M(atrix) Theory on an Orbifold and Twisted Membrane

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Abstract

M(atrix) theory on an orbifold and classical two-branes therein are studied with particular emphasis to heterotic M(atrix) theory on $\mathbf{S}_1/\mathbf{Z}_2$ relevant to strongly coupled heterotic and dual Type IA string theories. By analyzing orbifold condition on Chan-Paton factors, we show that three choice of gauge group are possible for heterotic M(atrix) theory: $SO(2N)$, $SO(2N + 1)$ or $USp(2N)$. By examining area-preserving diffeomorphism that underlies the M(atrix) theory, we find that each choices of gauge group restricts possible topologies of two-branes. The result suggests that only the choice of $SO(2N)$ or $SO(2N + 1)$ groups allows open two-branes, hence, relevant to heterotic M(atrix) theory. We show that requirement of both *local* vacuum energy cancellation and of worldsheet anomaly cancellation of resulting heterotic string identifies supersymmetric twisted sector spectra with sixteen fundamental representation spinors from each of the two fixed points. Twisted open and closed two-brane configurations are obtained in the large N limit.

1 Introduction

Witten [1] has made an important observation that strong coupling limit of perturbative string theories are unified into M-theory with an eleven-dimensional Lorentz invariance. The massless excitations are described by eleven-dimensional supergravity supplemented by additional interactions necessary for anomaly cancellation. BPS spectra and duality symmetries of various string theories have been identified by studying macroscopic BPS two-branes and five-branes wrapped on compactified space [2]. Despite such progress, a complete understanding of the M-theory has remained elusive.

Recently, an interesting proposal [3] have been put forward for a non-perturbative definition of the M-theory. This, so-called M(atrix) theory provides a partonic description of M-theory, in which the partons are type IIA 0-branes and open strings connecting them. In the infinite momentum limit, the total energy of 0-brane partons grow arbitrarily large, hence, its number and transverse density increases indefinitely. Correspondingly, the open strings connecting them become arbitrarily short and leaves only massless gauge fields. Thus, M(atrix) theory is a gauge theory of area-preserving diffeomorphism among the 0-brane partons. As such, two-branes arise quite naturally as collective excitations of 0-branes in a form of an incompressible fluid.

Horava and Witten[4] have shown that M-theory compactified on orbifold $\mathbf{S}_1/\mathbf{Z}_2$ along 11-th direction is a strongly coupled $E_8 \times E_8$ heterotic string theory. At the two fixed points are located the nine-branes or "ends of the world". Twisted sector of the M-theory consists of twisted two-branes whose open ends are attached to the nine-branes. By compactifying further on S_1 and interchanging the two compact directions, the heterotic string theory turns into S-dual, Type I string theory. Equally interesting situation is when M-theory is compactified on orbifold $(\mathbf{S}_1)^5/\mathbf{Z}_2$ with 16 five-branes at fixed points [5]. Townsend [6] and Strominger [7] have shown that in M-theory two-branes can also end on five-branes. In this case, the boundary of open two-brane is a self-dual string propagating on the five-brane world-volume. If the separation between the two ends shrinks, the tension of self-dual string decreases and a non-critical tensionless string emerges.

If the M(atrix) theory provides a non-perturbative definition of M-theory, the situations alluded above point to a problem of formulating M(atrix) theory on an orbifold and in the presence of five- and/or nine-branes. In particular, M(atrix) theory should support twisted two-branes with open ends attached to these branes. An interesting and important question is M(atrix) theory explanation to the origin and the enhancement of gauge symmetries from the twisted sector. In this regard, M(atrix) theory defined on an orbifold may shed new non-perturbative understanding to the dynamics of $D = 10$ strongly coupled heterotic string and of $D = 6$ non-critical self-dual string.

In this paper, with these motivations, we investigage M(atrix) theory defined on an orbifold $\mathbf{S}_1/\mathbf{Z}_2$ with particular emphasis on the strongly coupled heterotic string theory. As such, we will call it as heterotic M(atrix) theory. In section 2, we analyze Chan-Paton factors of open strings connecting 0-brane partons. We show that the orbifold conditions that reverse both the spacetime parity and the string orientation allow $SO(2N), SO(2N + 1)$ and $USp(2N)$ as possible gauge groups of the M(atrix) theory. It is also necessary, however, to check if appropriate two-branes in the twisted sector can arise from a chosen gauge group. In section 3, we find that possible topologies of two-brane are intimately related to the choice of M(atrix)

theory gauge group. Analysis of area-preserving diffeomorphism gauge symmetry indicates that only $SO(2N)$ and $SO(2N+1)$ groups are capable of describing both open and closed two-branes, while $USp(2N)$ group describes only closed two-branes. In section 4, we construct heterotic M(atrix) theory. We identify the twisted sector spectra that cancels *locally* both one-loop vacuum energy and worldsheet anomaly of resulting heterotic string. We also find macroscopic M(atrix) open and closed two-branes and derive correspondence rules in the large N limit.

Our notation of M(atrix) theory is as follows. Regularizing zero-momentum limit [2] by compactifying the longitudinal direction on a circle of radius R, the M(atrix) theory action is given by a matrix quantum mechanics of $SU(N)$ gauge group [3] :

$$S_M = \text{Tr}_N \int d\tau \left(\frac{1}{2R}(D_\tau \mathbf{X}^I)^2 + \frac{R}{4}[\mathbf{X}^I, \mathbf{X}^J]^2 + \mathbf{\Theta}^T D_\tau \mathbf{\Theta} + iR\mathbf{\Theta}^T \Gamma_I [\mathbf{X}^I, \mathbf{\Theta}] \right). \tag{1}$$

Here, $\mathbf{X}^I$ and $\mathbf{\Theta}^\alpha$ denote 9 bosonic and 16 spinor coordinates of 0-brane partons ($I = 1, \cdots, 9$ and $\alpha = 1, \cdots, 16$). The Majorana spinor conventions are such that Γ_I's are real and symmetric and $i\overline{\mathbf{\Theta}}\Gamma_- \equiv \mathbf{\Theta}^T$:

$$\Gamma_i = \begin{pmatrix} 0 & \sigma^i_{a\dot{a}} \\ \sigma^i_{\dot{a}a} & 0 \end{pmatrix} \quad i = 1, \cdots, 8; \qquad \Gamma_9 = \begin{pmatrix} -\delta_{ab} & 0 \\ 0 & +\delta_{\dot{a}\dot{b}} \end{pmatrix}. \tag{2}$$

The non-dynamical gauge field A_τ that enters through covariant derivatives $D_\tau \mathbf{X}^I \equiv \partial_\tau \mathbf{X}^I - i[A_\tau, \mathbf{X}^I]$ and $D_\tau \mathbf{\Theta}^\alpha \equiv \partial_\tau \mathbf{\Theta}^\alpha - i[A_\tau, \mathbf{\Theta}^\alpha]$ projects the physical Hilbert space to a gauge singlet sector and ensures invariance under area-preserving diffeomorphism transformation. Thus, M(atrix) theory consists of $9(N^2-1) - (N^2-1)$ bosonic degrees of freedom and $16(N^2-1)/2 = 8(N^2-1)$ spinor degrees of freedom. Hamiltonian in the infinite momentum limit is given by

$$H_M = R\,\text{Tr}_N \left(\frac{1}{2}\mathbf{\Pi}_I^2 - \frac{1}{4}[\mathbf{X}^I, \mathbf{X}^J]^2 + i\mathbf{\Theta}^T \Gamma_I [\mathbf{X}^I, \mathbf{\Theta}] \right). \tag{3}$$

The M(atrix) theory is invariant under the following supersymmetry transformations

$$\begin{aligned} \delta \mathbf{X}^I &= -2\epsilon^T \Gamma^I \mathbf{\Theta} \\ \delta \mathbf{\Theta} &= \frac{i}{2}\left(\Gamma_I D_\tau \mathbf{X}^I + \frac{1}{2}\Gamma_{IJ}[\mathbf{X}^I, \mathbf{X}^J] \right)\epsilon + \xi \\ \delta A_\tau &= -2\epsilon^T \mathbf{\Theta}. \end{aligned} \tag{4}$$

Here, $i\epsilon$ is a 16-component spinor generator of local supersymmetry, while ξ is a 16-component spinor generator of rigid translation[3]. As such, the sixteen dynamical and sixteen kinematical supersymmetry charges are given by:

$$\begin{aligned} \mathbf{Q}_\alpha &= \sqrt{R}\text{Tr}\left(\Gamma^I \mathbf{\Pi}_I + \frac{i}{2}\Gamma_{IJ}[\mathbf{X}^I, \mathbf{X}^J] \right)_{\alpha\beta} \mathbf{\Theta}_\beta, \\ \mathbf{S}_\alpha &= \frac{2}{\sqrt{R}}\text{Tr}\mathbf{\Theta}_\alpha \end{aligned} \tag{5}$$

respectively.

---

[2] It should be noted that non-trivial vacuum phenomena such as axial anomaly requires careful treatment of zero-momentum components[8] in the infinite momentum frame.

[3] Invariance under the kinematical supersymmetry can be checked easily using cyclic property of the trace operation.

2 Chan-Paton Factor Analysis

We first determine what gauge groups are capable of describing 0-brane parton interactions on an orbifold and in the presence of spatial boundaries. In the next section we will examine compatibility with the area-preserving diffeomorphism invariance and topology of classical two-branes that arise as a collective excitation in the respective M(atrix) theory. We first try to describe the group theory aspects which arise when we consider orbifolds in M(atrix) theory. As we will see, the allowed M(atrix) theory gauge groups are $SO(2N), SO(2N + 1)$ or $USp(2N)$.

A convenient way of describing an orbifold is in terms of covering space. The effect of orbifold is then to put both original and 'image' 0-brane partons together in the covering space and impose appropriate projection condition. This projection is a proper sub-group of the original M(atrix) theory. Therefore, in this description, coordinate and spinor matrices are $2N \times 2N$ in size. Denoting coordinate matrices parallel and perpendicular to a boundary by $\mathbf{X}_\parallel$ and $\mathbf{X}_\perp$ respectively, the Chan-Paton conditions to coordinate and spinor matrices are given by:

$$
\begin{aligned}
\mathbf{X}_\perp &= -M \cdot \mathbf{X}_\perp^{\mathrm{T}} \cdot M^{-1}, \\
\mathbf{X}_\parallel &= +M \cdot \mathbf{X}_\parallel^{\mathrm{T}} \cdot M^{-1}, \\
\mathbf{\Theta} &= \Gamma_\perp M \cdot \mathbf{\Theta}^{\mathrm{T}} \cdot M^{-1}, \qquad \Gamma_\perp \equiv \prod_{I \in \perp} \Gamma_I.
\end{aligned}
\tag{1}
$$

(The same boundary conditions have been observed previously [9, 10] but with a different interpreation or in a limited generality.) Here, $\Gamma_\perp$ denotes an ordered product of perpendicular direction gamma matrices. Note that $\{\Gamma_\perp, \Gamma_\parallel\} = 0$ for any parallel direction gamma matrices.

One can easily check that the orbifold mapping Eq.(1) is a symmetry of the M(atrix) theory Hamiltonian Eq.(3), hence, allows an orbifold projection. The symmetry is related to the type IIA symmetry that reverses the spacetime parity and the string orientation simultaneously. In M-theory, the corresponding symmetry is a simultaneous spatial reflection and reversal of antisymmetric 3-form tensor potential $A_{MNP} \to -A_{MNP}$.

The matrix M relates the original and the 'image' 0-brane partons. Hermiticity of the coordinate and spinor matrices requires that $M^{\mathrm{T}} \cdot M^{-1} = \pm \mathbf{I}_{2N \times 2N}$, viz. M is either a symmetric or an anti-symmetric matrix.

It is straightforward to check that the $\pm$ signs in the parallel and the perpendicular coordinates manifest orbifold projections: orignal and 'image' 0-brane should have opposite perpendicular coordinates but the same parallel coordinates. To see this, consider a characteristic polynomial $P(x)$ of eigenvalues of $\mathbf{X}$:

$$
\begin{aligned}
P(x) \equiv \det(x\mathbf{I} - \mathbf{X}) &= \det(x\mathbf{I} \pm M \cdot \mathbf{X}^{\mathrm{T}} \cdot M^{-1}) \\
&= \det(\mp x\mathbf{I} - \mathbf{X}) = P(\mp x)
\end{aligned}
\tag{2}
$$

for perpendicular and transverse coordinates respectively. Note that the result depends only on the overall sign and independent of the matrix M.

The matrix M imposes one-to-one relation between an original parton and its 'image' parton. We have shown that M is either symmetric or anti-symmetric matrix. To analyze the resulting gauge group of M(atrix) theory for each possible choices of M, it is convenient to consider $\mathbf{X}$ as an assemble of 4 blocks $N \times N$ matrices. The possible choices of M can be written as:

$$
(M_\mu)_{2N \times 2N} = \mathbf{I}_{N \times N} \otimes \sigma_\mu
\tag{3}
$$

where $\sigma_\mu \equiv (1, \sigma_1, \sigma_2, \sigma_3)$ is a complete set of basis of 2×2 matrices. Symmetric and anti-symmetric choices of M correspond to M_0, M_1, M_3 and M_2 respectively. For $M = M_1$ choice, one finds that

$$\mathbf{X}_\perp = \begin{pmatrix} Y & A \\ -A^* & -Y^T \end{pmatrix} \tag{4}$$

where Y is a Hermitian matrix and A is an anti-symmetric, complex matrix. The total number of independent elements is $N^2 + 2 \cdot N(N-1)/2 = 2N(2N-1)/2$. On the other hand, the parallel coordinate matrices take a form of

$$\mathbf{X}_\parallel = \begin{pmatrix} Z & S \\ S^* & Z^T \end{pmatrix} \tag{5}$$

where Z is an unconstrained Hermitian matrix and S is a symmetric, complex matrix. The total number of independent elements is $N^2 + 2 \cdot N(N+1)/2 = 2N(2N+1)/2$.

By unitary transformation, the above matrices can be brought into anti-symmetric and symmetric, Hermitian matrices. Consider $\mathbf{X} \to U \cdot \mathbf{X} \cdot U^\dagger$ where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{6}$$

Then,

$$
\begin{aligned}
\mathbf{X}_\perp &\to \frac{1}{2} \begin{pmatrix} (Y - Y^{\mathrm{T}}) - i(A + A^*) & -i(Y + Y^{\mathrm{T}}) + (A - A^*) \\ i(Y + Y^{\mathrm{T}}) + (A - A^*) & (Y - Y^{\mathrm{T}}) + i(A + A^*) \end{pmatrix} \\
\mathbf{X}_\parallel &\to \frac{1}{2} \begin{pmatrix} (Z + Z^{\mathrm{T}}) - i(S - S^*) & -i(Z - Z^{\mathrm{T}}) + (S + S^*) \\ +i(Z - Z^{\mathrm{T}}) + (S + S^*) & (Z + Z^{\mathrm{T}}) + i(S - S^*) \end{pmatrix}.
\end{aligned}
\tag{7}
$$

It clearly displays that $\mathbf{X}_\perp$ and $\mathbf{X}_\parallel$ are Hermitian, anti-symmetric and symmetric matrices. As such, they form adjoint and symmetric representations of $SO(2N)$ respectively.

Other symmetric choices $M = M_0, M_3$ give similar results: while block structure depends on a specific choice of M, the perpendicular and the parallel coordinate matrices are matrices having $2N(2N-1)/2$ and $2N(2N+1)/2$ independent elements respectively. They are precisely adjoint and symmetric representations of $SO(2N)$. We conclude that the M(atrix) theory on an orbifold with symmetric choice of M is described by a matrix quantum mechanics with gauge group $SO(2N)$.

Next, consider the choice of anti-symmetric matrix $M = M_2$. In this case, one finds that the perpendicular and the parallel coordinates take block matrix structure of:

$$
\begin{aligned}
\mathbf{X}_\perp &= \begin{pmatrix} Y & S \\ S^* & -Y^{\mathrm{T}} \end{pmatrix}, \\
\mathbf{X}_\parallel &= \begin{pmatrix} Z & A \\ -A^* & Z^{\mathrm{T}} \end{pmatrix}.
\end{aligned}
\tag{8}
$$

In this case, the perpendicular coordinate has $N^2 + 2 \cdot N(N+1)/2 = 2N(2N+1)/2$ independent elements, while the parallel coordinates have $N^2 + 2 \cdot N(N-1)/2 = 2N(2N-1)/2$ independent elements. As such, they may be identified with adjoint and antisymmetric representations of

4

$USp(2N)$ group. In fact, with $M_2 = \mathbf{I}_{N \times N} \otimes \sigma_2$, the orbifold condition for the transverse coordinate

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{pmatrix} = 0 \tag{9}$$

is nothing but the condition of $Sp(2N, \mathbf{C})$ for $SU(2N)$ Hermitian matrices. We conclude that M(atrix) theory on a space with boundaries with an anti-symmetric choice of M is described a by matrix quantum mechanics with gauge group $USp(2N)$.

Finally, we analyze the fermion coordinate matrices. Due to the light-cone projection $\Gamma_+ \boldsymbol{\Theta} = 0$, there are only 16-components. They form a spinor representation of $SO(9)$ of the transverse space. Consider a pair of longitudinal nine-branes ("end of the world") located perpendicular to the 9-th direction, as is relevant for $\mathbf{S}_1/\mathbf{Z}_2$. The residual transverse space manifests $SO(8)$ rotational symmetry. The 16-component spinor can be further decomposed into two inequivalent chiral spinors of $SO(8)$. Thus,

$$\boldsymbol{\Theta} = \mathbf{8}_s \oplus \mathbf{8}_c \equiv \mathbf{S}_a \oplus \mathbf{S}_{\dot{a}} \tag{10}$$

where

$$\Gamma_9 \mathbf{S}_a = -\mathbf{S}_a; \quad \Gamma_9 \mathbf{S}_{\dot{a}} = +\mathbf{S}_{\dot{a}}. \tag{11}$$

Under the orbifold condition, for gauge group $SO(2N)$, $\mathbf{S}_a, \mathbf{S}_{\dot{a}}$ are adjoint representation and symmetric representations respectively. For the gauge group $USp(2N)$, $\mathbf{S}_a$ is an adjoint representation while $\mathbf{S}_{\dot{a}}$ is an anti-symmetric representation. Altogether, the 16-components of the spinors $\mathbf{8}_s \oplus \mathbf{8}_c$ have total $16N^2$ on-shell degrees of freedom for both $SO(2N)$ and $USp(2N)$ gauge groups. Compared to the total bosonic on-shell degrees of freedom $2N(2N \mp 1)/2 + 8 \cdot 2N(2N \pm 1)/2 - 2N(2N \mp 1)/2$, there are $16N$ fermionic or $8N$ bosonic degrees of freedom deficit for $SO(2N)$ and $USp(2N)$ gauge groups respectively. It should be noted that center of mass degrees of freedom consist of 8 bosonic and 8 fermionic modes and the deficit degrees of freedom originate from the internal relative motion among the 0-brane partons.

The mismatch of Bose-Fermi degrees of freedom is consistent with the surviving supersymmetries of the orbifold projection and $SO(2N)$ or $USp(2N)$ gauge symmetries. As will be discussed in section 4, however, mismatch of the degrees of freedom leads to non-trivial one-loop renormalization of the moduli space metric and linear potential for the 0-brane. The radiatively induced potential amounts to dilaton tadpole amplitude, hence, local cosmological constant. Such a potential can be cancelled *locally* by introducing a twisted sector of the orbifold. From the above counting, we see that the twisted sector should consist of 8 fermions (4 bosons) in the fundamental representation of $SO(2N)$ $(USp(2N))$ gauge group. For now, it is sufficient to notice that the deficit degrees of freedom originates from the orbifold conditions that eliminates $8N$ fermionic or bosonic 0-brane partons excessively along the diagonal entries.

As a final remark, it is also possible to have 0-brane partons attached to the orbifolds. They are interpreted as a mirror pair of 'half' 0-brane partons moved to the orbifold fixed point. Clearly such a situation is described by $SO(2N + 1)$ M(atrix) theory uniquely.

3   Area-Preserving Diffeomorphism Analysis

It has been shown that the M(atrix) theory contains two-brane as a collective excitation of Landau-level orbiting 0-brane partons forming an incompressible two-dimensional fluid. The

existence of two-brane may be traced down to the area-preserving diffeomorphism invariance of the M(atrix) theory in the $N \to \infty$ limit. In other words, the M(atrix) theory is a gauge theory of area-preserving diffeomorphism group [11].

In the previous section, we have identified $SO(2N)$, $SO(2N + 1)$ and $USp(2N)$ as possible gauge groups of the M(atrix) theory on an orbifold. Several possible choices of M(atrix) theory gauge groups stems from the fact that the Chan-Paton factor analysis was purely kinematical. If M(atrix) theory based on these gauge groups were to describe open two-branes attached at the orbifold fixed points, then the M(atrix) theory should possess area-preserving diffeomorphism invariance relevant to their topologies. It is clear that this requirement is more stringent than the Chan-Paton factor analysis, hence, renders more refined information for the choice of M(atrix) theory gauge group.

In this section, with the above motivation, we analyze the area-preserving diffeomorphism invariance relevant to two-branes of lower genus topologies. It is well-known that M(atrix) theory with $SU(N)$ group is a gauge theory of area-preserving diffeomorphism for both spherical and toroidal two-branes, $\mathrm{Diff}_0(S_2)$ and $\mathrm{Diff}_0(T_2)$. The area-preserving diffeomorphism of open two-branes of disk D_2 and cylinder C_2 topologies are then obtained by truncating generators of $\mathrm{Diff}_0(S_2)$ or $\mathrm{Diff}_0(T_2)$ appropriately. In fact, all other topologies related to S_2 and T_2 by involution, Möbius strip M_2, Klein bottle K_2 and real projective space RP_2 are also obtained in this manner. Some of these unoriented and closed topologies have been studied previously by Pope and Romans[13] and by Fairlie, Fletcher and Zachos[14]. For completeness, we include their results briefly in the foregoing list.

As will be shown below, $SO(2N)$ M(atrix) theory describes disk D_2, cylinder C_2, Klein bottle K_2, and Möbius strip M_2. $SO(2N + 1)$ M(atrix) theory describes disk D_2, cylinder C_2 and Möbius strip M_2, hence, open branes only. On the other hand, $USp(2N)$ M(atrix) theory yields real-projective space RP_2, and Klein bottle K_2, hence, unoriented closed two-branes only (The result of our analysis is summarized in the table at the end of this section). We thus conclude that only $SO(2N)$ and $SO(2N + 1)$ M(atrix) theories are capable of having open two-branes.

3.1   Genus 0 and 1/2 Two-Branes

By involution of spherical two-brane with $\mathrm{Diff}_0(S_2)$, it is possible to construct disk D_2 and real-projective plane RP_2 two-branes. We now show that the M(atrix) theory describing these two-branes, hence, preserving $\mathrm{Diff}_0(D_2)$ and $\mathrm{Diff}_0(RP_2)$ area-preserving diffeomorphism invariance is given by $SO(2N)$, $SO(2N + 1)$ or $USp(2N)$ gauge groups respectively.

3.1.1   Spherical Two-Brane and $\mathrm{Diff}_0(S_2)$

To describe area-preserving diffeomorphism of sphere S_2, following Ref.[13], let us introduce a complete set of scalar spherical harmonics

$$Y_{\mathbf{lm}}(\mathbf{x}) \equiv C_{i_1,\cdots,i_l} x^{i_1} \cdots x^{i_l} \tag{1}$$

in terms of embedding space coordinates $\mathbf{x} \equiv (x_1, x_2, x_3)$ satisfying

$$\mathbf{x} \cdot \mathbf{x} = 1. \tag{2}$$

In Eq.(1), $C_{i_1,\cdots,i_n}$ are symmetric, traceless, tensor coefficients. The 'magnetic' quantum number ranges over $\mathbf{m}$: $-\mathbf{l} \leq \mathbf{m} \leq \mathbf{l}$. The S_2 area-preserving diffeomorphism algebra is encoded into the Poisson bracket algebra among the spherical harmonics

$$
\begin{aligned}
\{Y_{\mathbf{lm}},, Y_{\mathbf{l'm'}}\} &\equiv \epsilon_{ijk} x^i (\partial_j Y_{\mathbf{lm}})(\partial_k Y_{\mathbf{l'm'}}) \\
&= \bigoplus_{\mathbf{j}=|\mathbf{l}-\mathbf{l'}|}^{\mathbf{l}+\mathbf{l'}-1} [\, Y_{\mathbf{j(m+m')}} \,],
\end{aligned} \tag{3}
$$

viz. a sum of irreducible polynomials of scalar harmonics in the range $|\mathbf{l}-\mathbf{l'}| \leq \mathbf{j} \leq (\mathbf{l}+\mathbf{l'}-1)$.

The above construction of area-preserving diffeomorphism algebra is in one-to-one correspondence with $SU(N)$ Lie algebra expressed in terms of maximal embedding of $SU(2)$. Under maximal embedding, the generators of $SU(N)$ can be expressed as products of $SU(2)$ generators Σ_i in the N-dimensional representation:

$$
\begin{aligned}
T^{(1)} &= C_i \Sigma_i \\
T^{(2)} &= C_{ij} \Sigma_i \Sigma_j \\
&\cdots \\
T^{(N-1)} &= C_{i_1 i_2 \cdots i_{N-1}} \Sigma_{i_1} \cdots \Sigma_{i_{N-1}}.
\end{aligned} \tag{4}
$$

Here, the coefficients $C_{ijk\ldots}$ are the same symmetric, traceless, tensor coefficients as in Eq.(1). The above form of $SU(N)$ generators simply express that, using the fact that the fundamental N-dimensional representation of $SU(N)$ remains irreducible in $SU(2)$, the adjoint representation of $SU(N)$ is decomposed into $N^2 - 1 = 3 + 5 + \cdots + (2N - 1)$ representations of $SU(2)$. The $T^{(i)}$ matrices form a complete set of traceless, Hermitian $N \times N$ matrices, hence, provide a basis for $SU(N)$.

The Σ_i generators in the N-dimensional representation of $SU(2)$ can be represented by a totally symmetrized $2^{(N-1)} \times 2^{(N-1)}$ matrices:

$$
\Sigma_i = \mathrm{Sym}[\sigma_i \otimes \mathbf{I} \otimes \cdots \otimes \mathbf{I} + \mathbf{I} \otimes \sigma_i \otimes \mathbf{I} \otimes \cdots \otimes \mathbf{I} + \cdots + \mathbf{I} \otimes \mathbf{I} \otimes \cdots \otimes \sigma_i], \quad i = 1, 2, 3 \tag{5}
$$

in which $\sigma_i, i = 1, 2, 3$ are the Pauli matrices.

Comparison of Eq.(1) and Eq.(4) shows that the commutation relation among $SU(N)$ generators $T_{(i)}$ is isomorphic to the Poisson bracket relation Eq.(3) among the spherical harmonics $Y_{\mathbf{l}}(\mathbf{x})$. In the large N limit, the non-commutativity among $T^{(i)}$'s becomes irrelevant. Thus, the area-preserving diffeomorphism of S_2, $\mathrm{Diff}_0(S_2)$ is realized by $SU(N)$ algebra, reproducing well-known result[12].

3.1.2 $R/\mathbf{Z}_2$ Two-Brane and $\mathrm{Diff}_0(S_2)$

Two-brane attached on a boundary at fixed point of $R/\mathbf{Z}_2$ is topologically equivalent to disk D_2 two-brane. The latter is obtained by an involution

$$
x^3 \to -x^3 \tag{6}
$$

from the S_2 two-brane constructed above. In terms of polar angle (θ, ϕ), the involution Eq.(6) is equivalent to simultaneous parity and $\phi \to \phi + \pi$ rotation, under which

$$
Y_{\mathbf{lm}}(\mathbf{x}) \to (-)^{\mathbf{l}+\mathbf{m}} Y_{\mathbf{lm}}(\mathbf{x}). \tag{7}
$$

The vector harmonics that form a basis of generators of area-preserving diffeomorphism transformation are obtained as parity-odd combinations:

$$L_{\mathbf{lm}} \equiv \{Y_{\mathbf{lm}} : Y_{\mathbf{lm}} - (-)^{\mathbf{l+m}} Y_{\mathbf{lm}}\}. \tag{8}$$

Since only odd values of $(\mathbf{l+m})$ are selected as the basis, the total number of vector harmonics generators are given by

$$L_{\mathbf{lm}} = \{Y_{1,0}, \ Y_{2,+1}, \ Y_{2,-1}, \ Y_{3,+2}, \ Y_{3,0}, \ Y_{3,-2}, \cdots\}, \tag{9}$$

hence, yield $1 + 2 + 3 + \cdots + (2N - 1) = 2N(2N - 1)/2$ generators. This equals precisely to the number of generators of $SO(2N)$ group, and suggests that area-preserving diffeomorphism on D_2 is described by $N \to \infty$ limit of $SO(2N)$ Lie algebra [4].

We now show that the above construction of D_2 area-preserving diffeomorphism is in one-to-one correspondence with the Lie algebra of $SO(N)$. Again, using the maximal embedding of $SU(2)$ in $SU(N)$ and corresponding representation of the basis Eq.(4), it remains to show that the generators are Hermitian and antisymmetric. The symmetric tensor $C_{i_1 \cdots i_n}$ relevant for vector harmonics satisfying the involution Eq.(6) allows only odd numbers of Σ_3. It is now convenient to make $(\Sigma_1, \Sigma_2, \Sigma_3)_{D_2} = (\Sigma_3, \Sigma_1, \Sigma_2)_{S_2}$. This cyclic permutation of N-dimensional $SU(2)$ generators redefines Σ_3 naturally into an anti-symmetric matrix :

$$\Sigma_3 = \text{Sym}[\sigma_2 \otimes \mathbf{I} \otimes \cdots \otimes \mathbf{I} + \mathbf{I} \otimes \sigma_2 \otimes \mathbf{I} \otimes \cdots \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{I} \otimes \cdots \otimes \sigma_2]. \tag{10}$$

Noting that the constant tensor $C_{ijk\ldots}$'s are completely symmetric, we find that only a set of generators left over are $2N(2N - 1)/2$ independent, $2N \times 2N$ Hermitian, anti-symmetric matrices. They are the generators of $SO(2N)$.

Again, in the large N limit, the non-commutativity among the surviving $T_{(i)}$'s die off sufficiently fast that the resulting $SO(N)$ Lie algebra is exactly the same as the area-preserving diffeomorphism algebra $\text{Diff}_0(D_2)$.

3.1.3 Real-Projective Two-Brane and $\text{Diff}_0(RP_2)$

The real-projective plane is described by an involution $\mathbf{x} \to -\mathbf{x}$ of the sphere S_2. Under the involution the spherical harmonics maps as $Y_{\mathbf{lm}} \to (-)^{\mathbf{l}} Y_{\mathbf{lm}}$. Hence, a complete set of vector harmonics that generate the area-preserving diffeomorphism $\text{Diff}_0(RP_2)$ are the odd-parity subset of S_2 spherical harmonics Eq.(1):

$$L_{\mathbf{lm}} = \{Y_1, \ Y_3, \ Y_5, \ \cdots, \ Y_{2N-1}\} \tag{11}$$

Hence, the Poisson algebra among these subset of harmonics is isomorphic to sub-algebra that closes among the generators:

$$T^{(1)}, \ T^{(3)}, \ T^{(5)}, \cdots, T^{(2N-1)}. \tag{12}$$

As shown by Pope and Romans[13], this sub-algebra forms $Sp(2N, \mathbf{C}) \cap SU(2N) = USp(2N)$ group. To see this, consider a totally anti-symmetric $(2^{(2N-1)} \times 2^{(2N-1)})$ matrix $\mathcal{M}$:

$$\mathcal{M} \equiv \text{Sym}[\sigma_2 \otimes \sigma_2 \otimes \cdots \otimes \sigma_2]. \tag{13}$$

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[4] A similar analysis shows that $SO(2N + 1)$ subgroup is also possible by recalling that $1 + 2 + \cdots + (N - 1) = N(N - 1)/2$.

It is straightforward to check that the $SU(2)$ generators Σ_i in Eq.(5) of the $2N$-dimensional representation satisfies:

$$\Sigma_i \cdot \mathcal{M} + \mathcal{M} \cdot \Sigma_i^{\mathrm{T}} = 0, \tag{14}$$

hence, for $i = 1, 3, \cdots, (2N-1)$,

$$T_{(i)} \cdot \mathcal{M} + \mathcal{M} \cdot T_{(i)}^{\mathrm{T}} = 0. \tag{15}$$

Therefore, the subset of generators Eq.(12) forms an $Sp(2N, \mathbf{C})$ algebra. Since they are Hermitian as well, the generators actually closes under $Sp(2N, \mathbf{C}) \cap SU(2N) = USp(2N)$. This establishes that M(atrix) theory with a gauge group $USp(2N)$ is relevant to real-projective plan two-brane and $\mathrm{Diff}_0(RP_2)$ thereof.

3.2 Genus-1 Two-Branes

Starting from toroidal two-brane, open two-branes of cylindrical and Möbius strip topologies, and closed and unoriented two-brane of Klein-bottle topology are obtained by involution.

3.2.1 Toroidal Two-Brane and $\mathrm{Diff}_0(T_2)$

Consider a torus defined in terms of coordinates (x, y)

$$\mathbf{x} \equiv (x, y) \simeq (x + 2\pi, y) \simeq (x, y + 2\pi). \tag{16}$$

A complete set of scalar harmonics on T_2 are Fourier modes:

$$Y_{\mathbf{m}}(\mathbf{x}) \equiv \exp[i\mathbf{m} \cdot \mathbf{x}]; \qquad \mathbf{m} = (m, n) \in \mathbf{Z}^2. \tag{17}$$

The area-preserving diffeomorphisms are generated by vector fields

$$\begin{aligned} T_{\mathbf{m}} &= -(\partial_a Y_{\mathbf{m}})\epsilon^{ab}\partial_b \\ &= Y_{\mathbf{m}}\mathbf{m} \times (-i\nabla_{\mathbf{x}}). \end{aligned} \tag{18}$$

They satisfy $\mathrm{Diff}_0(T_2)$ algebra

$$[T_{\mathbf{m}}, T_{\mathbf{n}}] = (\mathbf{m} \times \mathbf{n})T_{\mathbf{m}+\mathbf{n}}. \tag{19}$$

As is well-known[12, 14], finite matrix algebra realizes the above $\mathrm{Diff}_0(T_2)$ in the $N \to \infty$ limit. Specifically, one first picks a primitive N-th root of unity $\omega = \exp(2\pi i/N)$ and, following 't Hooft, a pair of unitary and traceless matrices U, V that are primitive N-th root of unity:

$$UV = \omega VU; \quad U^N = V^N = 1. \tag{20}$$

It then follows that a set of matrices

$$J_{\mathbf{m}} \equiv \omega^{-mn/2} U^m V^n \tag{21}$$

for $1 \le m, n \le N$ are linearly independent, complete set of basis for the $N \times N$ Hermitian matrices of $SU(N)$ (excluding the rigid mode generator J_{00}). From the composition rule:

$$J_{\mathbf{m}}J_{\mathbf{n}} = \omega^{\mathbf{m} \times \mathbf{n}/2}J_{\mathbf{m}+\mathbf{n}}, \tag{22}$$

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the commutators of $\mathbf{T_m} \equiv N/2\pi i J_\mathbf{m}$ are closed:

$$[\mathbf{T_m}, \mathbf{T_n}] = \frac{N}{\pi} \sin\left(\frac{\pi}{N}\mathbf{m} \times \mathbf{n}\right)\mathbf{T_{m+n}}. \tag{23}$$

In the large N limit, the commutator algebra Eq.(23) agrees with Eq.(19), the area-preserving diffeomorphism algebra $\mathrm{Diff}_0(T_2)$. Correspondingly, M(atrix) theory with gauge group $SU(N)$ possesses $\mathrm{Diff}_0(T_2)$ and describes T_2 two-branes as collective excitations of 0-brane partons.

3.2.2 $S_1/\mathbf{Z}_2$ Two-Branes and $\mathrm{Diff}_0(C_2)$

Cylindrical two-brane is obtained by an involution of T_2:

$$(x, y) \simeq (x, 2\pi - y), \tag{24}$$

so that the fundamental cell $[0, 2\pi] \times [0, \pi]$ is double-covered by T_2. Under the involution, the scalar harmonics transforms as

$$Y_\mathbf{m} \to Y_{\tilde{\mathbf{m}}}; \qquad \tilde{\mathbf{m}} \equiv (m, -n). \tag{25}$$

Therefore, the vector harmonics of the area-preserving diffeomorphism on C_2 is spanned by the subset of basis:

$$C_\mathbf{m} \equiv \{Y_\mathbf{m} : Y_\mathbf{m} - Y_{\tilde{\mathbf{m}}}\}. \tag{26}$$

They satisfy the $\mathrm{Diff}_0(C_2)$ algebra:

$$[C_\mathbf{k}, C_\mathbf{m}] = (\mathbf{k} \times \mathbf{m})C_\mathbf{k+m} - (\mathbf{k} \times \tilde{\mathbf{m}})C_{\mathbf{k}+\tilde{\mathbf{m}}}. \tag{27}$$

We now realize the above $\mathrm{Diff}_0(C_2)$ algebra in terms of large N limit matrix algebra. Using the commutation relation Eq.(23), it is easy to show that the set of matrices

$$\mathbf{C}_{mn} \equiv \frac{N}{2\pi i}[J_\mathbf{m} - J_{\tilde{\mathbf{m}}}] \tag{28}$$

are closed and form a subalgebra of $SU(N)$. Let us take U and V as

$$U = \begin{pmatrix} \omega & & & & \\ & \omega^2 & & & \\ & & \omega^3 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad V = \begin{pmatrix} & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}, \tag{29}$$

which have the property as $U^T = U$ and $V^T = V^{-1}$. We find that $J_\mathbf{m}^T = J_{\tilde{\mathbf{m}}}$ and matrices $\mathbf{C_m}$ are anti-symmetric. Being Hermitian, they span $SO(2N)$ or $SO(2N+1)$ Lie algebras.

Incidentally, there exists an alternative involution possible. Involution of T_2 as:

$$(x, y) \simeq (x, \pi - y) \tag{30}$$

yields the fundamental cell as $[0, 2\pi] \times [\pi/2, 3\pi/2]$. In this case, the generator matrices are projected to a subset consisting of $J_\mathbf{m} - (-)^n J_{\tilde{\mathbf{m}}}$. They span $SO(2N) \in U(2N)$.

We conclude that M(atrix) theory defined with a gauge group $SO(2N)$ or $SO(2N+1)$ possesses $\mathrm{Diff}_0(C_2)$ local symmetry and is capable of describing twisted two-branes of cylinder topology. Precisely such a two-brane stretched between the two ten-dimensional "end of the world" is expected to describe strongly coupled heterotic and dual type IA strings.

3.2.3   Möbius Strip Two-Brane and $\text{Diff}_0(M_2)$

The Möbius strip is obtained from the torus by an involution:

$$(x, y) \simeq (y, x). \tag{31}$$

The fundamental cell is a square made by 4 intersecting lines, i.e. $y = \pm x$ and $y = \pm(x - \pi)$. $y = -x$ and $y = -x + \pi$ are glued together with a twist, thus giving a Möbius strip. Under this involution the scalar harmonics transform as:

$$Y_{\mathbf{m}} \to Y_{\hat{\mathbf{m}}}; \qquad \hat{\mathbf{m}} \equiv (n, m). \tag{32}$$

The parity-odd harmonics are generated by:

$$M_{\mathbf{m}} = Y_{\mathbf{m}} - Y_{\hat{\mathbf{m}}}. \tag{33}$$

They are closed under commutators:

$$[M_{\mathbf{k}}, M_{\mathbf{m}}] = (\mathbf{k} \times \mathbf{m}) M_{\mathbf{k}+\mathbf{m}} - (\mathbf{k} \times \hat{\mathbf{m}}) M_{\mathbf{k}+\hat{\mathbf{m}}}. \tag{34}$$

This basis form $SO(N)$, for N can be both even and odd.

Consider the matrices spanned by

$$U = \omega^{-\frac{N-1}{4}} \begin{pmatrix} \omega^{1/2} & & & & \\ & \omega & & & \\ & & \omega^{3/2} & & \\ & & & \ddots & \\ & & & & \omega^{\frac{N-1}{2}} \\ 1 & & & & \end{pmatrix}, \quad V = U^T. \tag{35}$$

We have

$$J_{\mathbf{m}} = J_{\hat{\mathbf{m}}}^T, \tag{36}$$

hence, the matrices $M_{\mathbf{m}} = J_{\mathbf{m}} - J_{\hat{\mathbf{m}}}^{\mathrm{T}}$ are antisymmetric. Being Hermitian at the same time, they span $SO(2N)$ and $SO(2N+1)$ Lie algebra. We conclude that M(atrix) theory with these groups possess $\text{Diff}_0(M_2)$ and is capable of describing twisted two-branes of Möbius strip topology.

3.2.4   Klein-Bottle Two-Brane and $\text{Diff}_0(K_2)$

The Klein bottle K_2 is obtained from the torus T_2 by an involution of simultaneous reflection and shift:

$$(x, y) \simeq (x + \pi, 2\pi - y), \tag{37}$$

viz. the fundamental domain $[0, \pi] \times [0, 2\pi]$ is double-covered by T_2. Under the involution Eq.(37), the scalar harmonics transforms as:

$$Y_{\mathbf{m}} \to (-)^m Y_{\tilde{\mathbf{m}}}; \qquad \tilde{\mathbf{m}} = (m, -n). \tag{38}$$

Therefore, the parity-odd vector harmonics are generated by a subset of harmonics:

$$K_{\mathbf{m}} = L_{\mathbf{m}} - (-)^m L_{\tilde{\mathbf{m}}}. \tag{39}$$

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In addition, there is one global vector fields and associated rigid generator. The harmonics $K_{\mathbf{m}}$ in Eq.(39) are closed under commutators:

$$[K_{\mathbf{k}}, K_{\mathbf{m}}] = (\mathbf{k} \times \mathbf{m}) K_{\mathbf{k+m}} - (-)^n (\mathbf{k} \times \tilde{\mathbf{m}}) K_{\mathbf{k+\tilde{m}}}. \tag{40}$$

The matrix algebra realization of the above $\mathrm{Diff}_0(K_2)$ is proceeded by replacing $L_{\mathbf{m}}$ in the expression Eq.(39) by $(N/2\pi i)J_{\mathbf{m}}$ of Eq.(21). It turns out that the matrix algebra closes only for even N. With the same choice for U, V as was done for cylinder, we have

$$\begin{aligned} M U^T M^{-1} &= -U, \\ M V^T M^{-1} &= V^{-1}, \end{aligned} \tag{41}$$

where $M = \sigma_1 \otimes \mathbf{1}$. Using Eq.(41), we have $M K_{\mathbf{m}}^T M^{-1} = -K_{\mathbf{m}}$. We already know that with symmetric matrix M, above constraint produce $SO(2N)$, as subalgebra of $U(2N)$. With slightly different choice, we can also relate Klein bottle to $USp(2N)$. Take [5]

$$U = \omega^{1/2} U_{\mathrm{C}_2}, \qquad V = \left(\begin{array}{ccc|ccc} 1 & & & & & i \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & i & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array} \right). \tag{42}$$

This choice satisfies Eq.(41) again, with $M = \sigma_2 \otimes \mathbf{1}$. Since M is antisymmetric now, we have reduction to $USp(2N)$. Hence, both $SO(2N)$ and $USp(2N)$ can describe Klein-bottle two-brane.

To conclude, we summarize relation between the choice of M(atrix) theory gauge group and twisted two-branes that can be described by each choice.

M(atrix) Theory Gauge Group	Matrix Generators	Two-brane Topology
SO(2N) and SO(2N+1)	$Y_{\mathbf{lm}} - (-)^{\mathbf{l+m}} Y_{\mathbf{lm}}$	Disk
USp(2N)	$Y_{\mathbf{lm}} - (-)^{\mathbf{l}} Y_{\mathbf{lm}}$	Projective plane
SO(2N) and SO(2N+1)	$J_{\mathbf{m}} - J_{\tilde{\mathbf{m}}}$	Cylinder
SO(2N)	$J_{\mathbf{m}} - (-)^n J_{\tilde{\mathbf{m}}}$	Cylinder
SO(2N) and USp(2N)	$J_{\mathbf{m}} - (-)^m J_{\tilde{\mathbf{m}}}$	Klein bottle
SO(2N) and USp(2N)	$J_{\mathbf{m}} - (-)^{m+n} J_{\tilde{\mathbf{m}}}$	Klein bottle
SO(2N) and SO(2N+1)	$J_{\mathbf{m}} - J_{\hat{\mathbf{m}}}$	Möbius strip
SO(2N)	$J_{\mathbf{m}} - (-)^{m+n} J_{\hat{\mathbf{m}}}$	Möbius strip

From this result, it should be clear that the twisted two-brane with open ends at the 9-brane (end of the world) is possible only for the choice of M(atrix) theory gauge group $SO(2N)$ and $SO(2N+1)$.

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[5] This choice satisfies $U^{2N} = V^{2N} = -1$, which is different from usual convention.

4 Heterotic M(atrix) Theory

4.1 Supersymmetry and Twisted Sector

Based on the results of analysis on Chan-Paton factor and of area-preserving diffeomorphism gauge symmetries, we have found that heterotic M(atrix) theory is described by $SO(2N)$ or $SO(2N+1)$ matrix quantum mechanics[6]. The thirty-two supersymmetries of the underlying $SU(2N)$ M(atrix) theory is broken to sixteen supersymmetries due to the orbifold projections. The field contents and gauge quantum numbers are given as follows. For bosonic fields, 8 parallel coordinates $\mathbf{X}^i$, $i = 1, \cdots, 8$ are in the symmetric representations while transverse coordinate A_9 is in the adjoint representation. The non-dynamical gauge field A_0 is also in the adjoint representation. For spinor fields, eight component $\mathbf{S}_a$ are in the adjoint representations while eight component $\mathbf{S}_{\dot{a}}$ are in the symmetric representations. The surviving eight supersymmetries are $\mathbf{8}_s$ components. The heterotic M(atrix) theory Lagrangian is given by:

$$L_{\text{untwisted}} = \text{Tr}\Big(\ \frac{1}{2R}(D_\tau \mathbf{X}^i)^2 - \frac{1}{2R}(D_\tau A_9)^2 - \frac{R}{2}[A_9, \mathbf{X}^i]^2 + \frac{R}{4}[\mathbf{X}^i, \mathbf{X}^j]^2$$
$$- \ \mathbf{S}_a D_\tau \mathbf{S}_a + R\,\mathbf{S}_a[A_9, \mathbf{S}_a] + \mathbf{S}_{\dot{a}} D_\tau \mathbf{S}_{\dot{a}} + R\,\mathbf{S}_{\dot{a}}[A_9, \mathbf{S}_{\dot{a}}] - 2iR\,\mathbf{X}^i \sigma^i_{a\dot{a}}\{\mathbf{S}_a, \mathbf{S}_{\dot{a}}\}\ \Big)(1)$$

We take a convention that all the fields are Hermitian matrices. This heterotic M(atrix) theory is invariant under eight dynamical and eight kinematical supersymmetries. In terms of the conjugate momenta $\mathbf{\Pi}_i$ and E_9 to $\mathbf{X}^i$ and A_9 respectively, the eight dynamical supersymmetry charges are given by

$$\mathbf{Q}_a = \sqrt{R}\,\text{Tr}\Big[\Big(\mathbf{\Pi}_i \sigma^i_{a\dot{a}} \mathbf{S}_{\dot{a}} + E_9 \mathbf{S}_a\Big) + \frac{1}{2}\Big(\ \mathbf{X}^i \sigma^{ij}_{ab}[\mathbf{S}_b, \mathbf{X}^j] + [A_9, \mathbf{X}^i]\sigma^i_{a\dot{a}}\mathbf{S}_{\dot{a}}\ \Big)\Big]. \qquad (2)$$

It is easy to see that the $\mathbf{Q}_a$ supercharges map symmetric and adjoint representation fields of $SO(2N)$ back into symmetric and adjoint representation fields. For the other eight broken supersymmetries, it is not even possible to write down a gauge invariant expression of the 'would-be' supercharges $\mathbf{Q}_{\dot{a}}$. This is an expected result: while the covering space M(atrix) theory allows $SU(2N)$ gauge invariant $\mathbf{8}_c$ supercharges, orbifold projection reduces the gauge group to $SO(2N)$ with the above field content. In terms of the new field content, the $\mathbf{8}_c$ supercharges become no longer gauge invariant.

While the above heterotic M(atrix) theory is perfectly supersymmetric, the orbifold projection apparently does not lead to equal numbers of bosonic and spinor degrees of freedom. Counting on-shell degrees of freedom, there are $2N(2N-1)/2 + 8 \cdot 2N(2N+1)/2 - 2N(2N-1)/2 = 16N^2 + 8N$ bosons and $4 \cdot 2N(2N-1)/2 + 4 \cdot 2N(2N+1)/2 = 16N^2$ spinors. As alluded previously, this originates from the excess projection of the spinor coordinates along the diagonal entries in the $\mathbf{8}_s$. On the other hand, the deficit fermionic degrees of freedom can be matched by adding eight (plus eight mirror) spinors in the fundamental representation of the $SO(2N)$ gauge group for each orbifold fixed points, which we call the 'twisted sector'. The twisted sector spinors are neutral under the surviving supersymmetries of the orbifold projection. As such, each of the untwisted and the twisted sectors are manifestly supersymmetric.

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[6]$SO(2N+1)$ matrix quantum mechanics was studied first in a closely related context in Ref. [15] based on earlier work Ref. [9].

While the deficit of degrees of freedom seems unavoidable consequence of the orbifold projection, it is gratifying that the M(atrix) theory tells us indirectly what the field contents of the twisted sector consists of. The deficit spinor degrees of freedom are attributed to the $\mathbf{8}_s$ representation. As such, the fermions in the twisted sector have quantum numbers $(\mathbf{8}, 2N)$ under $SO(8)$ global and $SO(2N)$ gauge symmetries. Given these quantum numbers, interaction of the twisted sector fermions to the untwisted sector is uniquely determined [7]:

$$L_{\text{twisted}} = \sum_{a=1}^{8} \sum_{A,B=1}^{2N} \overline{\chi}_{1a}^A (D_\tau + \gamma_9 R A_9)_{AB} \chi_{1a}^B + \overline{\chi}_{2a}^A (D_\tau + \gamma_9 R(A_9 - i\pi R_9 \sigma_2))_{AB} \chi_{2a}^B. \qquad (3)$$

In this expression, the two sets of fermions χ_1 and χ_2 originate from each of the two fixed points on $\mathbf{S}_1/\mathbf{Z}_2$ orbifold. We also have introduced a notion of 'chirality' operator γ_9 so that $\gamma_9 \chi_a^A \equiv -\chi_a^A$, which amounts to a choice of relative sign between the A_0 and A_9 field couplings in Eq.(3). This 'chirality' operation is motivated by the observation that the twisted sector fermions originate from to the deficit $\mathbf{8}_s$ degrees of freedom in the untwisted sector.

The γ_9 'chirality' prescription given as above is not arbitrary but follows from consistency conditions. First, the twisted sector interactions should be invariant under the residual supersymmetry. With $\gamma_9 = -1$, the supersymmetry variations of A_0 and A_9 cancel out each other. Second, as will be shown below, with $\gamma_9 = -1$ 'chirality', the twisted sector fermions become the sixteen worldsheet fermions whose Kac-Moody currents generate the spacetime gauge symmetry in the T-dual heterotic string.

The complete heterotic M(atrix) theory Lagrangian is a sum of Eq.(1) and Eq.(3). Upon T-duality, the heterotic M(atrix) theory is related to the D-string in type I string theory. The latter was interpreted as the heterotic string theory itself [16]. As such, it would be interesting to see how the worldsheet theory of the heterotic string may be recovered from the heterotic M(atrix) theory. The T-duality turns the heterotic M(atrix) theory into an $(1+1)$-dimensional $SO(2N)$ gauge theory on a dual orbifold:

$$\begin{aligned}
L_{(1+1)} = \oint \frac{d\sigma}{2\pi} \Big(\ \text{Tr} \ \Big(\ &-\frac{1}{4R} F_{\alpha\beta}^2 + \frac{1}{2R}(D_\alpha \mathbf{X}^i)^2 + \frac{R}{4}[\mathbf{X}^i, \mathbf{X}^j]^2 \\
&- \ \mathbf{S}_a \not{D} \mathbf{S}_a + \mathbf{S}_{\dot a} \not{D} \mathbf{S}_{\dot a} - 2iR\mathbf{X}^i \sigma_{a\dot a}^i \{\mathbf{S}_a, \mathbf{S}_{\dot a}\} \Big) \\
&+ \ \sum_{a=1}^{8} \sum_{A=1}^{2N} \overline{\chi}_{1a}^A \not{D}_{AB} \chi_{1a}^B + \overline{\chi}_{2a}^A \not{D}_{Ab} \chi_{2a}^B \quad \Big),
\end{aligned} \qquad (4)$$

where $i, j = 1, \cdots, 8$ and all the fields depend on $(\tau, \sigma = RX_9)$. The covariant derivatives are defined as $D_\alpha \mathbf{X}^i \equiv (D_\tau \mathbf{X}^i, D_\sigma \mathbf{X}^i)$, $D_\sigma \equiv \partial_\sigma - i[A_9,]$, $\not{D}\mathbf{S}_a \equiv (D_\tau + \Gamma^9 D_9)\mathbf{S}_a$ etc. and $\mathcal{D}_{AB} \equiv (D_\tau - D_9)_{AB}$ respectively. We have also made a finite shift $A_9 \to A_9 + i\pi R_9 \sigma_2$ by making the boundary conditions of χ_2 fermions opposite to those of χ_1.

Consider the simplest case $N = 1/2$, viz. a single D-parton attached to the orientifold. Let us take the 'light-cone gauge' $A_0 = A_9$. There are no components to fields of adjoint representations, while fields of symmetric representations are (1×1) fields, $\mathbf{X}^i = X^i(\sigma, \tau) \in \mathbf{8}_v$ and $\mathbf{S}_{\dot a} = S_{\dot a}(\sigma, \tau) = S_{\dot a}(\tau - \sigma) \in \mathbf{8}_c$. They are the eight bosonic and eight spinor spacetime coordinates of the Type I D-string in the Green-Schwarz formalism. In the 'light-cone gauge',

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[7]A similar twisted sector fermion interaction was introduced previously in Ref. [15]. Our derivation offers a detailed account for the M(atrix) theory origin and dynamics governing them.

noting the γ_9 sign choice of the twisted sector fermions, $\chi^A_{1,2} = \chi^A_{1,2}(\sigma, \tau) = \chi^A_{1,2}(\tau + \sigma)$, we find precisely the worldsheet structure of the heterotic string in which the spacetime supersymmetry is realized in the light-cone Green-Schwarz formulation and the spacetime gauge symmetry are generated by worldsheet Kac-Moody current algebra of the 16 + 16 twisted sector fermions from each fixed points. Noting that they have opposite boundary conditions each other, the resulting gauge symmetry is expected to be $E_8 \times E_8$ rather than $SO(32)$.

The necessity of the twisted sector consisting of 16 spinors may be understood from another point of view as well. First, without the twisted sector fermions, it is straightforward to see that there arise non-trivial renormalization of moduli space metric and potential to the heterotic M(atrix) theory *at one loop* . Non-vanishing *static* potential represents M(atrix) theory manifestation of non-vanishing cosmological constant, viz. uncancelled dilaton tadpole amplitude. The presence of one-loop induced cosmological constant can be probed by scattering a 0-brane parton off the orientifold. The orbifold projection arranges an 'image' 0-brane at the opposite side of the orientifold. Therefore, integrating out massive modes, we should expect to reproduce the result of two 0-brane scattering proportional to v^4/r^7. In particular, for a consistent description of heterotic M(atrix) theory, a 0-brane parton *at rest* should be a stable configuration, viz. *static* potential between the 0-brane parton and its image should be absent. The relevant calculation has been considered in Ref. [9]. For a 0-brane 'head-on' collision to one of the orientifold, the potential [8] in the Born-Oppenheimer approximation is calculated by integrating out massive modes. In the background gauge $A_0 = 0$ and $A_9 = (i/2)\sigma^2 vt$, massive excitations consist of 16 bosonic modes from $\mathbf{X}^i$ and 8 complex fermionic modes from $\mathbf{S}_{\dot{a}}$. The 0-brane potential due to orbifold fixed point is calculated from one-loop Feynman integral (vacuum tadpole) over these massive modes. Using proper-time regularization of the one-loop integral, we find:

$$\mathcal{A}_{\text{orientifold}} = \int_{-\infty}^{+\infty} dt \, \mathcal{V}_{\text{orientifold}}(v, vt) = \det{}^{-8}(-\partial_\tau^2 + v^2\tau^2) \det{}^{+4} \begin{pmatrix} \partial_\tau & +v\tau \\ +v\tau & \partial_\tau \end{pmatrix},$$

$$\mathcal{V}_{\text{orientifold}}(v, r) = v \int_0^\infty \frac{ds}{\sqrt{\pi s}} e^{-sr^2} \frac{4 - 2\cos vs}{\sin vs} = [-4r + \frac{v^2}{2r^3} + \cdots]. \tag{5}$$

The result displays a linear, *velocity-independent*, one-loop vacuum energy, which arises due to the fact that boson and spinor coordinates are under different $SO(2N)$ representations, hence, mismatch of boson–fermion degrees of freedom. The linear potential is not compatible with the expected stability for a 0-brane parton at rest at a distance away from the orientifold. In the closed string channel, this is an exchange of dilaton between the fixed point and the 0-brane parton, viz. dilaton tadpole amplitude induced by non-vanishing local cosmological constant. However, the cosmological constant is cancelled *locally* by the twisted sector fermions [9] at each location along $A_9 = (i/2)\sigma^2 r$. It is important to recall that the twisted sector fermions are in the fundamental representation of the M(atrix) gauge group $SO(2N)$ [10] and the elementary

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[8] We have checked that the same result follows also for the 'light-cone gauge' $A_0 = A_9 = (i/2)\sigma^2 vt$.

[9] It is in fact sufficient to cancel the cosmological constant and dilaton tadpole amplitude only *globally*. In that case, however, it is not possible to identify field content of the twisted sector from the heterotic M(atrix) theory. Moreover, the only stable configuration of 0-brane partons is only when they are all located at the orbifold fixed point.

[10] We disagree the result of Ref. [9], in which the twisted sector fermions were apparently taken to be in symmetric representation, not in fundamental representation. This error has caused the author of Ref. [9] to conclude incorrectly that the total number of twisted sector fermions to be sixteen instead of thirty-two.

but crucial fact that the distance between probe 0-brane and its mirror is the *twice* the distance between probe 0-brane and orientifold. The 0-brane potential due to twisted sector is generated by integrating out massive fermionic modes from 16 complex, twisted sector fermions. Using proper-time regularization, we find:

$$\mathcal{A}_{\text{twist}} = \int_{-\infty}^{+\infty} dt\, \mathcal{V}_{\text{twist}}(v, vt) = \det^{+8} \begin{pmatrix} \partial_\tau & \frac{v\tau}{2} \\ \frac{v\tau}{2} & \partial_\tau \end{pmatrix},$$

$$\mathcal{V}_{\text{twist}}(v, r) = -\frac{v}{2} \int_0^\infty \frac{ds}{\sqrt{\pi s}} e^{-s\frac{r^2}{4}} \frac{4\cos\frac{vs}{2}}{\sin\frac{vs}{2}} = [+4r + \frac{4v^2}{3r^3} + \cdots]. \tag{6}$$

From the above local analysis, we clearly see that each orientifolds carry 8 (plus 8 'image') negative units of D8-brane RR charge. It shows that the twisted sector consists of eight fermion representing D8-branes and their 'mirror' images has to be located at the orientifold location in order to cancel *local* cosmological constant.

A few remarks are in order. Once the twisted sector spectra is determined as above, moduli space of twisted sector can be further explored by deformation and displacement of each of the D8-branes. In addition, while the above result is essentially a requirement of dilaton tadpole cancellation, we also expect to reach the same conclusion from the anomaly cancellation requirement of (1+1)-dimensional worldsheet gauge theory Eq.(4) of the T-dual Type I D-string that was identified as a heterotic string [16]. Noting that $\mathbf{S}_a$ and $\mathbf{S}_{\dot{a}}$ transform as left-handed adjoint and right-handed symmetric representations, we find that the $SO(2N)$ gauge anomaly $8C_2(\texttt{adjoint}) - 8C_2(\texttt{symmetric}) = 8((N + 2) - (N - 2))C_2(\texttt{fundamental})$ due to untwisted sector fermions is cancelled precisely by thirty-two twisted sector fermions transforming as left-handed fundamental representations. Finally, even though the *local* cosmological constant is cancelled completely by the twisted sector fermions at the orbifold fixed point, there still remains a nontrivial one-loop effect: moduli space metric of the 0-brane partons is renormalized nontrivially both by the orientifold and the twisted sector fermions to $g_{99} = (1 + 11R/3r^3)$.

4.2 Large–N Twisted and Untwisted Two-Branes

By construction, the heterotic M(atrix) theory Eqs.(1, 3) is a gauge theory of area-preserving diffeomorphism relevant to open two-branes. It is therefore of interest whether macroscopic matrix two-brane configurations can be constructed in the large N limit.

The simplest open two-brane of direct relevance to the strongly coupled heterotic string is a cylindrical two-brane whose ends are attached to the two nine-branes. We have found such a two-brane configuration as follows. Let the open two-brane is extended along 1-9 direction in which the 1-direction is infinitely extended. It is useful to recall the block matrix structures of the $SO(2N)$ adjoint and symmetric representations. Noting that the block diagonal matrices Y, Z are $(N \times N)$ Hermitian matrices, a matrix open two-brane configuration is easily constructed as:

$$\mathbf{X}_1 = \frac{R_1}{\sqrt{2}} \begin{pmatrix} P & 0 \\ 0 & P^{\text{T}} \end{pmatrix}, \quad A_9 = \frac{R_9}{\sqrt{2}} \begin{pmatrix} Q & 0 \\ 0 & -Q^{\text{T}} \end{pmatrix}, \tag{7}$$

where P, Q are $(N \times N)$ $(N \to \infty)$ Hermitian matrices satisfying a commutator $[Q, P] = i$. The configuration yields

$$\frac{1}{R_1 R_9}[A_9, \mathbf{X}_1] = \frac{1}{2}[Q, P] \oplus [-Q^{\text{T}}, P^{\text{T}}] = \frac{i}{2}\mathbf{I}_{2N \times 2N}. \tag{8}$$

Due to the reflection symmetry of A_9 configuration, the range of eigenvalues for $A_9, \mathbf{X}_1$ is $[0, 2\pi R_1] \otimes [0, \pi R_9]$. By permuting the 9-th and the 11-th (longitudinal) directions, the two ends of cylindrical two-brane become precisely the heterotic string propagating at each nine-branes.

In the background of the above open two-brane, it is also possible for the twisted sector fermions to show a nontrivial, macroscopic configuration. Since $A_0 = 0$ for the macroscopic matrix two-brane, the equation of motion for χ_a^A coupled to the two-brane is given by:

$$\left[\begin{pmatrix} \partial_\tau & 0 \\ 0 & \partial_\tau \end{pmatrix} + i \begin{pmatrix} Q & 0 \\ 0 & -Q^T \end{pmatrix} \right] \begin{pmatrix} \chi_a^{(1)} \\ \chi_a^{(2)} \end{pmatrix} = 0, \tag{9}$$

where we have used the chirality property $\gamma_9 \chi_a^A \equiv -\chi_a^A$ and decomposed the spinors explicitly into 8N partons $\chi_a^{(1)}$ and their mirror image partons $\chi_a^{(2)}$. Since the heterotic string as the boundary of open two-brane is extended along 1-direction, it is convenient to represent $P = \sigma$, a c-number variable on a range $[0, 2\pi]$, and $Q = i\partial_\sigma$. In this representation, by combining the eight original and the eight mirror fermions, a nontrivial configuration is found straightforwardly:

$$\chi^I = E^I \exp[i(\tau + \sigma)] \tag{10}$$

for a non-vanishing, normalized lattice vector $E^I \in SO(16)$. The configuration represents twisted sector fermions propagating chirally at each end of the open two-branes, viz. heterotic string [11]. Similar construction is possible for a disk or a cylindrical two-brane attached to either of the two nine-branes. Even though topologically not stable, at least for the ones with a macroscopic size, classical description should be valid.

In addition to the twisted open two-branes, the heterotic M(atrix) theory should possess macroscopic, matrix two-branes of closed topology. In fact, it is straightforward to recognize that the heterotic M(atrix) theory contains a subsector with sixteen supersymmetry charges. Recall that the orbifold projection in the heterotic M(atrix) theory has reduced the supersymmetry into half that of the covering space M(atrix) theory. The 'would-be' $\mathbf{8}_c$ supercharges were no longer $SO(2N)$ gauge-invariant.

On the other hand, away from the orbifold fixed points, strings connect among partons but not to their images. In terms of the block matrix structures given in Eqs.(4,5) of section 2, only the diagonal block ($N \times N$) matrices describe the parton interactions. These diagonal block matrices are, however, $unconstrained$ ($N \times N$) Hermitian matrices of $SU(N) \in SO(2N)$ for both adjoint and symmetric representations. Once all the matrix fields are truncated to the diagonal block matrices, it is then easy to recognize that the heterotic M(atrix) theory Eq.(1) reduces to a $SU(N)$ M(atrix) theory (and its mirror) in which all the fields are adjoint representations of $SU(N)$. This subsector should preserve sixteen supersymmetries. The $\mathbf{8}_c$ supercharges are $SU(N)$ gauge invariant even though non-invariant under $SO(2N)$, hence, were projected out in the latter. Since this subsector is equivalent to the $SU(N)$ M(atrix) theory, closed, oriented two-branes should be present as well. For definiteness, consider a toroidal two-brane extended along (1-2) directions. The matrix configuration is given by:

$$\mathbf{X}_1 = R_1 \begin{pmatrix} Q & 0 \\ 0 & Q^{\mathrm{T}} \end{pmatrix}; \quad \mathbf{X}_2 = R_2 \begin{pmatrix} P & 0 \\ 0 & P^{\mathrm{T}} \end{pmatrix}. \tag{11}$$

---

[11]The solution is valid, however, strictly in the large N limit. At finite but large N, the Dirac equation is a set of 2N-coupled equations, which can be solved iteratively. The large N limit then reduces to the above solution.

in the notation given above. Hence, the two-brane wrapping number is found:

$$
\begin{aligned}
[\mathbf{X}_1, \mathbf{X}_2] &= i\sigma_3 \otimes \mathbf{I}_{N\times N}, \\
\mathcal{Z}_{2\times 2} &= \frac{1}{N}\mathrm{Tr}_{N\times N}\Big(i[X_1, X_2]\Big) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}
\tag{12}
$$

The relative negative sign between the two-brane and its image correctly counts for the orientation reversal of the two-brane or, equivalently, of the antisymmetric tensor field $A_{MNP} \to -A_{MNP}$ under the orbifold condition.

4.3 Large N Limit

We finally discuss the large N limit of the heterotic M(atrix) theory. In this limit, we expect that the M(atrix) theory becomes a $(2+1)$ dimensional field theory coupled to a $(1+1)$-dimensional boundary field theory. Since the heterotic M(atrix) theory is constructed from an involution of $SU(2N)$ M(atrix) theory, we first recapitulate the large N limit recipe of the latter and then consistently impose the involution condition of $SO(2N)$ M(atrix) theory.

In this section, extending results of Ref. [11, 12, 14] we explain correspondence rules between the matrix fields and the continuum fields in the large N limit for open two-branes. For definiteness, we will consider cylindrical involution in what follows.

In the large N limit of M(atrix) theory, Hermitian $SU(N)$ matrices are expanded as $\mathbf{X}^I = \sum_{\mathbf{m}} X^I_{\mathbf{m}} J_{\mathbf{m}}$, where $J_{\mathbf{m}}$ is a complete set of basis for the $(N \times N)$ matrices introduced in Eq.(21) of section 3. Therefore, it is sufficient to study large N limit of the basis matrices $J^\dagger_{\mathbf{m}} = J_{-\mathbf{m}}$. In section 3.2.1, we have shown that $J_{\mathbf{m}}$'s satisfy the trigonometric Lie algebra Eq.(23). In the large N limit, the trigonometric Lie algebra is reduced to

$$
[\mathbf{T_m}, \mathbf{T_n}] = (\mathbf{m} \times \mathbf{n})\mathbf{T_{m+n}}
\tag{13}
$$

for $\mathbf{T_m} \equiv (N/2\pi i)J_{\mathbf{m}}$. This algebra is most conveniently described in terms of two non-commuting variables $\hat{p}, \hat{q}$ satisfying commutation relation $[\hat{q}, \hat{p}] = \frac{2\pi i}{N}$ so that

$$
U = e^{i\hat{p}}, \quad V = e^{i\hat{q}}; \qquad J_{\mathbf{m}} = e^{im\hat{p} + in\hat{q}}.
\tag{14}
$$

Since the non-commutativity is suppressed in the large N limit, the matrices $J_{\mathbf{m}}$ approach ordinary functions that depend on the classical phase–space variables $\mathbf{p} = (p, q)$:

$$
J_{\mathbf{m}} \to Y_{\mathbf{m}}(\mathbf{p}) \equiv \exp(i\mathbf{m} \cdot \mathbf{p}).
\tag{15}
$$

In the phase–space, the Poisson bracket algebra of $Y_{\mathbf{m}}$'s

$$
\{Y_{\mathbf{m}}, Y_{\mathbf{n}}\}_{\mathrm{PB}} = (\mathbf{m} \times \mathbf{n})Y_{\mathbf{m+n}}
\tag{16}
$$

becomes isomorphic to the large N commutator algebra Eq.(13). Therefore, by introducing doubly periodic functions $X^I(\mathbf{p}) \equiv \sum_{\mathbf{m}} X^I_{\mathbf{m}} Y_{\mathbf{m}}(\mathbf{p})$, in the large N limit, the matrix commutator algebra can be replaced by the Poisson bracket algebra:

$$
[\mathbf{X}_I, \mathbf{X}_J] \to \frac{2\pi i}{N}\{X_I(\mathbf{p}), X_J(\mathbf{p})\}_{\mathrm{PB}}.
\tag{17}
$$

18

Next, we replace trace over matrices into a phase space integral. To do so, it is convenient to recast the trace operation as an inner product:

$$\mathrm{Tr}\mathbf{X}_I^\dagger\mathbf{X}_J \equiv \langle\mathbf{X}_I|\mathbf{X}_J\rangle. \tag{18}$$

While we consider Hermitian matrices exclusively, we find it more convenient to retain the Hermitian conjugation operation and keep track of the inner product structure. Thus,

$$\mathrm{Tr}J_\mathbf{m}^\dagger J_\mathbf{k} = \mathrm{Tr}J_{-\mathbf{m}}J_\mathbf{k} = N\,\delta_{\mathbf{m},\mathbf{k}}. \tag{19}$$

On the other hand, for the classical basis functions on the classical phase space, we have

$$\oint_{-\pi}^\pi dp \oint_{-\pi}^\pi dq\; Y_\mathbf{m}^*(\mathbf{p})Y_\mathbf{k}(\mathbf{p}) = (2\pi)^2\delta_{\mathbf{m},\mathbf{k}}. \tag{20}$$

Thus we are led to the following identification

$$\mathrm{Tr}\mathbf{X}_I^\dagger\mathbf{X}_J \quad\to\quad \frac{N}{(2\pi)^2}\oint_{-\pi}^\pi dp \oint_{-\pi}^\pi dq X_I^*(\mathbf{p})X_J(\mathbf{p}). \tag{21}$$

Finally, consider the N-dimensional fundamental representations coupled to matrices, relevant to the twisted sector. Fundamental representations χ^A of $SO(N)$ are most conveniently expanded

$$\chi^I = \sum_n \chi_n^I \mathbf{e}_n \tag{22}$$

in terms of the basis vectors:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad, \mathbf{e}_N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{23}$$

The action of $SO(N)$ matrix basis U,V in section 3, Eq.(29) to the basis vectors is given by

$$\begin{aligned} U\,\mathbf{e}_n &= \omega^n\mathbf{e}_n, \\ V\,\mathbf{e}_n &= \mathbf{e}_{n+1} \end{aligned} \tag{24}$$

with an identification $\mathbf{e}_{N+n} = \mathbf{e}_n$. This algebra, with an inner product $<\mathbf{e}_k|\mathbf{e}_l> \equiv \mathbf{e}_k^T\cdot\mathbf{e}_l = \delta_{kl}$, can be represented by periodic functions and operators acting on them defined on a periodic interval $p \in [-\pi,\pi]$:

$$\begin{aligned} e^{i\hat{q}}\exp(imp) &= \omega^m\exp(imp), \\ e^{i\hat{p}}\exp(imp) &= \exp(i(m+1)p), \\ \int_{-\pi}^\pi dp\,(e^{ikp})^*e^{imp} &= 2\pi\delta_{km}, \end{aligned} \tag{25}$$

where $\hat{q} \equiv -\frac{2\pi i}{N}\partial_p$ and $\hat{p} \equiv p$. Thus, identifying $U = e^{\frac{2\pi}{N}\partial_p}$, $V = e^{ip}$, and $\mathbf{e}_n = e^{inp}$, we obtain the large N limit expresssion of twisted sector interactions:

$$\chi^I\mathbf{X}_{IJ}\chi^J \to \oint_{-\pi}^\pi dp\,\chi^*(p)X(p,-\frac{\partial}{\partial p})\chi(p) \tag{26}$$

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at each boundaries $q = 0, \pi$.

The basis of heterotic M(atrix) theory Eq.(28) of section 3 restricts the continuum fields to $q \in [0, \pi]$ with two fixed points at $q = 0, \pi$. As such, the continuum fields should be supplemented by boundary conditions. In the large N limit, the orbifold conditions section 2, Eq.(1) are replaced by reflection conditions:

$$
\begin{aligned}
X_\perp(p, q) &= -X_\perp(p, -q), \\
X_\parallel(p, q) &= +X_\parallel(p, -q), \\
S_a(p, q) &= -S_a(p, -q), \\
S_{\dot{a}}(p, q) &= +S_{\dot{a}}(p, -q).
\end{aligned}
\tag{27}
$$

Therefore, the appropriate boundary conditions are

$$
\begin{aligned}
X_\perp(p, q = 0, \pi) &= 0, \\
\partial_q X_\parallel(p, q = 0, \pi) &= 0, \\
S_a(p, q = 0, \pi) &= 0, \\
\partial_q S_{\dot{a}}(p, q = 0, \pi) &= 0.
\end{aligned}
\tag{28}
$$

The boundary condition agrees with the ones derived from the M-theory [18].

With the above correspondence rules Eqs.(17, 21, 26) and the boundary conditions Eq.(28), it is straightforward to find continuum expression of the heterotic M(atrix) theory Lagrangian.

5 Discussions

M(atrix) theory as a viable non-perturbative definition of M-theory has passed various consistency tests so far [19, 20, 21, 22, 23]. In particular, it has been shown that the M(atrix) theory supports two-brane [3] and longitudinal five-brane [19], which are shown to be BPS states made out of infinitely many 0-branes.

In this paper, we have initiated investigation of M(atrix) theory on an orbifold. We have shown that different choices of M(atrix) theory gauge group give rise to different topologies of classical two-branes. This is not surprising after all: the M(atrix) theory is nothing but a gauge theory of area-preserving diffeomorphism transformation. Interestingly, while Chan-Paton analysis constrains the possible choices of M(atrix) theory gauge groups, we have found that it is through the area-preserving diffeomorphism analysis that we uncover more refined information of the M(atrix) theory. We have found that, among $SO(2N)$, $SO(2N + 1)$ and $USp(2N)$ gauge groups allowed by Chan-Paton factor analysis for M(atrix) theory, only $SO(2N)$ or $SO(2N + 1)$ M(atrix) theory is capable of describing nontrivial twisted two-branes as BPS excitations.

As the simplest yet non-trivial example, we have considered the heterotic M(atrix) theory. This is a M(atrix) theory defined on an orbifold $\mathbf{S}_1/\mathbf{Z}_2$. We have shown that Bose-Fermi degrees of freedom matching, hence, cancellation of localized cosmological constants require an introduction of twisted sector consisting of 16 fermions (plus mirrors) transforming in the fundamental representation of $SO(2N)$ or $SO(2N + 1)$ M(atrix) gauge group. They are nothing but the 8 D8-branes (plus mirrors) located at each orbifold fixed points. We have constructed a classical BPS configuration of open two-brane stretched between the two fixed points. The two ends are nothing but the heterotic string moving freely in the ten-dimensional spacetime.

In addition, we also found a closed two-brane hovering around the full eleven-dimensional spacetime. By studying the large N limit, we have shown that the correct boundary conditions to the open two-brane are obtained for both bosonic and spinor coordinate fields.

In the forthcoming papers [24], we will report consistency checks of the heterotic M(atrix) theory and orbifold M(atrix) theories in higher-dimensional orbifolds, especially, $(\mathbf{S}_1)^5/\mathbf{Z}_2$ and $(\mathbf{S}_1)^9/\mathbf{Z}_2$ related by string dualities to compactified string theories.

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