixing can be separated into two $\Delta b = 1$ operators the $\theta_d \rightarrow \theta_d + \pi$ ambiguity is physical. Otherwise, it is not physical.

3. The $\rho - \eta$ Plane

The key point in the extraction of the CKM parameters is that the angle θ_d cancels in the following sum:

$$2(\alpha + \beta) = \arcsin(a_{\psi K_S}) + \arcsin(a_{\pi\pi}). \quad (3.1)$$

In other words, the angle γ can be determined (up to the discrete ambiguities discussed above). In the $\rho - \eta$ plane, a value for γ gives a ray from the origin, while a value for R_u gives a circle that is centered in the origin. The intersection point of the line and the circle gives (ρ, η) of the unitarity triangle and determines it completely.

A graphical way to carry out these calculations in the $\rho - \eta$ plane is the following (see Figure 1) [10]. One draws the four curves that correspond to eqs. (1.3), (1.4), (1.6) and (1.8) (even though only the latter is valid!). The next step is to draw the ray from the origin that passes through the intersection point of the β -ray and the α -circle: this is the *correct* γ -ray (see the dashed line in Figure 1). The intersection point of the γ -ray and the R_u -circle gives the *correct* vertex of the unitarity triangle, (ρ, η) , namely

$$\begin{aligned}\tan \beta &= \frac{\eta}{1 - \rho}, \\ R_u^2 &= \eta^2 + (1 - \rho)^2.\end{aligned}\tag{3.2}$$

The information about the new physics contribution to $B - \bar{B}$ mixing is found from the intersection point of the β -ray and the x_d -circle, (ρ', η') , namely

$$\begin{aligned}\theta_d &= \arctan \frac{\eta'}{1 - \rho'} - \arctan \frac{\eta}{1 - \rho}, \\ r_d^2 &= \frac{\eta'^2 + (1 - \rho')^2}{\eta^2 + (1 - \rho)^2}.\end{aligned}\tag{3.3}$$

4. The $\sin 2\alpha - \sin 2\beta$ Plane

A presentation of the various constraints in the $\sin 2\alpha - \sin 2\beta$ plane [7,13,14] is useful because the two angles are usually correlated [15]. The model independent analysis is demonstrated in Figure 2. The R_u constraint gives an eight-shaped curve on which the physical values have to lie. The various solutions for eq. (3.1) fall on two ellipses, the intersections of which with the R_u curve determine the allowed values of $\sin 2\alpha$ and $\sin 2\beta$. Note that these ellipses cross the eight-shaped curve in sixteen points but, as argued above, only eight of these points are true solutions. The inconsistent intersection points can be found by noting that the slopes of the ellipse at the consistent points should be $(\cos 2\alpha, -\cos 2\beta)$. The eight correct solutions are denoted by the filled circles in Figure 2.

In the above, we showed how to use measured values of the CP asymmetries $a_{\psi K_S}$ and $a_{\pi\pi}$ to find the allowed values for α and β . The presentation in the $\sin 2\alpha - \sin 2\beta$ plane is also useful for the opposite situation. Some models predict specific values for α and β . (Such predictions can arise naturally from horizontal symmetries.) On the other hand, the models often allow new contributions to $B - \bar{B}$ mixing of unknown magnitude and phase. In this case, the predicted value of $(\sin 2\alpha, \sin 2\beta)$ is just a point in the plane, and the ellipse (3.1) actually gives the allowed (and correlated) values of $(a_{\pi\pi}, a_{\psi K_S})$. Such an analysis was carried out in ref. [16].

More generally, even in models that make no specific predictions for CKM parameters, we usually have some constraints on the allowed range for α and β . For example, in this work we assume the validity of the limits on R_u from charmless semileptonic B decays which constrains the ratio $\sin \beta / \sin \alpha$ through (1.14). Note, however, that this constraint by itself cannot exclude any region in the $a_{\pi\pi} - a_{\psi K_S}$ plane. The reason is the following. For any value of R_u , neither α nor θ_d are constrained. (The angle β is constrained for any $R_u < 1$ and certainly by the present range, $0.27 < R_u < 0.45$.) Then any value of $a_{\psi K_S}$ can be accommodated by an appropriate choice of θ_d and any value of $a_{\pi\pi}$ can be fitted by further choosing an appropriate α . Obviously, to get predictions for the CP asymmetries beyond the Standard Model, one has to make some assumptions that go beyond our generic analysis.

For example, consider models where ε_K is dominated by the Standard Model box diagrams (while $B - \bar{B}$ mixing is not). Then, we know that $0 < \gamma < \pi$. This already excludes part of the allowed range. In particular, $(a_{\pi\pi}, a_{\psi K_S}) = (1, -1)$ or $(-1, 1)$ requires $\gamma = 0$ or π , and is therefore excluded in this class of models. More generally, in any class of models where $\sin^2 \gamma$ cannot assume any value between zero and one, some regions in the $a_{\pi\pi} - a_{\psi K_S}$ plane are excluded.

5. Final Comments

We argued that the most likely effect of new physics on CP asymmetries in neutral B decays into CP eigenstates will be a significant contribution to the mixing. This is because

we have concentrated on decays that are allowed at tree level in the Standard Model. Thus the new physics effects on the decay amplitudes and on CKM unitarity can be neglected in a large class of models¹. We explained that in this class of models, the unitarity triangle can be constructed model independently and the new physics contribution to the mixing can be disentangled from the Standard Model one.

However, the combination of hadronic uncertainties and discrete ambiguities puts serious obstacles in carrying out this calculation. In particular, there is an eightfold ambiguity in the construction of the triangle. In order to get useful results, it will be necessary to reduce this ambiguity.

One way to eliminate some of the allowed solutions can be provided by a rough knowledge of $\cos(2\alpha - 2\theta_d)$, $\cos(2\beta + 2\theta_d)$ or $\cos 2\gamma$ [17]. For example, $\cos(2\alpha - 2\theta_d)$ can be determined from the CP asymmetry in $B \rightarrow \rho\pi$ [18] and $\cos 2\gamma$ from $B \rightarrow DK$ [19]. While a precise measurement of either of these is not expected in the first stages of a B factory, a knowledge of the sign of the cosine is already useful for our purposes: knowing either of $\text{sign}[\cos 2(\alpha - \theta_d)]$, $\text{sign}[\cos 2(\beta + \theta_d)]$ or $\text{sign}[\cos 2\gamma]$ reduces the ambiguity in γ to fourfold. Knowing two of them reduces it to twofold. (Knowing the three of them, however, cannot be combined to completely eliminate the ambiguity.)

The ambiguity associated with the isospin analysis can be removed by measuring the time dependent CP asymmetry in $B \rightarrow \pi^0\pi^0$ [2]. Another way is by studying $B \rightarrow \rho\pi$ [18,17]. Here, due to interference between several amplitudes, isospin relations can be used to determine $\sin 2\alpha$ without penguin contamination, and without any discrete ambiguity.

A different approach is to make further assumptions about the new physics that is responsible for the effects discussed above. For example, there are many models where processes involving third generation quarks, such as $B-\bar{B}$ mixing, are significantly modified by the new physics, but processes with only light quarks, such as $K \rightarrow \pi\nu\bar{\nu}$, are not. Then measurements of $K^+ \rightarrow \pi^+\nu\bar{\nu}$ and $K_L \rightarrow \pi^0\nu\bar{\nu}$ will provide the true values of R_t or $|\eta|$, respectively. The unitarity triangle can be determined from these up to a fourfold ambiguity. The additional input of R_u reduces this to a twofold ambiguity. The

¹ The new physics effects may significantly alter the patterns of CP asymmetries in decays that are dominated by penguins in the Standard Model [6].

determination of γ by the methods described above will provide a *test* of this class of models. It will not resolve the twofold ambiguity.

In some models [20] there is a significant contribution to both B_d and B_s mixing but the ratio between the two obeys the Standard Model relation,

$$\frac{\Delta m_{B_d}}{\Delta m_{B_s}} = F_{SU(3)} \sin^2 \theta_C R_t^2, \quad (5.1)$$

where $F_{SU(3)}$ is an $SU(3)$ -isospin breaking parameter. Then, a measurement of Δm_{B_s} will provide the correct R_t and, again, the unitarity triangle can be determined, up to a twofold discrete ambiguity, from R_u and R_t . The determination of γ by our analysis is in this case, again, a test and will not resolve the twofold ambiguity. Note, however, that in most models where the *ratio* between B_d and B_s mixing obeys (5.1), the *phases* in the B_s, B_d mixing amplitudes are the same as in the Standard Model, namely $\theta_d = 0$. Then r_d is the only new parameter, and the whole analysis becomes trivial.

In a large class of models, ε_K has only small contributions from new physics. If dominated by the Standard Model, ε_K implies that all angles of the unitarity triangle are in the range $\{0, \pi\}$, and the ambiguity is reduced to fourfold.

Of course, one can combine several of these measurements and assumptions to get a better handle on the true form of the unitarity triangle. It is obvious however that the model independent construction of the triangle, while possible in principle, will pose a serious theoretical and experimental challenge.

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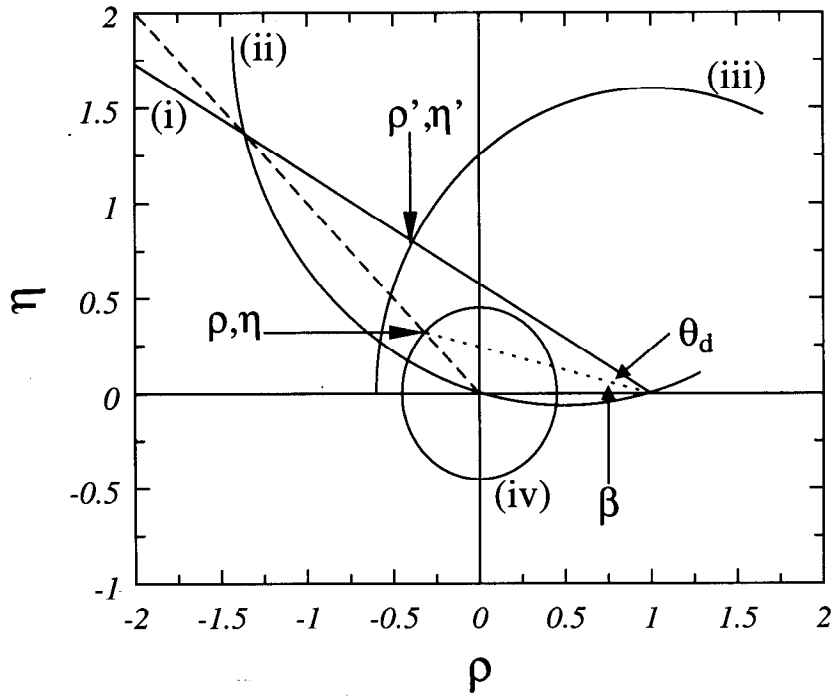


Figure 1. The model independent analysis in the $\rho - \eta$ plane: (i) The $a_{\psi K_S}$ ray; (ii) The $a_{\pi\pi}$ circle; (iii) The x_d circle; (iv) The R_u circle. The γ ray is given by the dashed line. The true β ray is given by the dotted line. Also shown are the true vertex of the unitarity triangle (ρ, η) and the (ρ', η') point that serves to find θ_d and r_d .

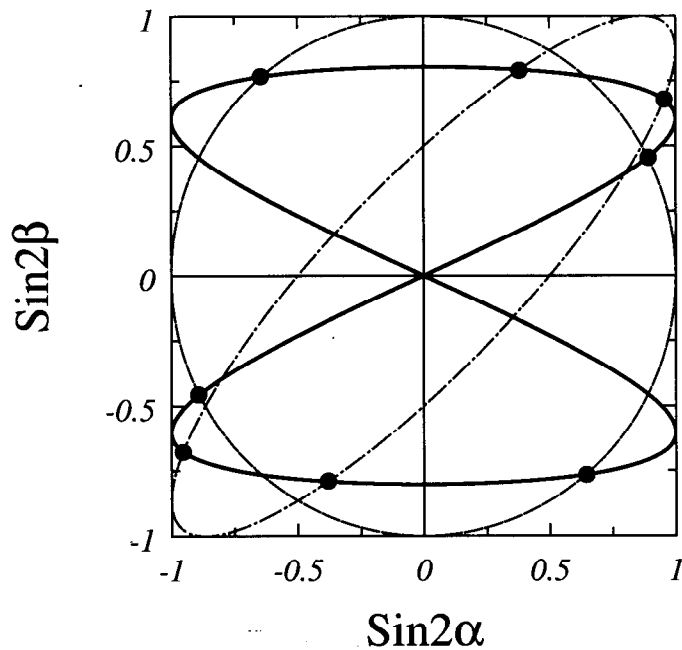


Figure 2. The $\alpha + \beta$ constraint (3.1) and the R_u constraint (1.14) in the $\sin 2\alpha - \sin 2\beta$ plane. The eight possible solutions for the unitarity triangle are given by the filled circles.