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One-Loop Self-Dual and N=4 Super Yang-Mills

Zvi Bern[♯]
Department of Physics, UCLA, Los Angeles, CA 90095, USA
bern@physics.ucla.edu

Lance Dixon[⋆]
Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA
lance@slac.stanford.edu

David C. Dunbar[†]
Department of Physics, University of Wales Swansea, Swansea, SA2 8PP, UK
d.c.dunbar@swan.ac.uk

and

David A. Kosower[‡]
Service de Physique Théorique, Centre d'Etudes de Saclay
F-91191 Gif-sur-Yvette cedex, France
kosower@spht.saclay.cea.fr

Abstract

We conjecture a simple relationship between the one-loop maximally helicity violating gluon amplitudes of ordinary QCD (all helicities identical) and those of $N = 4$ supersymmetric Yang-Mills (all but two helicities identical). Because the amplitudes in self-dual Yang Mills have been shown to be the same as the maximally helicity violating ones in QCD, this conjecture implies that they are also related to the maximally helicity violating ones of $N = 4$ supersymmetric Yang-Mills. We have an explicit proof of the relation up to the six-point amplitude; for amplitudes with more external legs, it remains a conjecture. A similar conjecture relates amplitudes in self-dual gravity to maximally helicity violating $N = 8$ supergravity amplitudes.

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[†]Research supported by PPARC, the Leverhulme trust and EEC contract ERBCHRXCT920069.
[‡]Laboratory of the *Direction des Sciences de la Matière* of the *Commissariat à l'Energie Atomique* of France.

1. Introduction

The development of sophisticated techniques [1] for computing one-loop helicity amplitudes in four-dimensional gauge theories has allowed various workers to obtain explicit expressions for a number of infinite sequences of such amplitudes [2,3,4,5]. In particular, the nonvanishing maximally helicity violating (MHV) one-loop gluon amplitudes in QCD (where all helicities are identical) and in $N = 4$ supersymmetric Yang-Mills (where all but two helicities are identical) are remarkably simple. This suggests that they may possess an additional symmetry beyond the gauge symmetry.

At tree level, Nair [6] has observed that the MHV n-gluon amplitudes [7] (also known as Parke-Taylor amplitudes) may be derived from a free-fermion Wess-Zumino-Witten model which contains an infinite-dimensional symmetry algebra. (The construction was actually for an $N = 4$ supersymmetric gauge theory, but the superpartners do not contribute at tree level, so the results also apply to ordinary QCD.) Duff and Isham [8], and more recently Bardeen [9], have pointed out that tree-level gluon currents with all identical helicities in ordinary QCD may be obtained from self-dual Yang-Mills. Selivanov has also produced similar results using a different ansatz [10]. Self-dual Yang-Mills is the prototypical integrable model and as such possesses an infinite-dimensional symmetry algebra [11]. In a spacetime of signature $(2, 2)$, it arises from the $N = 2$ string [12].

Recently, Cangemi [13,14] and Chalmers and Siegel [15] showed that a connection between amplitudes in self-dual Yang-Mills and the maximally helicity violating all-plus helicity amplitudes in QCD continues to hold at one-loop. Indeed, the one-loop amplitudes generated by various self-dual Yang-Mills actions [16,17,15] are identical to the QCD all-plus helicity amplitudes. It is intriguing that the action of Chalmers and Siegel leads to a perturbatively solvable theory: the *only* non-vanishing amplitudes in the perturbative expansion are the known all-plus helicity one-loop amplitudes in QCD! Bardeen has suggested that an anomaly in the symmetry algebra determines the structure of these amplitudes [9].

In this paper we examine the relationship between the one-loop MHV amplitudes in $N = 4$ supersymmetric Yang-Mills theory and the all-plus helicity QCD amplitudes (i.e., the self-dual Yang-Mills amplitudes). We conjecture a 'dimension shifting' relationship between the two sets of amplitudes, in which the all-plus amplitudes are given essentially by evaluating the loop integration for the $N = 4$ MHV amplitudes in a dimensions larger by four($D = 8$). We have explicitly verified the conjecture for amplitudes with up to six external legs, and have evidence that it holds for an *arbitrary* number of external legs. A similar conjecture can be made to link the one-loop n-point amplitudes of self-dual gravity [18,15] (the all-plus helicity graviton amplitudes), with MHV amplitudes in $N = 8$ supergravity. We have verified this conjecture for the four-point amplitude. The underlying symmetry responsible for the simplicity of these amplitudes, and their relation to each other, remains to be clarified.

2. Preliminaries

We now review two basic tools necessary to present the conjecture, namely color-ordering and the spinor helicity formalism. Further details may be found in review articles [19,1], whose normalizations and conventions we follow.

One-loop $SU(N_c)$ gauge theory amplitudes can be written in terms of independent color-ordered partial amplitudes multiplied by an associated color structure [20,21]. As a simple example, the decomposition of the four-gluon amplitude (with adjoint particles in the loop) is

$$\begin{aligned} \mathcal{A}_4(\{a_i,k_i,\varepsilon_i\}) = g^4 \sum_{\sigma} N_c \, \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}) A_{4;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \\ + g^4 \sum_{\rho} \mathrm{Tr}(T^{a_{\rho(1)}}T^{a_{\rho(2)}}) \mathrm{Tr}(T^{a_{\rho(3)}}T^{a_{\rho(4)}}) A_{4;3}(\sigma(1),\sigma(2);\sigma(3),\sigma(4)) \,, \end{aligned} \tag{1}$$

where we have abbreviated the arguments of the 'partial amplitudes', $A_{n;j}$, by the labels i of the legs and the T^{a_i} are fundamental representation matrices, normalized so that $\mathrm{Tr}(T^aT^b) = \delta^{ab}$. The ρ and σ permutation sums are over the ones which alter the color trace structure. The structure for any number of legs is similar, with no more than two color traces appearing in each term (at one loop). String theory suggests, and it has been proven in field theory, that the $A_{n;j>1}$ may be obtained from $A_{n;1}$ by an appropriate permutation sum [21,4,22]. Thus, we need only consider the $A_{n;1}$ — they contain the information necessary to reconstruct the full one-loop amplitude, and any identity proven for the $A_{n;1}$ extends automatically to the full amplitude.

The relations we find are for special choices of the external gluon helicities. In the helicity formalism of Xu, Zhang and Chang [23] the gluon polarization vectors are expressed in terms of Weyl spinors $|k^{\pm}\rangle$ as

$$\varepsilon^{+}_{\mu}(k;q) = \frac{\langle q^- |\, \gamma_\mu \,| k^- \rangle}{\sqrt{2}\, \langle q\, k\rangle}\,, \qquad\qquad \varepsilon^{-}_{\mu}(k;q) = \frac{\langle q^+ |\, \gamma_\mu \,| k^+ \rangle}{\sqrt{2}\, [k\, q]}\,, \tag{2}$$

where k is the gluon momentum and q is an arbitrary null 'reference momentum' which drops out of final gauge-invariant amplitudes. The plus and minus labels on the polarization vectors refer to the gluon helicities and we use the notation $\langle ij\rangle \equiv \langle k_i^- | k_j^+ \rangle\,, [ij] \equiv \langle k_i^+ | k_j^- \rangle$. These spinor products are anti-symmetric and satisfy $\langle i\, j\rangle\, [j\, i] = 2k_i\cdot k_j$.

When performing a calculation in dimensional regularization [24] it is convenient to choose a scheme which is compatible with the spinor helicity formalism. We use the four-dimensional helicity scheme [25] which is equivalent at one loop to a helicity form of Siegel's dimensional reduction scheme [26]. The conversion to the standard $\overline{\mathrm{MS}}$ scheme is discussed in refs. [25,27].

3. Previously obtained amplitudes.

The simplest one-loop QCD n-gluon helicity amplitude is the one with all identical helicities [2,3],

$$A_{n;1}^{\rm gluon}(1^+,2^+,\ldots,n^+) \;=\; -\frac{i}{48\pi^2}\sum_{1\le i_1<i_2<i_3<i_4\le n}\frac{{\rm tr}_-[i_1 i_2 i_3 i_4]}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}+\mathcal{O}(\epsilon)\,, \tag{3}$$

where ${\rm tr}_-[i_1 i_2 i_3 i_4]\equiv \frac{1}{2}{\rm tr}[(1-\gamma_5)\not{k}_{i_1}\not{k}_{i_2}\not{k}_{i_3}\not{k}_{i_4}]$ and the label 'gluon' denotes a gluon circulating in the loop. As indicated by the '+' superscripts on the gluon labels, we have chosen the all-plus helicity configuration; the all-minus helicity configuration is related by parity. This amplitude contains no poles in the dimensional regularization parameter $\epsilon=(4-D)/2$; it is both ultraviolet and infrared finite. In a supersymmetric theory identical helicity amplitudes vanish by a supersymmetry identity [28]. This implies that the contribution of a massless adjoint representation Weyl fermion or complex scalar circulating in the loop is the same up to a statistics factor [29,2],

$$A_{n;1}^{\rm scalar}(1^+,2^+,\ldots,n^+) = -A_{n;1}^{\rm fermion}(1^+,2^+,\ldots,n^+) = A_{n;1}^{\rm gluon}(1^+,2^+,\ldots,n^+)\,, \tag{4}$$

where the labels 'scalar' and 'fermion' again refer to the particle circulating in the loop.

The next simplest amplitude is the $N=4$ supersymmetric Yang-Mills MHV amplitude [4],

$$\begin{aligned}&A_{n;1}^{N=4}(1^+,2^+,\ldots,i^-,\ldots,j^-,\ldots n^+)\\ &\quad=\frac{i}{(4\pi)^{2-\epsilon}}\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\sum_{1\le i_1<i_2\le n}\frac{1}{4}{\rm tr}[\not{k}_{i_1}\not{P}_{i_1+1,i_2-1}\not{k}_{i_2}\not{P}_{i_2+1,i_1-1}]\,I_{4:i_1,i_2}^{D=4-2\epsilon}+\mathcal{O}(\epsilon)\,,\end{aligned} \tag{5}$$

where $P_{i,j}=\sum_{m=i}^{j}k_m$ and only legs i and j carry negative helicity. (For notational convenience we take the labels on the legs mod n.) The box integral functions $I_{4:i_1,i_2}^{D=4-2\epsilon}$ are depicted in fig. 1; i_1,i_2 on the integral function label the two diagonally opposite massless legs. The formal definition for the m-point integral functions is

$$I_m^D\equiv i(-1)^{m+1}(4\pi)^{D/2}\int\frac{d^D\ell}{(2\pi)^D}\frac{1}{\ell^2(\ell-K_1)^2\ldots(\ell-\sum_{i=1}^{m-1}K_i)^2}\,, \tag{6}$$

where the K_i are the external momenta for the integral, which are in general sums of adjacent external massless momenta k_i for the amplitude, as indicated in fig. 1. The explicit forms of the box integral functions appearing in eq. (5), evaluated to $\mathcal{O}(\epsilon^0)$ in terms of logarithms and dilogarithms, may be found in the appendices of refs. [30,4].

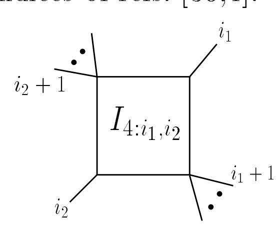

Figure 1. The scalar box integrals appearing in the $N=4$ MHV amplitudes.

The $N=4$ supersymmetric Yang-Mills MHV amplitudes (5) have some features in common with the all-plus helicity QCD amplitudes (3). Neither contains multi-particle poles. The appearance exclusively of two-particle poles is reminiscent of the 'Bethe ansatz' for integrable systems [9]. On the other hand, the $N=4$ supersymmetric Yang-Mills amplitudes contain infrared singularities as well as logarithms and dilogarithms which are not found in the all-plus helicity amplitudes. In this paper we will argue that up to an overall prefactor the two amplitudes are actually the same after an appropriate shift of the dimension D appearing in the loop integrals (6).

In refs. [13,15] it was shown that self-dual Yang-Mills generates the same amplitudes as the all-plus helicity QCD amplitudes. These comparisons were done on the actions and Feynman rules, so that the equivalence holds to all orders of the dimensional regularization parameter, assuming that we are using a form of dimensional regularization that modifies the dimension of the loop momentum [26,25], but preserves the number of physical states to their $D=4$ values. With this type of regularization we can define a simple analytic continuation of self dual-Yang-Mills (whose definition contains the four-dimensional Levi-Civita tensor) in the dimensional regularization parameter.

4. The conjecture.

The basic relationship we conjecture is,

$$A_{n;1}^{\text{gluon}}(1^+,2^+,\ldots,n^+) = -2\epsilon(1-\epsilon)(4\pi)^2 \left[\frac{A_{n;1}^{N=4}(1^+,2^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)}{\langle i\,j\rangle^4}\bigg|_{D\to D+4} \right], \quad (7)$$

where $D=4-2\epsilon$ and the dimension shift on the $N=4$ amplitude takes $\epsilon\to\epsilon-2$ and $I_m^D \to I_m^{D+4}$. It leaves the external momenta and helicities invariant (as well as the explicit prefactor of $\epsilon(1-\epsilon)$).

One can motivate the conjecture in the $\epsilon\to 0$ limit by recognizing that the box integral functions in the $N=4$ supersymmetric expression (5) have a common logarithmic ultraviolet divergence as $D\to 8$, $I_{4:i_1,i_2}^{D=8-2\epsilon} \sim 1/6\epsilon$ as $\epsilon\to 0$, which is canceled by the explicit ϵ on the right-hand-side of (7). One then uses

$$\begin{aligned} \sum_{1\le i_1<i_2\le n} \mathrm{tr}[\not{k}_{i_1} \not{P}_{i_1+1,i_2-1} \not{k}_{i_2} \not{P}_{i_2+1,i_1-1}] &= \sum_{1\le i_1<j<i_2<k\le n} \mathrm{tr}[i_1 j i_2 k] + \sum_{1\le k<i_1<j<i_2\le n} \mathrm{tr}[i_1 j i_2 k] \\ &= 2 \sum_{1\le i_1<i_2<i_3<i_4\le n} \mathrm{tr}[i_1 i_2 i_3 i_4] \end{aligned} \quad (8)$$

to see that that these terms generate the 'even' terms in $A_{n;1}^{\text{scalar}}$ (i.e., those terms obtained by neglecting the γ_5 in $\mathrm{tr}[(1-\gamma_5)\cdots]$ in eq. (3)). On the other hand, one cannot check the 'odd' (γ_5) terms in this way; we shall see (for $n=5,6$) that they come from $\mathcal{O}(\epsilon)$ terms in (5) which are promoted to $\mathcal{O}(\epsilon^0)$ through the dimension shift. In other words, because it involves a shift in ϵ, the dimension shift in (7) only makes sense when the amplitudes are expressed to all orders in ϵ. The previously calculated amplitudes (3) and (5) are valid only through $\mathcal{O}(\epsilon^0)$, so we must inspect the terms higher order in ϵ to fully check the conjecture.

The conjecture (7) may also be reformulated in terms of the loop momentum integration. The D-dimensional integration in eq. (6) may be broken up into four- and (-2ϵ)-dimensional parts, allowing us to define

$$I_m^D[\mu^{2r}] \equiv i(-1)^{m+1}(4\pi)^{D/2} \int \frac{d^4p}{(2\pi)^4} \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \frac{\mu^{2r}}{(p^2-\mu^2)\ldots((p-\sum_{i=1}^{m-1} K_i)^2-\mu^2)}\,, \tag{9}$$

where μ is the (-2ϵ)-dimensional part of the original loop momentum. (We follow the standard prescription that the (-2ϵ)-dimensional subspace is orthogonal to the four-dimensional one.) Explicit evaluation of the (-2ϵ)-dimensional parts of the integrals relates the integrals with powers of μ^2 in the numerator to higher-dimensional integrals (see, for example, appendix A.2 of ref. [31]), yielding

$$I_m^{D=4-2\epsilon}[\mu^{2r}] = -\epsilon(1-\epsilon)\cdots(r-1-\epsilon) I_m^{D=4+2r-2\epsilon}\,. \tag{10}$$

With the definition of the integrals (9) we may reformulate the conjecture (7) as

$$A_{n;1}^{\rm gluon}(1^+,2^+,\ldots,n^+) = 2\frac{A_{n;1}^{N=4}(1^+,2^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)[\mu^4]}{\langle i\,j\rangle^4}\,, \tag{11}$$

where the symbol '$[\mu^4]$' indicates that we insert an extra factor of μ^4 into every loop integrand before performing the integrals.

5. Evidence for the conjecture.

We shall present evidence for the conjecture (7), but first let us address a seeming puzzle with it. The all-plus helicity amplitude is invariant under a cyclic relabeling of the legs, whereas the cyclic invariance of the $N=4$ supersymmetric amplitude is not obvious, because the two negative helicities break the manifest invariance. However, the cyclic symmetry of the $N=4$ MHV amplitude, up to the overall prefactor of $\langle i\,j\rangle^4$, follows from a supersymmetry identity. To prove this, use standard supersymmetry identities [28] to relate the n-gluon amplitude to the two scalar, $(n-2)$ gluon amplitude. After interchanging the two scalars, which does not affect the amplitude, use the same supersymmetry identities to obtain an amplitude with the negative helicity gluon in a different position. This argument works for the $N=4$ multiplet because the two gluon helicity states are related by supersymmetry (without using a CPT transformation).

We have verified the conjecture for the four-, five- and six-point amplitudes by explicitly calculating both sides of eq. (7) to all orders in ϵ. To calculate the all-plus helicity amplitudes we use the unitarity-based method recently reviewed in ref. [1]. In this method the amplitudes are constructed from cut loop momentum integrals, depicted in fig. 2,

$$
A_{n;1}(1,2,\ldots,n)\Big|_{\rm cut} = \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}}\; \frac{i}{\ell_1^2}\, A^{\rm tree}_{m_2-m_1+3}(-\ell_1, m_1,\ldots,m_2,\ell_2)\, \frac{i}{\ell_2^2}\, A^{\rm tree}_{n-m_2+m_1+1}(-\ell_2, m_2+1,\ldots,m_1-1,\ell_1)\Big|_{\rm cut}\,, \tag{12}
$$

where ℓ is the loop momentum and ℓ_1 and ℓ_2 are the momenta crossing the cut. This equation is valid only for the cut channel. One then reconstructs the complete amplitudes by finding a function which has the correct cuts in *all* channels.

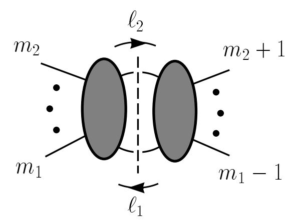

Figure 2. The cut amplitude corresponding to eq. (12).

From eq. (4), for the all-plus helicity amplitude we only need calculate the case of a complex scalar circulating in loop. Thus in eq. (12) we require the tree amplitudes with all-plus helicity gluons and two complex scalars, and the integration is over the momenta of the complex scalars. When working to all orders in ϵ, we must use tree amplitudes that are valid for D-dimensional cut momenta. In terms of the break-up of the loop momentum discussed in the previous section, the proper on-shell conditions on the cut legs are $\ell_1^2 - \mu^2 = 0$ and $\ell_2^2 - \mu^2 = 0$, where ℓ_1 and ℓ_2 are the four-dimensional components, and μ is the (-2ϵ)-dimensional component, of the loop momentum. For practical purposes we may think of μ^2 as a mass that gets integrated over.

Using recursive techniques [32,3] we find

$$
\begin{aligned}
A_4^{\rm tree}(-\ell_1^s, 1^+, 2^+, \ell_2^s) &= i\frac{\mu^2\,[1\,2]}{\langle 1\,2\rangle\,[(\ell_1-k_1)^2-\mu^2]}\,,\\
A_5^{\rm tree}(-\ell_1^s, 1^+, 2^+, 3^+, \ell_2^s) &= i\frac{\mu^2\sum_{j=1}^2 [3\,j]\,\langle j^-|\,\not{\ell}_1\,|1^-\rangle}{[(\ell_1-k_1)^2-\mu^2]\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,[(\ell_2+k_3)^2-\mu^2]}\,,\\
A_6^{\rm tree}(-\ell_1^s, 1^+, 2^+, 3^+, 4^+, \ell_2^s) &= i\frac{1}{[(\ell_1-k_1)^2-\mu^2]\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[(\ell_2+k_4)^2-\mu^2]}\\
&\quad\times\left[\mu^2\sum_{j=1}^{3}[4\,j]\,\langle j^-|\,\not{\ell}_1\,|1^-\rangle - \frac{\mu^4\,[1\,2]\,\langle 2\,3\rangle\,[3\,4]}{(\ell_1-k_1-k_2)^2-\mu^2}\right],
\end{aligned} \tag{13}
$$

where the superscript s on ℓ_1 and ℓ_2 indicates that these are the scalar lines.

Integrating these tree amplitudes according to eq. (12), and reconstructing the complete analytic form of the loop amplitudes we have

$$
A_{4;1}^{\rm scalar}(1^+,2^+,3^+,4^+) = \frac{2i}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\frac{\epsilon(1-\epsilon)}{(4\pi)^{2-\epsilon}}\times s_{12}s_{23}\,I_4^{D=8-2\epsilon}\,, \tag{14}
$$

$$A_{5;1}^{\rm scalar}(1^+,2^+,3^+,4^+,5^+) = \frac{i}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,1\rangle}\frac{\epsilon(1-\epsilon)}{(4\pi)^{2-\epsilon}}$$
$$\times\Big[s_{23}s_{34}I_4^{(1),D=8-2\epsilon}+s_{34}s_{45}I_4^{(2),D=8-2\epsilon}+s_{45}s_{51}I_4^{(3),D=8-2\epsilon} \tag{15}$$
$$+s_{51}s_{12}I_4^{(4),D=8-2\epsilon}+s_{12}s_{23}I_4^{(5),D=8-2\epsilon}+(4-2\epsilon)\varepsilon(1,2,3,4)I_5^{D=10-2\epsilon}\Big],$$

$$A_{6;1}^{\rm scalar}(1^+,2^+,3^+,4^+,5^+,6^+) = \frac{i}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,1\rangle}\frac{\epsilon(1-\epsilon)}{(4\pi)^{2-\epsilon}}$$
$$\times\frac{1}{2}\Big[-\sum_{1\le i_1<i_2\le 6}{\rm tr}[\not k_{i_1}\not P_{i_1+1,i_2-1}\not k_{i_2}\not P_{i_2+1,i_1-1}]I^{D=8-2\epsilon}_{4:i_1;i_2}+(4-2\epsilon)\,{\rm tr}[123456]\,I_6^{D=10-2\epsilon}$$
$$+(4-2\epsilon)\sum_{i=1}^{6}\varepsilon(i+1,i+2,i+3,i+4)I_5^{(i),D=10-2\epsilon}\Big], \tag{16}$$

where $s_{ij}=(k_i+k_j)^2$, the totally antisymmetric symbol is defined by

$$\varepsilon(i,j,m,n)\equiv 4i\varepsilon_{\mu\nu\rho\sigma}k_i^\mu k_j^\nu k_m^\rho k_n^\sigma = {\rm tr}[\gamma_5\not k_i\not k_j\not k_m\not k_n]\,, \tag{17}$$

and $I_n^{(i)}$ denotes the scalar integral obtained by removing the loop propagator between legs $i-1$ and i from the $(n+1)$-point scalar integral. It is easy to verify that each of the amplitudes (14)–(16) properly reduces to the expression in eq. (3), using values of the integrals in the $\epsilon\to 0$ limit,

$$\epsilon(1-\epsilon)I_4^{D=8-2\epsilon}\to\frac{1}{6}\,,\qquad \epsilon(1-\epsilon)I_5^{D=10-2\epsilon}\to\frac{1}{24}\,,\qquad \epsilon(1-\epsilon)I_6^{D=10-2\epsilon}\to 0\,. \tag{18}$$

We comment that these amplitudes may be converted to ones with a massive loop simply by performing the shift $\mu^2\to\mu^2+m^2$ [31]. Just as in the massless case, a supersymmetry identity implies that the all-plus helicity amplitude depends only on the number of statistics-weighted states circulating in the loop; thus the above conversion (4) also applies for a massive fermion in the loop. One may convert these amplitudes from QCD to QED simply by summing over permutations of the external legs.

We now compare the all-plus helicity amplitudes (14)–(16) with the $N=4$ MHV amplitudes. The $N=4$ four-point amplitude was first calculated by Green, Schwarz and Brink [33] using the low energy limit of superstring theory. We obtained the five-point amplitude by slightly modifying the string-based [25,34] calculation of ref. [35] to keep the terms higher order in ϵ. For the six-point amplitudes we used a string-motivated diagrammatic approach to ensure manifest supersymmetric cancellations [35,29,36], after which the diagrams were evaluated numerically. (Hexagon integrals I_6 with external momenta restricted to four-dimensions are linear combinations of the six pentagon integrals $I_5^{(i)}$ [37,30]; therefore we had to reduce the hexagon to pentagons before making any comparison.) In all these cases we find that the dimension-shifting formula (7) is satisfied, thus proving the conjecture up through $n=6$.

What evidence can we find for an arbitrary number of external legs? We noted above that if we start with the $N=4$ supersymmetric amplitudes (5) valid through $\mathcal{O}(\epsilon^0)$, perform the dimension shift, and then take the $\epsilon \to 0$ limit, we reproduce all 'even' terms in the all-plus helicity amplitudes (3). This check is not definitive since terms of $\mathcal{O}(\epsilon)$ can become terms of $\mathcal{O}(\epsilon^0)$ under the dimension shift. For example, present in the five-point $N=4$ amplitude is the $\mathcal{O}(\epsilon)$ 'odd' term

$$-2\epsilon\, \varepsilon(1,2,3,4) I_5^{D=6-2\epsilon}\,. \tag{19}$$

After shifting $D \to D+4$, and multiplying by the prefactor $-\epsilon(1-\epsilon)$ this becomes

$$-\epsilon(1-\epsilon) \times (4-2\epsilon)\varepsilon(1,2,3,4) I_5^{D=10-2\epsilon}\,, \tag{20}$$

which contributes at order ϵ^0 because the integral is ultraviolet divergent. From the explicit forms of the all-orders-in-ϵ five- and six-point amplitudes, it is clear that the 'odd' terms arise from integral functions not contributing through order ϵ^0 in $A^{N=4}$.

As a stronger check, we may appeal to the universal behavior [38] of the amplitudes as kinematic invariants vanish. Of particular utility is the behavior of amplitudes as two momenta become collinear [7,19,2,1]. In these limits an n-point amplitude must reduce to sums of $(n-1)$-point amplitudes multiplied by 'splitting functions' which are singular in the collinear limit. The constraints of factorization are sufficiently powerful that in many cases one may obtain the correct amplitude simply by finding a function that satisfies the constraints [2]. Since the conjecture (7) holds for up to six-point amplitudes, consistency of the collinear limits suggests that it will continue to hold for higher-point amplitudes. This argument is not a proof either, given the possible appearance of functions which are non-singular in all factorization limits; these limits do not constrain such functions. An example of such a function for the n-point amplitude (if n is even) is

$$\frac{\mathrm{tr}[123\cdots n]}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \cdots \langle n\,1\rangle} I_n^{D=n+4-2\epsilon}\,. \tag{21}$$

This function does appear in the six-point $(n=6)$ amplitude (16), but only at $\mathcal{O}(\epsilon)$. While collinear factorization does not prove the conjecture for $n>6$, it severely constrains terms which violate it.

Another way to check the conjecture (7) is to inspect the cuts (to all orders in ϵ) on both sides of the equation. This is convenient since the cut of a one-loop amplitude is a product of two tree amplitudes integrated over phase space. Tree amplitudes are in turn easier to manipulate than loop amplitudes. The cut relationship implied by the conjecture (7) is shown diagrammatically in fig. 3 for the case where the two negative helicities lie on the same side of the cut. (It is sufficient to check this case because of the supersymmetry identity proving the cyclic symmetry of the $N=4$

MHV amplitudes.) This may be expressed as

$$\int \mathrm{dLIPS}\ A^{\rm tree}(-\ell_1^s, m_1^+, \ldots, m_2^+, \ell_2^s)\ A^{\rm tree}(-\ell_2^s, (m_2+1)^+, \ldots, (m_1-1)^+, \ell_1^s) =$$
$$2\int \mathrm{dLIPS} \sum_f \frac{\mu^4}{\langle i\,j\rangle^4} A^{\rm tree}(-\ell_1^f, m_1^+, \ldots, i^-, \ldots, j^-, \ldots, m_2^+, \ell_2^f)\ A^{\rm tree}(-\ell_2^f, (m_2+1)^+, \ldots, (m_1-1)^+, \ell_1^f)\,, \tag{22}$$

where the summation is over the states f, of the $N=4$ multiplet and the integration, dLIPS is over Lorentz invariant phase space with ℓ_1, ℓ_2 on-shell.

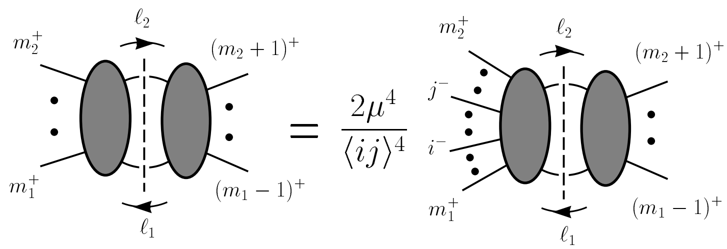

Figure 3. Equality needed for conjecture to be true. In the cut on the left, only scalars cross the cut; in the cut on the right, the entire $N=4$ supersymmetry multiplet appears.

A proof of this identity would lead directly to a proof of the conjecture (7). We offer no proof to all orders in μ; but as a first step, let us consider this equation to leading order in μ^2. The leading order on both sides is μ^4. On the $N=4$ side of the equation we can use the amplitudes to zeroth order in μ^2. These cuts were evaluated (to obtain those terms in the amplitudes which do not vanish as $\epsilon \to 0$) in ref. [4] with the result,

$$\begin{aligned}
&2\sum_f \frac{\mu^4}{\langle i\,j\rangle^4} A^{\rm tree}(-\ell_1^f, m_1^+, \ldots, i^- \ldots j^-, \ldots, m_2^+, \ell_2^f)\ A^{\rm tree}(-\ell_2^f, (m_2+1)^+, \ldots, (m_1-1)^+, \ell_1^f) \\
&\quad = 2\frac{\mu^4}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle (n-1)\,n\rangle\langle n\,1\rangle} \frac{\langle (m_1-1)\,m_1\rangle \langle \ell_1\,\ell_2\rangle^2 \langle m_2\,(m_2+1)\rangle}{\langle (m_1-1)\,\ell_1\rangle\langle \ell_1\,m_1\rangle\langle m_2\,\ell_2\rangle\langle \ell_2\,(m_2+1)\rangle} \\
&\quad = -2\frac{\mu^4}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle (n-1)\,n\rangle\langle n\,1\rangle} \\
&\qquad\qquad \times \frac{\mathrm{tr}_+[\not\ell_1 \not k_{m_1} \not k_{m_1-1} \not P_{m_1,m_2} \not\ell_2 \not k_{m_2+1} \not k_{m_2} \not P_{m_2+1,m_1-1}]}{[(\ell_1-k_{m_1})^2-\mu^2][(\ell_1+k_{m_1-1})^2-\mu^2][(\ell_2+k_{m_2})^2-\mu^2][(\ell_2-k_{m_2+1})^2-\mu^2]}\,,
\end{aligned} \tag{23}$$

where we used $\ell_2 = \ell_1 - P_{m_1,m_2} = \ell_1 + P_{m_2+1,m_1-1}$ and $\ell_i^2 = 0$ on the cut. We can now compare this result with the leading order in μ^2 for the all-plus helicity case. Recursive techniques [32,3] lead to the general form of the tree amplitudes for n plus-helicity gluons and two scalars,

$$A_n^{\rm tree}(-\ell_1^s, 1^+, \ldots, n^+, \ell_2^s) = i\frac{\mu^2 \sum_{j=1}^{n-1} [n\,j]\,\langle j^- |\,\ell_1\,| 1^-\rangle}{[(\ell_1-k_1)^2-\mu^2]\,\langle 1\,2\rangle\cdots\langle (n-1)\,n\rangle\,[(\ell_2+k_n)^2-\mu^2]} + \mathcal{O}(\mu^4)\,. \tag{24}$$

Using this expression to construct the cuts one reproduces eq. (23), so that eq. (22) is satisfied to leading order in μ^2. (The overall factor of 2 arises because complex scalars are composed of two states.) The agreement, even before performing the phase-space integrals, suggests that, in general, on the cuts the conjecture holds for the integrands.

6. Gravity.

String theory implies that gravity amplitudes are closely related to gauge theory amplitudes. This observation has been used to obtain gravity amplitudes at both tree level [39] and at loop level [40] and suggests that one can find conjectures similar to eq. (7), but for gravity.

Using the explicit results for four-graviton amplitudes obtained via string-based calculations [40], extended to all-orders in ϵ, we find

$$\begin{aligned} A_4^{\rm gravity}(1^+,2^+,3^+,4^+) &= -2\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)(4\pi)^4 \left[\left. \frac{A_4^{N=8}(1^-,2^-,3^+,4^+)}{\langle 1\,2\rangle^8} \right|_{D\to D+8} \right] \\ &= \frac{2A_4^{N=8}(1^-,2^-,3^+,4^+)[\mu^8]}{\langle 1\,2\rangle^8}\,, \end{aligned} \tag{25}$$

where the amplitude on the left is the pure gravity all-plus helicity amplitudes and the one on the right the $N=8$ supergravity amplitude. As in the QCD case, the all-plus amplitude is independent of the massless particle types circulating in the loop, but depends only on the number of states in the loop.

Following the QCD case, we may conjecture that the relation in eq. (25) continues to hold for an arbitrary number of external legs. For gravity the one-loop amplitudes are not known beyond four external legs. One can, however, argue [15] that the above amplitudes will also correspond to those for self-dual gravity [18].

7. Speculations.

In this paper we have provided evidence that two infinite sequences of maximally helicity violating gauge theory amplitudes, which at first sight seem quite different, are in fact closely related to each other through a "dimension shift". Is this result just a curiosity, or an indication of a deeper relation between a non-supersymmetric theory (self-dual Yang-Mills) and a supersymmetric one ($N=4$ super Yang-Mills)? We cannot yet answer this question directly. It may prove profitable to pursue the connection mentioned in the introduction, between maximal helicity violation and self-dual Yang-Mills theory [9,13,15], since the latter is known to possess an infinite-dimensional symmetry algebra [11]. (See ref. [14] for a review.) In two-dimensional integrable models, which are related to self-dual Yang-Mills theory through dimensional reduction, the extended symmetry algebra is responsible for a lack of multi-particle poles in the scattering amplitude. Bardeen has emphasized that the absence of multi-particle poles in the maximally helicity violating tree-level currents is reminiscent of the Bethe ansatz [9].

Thus it might be worthwhile to examine the other four-dimensional gauge theory amplitudes that lack multi-particle poles. The list of such amplitudes is quite limited. In non-supersymmetric QCD, beyond one loop all amplitudes with six or more legs contain multi-particle poles, as can be verified by checking their factorization onto a product of two one-loop amplitudes. On the other hand, the nonvanishing maximally helicity violating amplitudes in supersymmetric theories (those amplitudes with all but two helicities identical) do not develop multi-particle poles, to all orders of perturbation theory. (The residues of the would-be poles vanish by supersymmetry identities [28].) The simplest of the one-loop MHV supersymmetric amplitudes are the $N = 4$ amplitudes, which is why we chose to investigate their relationship to the self-dual Yang-Mills amplitudes in this letter. ($N = 1$ partial amplitudes [5] are more complicated and certainly do not possess the cyclic invariance of the $N = 4$ amplitudes.)

Finally, we speculate whether to take seriously the appearance of dimensions shifted upwards by four units (eight units for gravity) in the relations we have found. If we take $\epsilon \to 0$ so that the left-hand side of eq. (7) is in $D = 4$, we find that the self-dual gauge amplitudes are related to the one-loop ultraviolet divergences of an $N = 4$ supersymmetric gauge theory in $D = 8$. ($N = 4$ refers to the number of $D = 4$ supersymmetries.) Coincidentally, such theories have recently been considered in the context of certain $(7 + 1)$-brane configurations in string theory (also known as compactifications of F theory on $K3$) where they describe the low-energy world-volume theory [41,42]. The corresponding theory for the gravity relation (25) would be $N = 8$ supergravity in $D = 12$, which happens to be the “critical dimension” for F theory [41]. Perhaps it is also relevant that self-dual theories in four-dimensions (with signature $(2, 2)$) have been proposed for the world-volume dynamics of F theory [43]. At this stage, though, it is safest to say that the underlying reason for the relationships (7) and (25) remains to be clarified.

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References

[1] Z. Bern, L. Dixon and D.A. Kosower, to appear in *Annual Reviews of Nuclear and Particle Science* (1996), hep-ph/9602280.

[2] Z. Bern, G. Chalmers, L. Dixon and D.A. Kosower, Phys. Rev. Lett. 72:2134 (1994), hep-ph/9312333.

[3] G.D. Mahlon, Phys. Rev. D49 (1994) 2197, hep-ph/9311213; Phys. Rev. D49 (1994) 4438, hep-ph/9312276.

[4] Z. Bern, L. Dixon, D.C. Dunbar and D.A. Kosower, Nucl. Phys. B425:217 (1994), hep-ph/9403226.

[5] Z. Bern, L. Dixon, D.C. Dunbar and D.A. Kosower, Nucl. Phys. B435:59 (1995), hep-ph/9409265.

[6] V.P. Nair, Phys. Lett. B214:215 (1988).

[7] S.J. Parke and T.R. Taylor, Nucl. Phys. B269:410 (1986), Phys. Rev. Lett. 56:2459 (1986).

[8] M.J. Duff, C.J. Isham, Nucl. Phys. B162:271 (1980).

[9] W.A. Bardeen, preprint FERMILAB-CONF-95-379-T.

[10] K.G. Selivanov, preprint hep-ph/9604206.

[11] L. Dolan, Phys. Rep. 109:1 (1984), and references therein.

[12] H. Ooguri and C. Vafa, Nucl. Phys. B361:469 (1991); Nucl. Phys. B367:83 (1991); Nucl. Phys. B451:121 (1995), hep-th/9505183;
N. Berkovits and C. Vafa, Nucl. Phys. B433:123 (1995), hep-th/9407190.

[13] D. Cangemi, preprint hep-th/9605208.

[14] D. Cangemi, preprint hep-th/9610021.

[15] G. Chalmers and W. Siegel, preprint hep-th/9606061.

[16] C.N. Yang, Phys. Rev. Lett. 38:1377 (1977);
S. Donaldson, Proc. Lond. Math. Soc. 50:1 (1985);
V.P. Nair and J. Schiff, Phys. Lett. B246:423 (1990).

[17] A.N. Leznov, Theor. Math. Phys. 73:1233 (1988);
A.N. Leznov and M.A. Mukhtarov, J. Math. Phys. 28:2574 (1987);
A. Parkes, Phys. Lett. B286:265 (1992);
W. Siegel, Phys. Rev. D46:R3235 (1992).

[18] J.F. Plebanski, J. Math. Phys. 16:2395 (1975);
M.J. Duff, in *Proceedings of 1979 Supergravity Workshop*, ed. P. Van Nieuwenhuizen and D.Z. Freedman (North Holland, 1979);
W. Siegel, Phys. Rev. D47:2504 (1993), hep-th/9207043.

[19] M.L. Mangano and S.J. Parke, Phys. Rep. 200:301 (1991);
L. Dixon, hep-ph/9601359, in *Proceedings of Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 95)*, ed. D.E. Soper.

[20] J.E. Paton and H.M. Chan, Nucl. Phys. B10:516 (1969);
F.A. Berends and W.T. Giele, Nucl. Phys. B294:700 (1987);
M. Mangano, Nucl. Phys. B309:461 (1988).

[21] Z. Bern and D.A. Kosower, Nucl. Phys. B362:389 (1991).

[22] Z. Bern, L. Dixon and D.A. Kosower, Nucl. Phys. B437:259 (1995), hep-ph/9409393.

[23] F.A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans and T.T. Wu, Phys. Lett. 103B:124 (1981);

P. De Causmaeker, R. Gastmans, W. Troost and T.T. Wu, Nucl. Phys. B206:53 (1982);
R. Kleiss and W. J. Stirling, Nucl. Phys. B262:235 (1985);
J.F. Gunion and Z. Kunszt, Phys. Lett. 161B:333 (1985);
Z. Xu, D.-H. Zhang and L. Chang, Nucl. Phys. B291:392 (1987).

[24] G. 't Hooft and M. Veltman, Nucl. Phys. B44:189 (1972).

[25] Z. Bern and D.A. Kosower, Nucl. Phys. B379:451 (1992).

[26] W. Siegel, Phys. Lett. 84B:193 (1979);
D.M. Capper, D.R.T. Jones and P. van Nieuwenhuizen, Nucl. Phys. B167:479 (1980);
L.V. Avdeev and A.A. Vladimirov, Nucl. Phys. B219:262 (1983);
I. Jack, D.R.T. Jones and K.L. Roberts, Z. Phys. C63:151 (1994).

[27] Z. Kunszt, A. Signer and Z. Trócsányi, Nucl. Phys. B411:397 (1994), hep-ph/9305239;
S. Catani, M.H. Seymour and Z. Trócsányi, preprint hep-ph/9610553.

[28] M.T. Grisaru, H.N. Pendleton and P. van Nieuwenhuizen, Phys. Rev. D15:996 (1977);
M.T. Grisaru and H.N. Pendleton, Nucl. Phys. B124:81 (1977);
S.J. Parke and T. Taylor, Phys. Lett. 157B:81 (1985).

[29] Z. Bern, hep-ph/9304249, in *Proceedings of Theoretical Advanced Study Institute in High Energy Physics (TASI 92)*, eds. J. Harvey and J. Polchinski (World Scientific, 1993), hep-ph/9304249.

[30] Z. Bern, L. Dixon and D.A. Kosower, Nucl. Phys. B412:751 (1994), hep-ph/9306240.

[31] Z. Bern and A.G. Morgan, Nucl. Phys. B467:479 (1996), hep-ph/9511336.

[32] F.A. Berends and W.T. Giele, Nucl. Phys. B306:759 (1988);
D.A. Kosower, Nucl. Phys. B335:23 (1990);
G. Mahlon and T.-M. Yan, Phys. Rev. D47:1776 (1993), hep-ph/9210213;
G. Mahlon, Phys. Rev. D47:1812 (1993), hep-ph/9210214;
G. Mahlon, T.-M. Yan and C. Dunn, Phys. Rev. D48:1337 (1993), hep-ph/9210212;
C. Kim and V.P. Nair, preprint hep-th/9608156.

[33] M.B. Green, J.H. Schwarz and L. Brink, Nucl. Phys. B198:474 (1982).

[34] Z. Bern and D.A. Kosower, Phys. Rev. Lett. 66:1669 (1991);
Z. Bern, Phys. Lett. 296B:85 (1992);
Z. Bern and D.C. Dunbar, Nucl. Phys. B379:562 (1992).

[35] Z. Bern, L. Dixon and D.A. Kosower, Phys. Rev. Lett. 70:2677 (1993), hep-ph/9302280.

[36] Z. Bern and A.G. Morgan, Phys. Rev. D49:6155 (1994), hep-ph/9312218.

[37] D.B. Melrose, Il Nuovo Cimento 40A:181 (1965);
W. van Neerven and J.A.M. Vermaseren, Phys. Lett. 137B:241 (1984).

[38] Z. Bern and G. Chalmers, Nucl. Phys. B447:465 (1995), hep-ph/9503236.

[39] F.A. Berends, W.T. Giele and H. Kuijf, Phys. Lett. 211B:91 (1988).

[40] Z. Bern, D.C. Dunbar and T. Shimada, Phys. Lett. 312B:277 (1993), hep-th/9307001;
D.C. Dunbar and P.S. Norridge, Nucl. Phys. B433:181 (1995), hep-th/9408014; hep-th/9512084.

[41] C. Vafa, preprint hep-th/9602022.

[42] M.R. Douglas and M. Li, preprint hep-th/9604041.

[43] D. Kutasov and E. Martinec, preprint hep-th/9602049;
D. Kutasov, E. Martinec and M. O'Loughlin, preprint hep-th/9603116.