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Nonlinear Dynamics of Single Bunch Instability in Accelerators*

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Abstract

We develop a nonlinear theory of the weak single bunch instability in electron and positron circular accelerators and damping rings. A nonlinear equation is derived that governs the evolution of the amplitude of unstable oscillations with account of quantum diffusion effects due to the synchrotron radiation. Numerical solutions to this equation show a large variety of nonlinear regimes depending on the growth rate of the instability and the diffusion coefficient. Comparison with the observation in the SLC Damping Ring at SLAC shows qualitative agreement with the patterns observed in the experiment.

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1. Introduction

Microwave single bunch instability in circular accelerators has been known for many years. The instability usually arises when the number of particles in the bunch exceeds some critical value, N_c, which can vary depending on the parameters of the accelerating regime. Typically the instability leads to the growth of the bunch length ("turbulent bunch lengthening") and the increased energy spread of the beam [1]. The origin of the microwave instability is usually associated with unstable oscillations of the bunch caused by high-frequency part of the impedance of the vacuum chamber.

Recent observations in the SLC damping ring at SLAC [2] with a new low-impedance vacuum chamber revealed some new interesting features of the instability. It was found that in some cases, after initial exponential growth, the instability eventually saturated at a level that remained constant through the accumulation cycle. In other regimes, a relaxation-type oscillations were measured in nonlinear phase of the instability. In many cases, the instability was characterized by a frequency close to the second harmonic of the synchrotron oscillations. Similar effects have been observed in LEP for the oscillations of the bunch length [3].

A vast literature devoted to the microwave instability mostly focuses on the linear theory. The main objective of this theory is to predict the frequency, growth rate and the structure of the perturbation as a function of beam parameters. Especially important for the experiment is determination of the threshold of the instability for a given wake in the accelerator. Mathematically, the linear problem reduces to a set of integral equations whose solution usually invokes elaborate numerical methods [4-6].

A solution obtained in the linear theory, however, cannot explain the time development of the instability above the threshold. Several attempts have been made to address the nonlinear stage of the instability. Using numerical simulation method D'yachkov and Baartman studied a mechanism that generates sawtooth oscillations in a single bunch instability [7]. Simulation of the SLC damping ring instability that also showed nonlinear oscillations of the amplitude has been performed in Ref. [8]. Recently Heifets proposed a theory of nonlinear oscillations considering nonlinear phase of the instability as a new equilibrium around a nonlinear resonance [9]. However, being based on either computer simulations or some specific assumptions regarding the structure of the unstable mode, these works, in our view, do not give a consistent and universal description of the nonlinear stage of the instability.

An attempt of a more general consideration of the problem based on nonlinear Vlasov equation is carried out in this paper. We adopt an approach recently developed in plasma physics for analysis of nonlinear behavior of weakly unstable modes in dynamic systems [10,11]. Assuming that the growth rate of the instability is much smaller than its frequency, we find a time dependent solution to Vlasov equation and derive an equation for the complex amplitude of the oscillations valid in the nonlinear regime. This equation, after proper normalization, contains only two dimensionless parameters, and can be easily solved numerically. It turns out that even without detailed knowledge of the nature of the instability, we can qualitatively analyze and predict different patterns of the signal that can be observed in the experiment in a weakly nonlinear regime.

The paper is organized as follows. In Section 2 we formulate the stability problem in terms of Vlasov equation with a right hand side due to the effect of synchrotron radiation. In Section 3, a brief review of the linear theory for a single bunch instability is given. Section 4 contains a general derivation of an equation for the evolution of the amplitude of weakly unstable oscillations near the threshold of the instability. A detailed calculation of nonlinear part of the equation is presented in section 5. In Section 6 we include synchrotron radiation term into nonlinear equation and introduce dimensionless variables that minimize the number of free parameters in the equation. Analysis of the solutions and results of numerical computations are presented in Section 7, and in Section 8 we discuss the main results of the paper.

2. Basic Equations

We start from the equation of motion in longitudinal direction (see, e.g., Ref. [12]):

$$\dot{z} = -c\eta\delta, \quad \dot{\delta} = K(z,t), \tag{1}$$

where z is the longitudinal coordinate, δ is the relative energy deviation, η is the slip factor, the dot indicates differentiation with respect to time t, and

$$K(z,t) = \frac{\omega_{s0}^2}{\eta c} z - \frac{r_e}{T_0 \gamma} \int_z^\infty dz' n(z',t) w(z'-z). \tag{2}$$

In Eq. (2) ω_{s0} denotes the unperturbed synchrotron frequency, T_0 is the revolution period, r_e is the classical electron radius, γ is the relativistic factor, $n(z,t)$ is the longitudinal beam density, $\int_{-\infty}^{\infty} n(z,t) dz = N$, where N is the number of particles in the beam, and $w(z)$ is the longitudinal wake function. The first term in Eq. (2) corresponds to the potential of the accelerating voltage, and the second term describes the wakefield generated by the bunch.

Equations of motion (1) can be obtained from the following Hamiltonian:

$$H(z,-\delta,t) = \frac{1}{2} c\eta\delta^2 + \frac{\omega_{s0}^2}{2\eta c} z^2 - \frac{r_e}{T_0\gamma} \int_0^z dz \int_{z'}^\infty dz'' n(z'',t) w(z''-z'), \tag{3}$$

in which z plays a role of a coordinate, and $-\delta$ is the conjugate momentum.

We will use a distribution function $\psi(x,p,t)$ of the particles in the bunch such that integrating over δ gives the particle density

$$n(z,t) = N \int_{-\infty}^{\infty} \psi(z,\delta,t) d\delta. \tag{4}$$

This distribution function satisfies the Vlasov equation with a Fokker-Planck "collision" term on the right hand side,

$$\frac{\partial \psi}{\partial t} + \{H, \psi\} = R, \tag{5}$$

where we have the Poisson brackets on the left hand side, and R describes the effect of the synchrotron radiation (see, e.g., [13]),

$$R = \frac{\partial}{\partial \delta} \left(\gamma_D \psi \delta + \kappa \frac{\partial \psi}{\partial \delta} \right). \tag{6}$$

In Eq. (6) γ_D is the damping time for the amplitude of the synchrotron oscillations and κ is the diffusion coefficient associated with the quantum nature of the radiation.

In the equilibrium state the distribution function ψ and the Hamiltonian H do not depend on time. The equilibrium solution of Eq. (6) was given by Haissinski [14],

$$\psi(z,\delta) = \text{const} \times \exp\left(-\frac{H_0(z,-\delta)}{c\eta\sigma_E^2}\right), \tag{7}$$

where $\sigma_E = \sqrt{\kappa/\gamma_D}$ is the rms energy spread of the beam in the absence of the wake, and H_0 is the equilibrium Hamiltonian.

It is convenient to introduce dimensionless variables,

$$x = \frac{z}{\sigma_z},\ \ p = -\frac{\delta}{\sigma_E},\ \ \tau = t\omega_{s0},\ F = \sigma_z\psi\,, \tag{8}$$

where σ_z is the rms length of the beam without wake, $\sigma_z = \sigma_E \eta c/\omega_{s0}$. In these variables the Hamiltonian (3) takes the form

$$H(x,p,\tau) = \frac{1}{2}p^2 + U(x,\tau), \tag{9}$$

where the "potential energy" U is

$$U(x,\tau) = \frac{1}{2}x^2 - I\int_x^\infty dx' S(x'-x)\int_{-\infty}^\infty dp F(x',p,\tau), \tag{10}$$

with

$$I = \frac{Nr_e}{T_0\gamma\omega_{s0}\sigma_z\sigma_E}, \tag{11}$$

and

$$S(x) = \int_0^{x\sigma_s} dz w(z). \tag{12}$$

Note that the function S is a dimensionless function of its argument.

Let us now perform a canonical transform from x and p to action and angle variables, J and θ, of the equilibrium Hamiltonian H_0, and denote by $\tilde{V}$ the deviation of the potential energy from the equilibrium in Eq. (9). Since H_0 depends on J only, the total Hamiltonian $H(\theta,J,t)$ takes the form

$$H(\theta,J,\tau) = H_0(J) + \tilde{V}(\theta,J,\tau). \tag{13}$$

The Vlasov equation for F in terms of action – angle variables is

$$\frac{\partial F}{\partial \tau} + \omega_s \frac{\partial F}{\partial \theta} + \frac{\partial \tilde{V}}{\partial J}\frac{\partial F}{\partial \theta} - \frac{\partial \tilde{V}}{\partial \theta}\frac{\partial F}{\partial J} = R, \tag{14}$$

where $\omega_s = \omega_s(J)$ is the frequency of synchrotron oscillations with the wake taken into account, $\omega_s(J) = dH_0/dJ$. Suppose that $F_0(J)$ is the equilibrium distribution function, and $\delta F(J,\theta,\tau) = F - F_0(J)$ is its deviation from the equilibrium. Then δF satisfies the following equation,

$$\frac{\partial \delta F}{\partial \tau} + \omega_s \frac{\partial \delta F}{\partial \theta} - \frac{\partial \delta \tilde{V}}{\partial \theta}\frac{\partial F_0}{\partial J} - \frac{\partial \delta \tilde{V}}{\partial \theta}\frac{\partial \delta F}{\partial J} + \frac{\partial \delta \tilde{V}}{\partial J}\frac{\partial \delta F}{\partial \theta} = R, \tag{15}$$

where

$$\delta\tilde{V}(\theta,J,\tau) = -I\int dJ_1 d\theta_1 K(J,J_1,\theta,\theta_1)\delta F(J_1,\theta_1,\tau), \tag{16}$$

and $K(J,J_1,\theta,\theta_1) = S(x(J,\theta) - x(J_1,\theta_1))$.

We note that equations (15) and (16) are exact because we did not make any approximation in the derivation above.

3. Linear Theory

In linear theory, the last two terms on the left hand side in Eqs. (15) must be discarded. We assume that the perturbation of the distribution function oscillates with the frequency ω,

$$\delta F = f_1(J,\theta)e^{-i\omega\tau} + \text{c.c.}, \tag{17}$$

where for the sake of brevity we use the notation "c.c." to denote a complex conjugate of the first term.

The perturbation of the potential $\tilde{V}$ is

$$\tilde{V} = Ve^{-i\omega\tau} + \text{c.c.}. \tag{18}$$

Since V is a periodic function of θ, we can expand it in Fourier series,

$$V = \sum_{n=-\infty}^{\infty} v_n(J)e^{in\theta}. \tag{19}$$

For simplicity we will neglect here the effect of the synchrotron damping in the linear theory by dropping the R-term in Eq. (15). This greatly simplifies the linear analysis and is usually assumed in the literature. However, as we will see in Section 7, the effect of the synchrotron damping is crucial for the nonlinear stage of the instability and will later be included in the derivation of the nonlinear equations.

Substituting Eqs. (17) and (19) into Eq. (15) gives in linear approximation

$$-i\omega f_1 + \omega_s \frac{\partial f_1}{\partial \theta} = F_0' \sum_{n=-\infty}^{\infty} inv_n(J)e^{in\theta}, \tag{20}$$

where $F_0' = \partial F_0/\partial J$. A solution to Eq. (20) is

$$f_1 = -F_0' \sum_{n=-\infty}^{\infty} \frac{n v_n(J)}{\omega - n\omega_s} e^{in\theta} . \tag{21}$$

Now, substituting this equation into Eq. (16) yields an infinite set of integral equations that determine eigenfrequencies and eigenfunctions for the collective oscillations of the bunch:

$$v_n(J) = I \sum_{m=-\infty}^{\infty} \int_0^{\infty} dJ_1 F_0'(J_1) K_{nm}(J, J_1) \frac{m v_m(J_1)}{\omega - m\omega_s(J_1)}, \tag{22}$$

with the kernel given by

$$K_{nm}(J, J_1) = \frac{1}{2\pi} \int_0^{2\pi}\int_0^{2\pi} d\theta d\theta_1 e^{i(m\theta_1 - n\theta)} K(J, \theta, J_1, \theta_1). \tag{23}$$

The integral on the right hand side of Eq. (22) defines an analytical function in the upper half plane of the complex variable ω; for $\mathrm{Im}\,\omega \le 0$ the integral must be analytically continued into the lower half plane. For a real value of ω, the integration is performed along a contour in the complex plane which bypasses the singular points of the integrand below the pole (see, e.g., [12]). The residues of the integral (22) are associated with the Landau damping effect.

4. Nonlinear Theory

Let us assume that the instability has a threshold corresponding to a critical value of the parameter $I = I_c$ with the frequency at the threshold $\omega = \omega_c$ ($\mathrm{Im}\,\omega_c = 0$). We will be interested in the analysis of the nonlinear phase of the instability in the vicinity of the threshold when the growth rate of the instability, Γ, is much smaller than ω_c, $\Gamma \ll \omega_c$. In other words, we assume that the instability is weak and develops on a time scale which is much larger then the period of the oscillations. It turns out that in this case one can separate a "slow" time scale on which the amplitude evolves from "fast" oscillations with the frequency ω_c and derive nonlinear equations for the evolution of the amplitude of the instability by averaging over ω_c. In this section we will give a general description of the approach following a similar analysis in the theory of nonlinear plasma oscillations [15].

First, we rewrite the result of the previous section in a concise form,

$$\hat{L}(\omega, I) V_\omega = 0, \tag{24}$$

where the linear operator $\hat{L}$ represents a set of integral equations (22) and (23),

$$\hat{L}(\omega, I) V_\omega \equiv \sum_{n=-\infty}^{\infty} e^{in\theta} \left[v_n(J) - I \sum_{m=-\infty}^{\infty} \int dJ_1 d\theta_1 K F_0'(J_1) \frac{n v_n(J_1)}{\omega - n\omega_s(J_1)} \right], \tag{25}$$

and V_ω is a Fourier harmonic of the function $\tilde{V}$ corresponding to the frequency ω, $V_\omega = \sum v_n(J) e^{in\theta}$. Note that at this point we can also include in $\hat{L}$ a contribution from the Fokker-Planck term R. A particular form of the operator $\hat{L}$ is not essential for the analysis in this Section.

The frequency of the oscillations ω_c at the threshold and the corresponding eigenfunction $V_{\omega_c} \equiv u_c$ are determined by the equation

$$\hat{L}(\omega_c, I_c) u_c = 0. \tag{26}$$

We now need to define a scalar product of two functions u and w of the phase space variables J, θ. Let us denote this product by (u,w). Usually, scalar multiplication in Hilbert space is given in terms of an integration of the product uw^* over J and θ with some weight function. The exact choice of the weight function is not important for what follows, and we do not specify it here. For a given scalar product, we can define an operator $\hat{L}^+$ conjugate to $\hat{L}$ satisfying the following condition for two arbitrary functions u and w,

$$\left(u, \hat{L}w\right) = \left(w, \hat{L}^+ u\right). \tag{27}$$

We will assume that the operator $\hat{L}^+$ is known and together with the solution of Eq. (26) the solution w_c of the conjugate problem

$$\hat{L}^+(\omega_c, I_c) w_c = 0 \tag{28}$$

is available. Note that solution of Eq. (28) represents a linear problem and in each particular case can be accomplished by standard methods of numerical analysis.

We now consider a situation when I slightly exceeds the threshold, $I = I_c + \Delta I$, with $\Delta I \ll I_c$. Taking into account nonlinear terms in the Vlasov equation we will assume that they are much smaller than the linear ones. That is to say, we are expecting that the instability, after initial exponential growth, will eventually saturate at a level where the amplitude of the oscillations is relatively small. If this is not a case, and the instability evolves to a highly nonlinear regime, our theory will only be applicable for a relatively short period of time following the linear growth. Fortunately, as we will see in Section 7, in many cases the damping associated with synchrotron radiation indeed limits the growth of the instability, and the whole process is described within a framework of a weakly nonlinear approximation.

With nonlinear terms, the equation for the V_ω can now be written as

$$\hat{L}(\omega, I) V_\omega = \hat{N}_\omega, \tag{29}$$

where $\hat{N}_\omega$ is a Fourier transform of the nonlinear term neglected in the linear analysis. The operator $\hat{N}_\omega$ depends on the parameter I, and acts on the function V_ω.

Following a general prescription of nonlinear theory of oscillations [15], we will assume the following type of solution (in time representation) for Eq. (29)

$$\tilde{V} = \left[A(\tau) u_c e^{-i\omega_c \tau} + \text{c.c.}\right] + \Delta V(J, \theta, \tau). \tag{30}$$

where $|Au_c| \gg |\Delta V|$. The first term in Eq. (30) describes oscillations with the eigenfunction u_c, frequency ω_c and varying amplitude $A(\tau)$, and the second term is a correction due to the deviation of the exact eigenfunction from u_c. It is important to emphasize here that

$A(\tau)$ is supposed to be a slow function of time, $|\partial \ln A/\partial\tau| \ll \omega_c$. It also means that the spectrum A_ω of the function $A(\tau)$ is represented by a narrow peak (the width of the peak is much smaller than ω_c) localized near the zero frequency.

We now need to make a Fourier transform of Eq. (30) and substitute it into Eq. (29). Since we are interested in the frequency range close to ω_c, an approximate relation holds:

$$\tilde{V}_\omega \approx A_{\omega-\omega_c} u_c + \Delta V_\omega. \tag{31}$$

In Eq. (31) we neglected the term containing $A_{\omega+\omega_c}$ which is peaked around $\omega = -\omega_c$. Eq. (29) now reads

$$\hat{L}(\omega, I_c + \Delta I)\left(A_{\omega-\omega_c} u_c + \Delta V_\omega\right) = \hat{N}_\omega. \tag{32}$$

Making a Taylor expansion of the linear part and neglecting the product $\Delta I \Delta V_\omega$ one finds

$$\hat{L}(\omega_c, I_c)\Delta V_\omega + (\omega - \omega_c)A_{\omega-\omega_c}\frac{\partial \hat{L}}{\partial\omega}u_c + \Delta I A_{\omega-\omega_c}\frac{\partial \hat{L}}{\partial I}u_c = \hat{N}_\omega, \tag{33}$$

where the derivatives of the operator $\hat{L}$ are evaluated at $\omega = \omega_c$, $I = I_c$. We can annihilate the first term in Eq. (33) by a scalar multiplication with w_c and using Eqs. (27) and (28). The result is

$$(\omega - \omega_c)A_{\omega-\omega_c}\left(w_c, \frac{\partial \hat{L}}{\partial\omega}u_c\right) + \Delta I A_{\omega-\omega_c}\left(w_c, \frac{\partial \hat{L}}{\partial I}u_c\right) = \left(w_c, \hat{N}_\omega\right). \tag{34}$$

We now multiply Eq. (34) by $e^{i\omega_c\tau}$ and make an inverse Fourier transform to time τ:

$$\frac{\partial A}{\partial\tau} + i\Delta\omega A = -ie^{i\omega_c\tau}\left(w_c, \hat{N}\right)\left(w_c, \frac{\partial \hat{L}}{\partial\omega}u_c\right)^{-1}, \tag{35}$$

where

$$\Delta\omega = -\Delta I\left(w_c, \frac{\partial \hat{L}}{\partial I}u_c\right)\left(w_c, \frac{\partial \hat{L}}{\partial\omega}u_c\right)^{-1} \tag{36}$$

is a linear frequency shift due to the change of I. Note that in Eq. (35), after the inverse Fourier transform, N represents a function of time rather than ω.

Without the right hand side it follows from Eq. (35) that the amplitude A will vary with time as $\exp(-i\Delta\omega\tau)$ which is a trivial consequence of the fact that in linear theory $V \propto \exp(-i\omega\tau)$ with $\omega = \omega_c + \Delta\omega$. In the next section we will find the nonlinear term averaged over fast oscillations which adds nonlinear dynamics to Eq. (35).

5. Derivation of Nonlinear Equations

The nonlinear terms in our problem arise from the last term in kinetic equation (15). We need to approximately solve this equation and find N in Eq. (29). In order to simplify the derivation, we first consider the case when $R=0$. In the next section a generalization for $R\neq 0$ will be given.

Since nonlinear term is assumed to be small, it will be accurate enough to neglect ΔV term in its evaluation. Hence, $\tilde{V}\approx A(\tau)u_c e^{-i\omega_c\tau}+\text{c.c.}$ where u_c is decomposed into Fourier series over θ,

$$u_c=\sum_{n=-\infty}^{\infty}u_n(J)e^{in\theta}. \tag{37}$$

We will also represent the perturbation of the distribution function δF as

$$\delta F=\left[f_1(J,\theta,\tau)e^{-i\omega_c\tau}+\text{c.c.}\right]+f_0(J,\theta,\tau)+\left[f_2(J,\theta,\tau)e^{-2i\omega_c\tau}+\text{c.c.}\right], \tag{38}$$

where f_0, f_1 and f_2 are slow functions of time (as $A(\tau)$) in the sense that $\partial/\partial t \ll \omega_c$.

Substituting Eq. (38) into Eq. (15) we note that, as calculations show, the main contribution comes from the resonant terms in δF that are differentiated with respect to J. This allows us to neglect the last term in Eq. (15) to obtain

$$\frac{\partial f_0}{\partial\tau}+\omega_s\frac{\partial f_0}{\partial\theta}-\frac{\partial\tilde{V}}{\partial\theta}\frac{\partial f_1^*}{\partial J}-\frac{\partial\tilde{V}^*}{\partial\theta}\frac{\partial f_1}{\partial J}=0, \tag{39}$$

$$\frac{\partial f_2}{\partial\tau}-2i\omega f_2+\omega_s\frac{\partial f_2}{\partial\theta}-\frac{\partial\tilde{V}}{\partial\theta}\frac{\partial f_1}{\partial J}=0, \tag{40}$$

$$\frac{\partial f_1}{\partial\tau}-i\omega f_1+\omega_s\frac{\partial f_1}{\partial\theta}-\frac{\partial\tilde{V}}{\partial\theta}\frac{\partial F_0}{\partial J}-\frac{\partial\tilde{V}}{\partial\theta}\frac{\partial f_0}{\partial J}-\frac{\partial\tilde{V}^*}{\partial\theta}\frac{\partial f_2}{\partial J}=0, \tag{41}$$

where the asterisk indicates complex conjugating. The last two terms in Eq. (41) imply that we can split the function f_1 into linear (*L*) and nonlinear (*NL*) parts,

$$f_1=f_1^L+f_1^{NL}, \tag{42}$$

where f_1^L satisfies the equation of linear theory,

$$\frac{\partial f_1^L}{\partial\tau}-i\omega f_1^L+\omega_s\frac{\partial f_1^L}{\partial\theta}-\frac{\partial\tilde{V}}{\partial\theta}\frac{\partial F_0}{\partial J}=0, \tag{43}$$

and f_1^{NL} is the nonlinear correction arising from the higher order terms in the kinetic equation,

$$\frac{\partial f_1^{NL}}{\partial\tau}-i\omega f_1^{NL}+\omega_s\frac{\partial f_1^{NL}}{\partial\theta}-\frac{\partial\tilde{V}}{\partial\theta}\frac{\partial f_0}{\partial J}-\frac{\partial\tilde{V}^*}{\partial\theta}\frac{\partial f_2}{\partial J}=0. \tag{44}$$

In equations for f_0 and f_2 we can substitute f_1^L for f_1.

Let us consider first Eq. (43) for the linear part of the distribution function. This is in fact the same equation as Eq. (20), however, we now want to find its solution in time domain rather than in frequency domain. We expand f_1^L in Fourier series in θ,

$$f_1^L = \sum_{n=-\infty}^{\infty} g_n(J,\tau) F_0' e^{in\theta}, \qquad (45)$$

and find from Eq. (43) an equation for g_n,

$$\frac{\partial g_n}{\partial \tau} - i(\omega - n\omega_s) g_n = inA(\tau) u_n. \qquad (46)$$

This equation can be easily solved,

$$g_n = inu_n(J) \int_0^{\tau} A(\tau_1) e^{i(\tau-\tau_1)(\omega - n\omega_s)} d\tau_1. \qquad (47)$$

We now consider equation (39) for f_0. The dominant terms in this equation will be those that do not depend on θ; nonzero n terms will cause only small oscillations in f_0 at the frequency $n\omega_s$, without systematic changing of its amplitude. Keeping only $n=0$ terms we have

$$\frac{\partial f_0}{\partial \tau} = \sum_{n=-\infty}^{\infty} inAu_n \frac{\partial}{\partial J} g_n^* \frac{\partial F_0}{\partial J} + \text{c.c.}. \qquad (48)$$

When differentiating with respect to J in Eq. (48), it is sufficient to differentiate the exponential term $\exp[i(\tau-\tau_1)(\omega - n\omega_s(J))]$ in the solution (47) only; all other terms will be relatively small because we assume that the time scale on which the nonlinear effects become essential is such that $\tau\omega_s \gg 1$,

$$\frac{\partial f_0}{\partial \tau} = \sum_{n=-\infty}^{\infty} in^3 A(\tau) u_n u_n^* \omega_s' F_0' \int_0^{\tau} d\tau_1 (\tau - \tau_1) A^*(\tau_1) e^{-i(\tau-\tau_1)(\omega - n\omega_s)} + \text{c.c.}. \qquad (49)$$

Now we can integrate this equation, yielding

$$f_0 = 2\,\mathrm{Re} \sum_{n=-\infty}^{\infty} in^3 \omega_s' F_0' u_n u_n^* \int_0^{\tau} d\tau_1 A(\tau_1) \int_0^{\tau_1} d\tau_2 (\tau_1 - \tau_2) A^*(\tau_2) e^{-i(\tau_1-\tau_2)(\omega - n\omega_s)}. \qquad (50)$$

In a similar fashion the following equation can be obtained for f_2,

$$\frac{\partial f_2}{\partial \tau} - 2i\omega f_2 + \omega_s \frac{\partial f_2}{\partial \theta} = \sum_{n=-\infty}^{\infty} in^3 A(\tau) u_n^2 \omega_s' F_0' e^{2in\theta} \int_0^{\tau} d\tau_1 A(\tau_1) e^{-i(\tau-\tau_1)(\omega - n\omega_s)}, \qquad (51)$$

with the solution

$$f_2 = \sum_{n=-\infty}^{\infty} in^3 \omega_s' F_0' u_n^2 e^{2in\theta} \int_0^{\tau} d\tau_1 A(\tau_1) \int_0^{\tau_1} d\tau_2 (\tau_1 - \tau_2) A(\tau_2) e^{-i(2\tau-\tau_1-\tau_2)(\omega - n\omega_s)}. \qquad (52)$$

We now have to substitute f_0 and f_2 into Eq. (44). As calculations show, the leading contribution to f_1^{NL} comes from f_0; nonlinear terms arising from f_2 turns out to be small in parameter Γ/ω_s. Keeping only f_0 and performing differentiation with respect to J in the exponential terms only we find

$$\begin{aligned} &\frac{\partial f_1^{NL}}{\partial \tau} - i\omega f_1^{NL} + \omega_s \frac{\partial f_1^{NL}}{\partial \theta} = \frac{\partial \tilde{V}}{\partial \theta}\frac{\partial f_0}{\partial J} \\ &= -\sum_{n=-\infty}^{\infty} in^5 (\omega_s')^2 u_n u_n^* F_0' e^{in\theta} 2\,\mathrm{Re}\int_0^{\tau} d\tau_1 A(\tau_1) \int_0^{\tau_1} d\tau_2 (\tau_1-\tau_2)^2 A^*(\tau_2) e^{-i(\tau_1-\tau_2)(\omega-n\omega_s)}, \end{aligned} \tag{53}$$

with the solution

$$\begin{aligned} f_1^{NL} &= -\sum_{n=-\infty}^{\infty} in^5 (\omega_s')^2 u_n^* u_n^2 F_0' \int_0^{\tau} d\tau_1 A(\tau_1) \\ &\times 2\,\mathrm{Re}\int_0^{\tau_1} d\tau_2 A(\tau_2) \int_0^{\tau_2} d\tau_3 (\tau_2-\tau_3)^2 A^*(\tau_3) e^{i(\tau-\tau_1-\tau_2+\tau_3)(\omega-n\omega_s)}. \end{aligned} \tag{54}$$

Finally, since time τ is supposed to be much larger than ω_s^{-1}, one can use the following mathematical identity when integrating over τ,

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{ixy} = 2\pi\delta(x)\delta(y), \tag{55}$$

which in application to Eq. (54) after changing the variables $\sigma = \tau_1 - \tau_2$, $\zeta = \tau - \tau_1$, yields

$$\begin{aligned} f_1^{NL} &= -2\pi i \sum_{n=-\infty}^{\infty} n^5 (\omega_s')^2 \delta(\omega - n\omega_s) u_n^* u_n^2 e^{in\theta} \\ &\times \frac{\partial F_0}{\partial J} \int_0^{\tau/2} d\zeta A(\tau-\zeta)\zeta^2 \int_0^{\tau-2\zeta} d\sigma A(\tau-\zeta-\sigma) A^*(\tau-2\zeta-\sigma). \end{aligned} \tag{56}$$

We have found a nonlinear part of the perturbation of the distribution function f_1^{NL}. As we see, this function is proportional to the third order of the amplitude A. On the linear stage of the instability, when A is small, f_1^{NL} can be neglected, however as A grows, the nonlinear term becomes more important and eventually competes with the linear part f_1^L. Notice also, that due to the presence of delta-function $\delta(\omega - n\omega_s)$, the nonlinear term is peaked at the resonant values of the action J_n, such that $n\omega_s(J_n) = \omega$.

6. Effect of Synchrotron Damping and Nonlinear Equation for the Amplitude

In the previous section we neglected the effect of the synchrotron radiation in the Vlasov equation. To include the R-term we need to transform it first to $J-\theta$ variables. In doing so we notice that, because of strong localization near the resonant values J_n of the perturbed distribution function, the leading term in R will be the one containing the second derivative with respect to J. In other words, the most important effect of the synchrotron radiation will be the quantum diffusion of particles in the phase space rather than energy loss. Keeping only the second derivative in R gives

$$R = D(J)\frac{\partial^2 F}{\partial J^2}. \tag{57}$$

The diffusion coefficient D was found in Ref. [14] and equals

$$D(J) = J\gamma_D / \omega_s(J). \tag{58}$$

The derivation of f_1^{NL} given in Section 5 can now be repeated with the diffusion term R on the right hand side of the Vlasov equation. For the sake of brevity we will omit this derivation here referring the reader to Ref. [16] where a similar problem was worked out for nonlinear plasma oscillations problem. In our case, the inclusion of the diffusion reduces formally to appearing of a exponential factor in the integrand of Eq. (56),

$$\begin{aligned} f_1^{NL} &= -2\pi i \sum_{n=-\infty}^{\infty} n^5 (\omega_s')^2 \delta(\omega - n\omega_s) u_n^* u_n^2 e^{in\theta} \\ &\times \frac{\partial F_0}{\partial J} \int_0^{\tau/2} d\zeta A(\tau-\zeta)\zeta^2 \int_0^{\tau-2\zeta} d\sigma A(\tau-\zeta-\sigma) A^*(\tau-2\zeta-\sigma) e^{-B_n\zeta^2\left(\sigma+\frac{2}{3}\zeta\right)} \end{aligned} \tag{59}$$

where $B_n = n^2(\omega_s')^2 D(J_n)$, and J_n is the value of the action at the n-th resonance, $n\omega_s(J_n) = \omega$.

We are now in position to find the nonlinear term $\hat{N}$ in Eq. (35). Since it will be multiplied by $\exp(i\omega_c\tau)$, we need a component in $\hat{N}$ that oscillates as $\exp(-i\omega_c\tau)$, so that the right hand side in Eq. (35) would be a slowly varying function of time. From Eq. (16) and Eq. (38) we see that such a term in $\tilde{V}$ is

$$\delta\tilde{V} = -I_c e^{-i\omega_c\tau} \int dJ_1 d\theta_1 K(J, J_1, \theta, \theta_1)\left(f_1^L(J_1,\theta_1,\tau) + f_1^{NL}(J_1,\theta_1,\tau)\right), \tag{60}$$

which gives for $\hat{N}$

$$N = -I_c e^{-i\omega_c\tau} \int dJ_1 d\theta_1 K(J, J_1, \theta, \theta_1) f_1^{NL}(J_1,\theta_1,\tau). \tag{61}$$

With this expression, the right hand side of Eq. (35) becomes

$$\begin{aligned} &2\pi I_c \sum_{n=-\infty}^{\infty} K_n(J_n) F' n^4 |\omega_s'| \\ &\quad\times \int_0^{\tau/2} d\zeta A(\tau-\zeta)\zeta^2 \int_0^{\tau-2\zeta} d\sigma A(\tau-\zeta-\sigma) A^*(\tau-2\zeta-\sigma) e^{-B_n\zeta^2\left(\sigma+\frac{2}{3}\zeta\right)}, \end{aligned} \tag{62}$$

where

$$K_n(J) = \left(w_c, \frac{\partial \hat{L}}{\partial \omega} u_c\right)^{-1} \int d\theta e^{in\theta} (w_c, K) u_n^* u_n^2, \tag{63}$$

and the scalar product $\left(w_c, K\right)$ in Eq. (63) is performed with respect to variables J and θ in $K\left(J, J_1, \theta, \theta_1\right)$.

To further simplify the analysis we will assume here that only one term dominates in the sum of Eq. (62). This assumption is correct if the variation of the frequency $\omega_s(J)$ within a distribution function is not very large so that equation $n\omega_s\left(J_n\right)=\omega_c$ has a solution only for one value of n. Omitting the sum sign in Eq. (62) gives the following nonlinear equation for the amplitude A,

$$\begin{aligned}\frac{\partial A}{\partial \tau}+i\Delta\omega A &= 2\pi I_0 K_n\left(J_n\right)F_0' n^4\left|\omega_s'\right| \\ &\times \int_0^{\tau/2} d\zeta A(\tau-\zeta)\zeta^2 \int_0^{\tau-2\zeta} d\sigma A(\tau-\zeta-\sigma)A^*(\tau-2\zeta-\sigma)e^{-B_n\zeta^2\left(\sigma+\frac{2}{3}\zeta\right)}.\end{aligned} \tag{64}$$

In this form, Eq. (64) contains two complex and one real parameters. For numerical solution it is convenient to reduce the number of the parameters by choosing new variables. First, we denote the real part of the coherent frequency shift by Ω, $\Delta\omega=\Omega+i\Gamma$, and introduce the absolute value ρ and the phase ϕ of the complex factor in front of the integral so that $2\pi I_0 K_n\left(J_n\right)F_0' n^4\left|\omega_s'\right|=-\rho e^{i\phi}$. With new variables

$$a=A\frac{\sqrt{\rho}}{B_n^{5/6}}e^{i\Omega\tau},\ g=\frac{\Gamma}{B_n^{1/3}},\ \xi=B_n^{1/3}\tau, \tag{65}$$

equation (64) becomes

$$\frac{\partial a}{\partial \xi}-ga=-e^{i\phi}\int_0^{\xi/2} d\zeta a(\xi-\zeta)\zeta^2\int_0^{\xi-2\zeta} d\sigma a(\xi-\zeta-\sigma)a^*(\xi-2\zeta-\sigma)e^{-\zeta^2\left(\sigma+\frac{2}{3}\zeta\right)}. \tag{66}$$

The parameter g here plays a role of dimensionless growth rate of the instability that is measured in time units related to the synchrotron damping rate. Note that now Eq. (66) contains only two real parameters, g and ϕ.

7. Analysis and Solutions of Nonlinear Equation

A complete analysis of nonlinear dynamics of the instability in any particular case requires computing of the coefficients in Eq. (66) which can only be done based on the solution of the linear problem described in Section 3. In the general case, this constitutes a major computational task, which lies beyond the scope of the present paper. Rather than trying to find a particular solution to nonlinear problem for a given set of beam parameters we will outline here possible scenarios by numerically solving Eq. (66) for different values of g and ϕ.

First, note that equation (66) admits an asymptotic solution in the form of $a=\mathrm{const}\times\exp(i\lambda\xi)$ that corresponds to oscillations with a constant amplitude and a coherent frequency shift λ. This solution is valid in the limit $\xi\to\infty$ and exists only if $|\phi|<\pi/2$. It is given by the following formula that can be easily verified by direct substitution into Eq. (66),

$$a = 18^{1/6} g^{1/2} \frac{1}{\sqrt{\Gamma(1/3)\cos\phi}} e^{-i\xi\tan\phi}, \qquad (67)$$

where $\Gamma(1/3)$ stands for the gamma function. According to this solution, the steady state amplitude $|a|$ increases in proportion to the square root of the dimensionless growth rate, $g^{1/2}$. It turns out however, that this solution is only stable for relatively small values of the parameter g [10].

We have solved numerically Eq. (66) for several sets of g and ϕ. The results are presented in Figs. 1 – 3.

In Fig. 1 we show solutions for $\phi = 0$ and various values of g starting with a sufficiently small value of a so that initially the nonlinear term is unimportant. For small values of g, $g < 0.4$, we see that the solution, after initial exponential growth, reaches the equilibrium after several oscillations. With increasing g, the oscillations become more pronounced, and finally at $g = 0.48$ a steady state solution with periodic oscillating amplitude sets up. Further increasing g beyond the value of 0.5 causes the period of those oscillations to break up which, after initial transient period, results in a relaxation-type behavior of the amplitude . For even larger g, $g > 0.8$, the nonlinear term cannot stabilize the system any more and the amplitude starts to grow without limit.

Fig. 2 shows solutions for $\phi = \pi/4$. In this case the amplitude oscillations appear to be less stable and runaway solution develops already for $g = 0.5$.

As was mentioned above, a stable asymptotic solution exists only if $|\phi| < \pi/2$. Numerical solutions indicate that for $|\phi| > \pi/2$ all the solutions diverge with unlimited growth as $\xi \to \infty$. An example of such a solution is shown in Fig. 3. We see that, in this case, the nonlinear term cannot stop the instability whose amplitude continues to grow and eventually goes beyond the limit of applicability of the present theory.

8. Conclusion

In this paper we applied the theory of weakly nonlinear unstable oscillations to the case of a single bunch instability in circular accelerators. We derived an equation which describes evolution of the amplitude of the instability and depends only on two dimensionless parameters – a normalized linear growth rate of the instability g, and a phase of nonlinear term ϕ. We found that for small values of ϕ the nonlinear term has a stabilizing effect and, for not very large values of g, results in the saturation of the instability at some level. Larger values of g lead to relaxation type oscillations of the amplitude. In the case of $\phi > \pi/2$, within the limits of the applicability of our theory, the nonlinear term does not prevent the growth of the amplitude.

As was mentioned before, a complete comparison of our theory with the experiment requires solution of equations of the linear theory and determination of the parameters in the nonlinear equation. Due to computational complexity of this problem we did not attempt to solve it in this paper. However, even without knowing the exact parameters, we can try to compare different patterns of the signal that have been measured in the experiment with solutions obtained in the theory. In such a comparison we only pay attention to qualitative behavior of the amplitude such as growth, oscillation and saturation at some level.

Even visual comparison of the instability signal from Ref. [2] shows a clear resemblance to our curves. In one case (Fig. 5 of Ref. [2]), after injection in the ring, the amplitude of signal from spectrum analyzer tuned to a sideband frequency began to grow monotonically and after some time of the order of synchrotron damping time saturated at approximately constant level. This situation is very similar to our Fig. 1a. In another case (Fig. 4 of Ref. [2]), oscillations with decreasing amplitude were observed, which can be identified with Fig. 1a or 1b. In later measurements [17], amplitude oscillations with approximately constant modulation were measured. This situation reminds our Fig. 1e. Unfortunately, at this time we are not able to compare with the experiment theoretical predictions for the period of the nonlinear oscillation, although preliminary crude estimates indicate they are about of the same order.

In conclusion, our theory shows qualitative agreement with the signals observed in the SLC Damping Ring single bunch instability. Further work is planned to make a more definite comparison of the theory and the experiment.

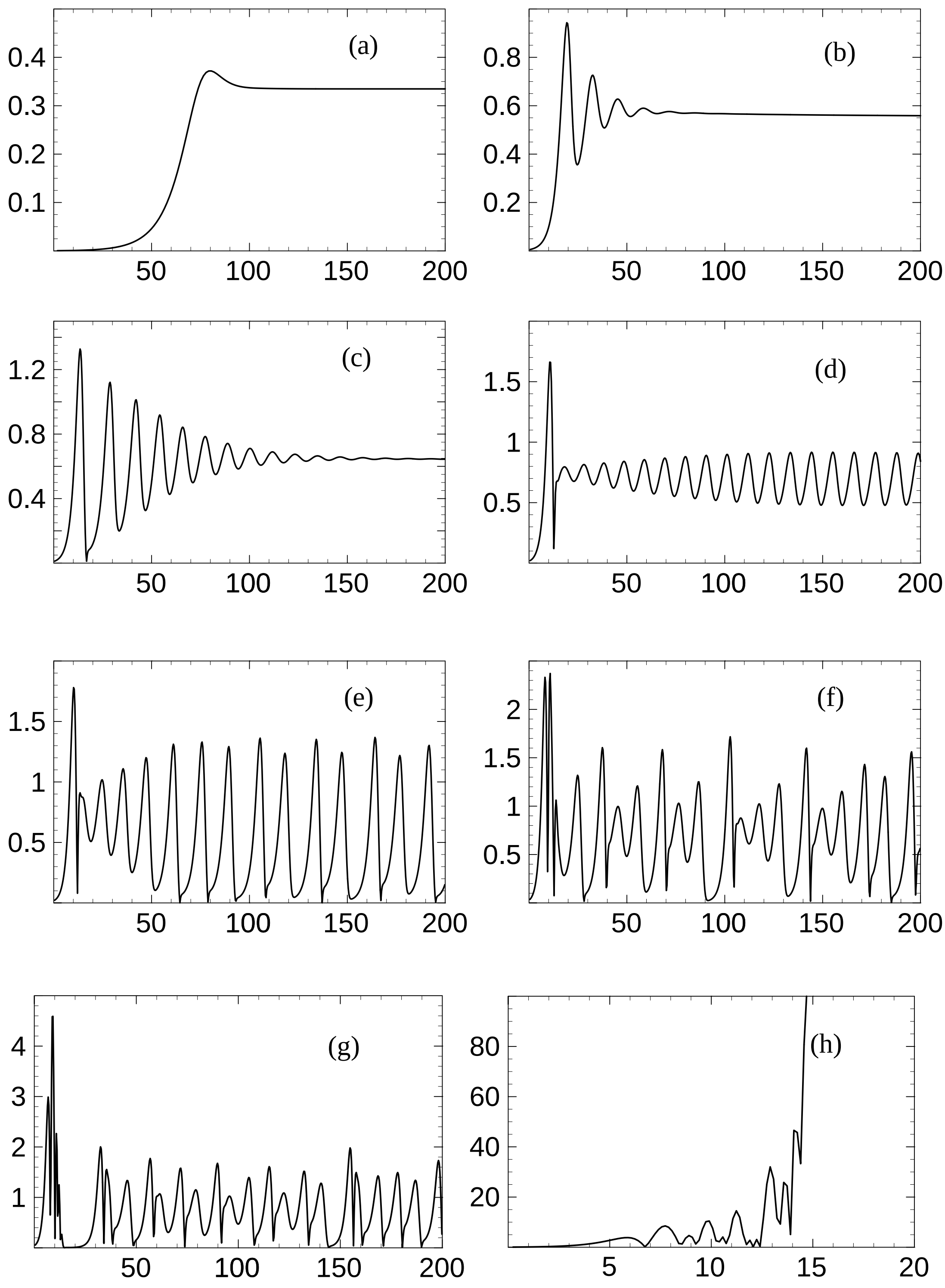

Fig. 1. Plots of the absolute value of the amplitude, $|a|$, versus time ξ for $\phi = 0$. a – $g = 0.1$, b – $g = 0.3$, c – $g = 0.4$, d – $g = 0.48$, e – $g = 0.5$, f – $g = 0.6$, g – $g = 0.7$, h – $g = 0.8$.

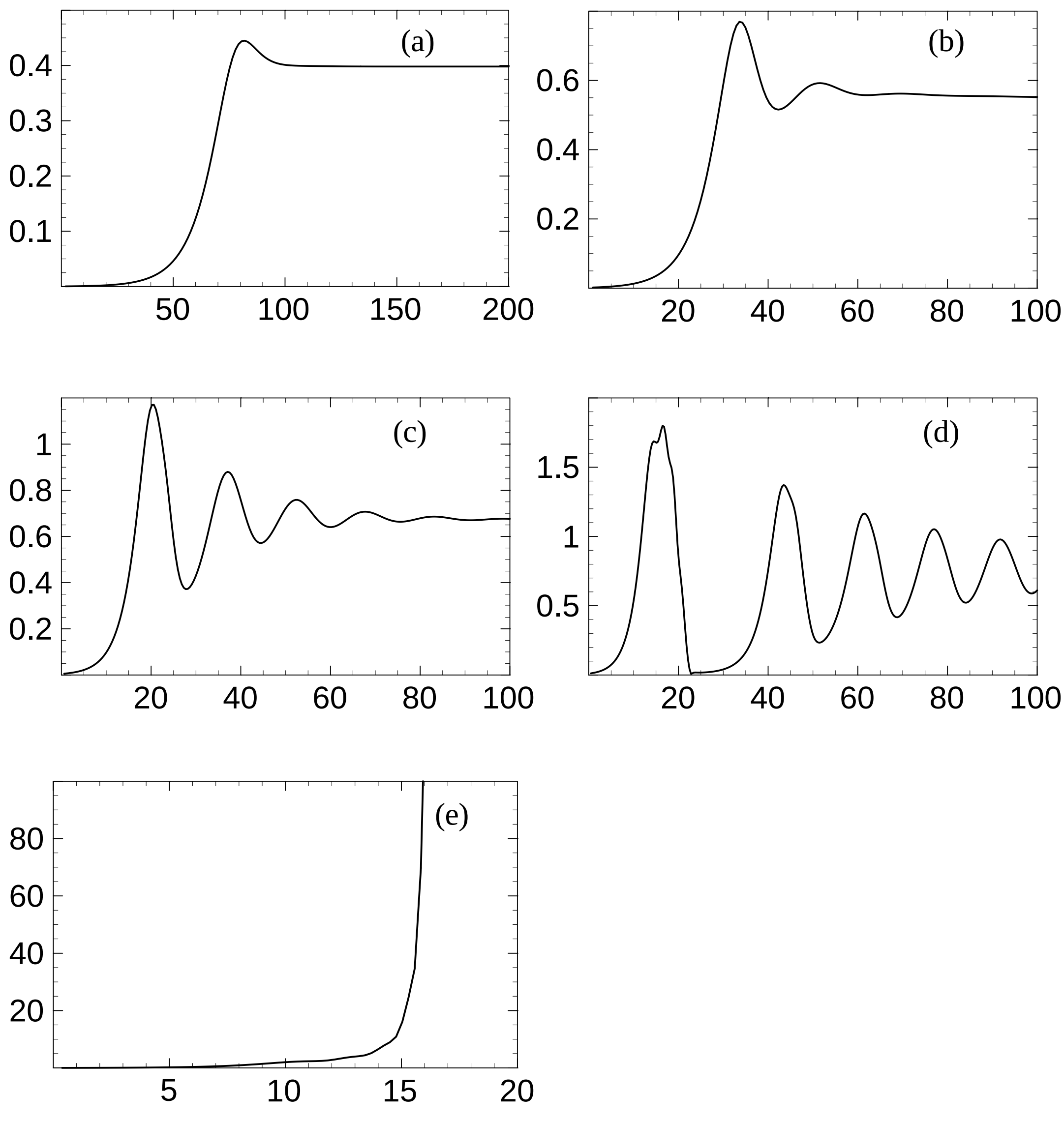

Fig. 2. Plots of the absolute value of the amplitude, $|a|$, versus time ξ for $\phi = \pi/4$. a – $g = 0.1$, b – $g = 0.2$, c – $g = 0.3$, d – $g = 0.4$, e – $g = 0.5$.

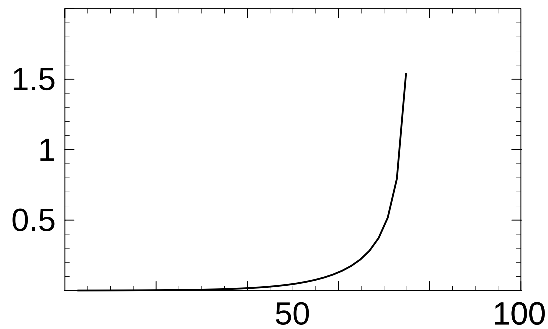

Fig. 3. Plot of the absolute value of the amplitude, $|a|$, versus time ξ for $\phi = \pi$ and $g = 0.1$.

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